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BREGMAN SUPERQUANTILES. ESTIMATION METHODS AND APPLICATIONS

T. LABOPIN-RICHARD, F. GAMBOA, A. GARIVIER, AND B. IOOSS

Abstract. In this work, we extend some quantities introduced by Rockafellar to the case where the proximity between real numbers is measured by using a Bregman divergence. This leads to the definition of the Bregman superquantile. Axioms of a coherent measure of risk discussed previously by Rockafellar are studied in the case of Bregman superquantile. Furthermore, we deal with asymptotic properties of a Monte Carlo estimator of the Bregman superquantile. Several numerical tests confirm the theoretical results and an application illustrates the potential interests of the Bregman superquantile.

1. Introduction

1.1. **Aim and scope.** The aim of this article is to define and to study properties and estimation procedures for Bregman extension of the superquantile defined in [25] (see also [18], [21] and references therein). In the introduction we first recall the necessary conditions for a measure of risk to be coherent. Further in Section 2 we present the superquantile as a partial response to this problem. We also introduce the Bregman superquantile and study axioms of a coherent measure of risk for this quantity. In Section 3 we seek to estimate this Bregman superquantile, we introduce a plug-in estimator and study its convergence and asymptotic normality. Some numerical simulations are shown in Section 4. An application on real data of radiological exposure is given in Section 5. All the proofs are postponed to section 6.

1.2. **Coherent measures of risk.** Let X be a real-valued random variable and let F_X be its cumulative distribution function. We define for $u \in]0,1[$, the quantile function

$$F_X^{-1}(u) := \inf\{x : F_X(x) \geq u\}.$$

A usual way to quantify the risk associated with X is to consider, for a given number $\alpha \in]0,1[$ close to 1, its lower quantile $q_\alpha^X := F_X^{-1}(\alpha)$.

Nevertheless, the quantile is not a sub-additive function of X, a major property in some applications (e.g finance, see [2]). Thus Rockafellar introduces in [22] the new quantity called therein superquantile that satisfies this property. The superquantile is defined by

$$Q_\alpha = \mathbb{E}(X|X \geq q_\alpha) = \mathbb{E}(X|X \geq F_X^{-1}(\alpha)) = \mathbb{E}\left(\frac{X\mathbf{1}_{X \geq F_X^{-1}(\alpha)}}{1-\alpha}\right).$$

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Notice that this quantity is also called conditional value at risk in older references ([25], [24], [21]).

Sub-additivity is not the sole interesting property for a measure of risk (for example for financial application). Following Rockaffelar in [22] we define:

Definition 1.1. *Let $\mathcal{R}$ be a measure of risk that is a numerical function defined on random variables and X and X' be two real-valued random variables. We say that $\mathcal{R}$ is coherent if, and only if, it satisfies the five following properties :*

i) *Constant invariance : let $C \in \mathbb{R}$, if $X = C$ (a.s.) then $\mathcal{R}(C) = C$.*
ii) *Homogeneity : $\forall \lambda > 0$, $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$.*
iii) *Subaddidivity : $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$.*
iv) *Non decreasing : If $X \leq X'$ (a.s.) then $\mathcal{R}(X) \leq \mathcal{R}(X')$.*
v) *Closeness : Let $(X_h)_{h\in\mathbb{R}}$ be a collection of random variables. If $\mathcal{R}(X_h) \leq 0$ and $\lim_{h\to 0} ||X_h - X||_2 = 0$ then $\mathcal{R}(X) \leq 0$.*

The superquantile is a coherent measure of risk (see [17], [16], [1]).

2. Bregman superquantiles

In this section the aim is to build a general measure of risk that satisfies some of the regularity axioms stated in Definition 1.1. These quantities will be built by using a dissimilarity measure beetween real numbers, the Bregman divergence (see [5]).

2.1. Bregman divergence, mean and superquantile.

In this section we first recall the definition of the Bregman mean of a probability measure μ (see [4]) and define the measure of risk that we will study. To begin with, we recall the definition of the Bregman divergence that will be used to build the Bregman mean. Let γ be a strictly convex function, $\overline{\mathbb{R}}$-valued on $\mathbb{R}$. As usual we set

$$\mathrm{dom}\gamma := \{x \in \mathbb{R} : \gamma(x) < +\infty\}.$$

For sake of simplicity we assume that $\mathrm{dom}\gamma$ is a non empty open set and that γ is a closed proper differentiable function on the interior of $\mathrm{dom}\gamma$ (see [19]). From now we always consider function γ satisfying this assumption. The Bregman divergence d_γ associated to γ (see [5]) is a function defined on $\mathrm{dom}\gamma \times \mathrm{dom}\gamma$ by

$$d_\gamma(x, x') := \gamma(x) - \gamma(x') - \gamma'(x')(x - x') \ , \ (x, x' \in \mathrm{dom}\gamma).$$

The Bregman divergence is not a distance as it is not symmetric. Nevertheless, as it is non negative and vanishes, if and only if, the two arguments are equal, it quantifies the proximity of points in $\mathrm{dom}\gamma$. Let us recall some classical examples of such a divergence.

- Euclidean. $\gamma(x) = x^2$ on $\mathbb{R}$, we obviously obtain, for $x, x' \in \mathbb{R}$,
$$d_\gamma(x, x') = (x - x')^2.$$
- Geometric. $\gamma(x) = x\ln(x) - x + 1$ on $\mathbb{R}_+^*$ we obtain, for $x, x' \in \mathbb{R}_+^*$,
$$d_\gamma(x, x') = x \ln \frac{x}{x'} + x' - x.$$

- Harmonic. $\gamma(x) = -\ln(x) + x - 1$ on $\mathbb{R}_+^*$ we obtain, for $x, x' \in \mathbb{R}_+^*$,

$$d_\gamma(x, x') = -\ln\frac{x}{x'} + \frac{x}{x'} - 1.$$

Let μ be a probability measure whose support is included in domγ and that does not weight the boundary of domγ. Assume further that γ' is integrable with respect to μ. Following [4], we first define the Bregman mean as the unique point b in the support of μ satisfying

$$\int d_\gamma(b, x)\mu(dx) = \min_{m \in \mathrm{dom}\gamma} \int d_\gamma(m, x)\mu(dx). \tag{1}$$

In fact, we replace the L^2 minimization in the definition of the mathematical classical expectation by the minimization of the Bregman divergence. Existence and uniqueness come from the convexity properties of d_γ with respect to its first argument. By differentiating it is easy to see that

$$b = \gamma'^{-1}\left[\int \gamma'(x)\mu(dx)\right].$$

Hence, coming back to our three previous examples, we obtain the classical mean in the first example (Euclidean case), the geometric mean ($\exp\int \ln(x)\mu(dx)$), in the second one and the harmonic mean ($[\int x^{-1}\mu(dx)]^{-1}$), in the third one. Notice that, as the Bregman divergence is not symmetric, we have to pay attention to the definition of the Bregman mean. Indeed, we have

$$\int d_\gamma(x, \mathbb{E}(X))\mu(dx) = \min_{m \in \mathrm{dom}\gamma} \int d_\gamma(x, m)\mu(dx).$$

We turn now to the definition of our new measure of risk.

Definition 2.1. *Let $\alpha \in]0,1[$, the Bregman superquantile $Q_\alpha^{d_\gamma}$ is defined by*

$$Q_\alpha^{d_\gamma} := \gamma'^{-1}\Big(\mathbb{E}(\gamma'(X)|X \geq F_X^{-1}(\alpha))\Big) = \gamma'^{-1}\left[\mathbb{E}\left(\frac{\gamma'(X)\mathbf{1}_{X \geq F_X^{-1}(\alpha)}}{1-\alpha}\right)\right].$$

In words $Q_\alpha^{d_\gamma}$ satisfies (1) taking for μ the distribution of X conditionally to $X \geq F_X^{-1}(\alpha)$. We now denote $Q_\alpha^{d_\gamma}$ the Bregman superquantile of the law X when there is no ambiguity and $Q_\alpha^{d_\gamma}(X)$ if we need to distinguish Bregman superquantile of different laws.

2.2. **Coherence of Bregman superquantile.** The following proposition gives some conditions under which the Bregman superquantile is a coherent measure of risk.

Proposition 2.1. *Fix α in $]0,1[$.*

i) *Any Bregman superquantile always satisfies the properties of constant invariance and non decreasing.*

ii) *The Bregman superquantile associated to the function γ is homogeneous, if and only if,*
$\gamma''(x) = \beta x^\delta$ *for some real numbers $\beta >$ and δ (as γ is convex, if the support of γ is strictly included in $\mathbb{R}^{+,*}$ there is no condition on δ but if not, δ is an even number).*

iii) *If γ' is concave and sub-additive, then subadditivity and closeness axioms both hold.*

The proof of this proposition, like all the others, is differed to Section 5.

To conclude, under some regularity assumptions on γ, the Bregman superquantile is a coherent measure of risk. Let us take some examples.

2.2.1. *Examples and counter-examples.*

- **Example 1 :** $x \mapsto x^2$ satisfies all the hypothesis but it is already known that the classical superquantile is sub-addtive.
- **Example 2 :** The Bregman geometric and harmonic functions satisfies the assumptions i) and ii). Moreover, their derivatives are respectively $x \mapsto \gamma'(x) = \ln(x)$ and $x \mapsto \gamma'(x) = \frac{x-1}{x}$ which are concave but sub-additive only on $[1, +\infty[$. Then the harmonic and geometric functions satisfy iii) not for all pairs of random variables but only for pairs (X, X') such that, denoting $Z := X + X'$ we have
$$\min\left(q_\alpha^X(\alpha), q_\alpha^{X'}(\alpha), q_\alpha^Z(\alpha)\right) > 1.$$
- **Counter-example 3 :** The sub-additivity is not true in the general case. Indeed, let $\gamma(x) = \exp(x)$ and assume that $X \sim \mathcal{U}([0,1])$.
$$\mathbb{E}(\gamma'(X)\mathbf{1}_{X \geq F_X^{-1}(\alpha)}) = \int_\alpha^1 \exp(x)dx = e - \exp(\alpha).$$
Then
$$\mathcal{R}(X) = \ln\left(\frac{e - \exp(\alpha)}{1-\alpha}\right).$$
Moreover,
$$\mathbb{E}\left(\gamma'(\lambda X)\mathbf{1}_{\lambda X \geq F_{\lambda X}^{-1}(\alpha)}\right) = \int_\alpha^1 \exp(\lambda x)dx = \frac{\exp(\lambda) - \exp((\alpha)\lambda)}{\lambda}.$$
So
$$\mathcal{R}(\lambda X) = \ln\left(\frac{\exp(\lambda) - \exp((\alpha)\lambda)}{\lambda(1-\alpha)}\right).$$
For $\alpha = 0.95$ and $\lambda = 2$, we obtain
$$\mathcal{R}(2X) - 2\mathcal{R}(X) = \mathcal{R}(X+X) - (\mathcal{R}(X) + \mathcal{R}(X)) = 0.000107 > 0,$$
and subadditivity fails.

We can also notice that for $\lambda = 4$

$$\frac{\mathcal{R}(4X)}{4\mathcal{R}(X)} = 1,000321,$$

and the homogeneity is not true. It is coherent with the Proposition 2.1 since the derivative of γ does not fulfill the assumption.

3. Estimation of the Bregman superquantile

In this section the aim is to make estimation of the Bregman superquantile. We introduce a Monte Carlo estimator and study its asymptotics properties. Under regularity assumptions on the functions γ and F_X^{-1}, the Bregman superquantile is consistent and asymptotically Gaussian. All along this section, we consider a function γ satisfying our usual properties and a real-valued random variable X such that $\gamma'(X)\mathbf{1}_{X\geq 0}$ is integrable.

3.1. **Monte Carlo estimator.** Assume that we have at hand $(X_1, \dots, X_n)$ an i.i.d sample with same distribution as X. If we wish to estimate $Q_\alpha^{d_\gamma}$, we may use the following empirical estimator :

$$\widehat{Q_\alpha^{d_\gamma}} = \gamma'^{-1}\left[\frac{1}{1-\alpha}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\gamma'(X_{(i)})\right)\right],$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ is the re-ordered sample built with $(X_1, \dots, X_n)$.

3.2. **Asymptotics.** We give a theorem which study the asymptotic behaviour of the Bregman superquantile. The following assumptions will be used in our next theorem.

H1) γ is twice differentiable, and the derivative of $\left(\gamma' \circ F_X^{-1}\right)$ that we denote by l_γ satisfies $l_\gamma = o\left((1-t)^{-2}\right)$ and is non-decreasing when t goes to 1^-.

H2) γ is three times differentiable, and the second derivative of $\left(\gamma' \circ F_X^{-1}\right)$ that we denote L_γ satisfies $L_\gamma = O\left((1-t)^{-m_L}\right)$ and is non decreasing and when t goes to 1^-.

Theorem 3.1. *Let $\frac{1}{2} < \alpha < 1$ and X be a real-valued random variable such that the cumulative function F_X is absolutely continuous and the density function f_X is strictly non-negative on its support. Let $(X_1, \dots X_n)$ be an independant sample with the same distribution as X.*

i) *Under assumption **H1)**, the estimator $\widehat{Q_\alpha^{d_\gamma}}$ is consistent in probability.*

ii) *Under assumption **H2)**, the estimator $\widehat{Q_\alpha^{d_\gamma}}$ is asymptotically normal :*

$$\sqrt{n}\left(\widehat{Q_\alpha^{d_\gamma}} - Q_\alpha^{d_\gamma}(X)\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma_\gamma^2}{\left(\gamma''\left(Q_\alpha^{d_\gamma}(X)\right)\right)^2(1-\alpha)^2}\right),$$

where

$$\sigma_\gamma^2 := \int_\alpha^1\int_\alpha^1 \frac{(\min(x,y)-xy)}{f_Z(F_Z^{-1}(x))f_Z(F_Z^{-1}(y))}dxdy.$$

and $Z := \gamma'(X)$.

Remark 3.1. *Easy calculations show that we have the following equalities*

$$l_\gamma := \frac{\gamma'' \circ F_X^{-1}}{f_X \circ F_X^{-1}},$$

$$L_\gamma := \frac{\left(\gamma''' \circ F_X^{-1}\right) \times \left(f_X \circ F_X^{-1}\right) - \left(f - X' \circ F_X^{-1}\right) \times \left(\gamma'' \circ F_X^{-1}\right)}{\left(f_X \circ F_X^{-1}\right)^3}$$

and

$$f_Z = \frac{f_X \circ \gamma'^{-1}}{\gamma'' \circ \gamma'^{-1}}.$$

Remark 3.2. *The second part of the theorem shows the asymptotic normality of the Bregman superquantile empirical estimator. We can then use the Slutsky's lemma to find confident intervals. Indeed, since our estimator* $\widehat{Q_\alpha^{d\gamma}} := \gamma'^{-1}\left((n(1-\alpha))^{-1} \sum_{i=\lfloor n\alpha \rfloor+1}^{n} Z_{(i)}\right)$ *is consistent, we also have*

$$\frac{\sqrt{n}}{\left(\gamma'' \circ \widehat{Q_\alpha^{d\gamma}}\right)} \left(\widehat{Q_\alpha^{d\gamma}} - Q_\alpha^{d\gamma}(X)\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma_\gamma^2}{(1-\alpha)}\right).$$

To prove Theorem 3.1 we use the results on the asymptotic properties of the superquantile. For sake of completness we will show these properties. As a matter of fact we have

$$Q_\alpha^{d\gamma}(X) = \gamma'^{-1}\left(Q_\alpha(\gamma'(X)\right). \tag{2}$$

Indeed, as $\gamma'\left(F^{-1}_{(\gamma')^{-1}(Z)}(\alpha)\right) = F_Z^{-1}(\alpha)$, so that

$$\mathbb{E}\left(\gamma'(X)\mathbf{1}_{X > F_X^{-1}(\alpha)}\right) = \mathbb{E}\left(Z\mathbf{1}_{Z > F_Z^{-1}(\alpha)}\right).$$

Hence, we will first prove the following proposition on the asymptotic behavior of the superquantile Monte Carlo estimator (which is equivalent to deal with the Bregman superquantile when the function γ equals to identity). Then, we apply this proposition to the sample $(Z_1, \dots Z_n)$ where $Z_i := \gamma'(X_i)$. We conclude by applying the continuous map theorem for the consistency and the delta-method for the asymptotic normality (see for example [27]), both with the regular function γ'^{-1}.

For the next proposition we need the two following assumptions.

H3) The derivative of the quantile function F_X^{-1} denoting l satisfies $l = o((1-t)^{-2})$ and is non-decreasing when t goes to 1^-.

H4) The second derivative of the quantile function that we denote L satisfies $L = O\left((1-t)^{-m_L}\right)$ for an $1 < m_L < \frac{5}{2}$ and is non decreasing when t goes to 1^-.

Proposition 3.1. *Let* $\frac{1}{2} < \alpha < 1$, *and* X *be a real-valued random variable such that the cumulative function* F_X *is absolutely continuous and the density function* f_X *is strictly*

non-negative on its support.Let $(X_1, \dots X_n)$ *be an independent sample with the same distribution as* X.

i) *Under* ***H3)****, the estimator* $\widehat{Q_\alpha} := ((1-\alpha)n)^{-1} \sum_{i=\lfloor n\alpha \rfloor}^{n} X_{(i)}$ *of* Q_α^X *is consistent in probability.*

ii) *Under* ***H4)****, the estimator* $\widehat{Q_\alpha}$ *is asymptotically normal*

$$\sqrt{n}\left(\frac{1}{n(1-\alpha)} \sum_{i=\lfloor n\alpha \rfloor+1}^{n} X_{(i)} - Q_\alpha\right) \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2}{(1-\alpha)^2}\right),$$

where

$$\sigma^2 := \int_\alpha^1 \int_\alpha^1 \frac{(\min(x,y) - xy)}{f(F^{-1}(x))f(F^{-1}(y))}.$$

3.3. **Examples of asymptotic behaviors for the classical superquantile.** Our assumptions are easy to check in practice. Let us show some examples of the asymptotic behaviour of the superquantile by using the exponential distribution of parameter 1 and the Pareto distribution.

3.3.1. *Exponential distribution.* In this case, we have on $\mathbb{R}_*^+$ $f(t) = \exp(-t)$, $F(t) = 1 - \exp(-x)$. Then $F^{-1}(t) = -\ln(1-t)$.

- Consistency :
  $l(t) = (1-t)^{-1} = o\left((1-t)^{-2}\right)$ (when $t \mapsto 1^-$) and l is non decreasing near 1 so that the estimator of the superquantile is consistent.
- Asymptotic normality :
  $L(t) = (1-t)^{-2} = O\left((1-t)^{-m_L}\right)$ $(t \mapsto 1^-)$ for $2 < m_L < \frac{5}{2}$ and L is non-decreasing near 1. So the asymptotic normality holds.

3.3.2. *Pareto law.* Here, we consider the Pareto law of parameter $a > 0$: on $\mathbb{R}_*^+$, $F(t) = 1 - x^{-a}$, $f(t) = ax^{-a-1}$, and $F^{-1}(t) = (1-t)^{\frac{-1}{a}}$.

- Consistency :
  $l(t) = (a(1-t)^{-1-\frac{1}{a}})$ thus, $l(t) = o\left((1-t)^{-2}\right)$ (when $t \mapsto 1^-$) as soon as $a > 1$. Then, l is non-decreasing near 1. The consistency is true.
- Asymptotic normality :
  $L(t) = C(a)(1-x)^{-\frac{1}{a}-2}$ thus, as soon as $a > 2$, there exists $\frac{3}{2} < m_L < \frac{5}{2}$ such that $L(t) = O\left(\frac{1}{(1-t)^\delta}\right)$ (when $t \mapsto 1^-$) and L is non decreasing around 1. The asymptotic normality is true if and only if $a > 2$.

3.4. **Examples of asymptotic behaviour of the Bregman superquantile.** Let us now study the same examples for the Bregman superquantile. For the exponential distribution, the conclusion is the same. However, for the Pareto distribution, we can find a function γ such that the estimator of the Bregman superquantile is asymptotically normal without any condition on a. So, the Bregman superquantile is a more interesting measure of risk than the superquantile.

3.4.1. *Exponential law.* Let us show the example of the exponential distribution and the harmonic Bregman function. We have $\gamma'(x) = (x-1)x^{-1}$ and $F^{-1}(x) = -\ln(1-t)$. So that

$$F_Z^{-1}(t) = 1 + \frac{1}{\ln(1-t)},$$

- Consistency. In this case, we have,

$$l_\gamma(t) = \frac{1}{(1-t)(\ln(1-t))^2}.$$

  So, l_γ is non decreasing near 1 and $o\left((1-t)^{-2}\right)$ (when $t \mapsto 1^-$).

  The estimator $\widehat{Q_\alpha^{d\gamma}}$ is consistent.
- Asymptotical normality.

$$L_\gamma(t) = \frac{(\ln(1-t))^2 + 2\ln(1-t)}{(1-t)^2(\ln(1-t))^4},$$

  Then L_γ is non decreasing near 1 and $O\left((1-t)^{-m_L}\right)$ (when $t \mapsto 1^-$), for $2 < m_L < \frac{5}{2}$.

  Our estimator is asymptotically Gaussian.

3.4.2. *Pareto law.* Let us now study the case of the Pareto law with the geometric Bregman function. We have $F^{-1}(t) = (1-t)^{\frac{-1}{a}}$ and $\gamma'(t) = \ln(t)$. Then

$$F_Z^{-1}(t) = -\frac{1}{a}\ln(1-t).$$

- Consistency.

$$l_\gamma(t) = \frac{1}{a}\frac{1}{1-t} = o\left(\frac{1}{(1-t)^2}\right),$$

  and the non-decreasing is true. The estimator is consistent.
- Asymptotic normality.

$$L_\gamma(t) = \frac{1}{a}\frac{1}{(1-t)^2} = O\left((1-t)^{-m_L}\right),$$

  for $2 < m_L < \frac{5}{2}$.

  The estimator is consistent and normally asymptotic for every $a > 0$.

4. Numerical simulations

Our numerical tests consist in simulating values from a known theoretical distribution and computing the 0.95-quantiles and superquantiles. For each estimated quantity, the reference value is given via a 10^6-size random sample and a convergence study is performed from a 1000-size sample to a 10^5 size sample (with a step of 500). In order to annihilate the effect of randomness, 50 repetitions of each numerical experiment are made. Then, we compute

- The mean value of the 50 estimations to be compared to the reference value,

- The standard deviation of the 50 estimations. It allows to compute an experimental 95%-confidence interval (CI) to be compared to the theoretical 95%-CI (given by the central limit theorem).

Each is composed of four plots of convergence for the following quantities: quantile (up left), classical superquantile (up right), geometrical superquantile (bottom left) and harmonic superquantile (bottom right). Each superquantile convergence plot is composed of the following curves: Reference value (dotted black line), mean estimated values (red circles), theoretical 95%-CI (dashed black line) and experimental 95%-CI (solid blue line).

Figure 1 gives the results for an exponential distribution of parameter $\lambda = 1$. As predicted by the theory (see Section 3.3), for the three different superquantiles, the consistency is verified while the experimental CI perfectly fits the theoretical CI (given by the central limit theorem).

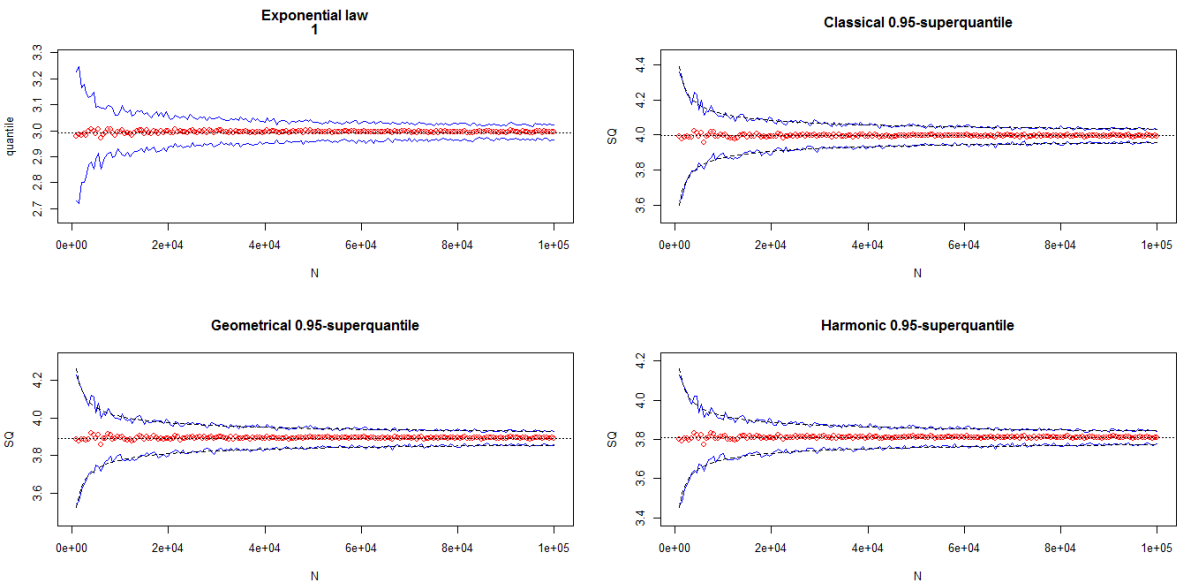

FIGURE 1. Numerical convergence test for the exponential distribution.

We then test the Pareto distribution (see Section 3.3) with three different shape parameters: $a = 0.5$, $a = 1.5$ and $a = 2.5$. Figures 2 ($a = 0.5$), 3 ($a = 1.5$) and 4 ($a = 2.5$) give the convergence results. For the geometrical and harmonic superquantiles, as predicted by the theory (see Section 3.3), the consistency of the Monte Carlo estimation is verified while the experimental CI perfectly fits the theoretical CI (asymptotic normality). For the classical superquantile, we distinguish three different behaviors:

- No consistency for $a = 0.5$ (Figure 2) (theory predicts consistency only if $a > 1$),
- Consistency but no asymptotic normality for $a = 1.5$ (Figure 3) (theory predicts asymptotic normality only if $a > 2$),
- Consistency and asymptotic normality for $a = 2.5$ (Figure 4),

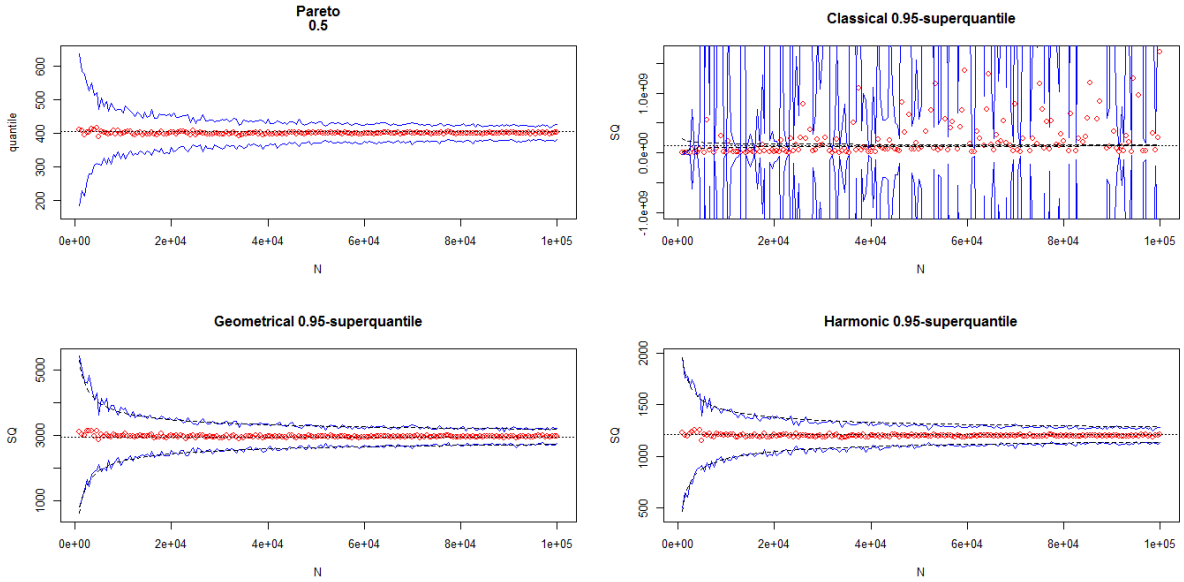

FIGURE 2. Numerical convergence test for the Pareto distribution ($a = 0.5$).

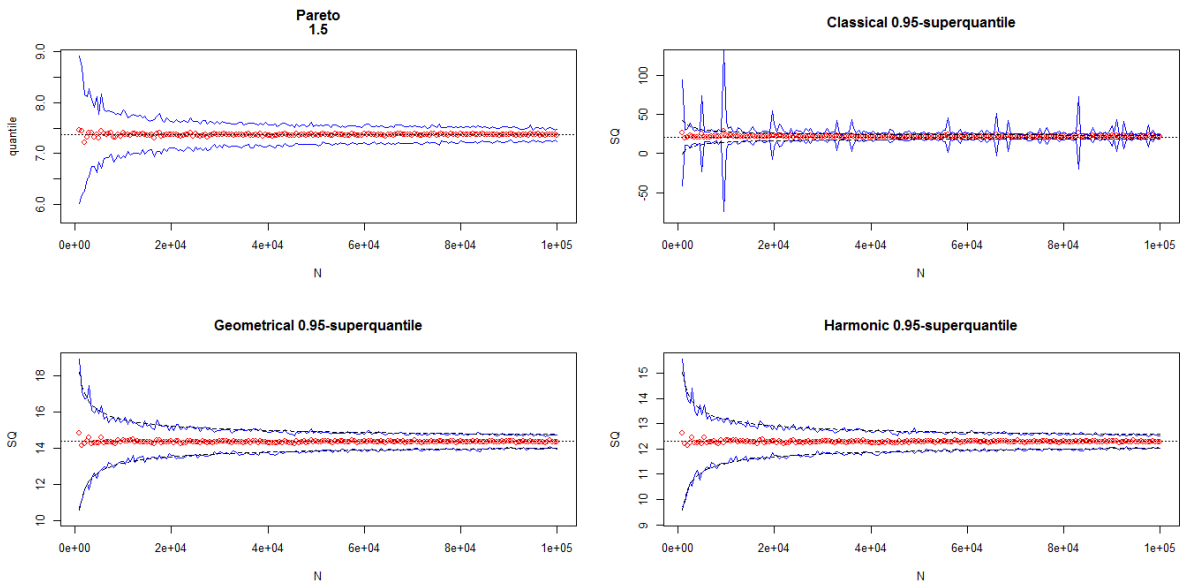

FIGURE 3. Numerical convergence test for the Pareto distribution ($a = 1.5$).

5. APPLICATIONS TO A NUCLEAR SAFETY EXERCISE

GASCON is a software developed by CEA (French Atomic Energy Commission) to study the potential chronological atmospheric releases and dosimetric impact for nuclear

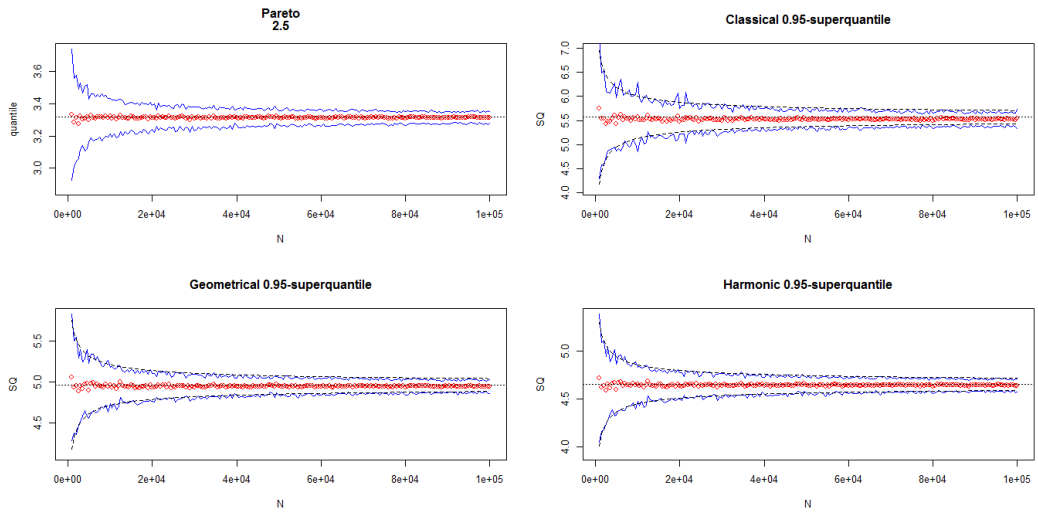

FIGURE 4. Numerical convergence test for the Pareto distribution ($a = 2.5$).

facilities safety assessment [13]. It evaluates, from a fictitious radioactive release, the doses received by a population exposed to the cloud of radionuclides and through the food chains. It takes into account the interactions that exist between humans, plants and animals, the different pathways of transfer (wind, rain, ...), the distance between emission and observation, and the time from emission.

As GASCON is relatively costly in computational time, [13] have built metamodels (of polynomial form) of GASCON outputs in order to perform uncertainty and sensitivity analysis. As in [14], we focus on one output of GASCON, the annual effective dose in ^{129}I received in one year by an adult who lives in the neighborhood of a particular nuclear facility. Instead of the GASCON software, we will use here the metamodel of this output which depends on 10 input variables, each one modelled by a log-uniform random variable (bounds are defined in [13]). The range of the model output stands on several decades (10^{-14} to 10^{-11} Sv/year) as shown by Figure 5 which represents the histogram (in logarithmic scale) of 10^6 simulated values.

For this kind of numerical simulation exercises, we can be typically interested by safety criteria as 95%-quantile and its associated superquantiles. The idea is to compare these values to regulatory limits or to results coming from other scenarios or from other tools. In practice, the number of simulations performed with the GASCON model is several hundreds. Table 1 gives the estimated values of the quantile and superquantiles for 1000 metamodel simulations. Figure 6 shows the relative errors (computed by averaging 1000 different estimations) which are made when estimating the superquantiles using 3 different Bregman divergences and with different sampling sizes. We observe that geometrical and harmonic superquantiles are clearly more precise than the classical one. Using such measures is therefore more relevant when performing comparisons.

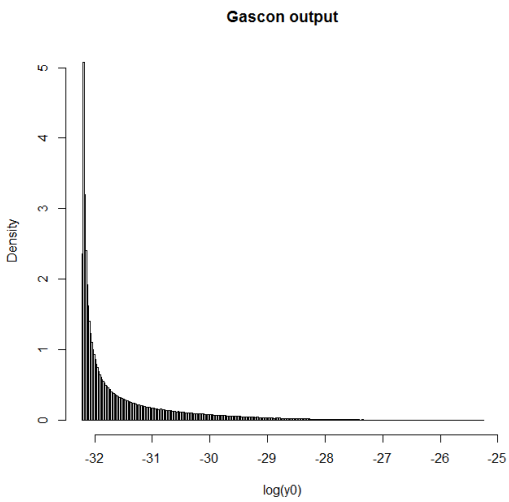

FIGURE 5. Distribution of the GASCON output variable.

TABLE 1. Estimated values of 95%-quantile and 95%-superquantiles for 1000 simulations.

Quantile	Classical superquantile	Geometrical superquantile	Harmonic superquantile
1.304×10^{-13}	4.769×10^{-13}	3.316×10^{-13}	2.637×10^{-13}

6. PROOFS

6.1. Proof of the Proposition 2.1 : coherence of the Bregman superquantile.

Proof. **Proof of i) :**

First we obviously have $\mathcal{R}(C) = \gamma'^{-1}(\gamma'(C)) = C$.

Let us show the non decreasing property. We first show that the superquantile is non-decreasing. Then, (2) and the monotony of γ'^{-1} and γ will allow us to conclude. In [16] it is shown that the Superquantile of the law of X satisfy

$$Q_\alpha = \min_{a\in\mathbb{R}}\{a + \frac{1}{1-\alpha}\mathbb{E}\left([X-a]^+\right)\} = q_\alpha + \frac{1}{1-\alpha}\mathbb{E}\left([X-q_\alpha]^+\right).$$

Let X and X' be two random variables such that $X \leq X'$ (a.s.). Using the previous results we have

$$Q_\alpha(X) \leq q_\alpha(X') + \frac{1}{1-\alpha}\mathbb{E}\left(\left[X - q_\alpha(X')\right]^+\right).$$

Then the monotony of the function $x \mapsto (x-c)^+$ when c is fixed gives

$$Q_\alpha(X) \leq q_\alpha(X') + \frac{1}{1-\alpha}\mathbb{E}\left(\left[X' - q_\alpha(X')\right]^+\right) = Q_\alpha(X').$$

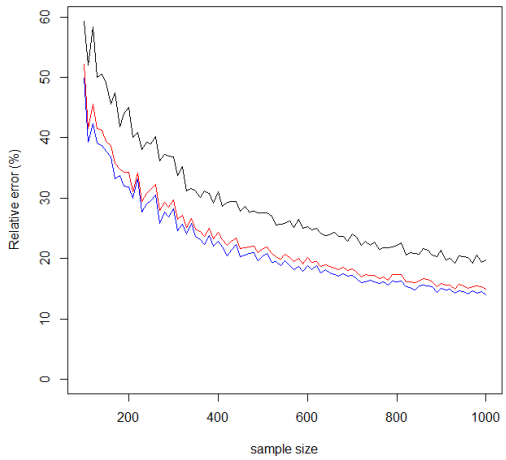

FIGURE 6. Evolution of the relative errors (mean square error divided by the reference value) on the estimated superquantiles in function of the sample size. In black: classical superquantile; in red: geometrical superquantile; in blue: harmonic superquantile. The reference value has been calculated with 10^7 simulations.

Proof of ii) :
For every (measurable) function f, we denote

$$\mathbb{E}_\alpha[f(X)] = \mathbb{E}[f(X)|X \geq F_X^{-1}(\alpha)].$$

Let X and X' be two real-valued random variable. The Bregman superquantile associated to γ is

$$\gamma'^{-1}\left(\mathbb{E}_\alpha[\gamma'(X)]\right) .$$

According to Definition 1.1, it is *homogeneous* if, for every $\lambda > 0$,

$$\gamma'^{-1}\left(\mathbb{E}_\alpha[\gamma'(\lambda X)]\right) = \lambda\gamma'^{-1}\left(\mathbb{E}_\alpha[\gamma'(X)]\right) .$$

As γ' and $x \mapsto (\gamma'(x) - \gamma'(1))/\gamma''(1)$ yield the same superquantiles, one may assume without loss of generality that $\gamma'(1) = 0$ and that $\gamma''(1) = 1$

First, it is easy to check that the condition given is sufficient. For simplicity, we write $\phi = \gamma'$. If $\phi(x) = (x^\beta - 1)/\beta$, then $\phi^{-1}(y) = (1+\beta y)^{1/\beta}$ and

$$\begin{aligned}\phi^{-1}\left(\mathbb{E}_\alpha[\phi(\lambda X)]\right) &= \left(1+\beta\mathbb{E}_\alpha\left[\frac{(\lambda X)^\beta - 1}{\beta}\right]\right)^{\frac{1}{\beta}} \\ &= \lambda\left(\mathbb{E}_\alpha\left[X^\beta\right]\right)^{1/\beta} \\ &= \lambda\left(1+\beta\mathbb{E}_\alpha\left[\frac{X^\beta - 1}{\beta}\right]\right)^{\frac{1}{\beta}} \\ &= \lambda\phi^{-1}\left(\mathbb{E}_\alpha[\phi(X)]\right)\,.\end{aligned}$$

If $\phi(x) = \ln(x)$, then $\phi^{-1}(y) = \exp(y)$ and

$$\begin{aligned}\phi^{-1}\left(\mathbb{E}_\alpha[\phi(\lambda X)]\right) &= \exp\left(\mathbb{E}_\alpha\left(\ln(\lambda X)\right)\right) \\ &= \exp\left(\mathbb{E}_\alpha\left(\ln(\lambda)\right) + \mathbb{E}_\alpha\left(\ln(X)\right)\right) \\ &= \lambda\exp\left(\mathbb{E}_\alpha\left(\ln(X)\right)\right) \\ &= \lambda\phi^{-1}\left(\mathbb{E}_\alpha[\phi(X)]\right)\,.\end{aligned}$$

For the other implication, let $y > 0$. Let Y be a random variable with distribution $\mathbb{P}$ such that, denoting $a = y \wedge 1$, $\mathbb{P}(du) = \alpha a^{-1} 1_{[0,a]}(u)du + (1-\alpha)p\delta_y + (1-\alpha)(1-p)\delta_1$. Its quantile of order α is $F_Y^{-1}(\alpha) = a$. The conditional distribution of Y given $Y \geq F_Y^{-1}(\alpha)$ is $(1-p)\delta_1 + p\delta_y$, and $\mathbb{E}_\alpha[\phi(Y)] = (1-p)\phi(1) + p\phi(y)$. The homogeneity property and the assumption $\phi(1) = 0$ imply that

$$\phi^{-1}\left((1-p)\phi(\lambda) + p\phi(\lambda y)\right) = \lambda\phi^{-1}\left(p\phi(y)\right)\,.$$

By assumption, the expressions on both sides are smooth in p and y. Taking the derivative in p at $p = 0$ yields

$$\frac{\phi(\lambda y) - \phi(\lambda)}{\phi'(\lambda)} = \lambda\frac{\phi(y)}{\phi'(1)}\,,$$

and hence, as $\phi'(1) = 1$,

$$\phi(\lambda y) - \phi(\lambda) = \lambda\phi'(\lambda)\phi(y)\,.$$

By differentiating with respect to y, one gets

$$\phi'(\lambda y) = \phi'(\lambda)\phi'(y)\,. \tag{3}$$

Let ψ be defined on $\mathbb{R}$ by $\psi(z) = \ln\left(\phi'(\exp(z))\right)$. One readily checks that Equation (3) yields

$$\psi\left(\ln(y) + \ln(\lambda)\right) = \psi\left(\ln(y)\right) + \psi\left(\ln(\lambda)\right)\,.$$

This equation holds for every $y, \lambda > 0$. This is well known to imply the linearity of ψ: there exists a real number β such that for all $z \in \mathbb{R}$,

$$\ln\left(\phi'(\exp(z))\right) = \psi(z) = \beta z\,.$$

Thus, $\phi'(\exp(z)) = \exp(\beta \exp(z))$, that is $\phi'(y) = y^\beta$ for all $y > 0$. For $\alpha = -1$, one obtains $\phi(y) = \ln(y)$. Otherwise, taking into account the constraint $\phi(1) = 0$, this yields

$$\phi(y) = \frac{y^{1+\beta} - 1}{1+\beta}\,.$$

Proof of iii) :

Let X and X' be two real-valued random variable. Since γ is convex, γ' is non-decreasing. So, to deal with expectation, it is the same thing to show the subadditivity or that

$$\gamma'\left(\frac{\mathcal{R}(X+X')}{2}\right) \leq \gamma'\left(\frac{\mathcal{R}(X)+\mathcal{R}(X')}{2}\right).$$

We set $S := X + X'$. Using the concavity of γ, we have

$$\begin{aligned}\gamma'\left(\frac{\mathcal{R}(X)+\mathcal{R}(X')}{2}\right) &\leq \frac{1}{2}\left[\gamma'\Big(\mathcal{R}(X)\Big) + \gamma'\left(\mathcal{R}(X')\right)\right] \\ &= \frac{1}{2(1-\alpha)}\mathbb{E}\left(\gamma'(X)\mathbf{1}_{X\geq q_\alpha^X} + \gamma'(X')\mathbf{1}_{X'\geq q_\alpha^{X'}}\right).\end{aligned}$$

But

$$\gamma'\left(\frac{\mathcal{R}(S)}{2}\right) = \frac{1}{2(1-\alpha)}\mathbb{E}\left(\gamma'(S)\mathbf{1}_{S\geq q_\alpha^S}\right).$$

So we want to show that

$$\mathbb{E}\left(\gamma'(X)\mathbf{1}_{X\geq q_\alpha^X} + \gamma'(X')\mathbf{1}_{X'\geq q_\alpha^{X'}} - \gamma'(S)\mathbf{1}_{S\geq q_\alpha^S}\right) \geq 0.$$

The sud-additivity hypothesis allows us to use the same argument as in [2] for the classical superquantile :

$$\begin{aligned}&\mathbb{E}\Big(\gamma'(X)\mathbf{1}_{X\geq q_\alpha^X} + \gamma'(X')\mathbf{1}_{X'\geq q_\alpha^{X'}} - \gamma'(S)\mathbf{1}_{S\geq q_\alpha^S}\Big) \\ &\geq \mathbb{E}\Big(\gamma'(X)\mathbf{1}_{X\geq q_\alpha^X} + \gamma'(X')\mathbf{1}_{X'\geq q_\alpha^{X'}} - \gamma'(X)\mathbf{1}_{S\geq q_\alpha^S} - \gamma'(X')\mathbf{1}_{S\geq q_\alpha^S}\Big) \\ &\geq \gamma'(q_\alpha^X)\mathbb{E}\Big(\mathbf{1}_{X\geq q_\alpha^X} - \mathbf{1}_{S\geq q_\alpha^S}\Big) + \gamma'(q_\alpha^{X'})\mathbb{E}\Big(\mathbf{1}_{X'\geq q_\alpha^{X'}} - \mathbf{1}_{S\geq q_\alpha^S}\Big) \\ &= 0.\end{aligned}$$

Finally, we show the closeness under the same assumption as just before. Let be $(X_h)_h > 0$ satisfying the hypothesis. By subadditivity we have

$$\mathcal{R}(X) \leq \mathcal{R}(X_h) + \mathcal{R}(X_h - X) \leq 0 + \mathcal{R}(X_h - X).$$

Then denoting $Y_h = X_h - X$, it is enough to show that

$$Y_h \underset{L^2,\, n\to+\infty}{\longrightarrow} 0 \Longrightarrow \mathcal{R}(Y_h) \underset{n\to+\infty}{\longrightarrow} 0$$

to conclude. Thanks to the concavity of γ' we can use Jensen inequality

$$\gamma'^{-1}\left[\frac{\mathbb{E}\Big(\gamma'(Y_h)\mathbf{1}_{Y_h\geq F_Y^{-1}(\alpha)}\Big)}{1-\alpha}\right]\leq\frac{\mathbb{E}\left(Y_h\mathbf{1}_{Y_h\geq F_{Y_h}^{-1}(\alpha)}\right)}{1-\alpha}.$$

We conclude with Cauchy-Schwartz inequality

$$\frac{\mathbb{E}\left(Y_h\mathbf{1}_{Y_h\geq F_{Y_h}^{-1}(\alpha)}\right)}{1-\alpha}\leq\mathbb{E}((Y_h)^2)^{\frac{1}{2}}\frac{\mathbb{E}\left(\mathbf{1}^2_{Y_h\geq F_{Y_h^{-1}(\alpha)}}\right)^{\frac{1}{2}}}{1-\alpha}=||Y_h||^2\sqrt{1-\alpha}\underset{h\to 0}{\longrightarrow}0.$$

□

6.2. **Proof of Proposition 3.1 : asymptotic behavior of the plug-in estimator of the superquantile.**

6.3. **Mathematical tools.** We first give some technical or classical results that we will use in the forthcoming proofs.

6.3.1. *Ordered statistics and Beta function.* Let us recall some results of ordered statistics (see [11]). Let $(Y_i)_{i=1\dots n+1}$ be an independent sample having the standard exponential distribution. It's well known that

$$U_{(i)}:=\sum_{j=1}^{i}Y_j\left(\sum_{j=1}^{n+1}Y_j\right)^{-1} \tag{4}$$

has the same distribution as the ith ordered statistics of an i.i.d sample of size n uniformly distributed on $[0,1]$, that is Beta distribution of parameters i and $n-i+1$ denoted $\mathcal{B}(i,n-i+1)$. It is also known that this equality in law holds

$$X_{(i)}\overset{\mathcal{L}}{=}F^{-1}(U_{(i)}).$$

Recall that the $\mathcal{B}(a,b)$ distribution has the following density

$$f_{\mathcal{B}(a,b)}(x)=\frac{x^{a-1}(x-1)^{b-1}}{B(\alpha,\beta)},\ a,b>0$$

where

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{5}$$

A classical property of the Beta function is

$$\forall(x,y)\in\mathbb{R}^+,\ B(x+1,y)=\frac{x}{x+y}B(x,y). \tag{6}$$

Generalizing the definition of the factorials, we set for $n\in\mathbb{N}^*$

$$\left(n-\frac{1}{2}\right)! := \left(n-\frac{1}{2}\right)\left(n-1-\frac{1}{2}\right)\dots\left(\frac{1}{2}\right),$$

we have for $i \in \mathbb{N}^*, n \geq i+2$

$$B(i, n-i-\frac{5}{2}+1) = \frac{(i-1)!\left(n-i-2-\frac{1}{2}\right)!}{\left(n-2-\frac{1}{2}\right)!}, \tag{7}$$

$$\left(n-\frac{1}{2}\right)! = \frac{(2n)!}{(2^n)^2 n!}. \tag{8}$$

Indeed, equation (7) comes directly from the definition (5) and to see the equation (8), we fix $k = \frac{1}{2}$ and notice that

$$2^n(n-k)! = (2n-1)(2n-3)\dots 3 \times 1 = \frac{(2n)!}{2n(2n-2)\dots 6 \times 4 \times 2} = \frac{(2n)!}{2^n n!}.$$

6.3.2. *Technical lemma.*

Lemma 6.1. *Let $\delta > 1$. Then $n^{-1}\sum_{i=\lfloor n\alpha\rfloor}^{n-1}(1-\frac{i}{n+1})^{-\delta} = \mathcal{O}(\sqrt{n})$ if and only if $\delta \leq \frac{3}{2}$.*

Proof. Let $\delta > 1$. We have to characterize the δ for which $n^{-\frac{3}{2}}\sum_{i=\lfloor n\alpha\rfloor}^{n-1}(1-\frac{i}{n+1})^{-\delta}$ is bounded when n goes to infinity. We set $j := n+1-i$. The sum becomes

$$n^{-\frac{3}{2}}\sum_{i=2}^{n+1-\lfloor n\alpha\rfloor}\left(\frac{j}{n+1}\right)^{\delta} \underset{n\to+\infty}{\sim} n^{\delta-\frac{3}{2}}\sum_{i=2}^{n+1-\lfloor n\alpha\rfloor}\frac{1}{j^{\delta}} \underset{n\to+\infty}{\sim} n^{\delta-\frac{3}{2}}\zeta(\delta),$$

where ζ denote the Zeta function. The conclusion of the lemma holds.

Finally, our problem is equivalent to characterize the set of δ which are superior than 1 and such that $n^{\delta-\frac{1}{2}}$ is bounded. This set is clearly $]1, \frac{3}{2}]$.

□

6.3.3. *A corollary of Lindenberg-Feller theorem.* To prove the asymptotic normality, we use a central limit theorem which is a corollary of the Lindeberg-Feller theorem (see lemma 1 in [8]).

Proposition 6.1. *Let $(Y_1, \dots, Y_n)$ be an independent sample of exponential variables of parameter 1 and $(\alpha_{j,n})_{j\leq n, n\geq 2}$ be a triangle array of real numbers.*

If $Q_n = n^{-1}\sum_{j=1}^{n}\alpha_{j,n}(Y_j-1)$ and $\sigma_n^2 = \frac{1}{n}\sum_{j=1}^{n}\alpha_{j,n}^2$, then

$$\frac{\sqrt{n}Q_n}{\sigma_n} \Longrightarrow \mathcal{N}(0,1)$$

if and only if $\max_{1\leq j\leq n}|\alpha_{j,n}| = o(n^{\frac{1}{2}}\sigma_n)$.

If furthermore σ_n converges in probability to σ then by Slutsky's lemma

$$\sqrt{n}Q_n \Longrightarrow \mathcal{N}(0,\sigma^2).$$

6.4. **Proof of i) of Theorem 3.1 : consistency of the plug-in estimator.**

Proof. We aim to show consistency of the estimator.

$$\frac{1}{(1-\alpha)n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)}.$$

Let us first notice that

$$Q_\alpha = \frac{\mathbb{E}\left(X\mathbf{1}_{X\geq F_X^{-1}(x)(\alpha)}\right)}{1-\alpha} = \frac{\int_{\mathbb{R}} x\mathbf{1}_{X\geq F_X^{-1}(\alpha)} f_X(x)dx}{1-\alpha} = \frac{\int_\alpha^1 F_X^{-1}(y)dy}{1-\alpha}.$$

Thus, we need to show that

$$\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)} - \int_\alpha^1 F_X^{-1}(y)dy \underset{n\to+\infty}{\longrightarrow} 0 \ a.s.$$

In the sequel we omit the index X in F_X^{-1} because there is no ambiguity. Let us introduce the two following quantities.

$$A_n = \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)} - \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}\left(\frac{i}{n+1}\right),$$

and

$$B_n = \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}\left(\frac{i}{n+1}\right) - \int_\alpha^1 F^{-1}(y)dy.$$

A_n converges to 0 in L^1. Indeed, denoting by F_n the empirical cumulative distribution function, we have $X_{(i)} = F_n^{-1}\left(\frac{i}{n}\right)$ and

$$\begin{aligned}\mathbb{E}\left(|A_n|\right) &= \mathbb{E}\left(\left|\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F_n^{-1}\left(\frac{i}{n+1}\right) - F^{-1}\left(\frac{i}{n+1}\right)\right|\right)\\ &\leq \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} \mathbb{E}\left(\left|F_n^{-1}\left(\frac{i}{n+1}\right) - F^{-1}\left(\frac{i}{n+1}\right)\right|\right).\end{aligned}$$

We know by recalling 6.3.1 that $X_{(i)} = F_n^{-1}\left(\frac{i}{n+1}\right)$ is distributed like the ith ordered statistic of a uniform sample. Thus, defining $U_{(i)}$ with law $\mathcal{B}(i, n+1-i)$, it holds that,

$$\mathbb{E}\left(|A_n|\right) = \mathbb{E}\left(\left|F^{-1}\left(U_{(i)}\right) - F^{-1}\left(\frac{i}{n+1}\right)\right|\right).$$

By the mean value theorem, there exists $w_{(i)}^n \in]U_{(i)}, \frac{i}{n+1}[$ such that for n large enough:

$$
\begin{aligned}
&\mathbb{E}\left(\left|F_n^{-1}\left(\tfrac{i}{n+1}\right)-F^{-1}\left(\tfrac{i}{n+1}\right)\right|\right) \leq \mathbb{E}\left(\left|U_{(i)}-\tfrac{i}{n+1}\right|\frac{1}{f\left(F^{-1}(w_{(i)}^n)\right)}\right)\\
&\leq \sqrt{\mathrm{Var}(U_{(i)})}\sqrt{E\left[\left(\frac{1}{f\left(F^{-1}(w_{(i)}^n)\right)}\right)^2\right]} \qquad (9)\\
&\leq \sqrt{\tfrac{i(n-i+1)}{(n+1)^2(n+2)}}\max\left(\sqrt{E\left[\left(\frac{1}{f\left(F^{-1}\left(\frac{i}{n+1}\right)\right)}\right)^2\right]},\sqrt{E\left[\left(\frac{1}{f\left(F^{-1}(U_{(i)})\right)}\right)^2\right]}\right).
\end{aligned}
$$

Indeed, l (the derivative of F^{-1}) in non decreasing in the neighborhood of 1. Let us now deal with the two terms in the maximum. As $l = o((1-t)^2)$ in the neighborhood of 1, for $\epsilon > 0$, there exists N such that for $n \geq N$ and $i \in [\lfloor n\alpha \rfloor, n]$

$$
\begin{aligned}
\sqrt{E\left[\left(\frac{1}{f(F^{-1}(U_{(i)})}\right)^2\right]} &\leq \epsilon\, \mathbb{E}\left[\frac{1}{(1-U_{(i)})^4}\right]\\
&= \frac{\epsilon}{B(i,n+1-i)}\int_0^1 x^{i-1}(1-x)^{n+1-i-4}dx.
\end{aligned}
$$

However, we have

$$
\begin{aligned}
\sqrt{\frac{i(n-i+1)}{(n+1)^2(n+2)}}\sqrt{E\left[\left(\frac{1}{f\left(F^{-1}(U_{(i)})\right)}\right)^2\right]} &= \sqrt{\frac{i(n-i+1)(i-1)!}{B(i,n+1-i)(n-4-(i-2))\dots(n-4)(n+1)^2(n+2)}}\\
&\underset{n\to+\infty}{\sim} \frac{\epsilon}{\sqrt{n}}\frac{1}{\sqrt{\frac{i}{n+1}}\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}.
\end{aligned}
$$

The second term of (9) can be calculated in the same way

$$
\sqrt{E\left[\left(\frac{1}{f(F^{-1})(\frac{i}{n})}\right)^2\right]} \leq \epsilon\, \mathbb{E}\left[\frac{1}{(1-\frac{i}{n})^4}\right] \underset{n\to+\infty}{\sim} \frac{\epsilon}{\sqrt{n}}\frac{\sqrt{\frac{i}{n+1}}}{\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}.
$$

Finally, the two sums

$$
P_{n,1} = \epsilon\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n}\frac{1}{\sqrt{n}\sqrt{\frac{i}{n+1}}\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}, \text{ and } P_{n,2} = \epsilon\,\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n}\frac{\sqrt{\frac{i}{n+1}}}{\sqrt{n}\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}
$$

converge to 0 when n goes to infinity thanks to Lemma 6.1.

So, A_n converges in L^1 to 0 when n goes to infinity, then it converges to 0 in probability.

Let us study the term $B_n = \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}\left(\frac{i}{n+1}\right) - \int_\alpha^1 F^{-1}(y)dy$ to show its a.s convergence to 0.

Remark 6.1. *To begin with, it is easy to show the convergence when the sum and the integral are truncated at $1-\epsilon$, $\forall 0<\epsilon<1$ (convergence of a Riemann sum for a continuous functions).*

Let us fix $\epsilon > 0$. We split the forthcoming sum in two parts denoting the first part S_n^1 and the second part S_n^2.

$$\frac{1}{n+1}\sum_{i=\lfloor n\alpha\rfloor}^{\lfloor n(1-\epsilon)\rfloor} F^{-1}(\frac{i}{n+1}) + \frac{1}{n+1}\sum_{\lfloor n(1-\epsilon)\rfloor+1}^{n-1} F^{-1}(\frac{i}{n+1}) := S_n^1 + S_n^2.$$

Since the quantile function is non-decreasing on $[\alpha, 1]$, we have :

$$\begin{aligned}
&\int_{\frac{\lfloor n\alpha\rfloor-1}{n+1}}^{\frac{\lfloor n(1-\epsilon)\rfloor}{n+1}} F^{-1}(t)dt + \int_{\frac{\lfloor n(1-\epsilon)\rfloor}{n+1}}^{\frac{n-1}{n+1}} F^{-1}(t)dt \\
&\leq \frac{1}{n+1}\sum_{i=\lfloor n\alpha\rfloor}^{\lfloor n(1-\epsilon)\rfloor} F^{-1}(\frac{i}{n+1}) + \frac{1}{n+1}\sum_{\lfloor n(1-\epsilon)\rfloor+1}^{n-1} F^{-1}(\frac{i}{n+1}) \\
&\leq \int_{\frac{\lfloor n\alpha\rfloor}{n+1}}^{\frac{\lfloor n(1-\epsilon)\rfloor}{n+1}} F^{-1}(t)dt + \int_{\frac{\lfloor n(1-\epsilon)\rfloor}{n+1}}^{\frac{n}{n+1}} F^{-1}(t)dt.
\end{aligned}$$

In the previous inequality, we denote C_n^1 the lower-bound of S_n^1 and D_n^1 its upper-bound. We use the same notation for S_n^2 with indexes 2. Then, we have :

$$(C_n^1 - S_n^1) + (C_n^2 - D_n^2) \leq (S_n^2 - D_n^2) \leq (D_n^1 - S_n^1)$$

Let us show that $C_n^2 - D_n^2$ converge to 0, to conclude (the convergence of $D_n^1 - S_n^1$ and $C_n^1 - S_n^1$ to 0 is true thanks to the Remark 6.1).

As in the neighborhood of 1 $l(t) = o\left((1-t)^{-2}\right)$, we also have $F^{-1}(t) = o\left((1-t)^{-1}\right)$.

Then, for $\epsilon > 0$, there exist N such that for $n \geq N$:

$$C_n^2 - D_n^2 = -\int_{\frac{n-1}{n+1}}^{\frac{n}{n+1}} F^{-1}(t)dt \leq \epsilon \int_{\frac{n-1}{n+1}}^{\frac{n}{n+1}} \frac{1}{1-t}dt = \epsilon \ln(2).$$

Finally, $S_n^2 - D_n^2$ converges to 0 a.s. So that, the same holds for B_n.

We have shown that $A_n + B_n$ converge to 0 in probability. So under our hypothesis, the superquantile is consistent in probability.

Remark 6.2. *Using the same arguments, we can show that under stronger hypothesis on the quantile function $F^{-1}(t) = o\left(\frac{1}{(1-t)^{\frac{1}{2}}}\right)$ (that is the case in ii) of Proposition 3.1), we have*

$$-\int_{\frac{n-1}{n+1}}^{\frac{n}{n+1}} F^{-1}(t)dt \leq \epsilon \int_{\frac{n-1}{n+1}}^{\frac{n}{n+1}} \frac{1}{(1-t)^{\frac{1}{2}}}dt = \epsilon - 2(1-\sqrt{2})\frac{1}{\sqrt{n}}.$$

Then

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}(\frac{i}{n+1}) - \int_{\alpha}^{1} F^{-1}(y)dy\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

We will use this result in the next part.

□

6.4.1. *Proof of ii) of Proposition 3.1 : asymptotic normality of the plug-in estimator.* Let us prove the asymptotic normality of the estimator of the superquantile. To begin with, we can make some technical remarks.

Remark 6.3. *The assumption on L implies that there exists $m_l < \frac{3}{2}$ and $m_{F^{-1}} < \frac{1}{2}$ such that $l(t) = O\left((1-t)^{-m_l}\right)$, and $F^{-1}(t) = O\left((1-t)^{-m_{F^{-1}}}\right)$. It also implies that in the neighborhood of 1, $L(t) = o\left((1-t)^{-\frac{5}{2}}\right)$.*

Proof. The proof stands in three steps. First we reformulate and simplify the problem and apply the Taylor Lagrange formula. Then, we show that the second order term converges to 0 in probability. In the third step, we identify the limit of the first order term.

Step 1 : Taylor-Lagrange formula

Let us first omit α^{-1}. We have to study the convergence in distribution of

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)} - \int_{\alpha}^{1} F^{-1}(y)dy\right).$$

We have already noticed (Remarks 6.2 and 6.3) that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}(\frac{i}{n+1}) - \int_{\alpha}^{1} F^{-1}(y)dy\right] \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus, Slutsky's lemma, allows us to study only the convergence in law of

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} X_{(i)} - \frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor}^{n} F^{-1}\left(\frac{i}{n+1}\right)\right].$$

The quantile function F^1 is $\mathcal{C}^2$ so that we may apply the order 1 Taylor-Lagrange formula. Using the same argument in the proof of i), we introduce $U_{(i)}$ a random variable distributed as a $\mathcal{B}(i, n+1-i)$. Considering an equality in law we then have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left[X_{(i)}-F^{-1}(\frac{i}{n+1})\right]\right)\overset{=}{\mathcal{L}}\sqrt{n}\left[\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left(U_{(i)}-\frac{i}{n+1}\right)\frac{1}{f\left(F^{-1}(\frac{i}{n+1})\right)}\right]$$
$$+\frac{1}{\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left[\int_{\frac{i}{n+1}}^{U_{(i)}}\frac{f'(F^{-1}(t))}{(f(F^{-1}(t)))^3}\left(U_{(i)}-t\right)dt\right].$$

Let us call $\sqrt{n}Q_n$ the first-order term and R_n the second-order one.

Step 2 : The second-order term converges to 0 in probability

Let us show that R_n converge to 0 in probability. Thanks to Markov's inequality, we have

$$\mathbb{P}(R_n>\epsilon)\leq\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\mathbb{E}\left(\left|\int_{\frac{i}{n+1}}^{U_{(i)}}L(t)(U_{(i)}-t)dt\right|\right).$$

Since the function L is non decreasing in the neighborhood of 1, for n large enough, we have

$$\begin{aligned}\mathbb{P}(R_n>\epsilon)&\leq\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\mathbb{E}\left(\max\left(\left|L(U_{(i)})\right|,\left|L(\frac{i}{n+1})\right|\right)\int_{\frac{i}{n+1}}^{U_{(i)}}\left(U_{(i)}-t\right)dt\right)\\&=\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\mathbb{E}\left(\max\left(\left|L(U_{(i)})\right|,\left|L(\frac{i}{n+1})\right|\right)\frac{\left(U_{(i)}-\frac{i}{n+1}\right)^2}{2}\right).\end{aligned}$$

As before, we study the two terms in the maximum separately. First, using the variance of the Beta distribution,

$$\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left|L(\frac{i}{n+1})\right|\mathbb{E}\left(\frac{\left(U_{(i)}-\frac{i}{n+1}\right)^2}{2}\right)=\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n}\left|L(\frac{i}{n+1})\right|\frac{i(n+1-i)}{2(n+1)^2(n+2)}.$$

But according to Remark 6.3, our assumption on L gives that for ϵ', there exists an integer N such that for $n\geq N$, for $i\in[\lfloor n\alpha\rfloor,n]$,

$$\begin{aligned}\mathbb{P}(R_n > \epsilon) &\leq \frac{1}{\epsilon\sqrt{n}} \sum_{i=\lfloor n\alpha\rfloor+1}^{n} \left| L(\frac{i}{n+1}) \right| \frac{i(n+1-i)}{2(n+1)^2(n+2)} \\ &\leq \frac{\epsilon'}{2\epsilon\sqrt{n}(n+2)} \sum_{i=\lfloor n\alpha\rfloor+1}^{n} \frac{\frac{i}{n+1}\left(1-\frac{i}{n+1}\right)}{\left(1-\frac{i}{n+1}\right)^{\frac{5}{2}}} \\ &= \frac{\epsilon'}{2\epsilon\sqrt{n}(n+2)} \sum_{i=\lfloor n\alpha\rfloor+1}^{n} \frac{\frac{i}{n+1}}{\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}.\end{aligned}$$

Further the convergence to 0 when n goes to infinity of the first term in the maximum holds thanks to the Lemma 6.1.

We have now to deal with the second term in the maximum. Using our assumptions, we have, for ϵ' and n large enough,

$$\frac{1}{\epsilon\sqrt{n}} \sum_{i=\lfloor n\alpha\rfloor+1}^{n} \mathbb{E}\left(\left|L(U_{(i)})\right| \frac{\left(U_{(i)} - \frac{i}{n+1}\right)^2}{2} \right) \leq \frac{\epsilon'}{2\epsilon\sqrt{n}} \sum_{i=\lfloor n\alpha\rfloor+1}^{n} \mathbb{E}\left(\frac{\left(U_{(i)} - \frac{i}{n+1}\right)^2}{\left(U_{(i)}-1\right)^{\frac{5}{2}}} \right).$$

As $U_{(i)}$ has a Beta distribution, we can write that,

$$\begin{aligned}\mathbb{E}\left(\frac{\left(U_{(i)} - \frac{i}{n+1}\right)^2}{\left(U_{(i)}-1\right)^{\frac{5}{2}}} \right) &= \frac{1}{B(i,n+1-i)} \int_0^1 x^{i-1}(1-x)^{n-i-\frac{5}{2}} \left(x - \frac{i}{n+1}\right)^2 dx \\ &= \frac{1}{B(i,n+1-i)} \Big(\int_0^1 x^{i+1}(1-x)^{n-i-\frac{5}{2}} dx - 2\frac{i}{n+1} \int_0^1 x^i (1-x)^{n-i-\frac{5}{2}} dx \\ &+ \left(\frac{i}{n+1}\right)^2 \int_0^1 x^{i-1}(1-x)^{n-i-\frac{5}{2}} \Big).\end{aligned}$$

Let us call this last quantity I_n^i. To have $n - i - \frac{5}{2} + 1$ non negative, we cut the sum and deal only with the terms for i from $\lfloor n\alpha \rfloor$ to $n-2$. Then, we obtain :

$$
\begin{aligned}
I_n^i &= \frac{1}{B(i,n+1-i)}\Big(\int_0^1 x^{i+1}(1-x)^{n-i-\frac{5}{2}}dx - 2\frac{i}{n+1}\int_0^1 x^i(1-x)^{n-i-\frac{5}{2}}dx \\
&+ (\frac{i}{n+1})^2\int_0^1 x^{i-1}(1-x)^{n-i-\frac{5}{2}}\Big) \\
&= \frac{1}{B(i,n+1-i)}\Big(B(i+2,n-i-\frac{5}{2}+1) - 2\frac{i}{n+1}B\left(i+1,n-i-\frac{5}{2}+1\right) \\
&+ (\frac{i}{n+1})^2 B\left(i,n-i-\frac{5}{2}+1\right).
\end{aligned}
$$

So that using (6) we obtain

$$
\begin{aligned}
I_n^i &= \frac{1}{B(i,n+1-i)}\Big(B\left(i+2,n-i-\frac{5}{2}+1\right) - 2\frac{i}{n+1}B\left(i+1,n-i-\frac{5}{2}+1\right) \\
&+ (\frac{i}{n+1})^2 B\left(i,n-i-\frac{5}{2}+1\right) \\
&= \frac{B\left(i,n-i-\frac{5}{2}+1\right)}{B(i,n+1-i)}\Big(\frac{i(i+1)}{(n-\frac{5}{2}+2)(n-\frac{5}{2}+1)} \\
&- 2\frac{i^2}{(n-\frac{5}{2}+1)(n+1)}\frac{i(i+1)}{(n-\frac{5}{2}+2)(n-\frac{5}{2}+1)} + (\frac{i}{n+1})^2\frac{i(i+1)}{(n-\frac{5}{2}+2)(n-\frac{5}{2}+1)}\Big).
\end{aligned}
$$

Let E_n^i be such that $I_n^i = \frac{B(i,n-i-\frac{5}{2}+1)}{B(i,n+1-i)}E_n^i$. Expanding E_i^n gives when n goes to infinity

$$
E_i^n \sim \frac{1}{n}\frac{i}{n+1}\left(1-\frac{i}{n+1}\right).
$$

Let us study the term $\frac{B(i,n-i-\frac{5}{2}+1)}{B(i,n+1-i)}$. Using (5) and (8), we obtain

$$
\begin{aligned}
\frac{B\left(i,n-i-\frac{5}{2}+1\right)}{B(i,n+1-i)} &= \frac{n!}{\left(n-2-\frac{1}{2}\right)!}\frac{\left(n-i-2-\frac{1}{2}\right)!}{(n-i)!} \\
&= \frac{n(n-1)}{(n-i-1)(n-i)}\frac{(2(n-i-2))!((n-2)!)^2 2^{2i}}{((n-i-2)!)^2(2(n-2))!}
\end{aligned}
$$

Since each i can be written as $i = \lfloor n\beta \rfloor$ with $\beta < 1$, $n-i$ goes to infinity when n goes to infinity and we can apply the Stirling formula:

$$\frac{(2(n-i-2))!}{((n-i-2)!)^2} \underset{n\to+\infty}{\sim} \frac{\sqrt{2(n-i-2)2\pi}\left(\frac{2(n-i-2)}{e}\right)^{2(n-i-2)}}{2\pi(n-i-2)\left(\frac{n-i-2}{e}\right)^{n-i-2}}$$
$$\underset{n\to+\infty}{\sim} \frac{2^{2(n-i-2)}}{\sqrt{\pi(n-i-2)}}.$$

Likewise,

$$\frac{(2(n-2))!}{((n-2)!)^2} \underset{n\to+\infty}{\sim} \frac{2^{2(n-2)}}{\sqrt{\pi(n-2)}}.$$

Then, when n goes to infinity

$$\frac{B\left(i, n-i-\frac{5}{2}+1\right)}{B(i, n+1-i)} \underset{n\to+\infty}{\sim} \frac{1}{(1-\frac{i}{n+1})^{\frac{5}{2}}}.$$

Hence we obtain

$$I_i^n \underset{n\to+\infty}{\sim} \frac{1}{n}\frac{\frac{i}{n+1}}{(1-\frac{i}{n+1})^{\frac{3}{2}}}.$$

Finally, for n large enough, we have

$$I_i^n \leq 2\frac{1}{n}\frac{\frac{i}{n+1}}{(1-\frac{i}{n+1})^{\frac{3}{2}}}.$$

and thanks to our assumptions on L, for ϵ' and n large enough, we have

$$\frac{1}{\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n-2}\mathbb{E}\left(\left|L(U_{(i)})\right|\frac{\left(U_{(i)}-\frac{i}{n+1}\right)^2}{2}\right) \leq \frac{\epsilon'}{2\epsilon\sqrt{n}}\sum_{i=\lfloor n\alpha\rfloor+1}^{n-2}\mathbb{E}\left(\frac{(U_{(i)}-\frac{i}{n+1})^2}{\left(U_{(i)}-1\right)^{\frac{5}{2}}}\right)$$
$$\leq \frac{\epsilon'}{\epsilon\sqrt{n}}\frac{1}{n}\sum_{i=\lfloor n\alpha\rfloor+1}^{n-2}\frac{\frac{i}{n+1}}{\left(1-\frac{i}{n+1}\right)^{\frac{3}{2}}}.$$

To conclude the second term in the maximum converges to 0 thanks the Lemma 6.1.

Remark 6.4. *The terms for $i=1$ and $i=n-1$ are of leaser order of the sum, so they converge to 0 too.*

Finally, the remaining term R_n converges to 0 in probability so we can now focus on the first order term.

Step 3 : Identification of the limit

Our goal is to find the limit of $\sqrt{n}Q_n$. Let us reorganize the expression of Q_n to have a more classical form (sum of independent random variables) and to allow the use of the Proposition 6.1.

Denoting by

$$\bar{Y} = \frac{\sum_{j=1}^{n+1} Y_j}{n+1},$$

we have thanks to (4)

$$\begin{aligned}
Q_n &= \frac{1}{n} \sum_{i=\lfloor n\alpha \rfloor+1}^{n} \left(\frac{\sum_{j=1}^{i} Y_j}{\sum_{j=1}^{n+1} Y_j} - \frac{i}{n+1} \right) l(\frac{i}{n+1}) \\
&= \frac{1}{n} \sum_{j=1}^{n} \left[\left(\frac{Y_j}{\sum_{j=1}^{n+1} Y_j} - \frac{1}{n+1} \right) \sum_{i=\sup(\lfloor n\alpha \rfloor+1, j)} l(\frac{i}{n+1}) \right] \\
&= \frac{n+1}{\sum_{j=1}^{n+1} Y_j} \frac{1}{n(n+1)} \left[\sum_{j=1}^{\lfloor n\alpha \rfloor+1} \left((Y_j - \bar{Y}) \sum_{i=\lfloor n\alpha \rfloor+1}^{n} l(\frac{i}{n+1}) \right) + \sum_{j=\lfloor n\alpha \rfloor+2}^{n} \left((Y_j - \bar{Y}) \sum_{i=j}^{n} l(\frac{i}{n+1}) \right) \right].
\end{aligned}$$

where we have permuted the two sums. The law of large numbers gives that $n+1 \left(\sum_{j=1}^{n+1} Y_j \right)^{-1}$ converges a.s to 1 when n goes to infinity. Then, thanks to Slutsky's lemma, we only need to study

$$\frac{1}{n(n+1)} \left[\sum_{j=1}^{\lfloor n\alpha \rfloor+1} \left((Y_j - \bar{Y}) \sum_{i=\lfloor n\alpha \rfloor+1}^{n} l(\frac{i}{n+1}) \right) + \sum_{j=\lfloor n\alpha \rfloor+2}^{n} \left((Y_j - \bar{Y}) \sum_{i=j}^{n} l(\frac{i}{n+1}) \right) \right].$$

We set $\forall j \leq n$, $G_j^n := \sum_{i=j}^{n} l(\frac{i}{n+1})$, $G_{n+1}^n := 0$, $H^n := \sum_{j=\lfloor n\alpha \rfloor+2}^{n} G_j$. Then

$$
\begin{aligned}
Q_n &= \frac{1}{n(n+1)}\left[\sum_{j=1}^{\lfloor n\alpha\rfloor+1}\left(\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n+1}\right)Y_j+\sum_{\lfloor n\alpha\rfloor+2}^{n+1}\left(G^n_j-\frac{H^n}{n+1}+G^n_{\lfloor n\alpha\rfloor+1}\frac{-1-\lfloor n\alpha\rfloor}{n+1}\right)Y_j\right]\\
&= \frac{1}{n(n+1)}\left[\sum_{j=1}^{\lfloor n\alpha\rfloor+1}\left(\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n+1}\right)(Y_j-1)\right.\\
&+\left.\sum_{\lfloor n\alpha\rfloor+2}^{n+1}\left(G^n_j-\frac{H^n}{n+1}+G^n_{\lfloor n\alpha\rfloor+1}\frac{-1-\lfloor n\alpha\rfloor}{n+1}\right)(Y_j-1)\right],
\end{aligned}
$$

because

$$
\begin{aligned}
&\frac{1}{n(n+1)}\left[\sum_{j=1}^{\lfloor n\alpha\rfloor+1}\left((\frac{n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n+1}\right)+\sum_{\lfloor n\alpha\rfloor+2}^{n+1}\left(G^n_j-\frac{H^n}{n+1}+G^n_{\lfloor n\alpha\rfloor+1}\frac{-1-\lfloor n\alpha\rfloor}{n+1}\right)\right]\\
&=\frac{1}{n(n+1)}\left[\left(\frac{G^n_{\lfloor n\alpha\rfloor}(n-\lfloor n\alpha\rfloor)-H^n}{n+1}\right)(\lfloor n\alpha\rfloor+1)\right.\\
&+\left.\left(\frac{G^n_{\lfloor n\alpha\rfloor}(-1-\lfloor n\alpha\rfloor)}{n+1}(n-\lfloor n\alpha\rfloor)-\frac{H^n}{n+1}(n-\lfloor n\alpha\rfloor)+H^n\right)\right]\\
&=0.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
Q_n &= \frac{1}{n(n+1)}\left[\sum_{j=1}^{\lfloor n\alpha+1}\left(\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n+1}\right)(Y_j-1)\right.\\
&+\left.\sum_{\lfloor n\alpha\rfloor+2}^{n+1}\left(G^n_j-\frac{H^n}{n+1}+G^n_{\lfloor n\alpha\rfloor+1}\frac{-1-\lfloor n\alpha\rfloor}{n+1}\right)(Y_j-1)\right]\\
&=\frac{1}{n+1}\sum_{j=1}^{n+1}\alpha_{j,n}(Y_j-1),
\end{aligned}
$$

where

$$
\alpha_{j,n}=\left(\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n(n+1)}\right),\ \forall j\le\lfloor n\alpha\rfloor+1
$$

and

$$
\alpha_{j,n}=\left(\frac{G^n_j(n+1)-H^n-G^n_{\lfloor n\alpha\rfloor+1}(1+\lfloor n\alpha\rfloor)}{n(n+1)}\right),\ \forall j\ge\lfloor n\alpha\rfloor+2.
$$

Let us check the assumptions of the Proposition 6.1. To begin with, let us show that σ_n^2 converges. We have

$$\begin{aligned}
\sigma_n^2 &= \frac{1}{n+1}\sum_{j=1}^{n+1}\alpha_{j,n}^2 \\
&= \frac{\lfloor n\alpha\rfloor+1}{n+1}\left[\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1}-H^n}{n(n+1)}\right]^2 \\
&+ \frac{1}{n+1}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\left(\frac{G_j^n(n+1)}{n(n+1)}\right)^2 + 2\frac{1}{n+1}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\frac{G_j^n(n+1)(-H^n-G^n_{\lfloor n\alpha\rfloor+1}(1+\lfloor n\alpha\rfloor))}{n^2(n+1)^2} \\
&+ \frac{1}{n+1}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\left(\frac{-H^n-G^n_{\lfloor n\alpha\rfloor+1}(1+\lfloor n\alpha\rfloor)}{n(n+1)}\right)^2.
\end{aligned}$$

Let us work with the two terms which depend on G_j^n. The first term can be expanded as

$$\begin{aligned}
\frac{1}{n+1}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\left(\frac{G_j^n(n+1)}{n(n+1)}\right)^2 &= \frac{1}{n^2(n+1)}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\left(G_j^n\right)^2 \\
&= \frac{1}{n^2(n+1)}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\left(\sum_{i=j}^{n} l\left(\frac{i}{n+1}\right)\right)^2 \\
&= \frac{1}{n(n+1)}\sum_{i_1=\lfloor n\alpha\rfloor+2}^{n+1}\sum_{i_2=\lfloor n\alpha\rfloor+2}^{n+1}\left(\frac{-1+(i_1\wedge i_2)-\lfloor n\alpha\rfloor}{n+1}\right) l\left(\frac{i_1}{n+1}\right) l\left(\frac{i_2}{n+1}\right).
\end{aligned}$$

The second term may be rewritten as

$$2\frac{1}{n+1}\sum_{j=\lfloor n\alpha\rfloor+2}^{n+1}\frac{G_j^n(n+1)(-H^n-G^n_{\lfloor n\alpha\rfloor+1}(1\lfloor n\alpha\rfloor))}{n^2(n+1)^2} = -2\frac{(H^n)^2}{n^2(n+1)^2} - 2\frac{(1+\lfloor n\alpha\rfloor)}{(n+1)^2n^2}H^nG^n_{\lfloor n\alpha\rfloor+1}.$$

Finally,

$$\sigma_n^2 = \frac{\lfloor n\alpha \rfloor + 1}{n+1}\left[\frac{(n-\lfloor n\alpha\rfloor)G^n_{\lfloor n\alpha\rfloor+1} - H^n}{n(n+1)}\right]^2$$
$$+ \frac{1}{n(n+1)}\sum_{i_1=\lfloor n\alpha\rfloor+2}^{n+1}\sum_{i_2=\lfloor n\alpha\rfloor+2}^{n+1}\left(\frac{-1}{n+1} + \frac{\min(i_1,i_2)}{n+1} - \frac{\lfloor n\alpha\rfloor}{n+1}\right) l(\frac{i_1}{n+1})l(\frac{i_2}{n+1})$$
$$- 2\frac{(H^n)^2}{n^2(n+1)^2} - 2\frac{(1+\lfloor n\alpha\rfloor)}{(n+1)^3n^2}H^nG^n_{\lfloor n\alpha\rfloor+1} + \frac{n-\lfloor n\alpha\rfloor-1}{n+1}\left(\frac{H^n + G^n_{\lfloor n\alpha\rfloor+1}(1+\lfloor n\alpha\rfloor)}{n(n+1)}\right)^2.$$

Let us first notice that, if we denote

$$K^n = \sum_{i_1=\lfloor n\alpha\rfloor+2}^{n+1}\sum_{i_2=\lfloor n\alpha\rfloor+2}^{n+1}\frac{\min(i_1,i_2)}{n+1} l\left(\frac{i_1}{n+1}\right) l\left(\frac{i_2}{n+1}\right)$$

and

$$T^n = \sum_{i=\lfloor n\alpha\rfloor}^{n}\frac{i}{n} l\left(\frac{i}{n+1}\right)$$

then

$$H^n = nT^n - (\lfloor n\alpha\rfloor + 1)G^n_{\lfloor n\alpha\rfloor+1}.$$

So that

$$\sigma_n^2 \underset{n\to+\infty}{\sim} \alpha\frac{(G^n-T^n)^2}{n^2} + \frac{K^n - \alpha(G^n)^2}{n^2} - \frac{-2(T^n-\alpha G^n)^2}{n^2} - 2\frac{\alpha(G^nT^n - \alpha(G^n)^2)}{n^2} + \frac{(1-\alpha)(T^n)^2}{n^2}$$
$$\underset{n\to+\infty}{\sim} \frac{K^n - (T^n)^2}{n^2}.$$

Let us show that this last quantity converges to $\sigma^2 = \int_\alpha^1\int_\alpha^1 \frac{\min(x,y)-xy}{f(F^{-1}(x))f(F^{-1}(y))} < \infty$. Indeed it is a generalized Rieman sum. First, we show that the function

$$g : (x,y) \mapsto \frac{\min(x,y)-xy}{f(F^{-1}(x))f(F^{-1}(y))}$$

is integrable on $]\alpha,1[\times]\alpha,1[$. Indeed, around 1,

$$g(x,y) = O\left(\frac{\min(x,y)-xy}{(1-x)^{m_l}(1-y)^{m_l}}\right)$$

which is integrable on this domain because for β close to 1

$$\int_\alpha^\beta\int_\alpha^\beta \frac{\min(x,y)-xy}{(1-x)^{m_l}(1-y)^{m_l}}dxdy \sim C(\alpha)\beta(1-\beta)^{\frac{-2}{m_l}+3}.$$

and $m_l < \frac{3}{2}$.

Remark 6.5. *Here, we need that* $L(t) = O\left((1-t)^{-m_L}\right)$ *with* $m_L < \frac{5}{2}$ *instead of* $L(t) = o\left((1-t)^{-\frac{5}{2}}\right)$. *Indeed, when* $m_L = \frac{5}{2}$ *and so* $m_l = \frac{3}{2}$ *the integral is not finite.*

As we have already seen, the results on Riemann's sum in dimension 2, give by the continuity of the function $(x,y) \mapsto \frac{\min(x,y)-xy}{f(F^{-1}(x))f(F^{-1}(y))}$ that for all $\beta < 1$:

$$\sigma^2_{n,\beta} := \frac{1}{n^2} \sum_{i_1=\lfloor n\alpha \rfloor}^{\lfloor n\beta \rfloor} \sum_{i_1=\lfloor n\alpha \rfloor}^{\lfloor n\beta \rfloor} \frac{\frac{\min(i_1,i_2)}{n} - \frac{i_1 i_2}{n^2}}{f(F^{-1}\left(\frac{i_1}{n+1}\right))f(F^{-1}\left(\frac{i_2}{n+1}\right))} \longrightarrow \int_\alpha^\beta \int_\alpha^\beta \frac{\min(x,y)-xy}{f(F^{-1}(x))f(F^{-1}(y))} dxdy.$$

We have to study the remaining part of the sum to conclude. Let us fix β close to 1 and deal with

$$r^2_{n,\beta} := \frac{1}{n^2} \sum_{i_1=\lfloor n\beta \rfloor}^{n+1} \sum_{i_1=\lfloor n\beta \rfloor}^{n+1} \frac{\frac{\min(i_1,i_2)}{n} - \frac{i_1 i_2}{n^2}}{f\left(F^{-1}\left(\frac{i_1}{n+1}\right)\right) f\left(F^{-1}\left(\frac{i_2}{n+1}\right)\right)}.$$

In this case, the monotony will allow us to conclude with the Lebesgue theorem. First of all, let us notice that

$$r^2_{n,\beta} = \int_\beta^1 \int_\beta^1 g\left(\frac{\lfloor nx \rfloor}{n}, \frac{\lfloor ny \rfloor}{n}\right) dxdy.$$

Then, we aim to permute integration and limit.

1) Let (x,y) be fixed in $[\beta,1[\times[\beta,1[$ and n. Then
$$g\left(\frac{\lfloor nx \rfloor}{n}, \frac{\lfloor ny \rfloor}{n}\right) \longrightarrow g(x,y)$$
by continuity. And g is integrable on $[\beta,1[\times[\beta,1[$ as we saw before.
2) Let (x,y) be fixed in $[\beta,1[\times[\beta,1[$ and n. Let us denote $x_n = \frac{\lfloor nx \rfloor}{n}$ and $y_n = \frac{\lfloor ny \rfloor}{n}$. By hypothesis
$$g\left(\frac{\lfloor nx \rfloor}{n}, \frac{\lfloor ny \rfloor}{n}\right) \leq C \frac{\min(x_n,y_n) - x_n y_n}{(1-x_n)^{m_l}(1-y_n)^{m_l}}.$$
By separating the two cases and using monotony we obtain that
$$g\left(\frac{\lfloor nx \rfloor}{n}, \frac{\lfloor ny \rfloor}{n}\right) \leq Ch(x,y)$$
where
$$h : (x,y) \mapsto \frac{\min(x,y)}{(1-\min(x,y))^{m_l}\left(1-\max(x,y)^{m_l-1}\right)}$$
is integrable on $[\beta,1[\times[\beta,1[$.

Then, the Lebesgue theorem allows us to permute integration and limit so that, we have shown that $\sigma_n^2 \longrightarrow \sigma^2$and the first assumption of Proposition 6.1 holds.

Let now deal with the second assumption about the maximum of the $\alpha_{i,n}$.
For $j \leq \lfloor n\alpha \rfloor + 1$, we have

$$\alpha_{j,n} = \frac{(n - \lfloor n\alpha \rfloor + 1)G^n_{\lfloor n\alpha \rfloor + 1} - H^n}{n(n+1)}.$$

Using the previous computations, for n large enough we have

$$\frac{(\alpha_{j,n})^2}{n\sigma_n^2} \underset{n\to+\infty}{\sim} \frac{(G^n_{\lfloor n\alpha \rfloor+1} - \frac{T^n}{n})^2}{K_n - \frac{T_n^2}{n^2}} \frac{1}{n},$$

But the convergence

$$\frac{\left(K^n - \left(\frac{T^n}{n}\right)\right)^2}{n^4} \underset{n\to+\infty}{\longrightarrow} \int_\alpha^1 \int_\alpha^1 (\min(x,y) - xy) l(x) l(y) dx dy$$

implies the convergence

$$\frac{(G^n - \frac{T^n}{n})^2}{n^4} \underset{n\to+\infty}{\longrightarrow} \int_\alpha^1 (1-x) l(x) dx.$$

Indeed

$$\int_\alpha^1 \int_\alpha^1 (\min(x,y) - xy) l(x) l(y) dx dy = \int_\alpha^1 \int_\alpha^1 (y(1-x)) l(x) l(y) dx dy + \int_\alpha^1 x l(x) \int_x^1 (1-y) l(y) dy dx.$$

So that,

$$\frac{(\alpha_{j,n})^2}{n\sigma_n^2} \underset{n\to+\infty}{\sim} \frac{C}{n} \underset{n\to+\infty}{\longrightarrow} 0$$

when n goes to infinity. If $j \geq \lfloor n\alpha \rfloor + 2$ the same property holds as

$$\alpha_{j,n} = \frac{(n+1)G^n_j - H^n - G^n_{\lfloor n\alpha \rfloor}(n\alpha \rfloor + 1)}{n(n+1)} \underset{n\to+\infty}{\sim} \frac{(n+1)G_{j^n} - T^n}{n^2}.$$

Hence, we may apply Proposition 6.1 and conclude that

$$\sqrt{n} Q_n \Longrightarrow \mathcal{N}(O, \sigma^2).$$

where $\sigma^2 = \int_\alpha^1 \int_\alpha^1 (\min(x,y) - xy) l(x) l(y) dx dy$. Finally, just multiply by $(1-\alpha)^{-1}$ to get the final result.

Step 4 : Conclusion
The Slutsky lemma allows to conclude using the results of steps 1 and 3.

□

7. Conclusion

The superquantile was introduced because the usual quantile was not subadditive. This quantity is interesting because it satisfies the axioms of a coherent measure of risk. In this paper, we have introduced a new coherent measure of risk with the help of the Bregman divergence associated to a strictly convex function γ. Those Bregman superquantiles are then rich tools because of the diversity of the functions γ that can be chosen according to the problem we study. Moreover, we have introduced a Monte Carlo estimator of the Bregman superquantile which is statistically powerful thanks to the strictly convex (and so settling) function γ.

The theoretical properties obtained in this paper are confirmed on several numerical test cases. More precisely, geometrical and harmonic superquantiles are more robust than the classical superquantile. This robustness is particularly important in in finance and risk assessment studies. For instance, in risk assessment, when dealing with real data, geometrical and harmonic statistics have been proved to be more relevant than classical statistics. For example [7] prove the usefulness of the geometrical mean and variance for the analysis of air quality measurements. As an illustration, we have applied the geometrical and harmonic superquantiles on real data coming from a radiological impact code used in the nuclear industry.

Further studies will try to apply these criteria in probabilistic assessment of physical components reliability using numerical simulation codes [12]. However, Monte Carlo estimators are no longer applicable in this context and efficient estimators have to be developed. Ideas involving response surface technique should be developed (see for example [6] for quantile estimation and [3] for rare event probability estimation).

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