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TWO FLOOR BUILDING NEEDING EIGHT COLORS

STÉPHANE BESSY, DANIEL GONÇALVES, AND JEAN-SÉBASTIEN SERENI

ABSTRACT. Motivated by frequency assignment in office blocks, we study the chromatic number of the adjacency graph of 3-dimensional parallelepiped arrangements. In the case each parallelepiped is within one floor, a direct application of the Four-Colour Theorem yields that the adjacency graph has chromatic number at most 8. We provide an example of such an arrangement needing exactly 8 colours. We also discuss bounds on the chromatic number of the adjacency graph of general arrangements of 3-dimensional parallelepipeds according to geometrical measures of the parallelepipeds (side length, total surface or volume).

1. INTRODUCTION

The Graph Colouring Problem for Office Blocks was raised by BAE Systems at the 53rd European Study Group with Industry in 2005 [1]. Consider an office complex with space rented by several independent organisations. It is likely that each organisation uses its own wireless network (WLAN) and ask for a safe utilisation of it. A practical way to deal with this issue is to use a so-called “stealthy wallpaper” in the walls and ceilings shared between different organisations, which would attenuate the relevant frequencies. Yet, the degree of screening produced will not be sufficient if two distinct organisations have adjacent offices, that is, two offices in face-to-face contact on opposite sides of just one wall or floor-ceiling. In this case, the WLANs of the two organisations need to be using two different channels (the reader is referred to the report by Allwright *et al.* [1] for the precise technical motivations).

This problem can be modeled as a graph coloring problem by building a *conflict graph* corresponding to the office complex: to each organisation corresponds a vertex, and two vertices are adjacent if the corresponding territories share a wall, floor, or ceiling area. The goal is to assign a color (frequency) to each vertex (organisation) such that adjacent vertices are assigned distinct colors. In addition, not every graph may occur as the conflict graph of an existing office complex. However, the structure of such conflict graphs is not clear and various fundamental questions related to the problem at hands were asked. Arguably, one of the most natural questions concerns the existence of bounds on the chromatic number of such conflict graphs. More specifically, which additional constraints one should add to the model to ensure “good” upper bounds on the chromatic number of conflict graphs? These additional constraints should be meaningful regarding the practical problem, reflecting as much as possible real-world situations. Indeed, as noted by Tietze [7], complete graphs of arbitrary size are conflict graphs, that is, for every integer n , there can be n organisations whose territories all are in face-to-face contact with each other. The reader is referred to the paper by Reed and Allwright [6] for a description of Tietze’s

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construction. Besicovitch [4] and Tietze [8] proved that this is still the case if the territories are asked to be convex polyhedra.

An interesting condition is when the territories are required to be *rectangular parallelepipeds* (sometimes called *cuboids*), that is, a 3-dimensional solid figure bounded by six rectangles aligned with a fixed set of Cartesian axes. For convenience, we shall call *box* a rectangular parallelepiped. When all territories are boxes, the *clique number* of any conflict graph, that is, the maximum size of a complete subgraph, is at most 4. However, Reed and Allwright [6] and also later Magnant and Martin [5] designed arrangements of boxes that yield conflict graphs requiring an arbitrarily high number of colours.

On the other hand, if the building is assumed to have floors (in the usual way) and each box is *1-floor*, i.e. restricted to be within one floor, then the chromatic number is bounded by 8: on each floor, the obtained conflict graph is planar and hence can be coloured using 4 colours [2, 3]. It is natural to ask whether this bound is tight. As noted during working sessions in Oxford (see the acknowledgments), it can be shown that up to 6 colours can be needed, by using an arrangement of boxes spanning three floors. Such a construction is shown in Figure 1.

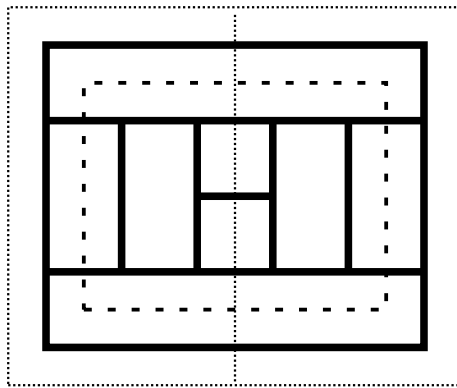


FIGURE 1. An arrangement of 1-floor boxes spanning three floors and requiring six colours. The solid, dotted, and dashed lines indicate the middle, top, and bottom floors, respectively.

The purpose of this note is to show that the upper bound is actually tight. More precisely, we shall build an arrangement of 1-floor boxes that spans two floors and yields a conflict graph requiring 8 colours. From now on, we shall identify a box arrangement with its conflict graph for convenience. In particular, we assign colors directly to the boxes and define the *chromatic number of an arrangement* as that of the associated conflict graph.

Theorem 1. *There exists an arrangement of 1-floor boxes spanning two floors with chromatic number 8.*

The boxes considered in Theorem 1 have one of their geometrical measures bounded: their height is at most one floor. We also discuss bounds on the chromatic number of box arrangements with respect to some other geometrical measures: the side lengths, the surface area and the volume. More precisely, assuming that boxes have integer coordinates, we obtain the following.

Theorem 2. *We consider a box arrangement A with integer coordinates.*

- (1) *If there exists one fixed dimension such that every box in A has length at most ℓ in this dimension, then A has chromatic number at most $4(\ell + 1)$.*
- (2) *If for each box, there is one dimension such that the length of this box in this dimension is at most ℓ , then A has chromatic number at most $12(\ell + 1)$.*
- (3) *If the total surface area of each box in A is at most s , then A has chromatic number at most $9\sqrt[3]{4s} + 13$.*
- (4) *If the volume of each box in A is at most v , then A has chromatic number at most $24\sqrt[4]{6v} + 13$.*

In the next section, we give the proof of Theorem 1 and in the last section we indicate how to obtain the bounds given in Theorem 2.

2. PROOF OF THEOREM 1

We shall construct an arrangement of 1-floor boxes that is not 7-colorable. Before that, we need the following definition. Consider an arrangement A and let A_1 , A_2 and A_3 be (non-necessarily disjoint) subsets of the boxes in A . Given a proper coloring c of A , let C_i be the set of colors used for the boxes in A_i , for $i \in \{1, 2, 3\}$. The *signature* $\sigma_A(c)$ of A with respect to c is defined to be $a_x b_y c$, where $a := |C_1|$, $x := |C_1 \cup C_2|$, $b := |C_2|$, $y := |C_2 \cup C_3|$, and $c := |C_3|$. The collection of all signatures can be endowed with a partial order: if $s = a_x b_y c$ and $s' = a'_x b'_y c'$ are two signatures, then $s \leq s'$ if $a \leq a'$, $x \leq x'$, $b \leq b'$, $y \leq y'$ and $c \leq c'$.

To build the desired arrangement, we use the arrangement X of 1-floor boxes described in Figure 2 as a building brick. The arrangement X has three specific regions, X_1 , X_2 and X_3 . We also abusively write X_1 , X_2 and X_3 to mean the subsets of boxes of X respectively intersecting the regions X_1 , X_2 and X_3 (note that some boxes may belong to several subsets). We start by giving some properties of the signatures of X with respect to proper colorings and according to the three subsets X_1 , X_2 and X_3 .

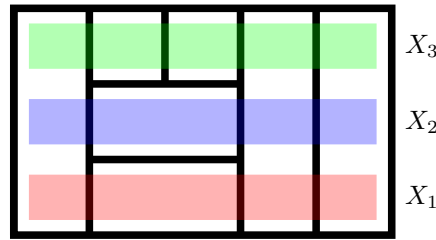


FIGURE 2. The gadget X with the regions X_1 , X_2 and X_3 .

Assertion 3. *For every proper coloring c of X ,*

- (1) $\sigma_X(c) \geq 3_3 2_4 4$,
- (2) $\sigma_X(c) \geq 3_4 3_3 2$, or
- (3) $\sigma_X(c) \geq 2_3 3_4 3$.

The proof of this assertion does not need any insight, we thus omit it. However, the reader interested in checking its accuracy should first note that one can restrict

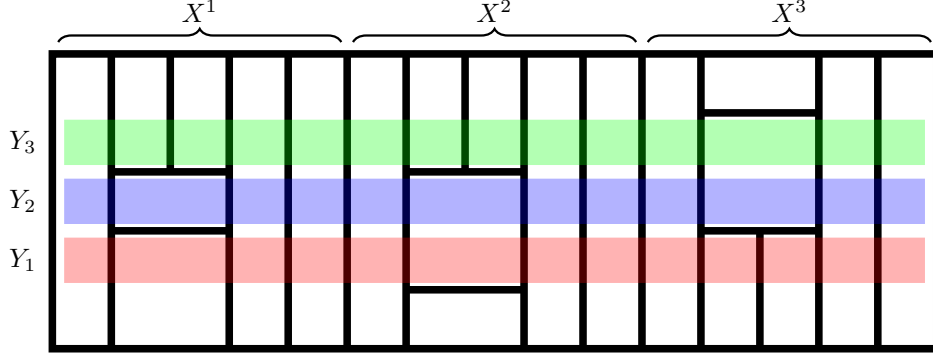


FIGURE 3. The gadget Y with the regions Y_1 , Y_2 and Y_3 .

to the cases where the three vertical (in Figure 2) boxes are respectively colored either 1, 2 and 1; or 1, 1 and 2; or 1, 2 and 3.

Now, the arrangement Y is obtained from three copies X^1 , X^2 and X^3 of the arrangement X . We define three regions Y_1 , Y_2 and Y_3 on Y as depicted in Figure 3. As previously, we also write Y_1 , Y_2 and Y_3 for the subsets of boxes intersecting the region Y_1 , Y_2 and Y_3 , respectively. We set $X_i^j := Y_i \cap X^j$ for $(i, j) \in \{1, 2, 3\}^2$.

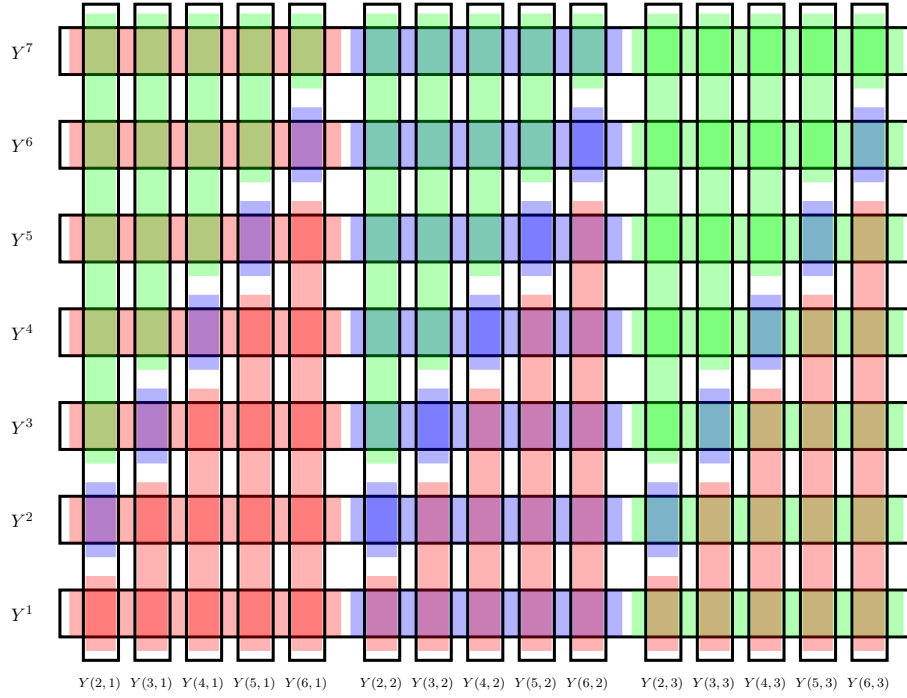
Assertion 4. *In any proper coloring of Y , at least four colors are used in one of the three regions.*

Proof. Suppose on the contrary that there is a proper coloring c of Y with at most three colors in each of Y_1 , Y_2 and Y_3 . For $i \in \{1, 2, 3\}$, the restriction of c to X^i is a proper coloring of X^i , which we identify to c . The condition on c implies that none of $\sigma_{X^1}(c)$, $\sigma_{X^2}(c)$ and $\sigma_{X^3}(c)$ fulfills inequality (1) of Assertion 3. In particular, note that exactly 3 different colours appear on X_2^1 , and they also appear on X_2^2 and on X_2^3 . Since $X_1^1 = X_2^2$, these three colours appear on Y_1 . Similarly, since $X_3^3 = X_2^2$, these three colours appear on Y_3 .

Assume now that $\sigma_{X^1}(c)$ satisfies (2). Then exactly three colours appear on X_1^1 , one of which does not appear on X_2^1 as $|c(X_1^1 \cup X_2^1)| \geq 4$. Thus in total at least four colours appear on $X_1^1 \cup X_2^1 \subset Y_1$, which contradicts our assumption on c . It remains to deal with the case where $\sigma_{X^1}(c)$ fulfills (3) of Assertion 3. Thanks to the symmetry of (2) and (3) with respect to the regions X_1 and X_3 , the same reasoning as above applied to X_3^3 instead of X_1^1 yields that four colours appear on Y_3 , a contradiction. \square

To finish the construction, we need the following definition. Consider two copies Y^1 and Y^2 of Y . The regions Y_i^1 and Y_j^2 *fully overlap* if every box in Y_i^1 is in face-to-face contact with every box in Y_j^2 . Observe that for every pair $(i, j) \in \{1, 2, 3\}$, there exists a 2-floor arrangement of Y^1 and Y^2 such that Y_i^1 and Y_j^2 fully overlap: it is obtained by rotating Y^2 ninety degrees, adequately scaling it (i.e. stretching it horizontally) and placing it on top of Y^1 .

We are now in a position to build the desired arrangement Z spanning two floors. To this end, we use several copies of Y . The first floor of Z is composed of seven parallel copies Y^1, \dots, Y^7 of Y (drawn horizontally in Figure 4). The second floor of Z is composed of fifteen parallel copies of Y (drawn vertically in Figure 4): for

FIGURE 4. Schematic view of the arrangement Z .

each $j \in \{1, 2, 3\}$ and each $i \in \{2, \dots, 6\}$, a copy $Y(i, j)$ of Y is placed such that the first region of $Y(i, j)$ fully overlap the regions Y_j^1, \dots, Y_j^{i-1} , the second region of $Y(i, j)$ fully overlaps the region Y_j^i , and the third region of $Y(i, j)$ fully overlaps the regions Y_j^{i+1}, \dots, Y_j^7 .

Consider a proper coloring of Z . Assertion 4 ensures that each copy of Y in Z has a region for which at least four different colours are used. In particular, there exists $j \in \{1, 2, 3\}$ such that three regions among Y_j^1, \dots, Y_j^7 are colored using four colours. Let these regions be $Y_j^{i_1}, Y_j^{i_2}$ and $Y_j^{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq 7$. Now, consider the arrangement $Y(i_2, j)$. By Assertion 4, there exists $k \in \{1, 2, 3\}$ such that the k -th region of $Y(i_2, j)$ is also colored using at least four different colors. Consequently, as this region and the region $Y_j^{i_k}$ fully overlap, they are colored using at least eight different colors. This concludes the proof.

3. BOUNDS WITH RESPECT TO GEOMETRICAL MEASURES

In this part, we provide bounds on the chromatic number of boxes arrangements provided that the boxes satisfy some geometrical constraints. Namely, we prove Theorem 2, which is recalled here for the reader's ease.

Theorem 2. *We consider a box arrangements A with integer coordinates.*

- (1) *If there exists one fixed dimension such that every box in A has length at most ℓ in this dimension, then A has chromatic number at most $4(\ell + 1)$.*
- (2) *If for each box, there is one dimension such the length of this box in this dimension is at most ℓ , then A has chromatic number at most $12(\ell + 1)$.*

- (3) *If the total surface area of each box in A is at most s , then A has chromatic number at most $9\sqrt[3]{4s} + 13$.*
- (4) *If the volume of each box in A is at most v , then A has chromatic number at most $24\sqrt[4]{6v} + 13$.*

Proof.

(1) The conflict graph corresponding to an arrangement where the boxes have height at most ℓ can be vertex partitioned into $\ell + 1$ planar graphs P_0, \dots, P_ℓ . Indeed if the distance between the levels of two boxes is at least $\ell + 1$, then these two boxes are not adjacent. So the planar graphs are obtained by assigning, for each x , all the boxes that have their floor at level x to be in the graph P_k where $k := x \bmod (\ell + 1)$. Consequently, the whole conflict graph has chromatic number at most $4(\ell + 1)$.

(2) The boxes can be partitioned into three sets according to the dimension in which the length is bounded. In other words, A is partitioned into U_1, U_2 and U_3 such that for each $i \in \{1, 2, 3\}$, all boxes in U_i have length at most ℓ in dimension i . Consequently, (1) ensures that each of U_1, U_2 and U_3 has chromatic number at most $4(\ell + 1)$ and, therefore, A has chromatic number at most $3 \cdot 4(\ell + 1) = 12(\ell + 1)$.

(3) For each box, the minimum length taken over all three dimensions is at most \sqrt{s} , and thus (2) implies that the chromatic number of A is $O(\sqrt{s})$. However, one can be more careful. Let us fix a positive integer ℓ , to be made precise later. The set of boxes is partitioned as follows. Let U be the set of boxes with lengths in every dimension at least ℓ and let R be the set all remaining boxes, that is, $R := A \setminus U$. By (2), the arrangement R has chromatic number at most 12ℓ . Now consider a box B in U with dimensions x, y and z , each being at least ℓ . We shall give an upper bound on the number of boxes of U that can be adjacent to B . The surface of a face of a box in U is at least ℓ^2 . So in U there are at most s/ℓ^2 that have a face totally adjacent to a face of B . Some boxes of U could also be adjacent to B without having a face totally adjacent to a face of B . In this case, such a box is adjacent to an edge of B . For an edge of length w , there are at most $w/\ell + 1$ such boxes. So the number of boxes of U adjacent to B but having no face totally adjacent to a face of B is at most $4(x + y + z)/\ell + 12$. Since $\ell \leq \min\{x, y, z\}$, we deduce that $2\ell(x + y + z) \leq 2xy + 2yz + 2xz \leq s$. Hence the total number of boxes in U that are adjacent to B is at most $s/\ell^2 + 2s/\ell^2 + 12 = 3s/\ell^2 + 12$. Consequently, by degeneracy, U has chromatic number at most $3s/\ell^2 + 13$. Therefore, A has chromatic number at most $12\ell + 3s/\ell^2 + 13$. Setting $\ell := \sqrt[3]{s/2}$ yields the upper bound $9\sqrt[3]{4s} + 13$.

(4) Once again, for a fixed parameter ℓ to be made precise later, the set of boxes is partitioned into two parts: the part U , composed of all the boxes with lengths in every dimension at least ℓ and the part R , composed of all the remaining boxes. By (2), we know that R has chromatic number at most 12ℓ . Let B be a box in U with dimensions x, y and z . Since $\ell \leq \min\{x, y, z\}$, the volume v_B of B satisfies that $6v \geq 6v_B = 6xyz \geq 2(\ell xy + \ell xz + \ell yz) = \ell s_B$, where s_B is the total surface area of B . So every box in U has total surface area at most $6v/\ell$ and thus (3) implies that U has chromatic number at most $9\sqrt[3]{4.6v/\ell} + 13$. Therefore, A has chromatic number at most $9\sqrt[3]{24v/\ell} + 12\ell + 13$. Setting ℓ to be $\sqrt[4]{3v/8}$ yields the upper bound $24\sqrt[4]{6v} + 13$. \square

In the previous theorem, we are mainly concerned with the order of magnitude of the functions of the different parameters. However, even in this context, we do not have any non trivial lower bound on the corresponding chromatic numbers.

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REFERENCES

- [1] D. ALLWRIGHT, T. GOULD, J. GRAVESEN, R. LEESE, AND H. PETERSEN, *Graph colouring for office blocks*, tech. rep., 53rd European Study Group with Industry, 2005.
- [2] K. APPEL AND W. HAKEN, *Every planar map is four colorable. I. Discharging*, Illinois J. Math., 21 (1977), pp. 429–490.
- [3] K. APPEL, W. HAKEN, AND J. KOCH, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math., 21 (1977), pp. 491–567.
- [4] A. S. BESICOVITCH, *On Crum's problem*, J. London Math. Soc., 22 (1947), pp. 285–287 (1948).
- [5] C. MAGNANT AND D. M. MARTIN, *Coloring rectangular blocks in 3-space*, Discuss. Math. Graph Theory, 31 (2011), pp. 161–170.
- [6] B. REED AND D. ALLWRIGHT, *Painting the office*, Mathematics-in-Industry Case Studies Journal, 1 (2008), pp. 1–8.
- [7] H. TIETZE, *Über nachbargebiete im raume*, Monatshefte für Mathematik und Physik, 16 (1905), pp. 211–216.
- [8] H. TIETZE, *Famous problems of mathematics. Solved and unsolved mathematical problems from antiquity to modern times*, Authorized translation from the second (1959) revised German edition. Edited by B. K. Hofstadter and H. Komm, Graylock Press, New York, 1965.

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