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Approximation of singularly perturbed linear hyperbolic systems

Ying TANG, Christophe PRIEUR and Antoine GIRARD

Abstract— **This paper is concerned with systems modelled by linear singularly perturbed partial differential equations. More precisely a class of linear systems of conservation laws with a small perturbation parameter is investigated. By setting the perturbation parameter to zero, the full system leads to two subsystems, the reduced system standing for the slow dynamics and the boundary-layer system representing the fast dynamics. The exponential stability for both subsystems are obtained by the stability of the overall system of conservation laws. However, the stability of the two subsystems does not imply the stability of the full system. The approximation of the solution for the overall system by the solution for the reduced system is validated via Lyapunov techniques.**

I. Introduction

The singular perturbation techniques occurred at the beginning of the 20th century. The interests in this method arose from many physical problems exhibiting both fast and slow dynamics. For example, DC-motor model and Voltage regulator in [15]. From late 1980s, the singularly perturbed partial differential equations (PDEs) have been considered in research works. This kind of systems is interesting for analysis since it describes numerous phenomenon in various fields, for instance, fluid dynamics, chemical-reactor, aerodynamics etc. (see [13]).

The model of fluid transport through a constant cross section tube from [2] provides the first motivation for this work. This model contains two time scales for propagation speed, which can be described by a singularly perturbed system of conservation laws. The decomposition of a singularly perturbed system into lower order subsystems, namely the reduced system and the boundary-layer system, provides a powerful tool for stability analysis in [11], [3], [10] and [4].

In this paper, we consider a class of linear systems of conservation laws with a small perturbation parameter ϵ. By setting $\epsilon = 0$, two subsystems, the reduced and boundary-layer systems, are computed. The exponential stability for singularly perturbed system of conservation laws implies both subsystems are exponentially stable. On the other hand the converse does not hold. The stability analysis for hyperbolic systems of conservation laws has been considered by many researchers. For instance, a stability criterion for linear hyperbolic systems by characteristics method has been given in [16] and the stability condition considered in [12] relies on the frequency domain. In [6], stability condition for the quasilinear systems of conservation laws is introduced by Lyapunov method. In this paper, it will be studied a Tikhonov like theorem for linear hyperbolic systems, the solutions of the full system can be approximated by that of the reduced system. The Tikhonov Theorem has been studied in many works for standard singular perturbation systems (ODEs) (e.g. [1], [20]). [14] shows the Tikhonov theorem based on the exponential stability criterion of both the reduced and boundary-layer systems. Moreover, to the best of our knowledge, this is the first paper dealing with singularly perturbed hyperbolic systems.

The paper is organized as follows. Section II introduces the linear singularly perturbed system of conservation laws. In Section III, the stability of both subsystems is presented. Precisely, the exponential stability of the full system of conservation laws implies that each of the two subsystems is stable. However, a counter-example is given to show that the stability of the two subsystems does not guarantee the stability of the full system. Section IV shows the approximation of solutions for the full system by that of the reduced system. In Section V, a numerical example is studied to illustrate the results. Finally, concluding remarks end the paper. Due to space limitation, some proofs are omitted.

Notation. Given a matrix A, A^{-1} and A^T represent the inverse and the transpose matrix of A respectively. For a symmetric matrix B, $\lambda_{min}(B)$ is the minimum eigenvalue of the matrix B. The symbol $\star$ in partitioned symmetric matrices stands for the symmetric block. For a positive integer n, I_n is the identity matrix in $\mathbb{R}^{n\times n}$. $|\ |$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\|\ \|$ is associated to the matrix norm. $\|\ \|_{L^2}$ denotes the associate norm in $L^2(0,1)$ space, defined by $\|\xi\|_{L^2} = \left(\int_0^1 |\xi|^2 dx\right)^{\frac{1}{2}}$ for all functions $\xi \in L^2(0,1)$. Similarly, the associate norm in $H^2(0,1)$ space is denoted by $\|\ \|_{H^2}$, defined for all functions $\psi \in H^2(0,1)$, by $\|\psi\|_{H^2} = \left(\int_0^1 |\psi|^2 + |\psi_x|^2 + |\psi_{xx}|^2 dx\right)^{\frac{1}{2}}$.

II. Linear singularly perturbed systems of conservation laws

Firstly, let us consider the following linear singularly perturbed system of conservation laws:

$$y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \tag{1a}$$

$$\epsilon z_t(x,t) + \Lambda_2 z_x(x,t) = 0, \tag{1b}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1]\times[0,+\infty) \to \mathbb{R}^n$, $z : [0,1]\times[0,+\infty) \to \mathbb{R}^m$, Λ_1 is a positive diagonal matrix in $\mathbb{R}^{n\times n}$, Λ_2 is a positive diagonal matrix in $\mathbb{R}^{m\times m}$, the

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perturbation parameter ϵ is a small positive value.
Moreover, we consider the following boundary conditions:

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = G \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \tag{2}$$

where $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ is a constant matrix in $\mathbb{R}^{(n+m)\times(n+m)}$ with the matrices G_{11} in $\mathbb{R}^{n\times n}$, G_{12} in $\mathbb{R}^{n\times m}$, G_{21} in $\mathbb{R}^{m\times n}$ and G_{22} in $\mathbb{R}^{m\times m}$.
Given two functions $y^0 : [0,1] \to \mathbb{R}^n$ and $z^0 : [0,1] \to \mathbb{R}^m$, the initial conditions are:

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y^0(x) \\ z^0(x) \end{pmatrix}, \quad x \in [0,1]. \tag{3}$$

Remark 2.1: Let us recall the existence of the solutions to the Cauchy problem (1)-(3). According to Section 2.1 in [5], for all $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in L^2(0,1)$, there exists a unique solution $\begin{pmatrix} y \\ z \end{pmatrix} \in C^0([0,+\infty), L^2(0,1))$ for the Cauchy problem (1)-(3). By Proposition 2.1 in [6], for every $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H^2(0,1)$ satisfying the following compatibility conditions:

$$\begin{pmatrix} y^0(0) \\ z^0(0) \end{pmatrix} = G \begin{pmatrix} y^0(1) \\ z^0(1) \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} \Lambda_1 y_x^0(0) \\ \Lambda_2 z_x^0(0) \end{pmatrix} = G \begin{pmatrix} \Lambda_1 y_x^0(1) \\ \Lambda_2 z_x^0(1) \end{pmatrix}, \tag{5}$$

the Cauchy problem (1)-(3) has a unique maximal classical solution $\begin{pmatrix} y \\ z \end{pmatrix} \in C^0([0,+\infty), H^2(0,1))$. ∘

Adapting the approach in [17] and [14] to the infinite dimensional case, let us compute the two subsystems, reduced and boundary-layer systems, for system (1)-(2). Setting $\epsilon = 0$ in system (1) yields

$$y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \tag{6a}$$

$$z_x(x,t) = 0. \tag{6b}$$

Substituting (6b) into the boundary conditions (2) yields

$$\begin{aligned} y(0,t) &= (G_{11} + G_{12}(I_m - G_{22})^{-1}G_{21})y(1,t), \\ z(.,t) &= (I_m - G_{22})^{-1}G_{21}y(1,t). \end{aligned} \tag{7}$$

Then, the reduced system is defined as

$$\bar{y}_t(x,t) + \Lambda_1 \bar{y}_x(x,t) = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \tag{8}$$

with the boundary condition

$$\bar{y}(0,t) = G_r \bar{y}(1,t), \quad t \in [0,+\infty), \tag{9}$$

where $G_r = G_{11} + G_{12}(I_m - G_{22})^{-1}G_{21}$, whereas the initial condition is given as

$$\bar{y}(x,0) = y^0(x), \quad x \in [0,1]. \tag{10}$$

To define the boundary-layer system, let first perform the change of variable

$$\bar{z} = z - (I_m - G_{22})^{-1}G_{21}y(1,t). \tag{11}$$

This shifts the equilibrium of z to the origin. Let us use a new time variable $\tau = \frac{t}{\epsilon}$. In the τ time scale, $y(1,t)$ in (11) is considered as a fixed parameter with respect to time. Then, the boundary-layer system is defined as

$$\bar{z}_\tau(x,\tau) + \Lambda_2 \bar{z}_x(x,\tau) = 0, \quad x \in [0,1], \quad \tau \in [0,+\infty), \tag{12}$$

with the boundary condition:

$$\bar{z}(0,\tau) = G_{22}\bar{z}(1,\tau), \quad \tau \in [0,+\infty), \tag{13}$$

whereas the initial condition is given as

$$\bar{z}(x,0) = z^0(x) - (I_m - G_{22})^{-1}G_{21}y^0(1), \quad x \in [0,1]. \tag{14}$$

III. Stability of reduced and boundary-layer systems

In this section, we will show how the stability of the singularly perturbed system of conservation laws (1)-(2) is related to the stability of the two subsystems, the reduced system (8)-(9) and the boundary-layer system (12)-(13).

Let us recall the following definition introduced in [6]:

Definition 3.1: *For all matrices $G \in \mathbb{R}^{(n+m)\times(n+m)}$,*

$$\rho_1(G) = \inf\{\|\Delta G \Delta^{-1}\|, \Delta \in D_{(n+m),+}\}, \tag{15}$$

where $D_{(n+m),+}$ denotes the set of diagonal positive matrix in $\mathbb{R}^{(n+m)\times(n+m)}$.

The following definition is adopted for the exponential stability of the linear singularly perturbed system of conservation laws (1)-(2) in L^2-norm.

Definition 3.2: *The linear system of conservation laws (1)-(2) is exponentially stable to the origin in L^2-norm if there exist $\gamma_1 > 0$ and $C_1 > 0$, such that for every $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in L^2(0,1)$, the solution to the system (1)-(2) satisfies*

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{L^2} \leqslant C_1 e^{-\gamma_1 t} \left\| \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \right\|_{L^2}, \quad t \in [0,+\infty).$$

Similarly the exponential stability of the linear system of conservation laws (1)-(2) in H^2-norm is defined as follows

Definition 3.3: *The linear system of conservation laws (1)-(2) is exponentially stable to the origin in H^2-norm if there exist $\gamma_2 > 0$ and $C_2 > 0$, such that for every $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H^2(0,1)$ satisfying the compatibility conditions (4)-(5), the solution to the system (1)-(2) satisfies*

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{H^2} \leqslant C_2 e^{-\gamma_2 t} \left\| \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \right\|_{H^2}, \quad t \in [0,+\infty).$$

In a similar way, we can define the exponential stability in L^2-norm and H^2-norm for the reduced and boundary-layer systems.

Let recall the following result for quasilinear hyperbolic system:

Theorem 1 ([6] and [7]): *If $\rho_1(G) < 1$ (resp. $\rho_1(G_r) < 1$, $\rho_1(G_{22}) < 1$), then the linear system (1)-(2) (resp. the reduced system (8)-(9), the boundary-layer system (12)-(13)) is exponentially stable to the origin in L^2-norm and H^2-norm.*

With the above theorem, we are ready to give a proposition which is about the stability of the reduced and the boundary-layer systems.

Proposition 3.4: *If $\rho_1(G) < 1$, then the reduced system (8)-(9) and the boundary-layer system (12)-(13) are exponentially stable to the origin in L^2-norm and H^2-norm.*

The stability criterion $\rho_1(G) < 1$ is thus a sufficient condition for stability of the reduced system (8)-(9) and the boundary-layer system (12)-(13). On the contrary, the stability of the two subsystems does not guarantee the stability of the overall system (1)-(2). To see this, let us consider the following counter-example.

Counter-example:
Let $\Lambda_1 = \Lambda_2 = 1$ in (1) with $n = m = 1$. The boundary condition of the singularly perturbed system in (2) is chosen as $G_{11} = 2.5$, $G_{12} = -1$, $G_{21} = 1$, $G_{22} = 0.5$. The boundary condition of the reduced system in (9) is computed as $G_r = 0.5$. Considering $\Delta = 1$ and (15), it holds $\rho_1(G_r) < 1$. By Theorem 1 the reduced system (8)-(9) is exponentially stable in L^2-norm and H^2-norm. The boundary condition of the boundary-layer system in (13) is $G_{22} = 0.5$. Considering the same $\Delta = 1$ and (15), it holds $\rho_1(G_{22}) < 1$. The boundary-layer system (12)-(13) is exponentially stable in L^2-norm and H^2-norm according to Theorem 1. Now let us check the stability condition $\rho_1(G) < 1$, which is equivalent to find a diagonal positive matrix Δ such that the following condition is satisfied

$$G^T \Delta^2 G < \Delta^2. \tag{16}$$

There is no loss of generality to consider $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. Straightforward computation shows that there is no such matrix Δ which satisfies the condition (16), thus $\rho_1(G) \geqslant 1$. Note that, Proposition 3.7 in [6] implies that $\rho_1(G) < 1$ is a necessary and sufficient condition for stability of linear hyperbolic systems with dimension 1 to 5. As this example is a linear singularly perturbed system of two conservation laws, it is not exponentially stable neither in L^2-norm nor in H^2-norm, although the reduced and boundary-layer systems are both exponentially stable. ◦

IV. Approximation theorem for linear singularly perturbed system of conservation laws

In this section, we present how solutions to the linear singularly perturbed system of conservation laws (1)-(2) can be approximated by solutions to the reduced system (8)-(9). It is based on the stability condition we considered in the previous section.

Theorem 2: *Consider the linear system of conservation laws (1)-(2). Assume that the boundary conditions G satisfy $\rho_1(G) < 1$, then, for all initial conditions $y \in H^2(0,1)$ satisfying the compatibility conditions $y^0(0) = G_r y^0(1)$, $\Lambda_1 y_x^0(0) = G_r \Lambda_1 y_x^0(1)$, and $z \in L^2(0,1)$, there exist positive values ϵ^*, C, C' and ω such that for all $0 < \epsilon < \epsilon^*$ and for all $t \geqslant 0$,*

$$\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2 \leqslant C\epsilon e^{-\omega t}, \tag{17}$$

$$\int_0^\infty \|z(.,t) - (I_m - G_{22})^{-1} G_{21} \bar{y}(1,t)\|_{L^2}^2 dt \leqslant C'\epsilon. \tag{18}$$

Sketch of proof: First, let us perform the change of variables: $\eta(x,t) = y(x,t) - \bar{y}(x,t)$, $\delta(x,t) = z(x,t) - (I_m - G_{22})^{-1} G_{21} \bar{y}(1,t)$. In the new variables (η, δ), the system is written as

$$\eta_t + \Lambda_1 \eta_x = 0, \tag{19a}$$

$$\epsilon \delta_t + \Lambda_2 \delta_x = \epsilon (I_m - G_{22})^{-1} G_{21} \Lambda_1 \bar{y}_x(1,t), \tag{19b}$$

with the boundary conditions

$$\begin{pmatrix} \eta(0,t) \\ \delta(0,t) \end{pmatrix} = G \begin{pmatrix} \eta(1,t) \\ \delta(1,t) \end{pmatrix}. \tag{20}$$

We consider the following Lyapunov function candidate for system (19)-(20)

$$V_\epsilon(\eta, \delta) = \int_0^1 e^{-\mu x} (\eta^T Q \eta + \epsilon \delta^T P \delta) dx, \tag{21}$$

with $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n \times n}$ and P a diagonal positive matrix in $\mathbb{R}^{m \times m}$. After computing the time derivative of $V_\epsilon(\eta, \delta)$ along (19)-(20) and integrating by parts, we obtain

$$\dot{V}_\epsilon(\eta, \delta) \leqslant BC - \alpha V_\epsilon(\eta, \delta) + \epsilon^2 \beta |\bar{y}_x(1,t)|^2, \tag{22}$$

where BC is the boundary term and α, β are positive constants. Since $\rho_1(G) < 1$, there exists a positive diagonal matrix Δ such that $\|\Delta G \Delta^{-1}\| < 1$, thus the boundary term BC is always non positive. Due to Proposition 3.4, $\rho_1(G) < 1$ implies the reduced system (8)-(9) is exponentially stable in H^2-norm. It is deduced from (22)

$$V_\epsilon(\eta, \delta) \leqslant e^{-\alpha t} V_\epsilon(\eta^0, \delta^0) + \epsilon^2 \gamma e^{-\alpha t} \|\bar{y}^0\|_{H^2}^2, \tag{23}$$

where γ is a positive constant value. Note that there exist positive values a, b, a' and b' such that

$$a\|\eta\|_{L^2}^2 + \epsilon b \|\delta\|_{L^2}^2 \leqslant V_\epsilon(\eta, \delta) \leqslant a'\|\eta\|_{L^2}^2 + \epsilon b' \|\delta\|_{L^2}^2. \tag{24}$$

Choosing the initial condition $y^0 = \bar{y}^0$ (i.e. $\eta^0 = 0$) yields

$$\|\eta(.,t)\|_{L^2}^2 \leqslant \epsilon C_1 e^{-\alpha t} \|\delta^0\|_{L^2}^2 + \epsilon^2 C_2 e^{-\alpha t} \|\bar{y}^0\|_{H^2}^2, \tag{25}$$

C_1 and C_2 are given positive values. Thus (17) is proved. Next, we rewrite (22) as follows

$$\dot{V}_\epsilon(\eta, \delta) \quad \leqslant \quad -\rho \|\delta(.,t)\|_{L^2}^2 + \epsilon^2 \iota \|\bar{y}^0\|_{H^2}^2, \tag{26}$$

for suitable positive values ρ and ι. Performing the time integration of both sides of (26), using $\lim_{t \to +\infty} V_\epsilon(\eta, \delta) = 0$, according to (24) and choosing initial condition $y^0 = \bar{y}^0$, we get

$$\int_0^\infty \|\delta(.,t)\|_{L^2}^2 dt \leqslant \epsilon C_3 \|\delta^0\|_{L^2}^2 + \epsilon^2 C_4 \|\bar{y}^0\|_{H^2}^2, \tag{27}$$

where C_3 and C_4 are given positive constants. Thus (18) holds. ■

V. Numerical example

In this section, we use a numerical example to illustrate the results that we get in the previous sections.

Let us consider a singularly perturbed system of two conservations laws (1) with $\Lambda_1 = \Lambda_2 = 1$, and the boundary conditions $G = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & -0.2 \end{pmatrix}$ in (2).

Use a two-step variant of the Lax-Wendroff method which is presented in [19] and the solver on Matlab in [18] to discretize the system. More precisely, we divide the space domain $[0, 1]$ into 100 intervals of identical length, and 10 as final time. We choose a time-step that satisfies the CFL condition for the stability and select the following initial functions:

$$y(x, 0) = 1 - \cos(6\pi x),$$
$$z(x, 0) = \sin(5\pi x),$$

for all $x \in [0, 1]$. Choosing $\epsilon = 0.003$, Figure 1 shows that the solution of the reduced system $\bar{y}$ converges to the origin as time increases which as expected from Proposition 3.4. Figures 2a and 2b give the time evolutions of η and δ which are the error between y of the full system and $\bar{y}$ of the reduced system, and the error between z and its equilibrium respectively. They decrease to 0 as time increases.

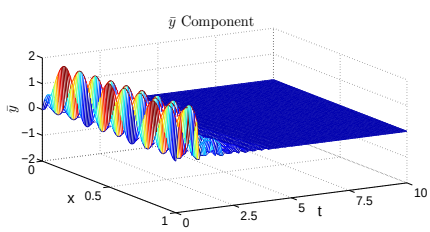
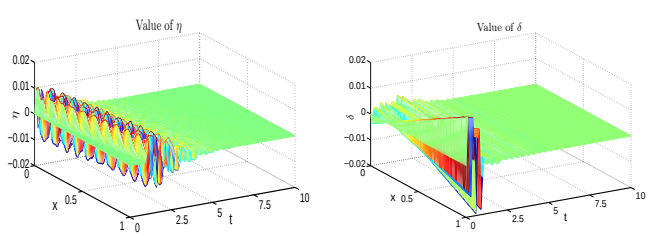


Fig. 1: Time evolution of the solution $\bar{y}$ of the reduced system



(a) Time evolution of η (b) Time evolution of δ

Fig. 2: Time evolutions of η and δ

TABLE I: Estimations of $\|\eta\|^2_{L^2}$ and $\int_0^{10} \|\delta\|^2_{L^2} dt$ for different ϵ

ϵ	0.001	0.002	0.003
$\|\|\eta(., t = 1)\|\|^2_{L^2}$	4.7×10^{-6}	1.8×10^{-5}	4.2×10^{-5}
$\int_0^{10} \|\|\delta\|\|^2_{L^2} dt$	6.9×10^{-6}	2.8×10^{-5}	6.2×10^{-5}

Table I gives estimations of square of L^2-norm of η and of the time integral of square of L^2-norm of δ with the different values of ϵ. It indicates that these two values are near to zero and decrease as ϵ decreases, as expected from Theorem 2.

VI. Conclusion

In this paper, a linear singularly perturbed system of conservation laws has been studied. The stability condition for the whole singularly perturbed system $\rho_1(G) < 1$ implies the two subsystems are exponentially stable. However, a counter-example has been given to show that the stability of the two subsystems does not guarantee the stability of the full system. A Tikhonov like theorem has been given under the stability condition $\rho_1(G) < 1$. The solution of the linear singularly perturbed system of conservation laws can be approximated by the solution of the reduced system.

This work leaves many open questions. It is natural to extend this work to systems with source terms. Another interesting point is to consider some physical applications, like open channels as considered in [9] and gas flow through pipelines in [8] or [2].

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