A Liouville theorem for the Degasperis-Procesi equation

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A LIOUVILLE THEOREM FOR THE DEGASPERIS-PROCESI EQUATION

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Abstract. We prove that the only global, strong, spatially periodic solution to the Degasperis-Procesi equation, vanishing at some point (t_0, x_0), is the identically zero solution. We also establish the analogue of such Liouville-type theorem for the Degasperis-Procesi equation with an additional dispersive term.

1. Introduction and main results

In this note we study spatially periodic solutions of the Degasperis-Procesi equation

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R} \\ u(t,x) = u(t,x+1), & t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Such equation attracted a considerable interest in the past few years, both for its remarkable mathematical properties (see, *e.g.* [5, 7]), and for its physical interpretation as an asymptotic model obtained from the water-wave system in shallow water regime. In this setting, the equation models moderate amplitude waves and u stands for a horizontal velocity of the water at a fixed depth, see [6, 10].

The associated potential $y = u - u_{xx}$ satisfies the equations

$$y_t + uy_x + 3u_x y = 0, \qquad y(t,x) = y(t,x+1), \qquad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

The Cauchy problem associated with (1.1) can be more conveniently reformulated as

$$\begin{cases} u_t + uu_x + \partial_x p * \left(\frac{3}{2}u^2\right) = 0, & t > 0, \quad x \in \mathbb{S} \\ u(0,x) = u_0(x), & x \in \mathbb{S} \end{cases} \tag{1.3}$$

where $\mathbb{S}$ is the circle and p the kernel of $(1 - \partial_x^2)^{-1}$, given by the continuous periodic function

$$p(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}. \tag{1.4}$$

It is well known (see, *e.g.*, [11]) that if $u_0 \in H^s(\mathbb{S})$, with $s > 3/2$, then the problem (1.3) possess a unique solution

$$u \in C([0,T), H^s(\mathbb{S})) \cap C^1([0,T), H^{s-1}(\mathbb{S})), \tag{1.5}$$

for some $T > 0$, depending only u_0.

The maximal time T^* of the above solution can be finite or infinite. For instance, if the initial potential $y_0 = y(0,\cdot)$ does not change sign, then it is known that $T^* = +\infty$, see [8]. On the

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other hand, several different blowup criteria were established, *e.g.*, in [8, 9, 11]: in the shallow water interpretation the finite time blowup corresponds to a wave breaking mechanism, as near the blowup time solutions remain bounded, but have an unbounded slope in at least one point.

The purpose of this note it to establish the following Liouville-type theorem:

Theorem 1.1. *The only global solution $u \in C([0,+\infty), H^s(\mathbb{S})) \cap C^1([0,+\infty), H^{s-1}(\mathbb{S}))$, with $s > 3/2$, to the Degasperis-Procesi equation vanishing at some point (t_0, x_0) is the identically zero solution.*

The Degasperis-Procesi equation is often written with the additional dispersive term $3\kappa u_x$ in the left-hand side of equation (1.1), where $\kappa \in \mathbb{R}$ is the dispersion parameter. In this more general setting the above theorem can be reformulated as follows:

Theorem 1.2. *Let $s > 3/2$. If $v \in C([0,+\infty), H^s(\mathbb{S})) \cap C^1([0,+\infty), H^{s-1}(\mathbb{S}))$ is a global solution to the Degasperis-Procesi equation with dispersion*

$$v_t + vv_x + \partial_x p * \left(\frac{3}{2}v^2 + 3\kappa v\right) = 0, \qquad t > 0, \quad x \in \mathbb{S}, \tag{1.6}$$

such that $v(t_0, x_0) = -\kappa$ at some point (t_0, x_0), then $v(t, x) \equiv -\kappa$ for all (t, x).

In fact, this second theorem can be reduced to the former, as $u(t, x) = v(t, x - \kappa t) + \kappa$, is a global solution of (1.3) if and only if v is a global solution of (1.6) with $u_0 = v_0 + \kappa$.

In the next section we will compare these theorems with earlier related results. The main idea of the present paper will be remark that, in the dispersionless case, for all time $t \in \mathbb{R}^+$, at least one of the two functions $x \mapsto e^{\pm\sqrt{3/2}\, q(t,x)} u(t, q(t, x))$, where $q(t, x)$ is the flow of u, must be monotonically increasing. As a byproduct of our approach, we get new simple and natural wave breaking criteria for periodic solutions, requiring no involved a priori estimate on spatial norms of the solution. See Proposition 1.3 below.

Proof of Theorem 1.1. Equation (1.1) is invariant under time translations and under the transformation $\tilde{u}(t, x) = -u(-t, x)$. Therefore, it is enough to prove that if $u_0(x_0) = 0$ at some point $x_0 \in \mathbb{S}$, but $u_0 \not\equiv 0$, then the solution $u \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ arising from u_0 must blow up in finite time. Let $\alpha \in (x_0, x_0 + 1)$ be such that $u_0(\alpha) \neq 0$.

We first consider the case $u_0(\alpha) > 0$. Let us introduce the map

$$\phi(x) = e^{\sqrt{\frac{3}{2}}x} u_0(x).$$

By the periodicity and the continuity of u_0, we can find an open interval $(\alpha, \beta) \subset (x_0, x_0 + 1)$ such that $\phi(x) > 0$ on the interval (α, β) and $\phi(\alpha) > 0$, $\phi(\beta) = 0$. An integration by parts gives

$$\int_\alpha^\beta e^{\sqrt{\frac{3}{2}}x} u_0'(x)\,\mathrm{d}x = \phi(\beta) - \phi(\alpha) - \int_\alpha^\beta \sqrt{\tfrac{3}{2}}\phi(x)\,\mathrm{d}x. \tag{1.7}$$

We deduce from this the existence of $a \in (\alpha, \beta)$ such that $u_0'(a) < -\sqrt{\frac{3}{2}}\, u_0(a) < 0$. Indeed, otherwise, we could bound the left-hand side in (1.7) from below by $\displaystyle\int_\alpha^\beta -\sqrt{\tfrac{3}{2}}\phi(x)\,\mathrm{d}x$, and get the contradiction $\phi(\alpha) \leq \phi(\beta)$.

The second case to consider is $u_0(\alpha) < 0$: introducing now the map $\psi(x) = e^{-\sqrt{\frac{3}{2}}x}u_0(x)$ and arguing as before, we get in this case the existence of a point a such that $u_0'(a) < \sqrt{\frac{3}{2}}\, u_0(a) < 0$. Notice that in both cases we get

$$\exists\, a \in \mathbb{S} \quad \text{such that} \quad u_0'(a) < -\sqrt{\tfrac{3}{2}}\, \big|u_0(a)\big|. \tag{1.8}$$

We thus reduced the proof of our claim to establishing the finite time blowup under condition (1.8), with $u_0 \in H^s(\mathbb{R})$. In fact, approximation u_0 with a sequence $(u_n) \subset H^3(\mathbb{S})$, we can assume without loss of generality that $u_0 \in H^3(\mathbb{S})$. (Indeed, the argument below will provide un upper bound for T^* independent on the parameter n).

Let us introduce the flow map

$$\begin{cases} q_t(t,x) = u(t,q(t,x)), & t \in (0,T^*), \quad x \in \mathbb{R} \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{1.9}$$

We also introduce the C^1 functions, defined on $(0,T^*)$,

$$f(t) = (-u_x + \sqrt{\tfrac{3}{2}}u)\big(t,q(t,a)\big),$$

and

$$g(t) = -(u_x + \sqrt{\tfrac{3}{2}}u)\big(t,q(t,a)\big).$$

Taking the spatial derivative in equation (1.3), recalling that $(1-\partial_x^2)p$ is the Dirac mass, we get

$$u_{tx} + uu_{xx} = -u_x^2 + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right).$$

Using the definition of the flow map (1.9) we obtain

$$\begin{aligned} f'(t) &= \Big[-(u_{tx} + uu_{xx}) + \sqrt{\tfrac{3}{2}}(u_t + uu_x)\Big](t,q(t,a)) \\ &= \Big[u_x^2 - \tfrac{3}{2}u^2 + (p - \sqrt{\tfrac{3}{2}}p_x) * (\tfrac{3}{2}u^2)\Big](t,q(t,a)). \end{aligned}$$

From expression (1.4) we easily get

$$(p \pm \beta p_x) \ge 0 \qquad \text{if and only if} \qquad |\beta| \le \coth(1/2),$$

and so, in particular

$$p \pm \sqrt{\tfrac{3}{2}}p_x \ge 0.$$

Hence we get

$$f'(t) \ge \big[u_x^2 - \tfrac{3}{2}u^2\big](t,q(t,a)).$$

Factorizing the right-hand side leads to the differential inequality

$$f'(t) \ge f(t)g(t), \qquad t \in (0,T^*). \tag{1.10}$$

A similar computation yields

$$g'(t) \ge f(t)g(t), \qquad t \in (0,T^*). \tag{1.11}$$

Let

$$h(t) = \sqrt{f(t)g(t)}.$$

We first observe that

$$h(0)=\sqrt{fg(0)}=\sqrt{u_0'(a)^2-\tfrac{3}{2}u_0(a)^2}\;>0.$$

Moreover, we deduce from the system (1.10)-(1.11), applying the geometric-arithmetic mean inequality, that

$$h'(t)\ge h^2(t),\qquad t\in(0,T^*).$$

This immediately implies $T^*\le 1/h(0)<\infty$. The theorem is completely established. □

As a byproduct of proof of Theorem 1.1 and the observation right after Theorem 1.2, we got the following new blowup criterion for periodic solutions the Degasperis-Procesi equation, with or without dispersion:

Proposition 1.3. *Let $v_0\in H^s(\mathbb{S})$, with $s>3/2$, be such that $v_0'(a)<-\sqrt{\frac{3}{2}}\,\big|v_0(a)+\kappa\big|$ for some $a\in\mathbb{S}$. Then the solution $v\in C([0,T^*),H^s(\mathbb{S}))\cap C^1([0,T^*),H^s(\mathbb{S}))$ of* (1.6) *arising from v_0 blows up in finite time.*

Our approach also reveals that global solutions must satisfy quite stringent pointwise estimates. Indeed, assume that $u\in C([0,\infty),H^s(\mathbb{S}))\cap C^1([0,\infty),H^{s-1}(\mathbb{S}))$ is a given global solution of (1.1). Then, by our theorem, sign(u) $=1,0$ or -1 is well defined and independent on (t,x). Moreover, $u'(t,x)\ge-\sqrt{\frac{3}{2}}|u(t,x)|$ for all $t\ge0$ and $x\in\mathbb{S}$. Then, arguing as in (1.7), we deduce that, for all $t\ge0$, the map

$$x\mapsto e^{\mathrm{sign}(u)\sqrt{\frac{3}{2}}x}\,u(t,x)\quad\text{is increasing.}$$

Combining this with the periodicity, we get the pointwise estimates for $u(t,x)$, for all $t\ge0$, all $\alpha\in\mathbb{R}$ and $\alpha\le x\le\alpha+1$:

$$e^{\mathrm{sign}(u)\sqrt{\frac{3}{2}}(\alpha-x)}u(t,\alpha)\le u(t,x)\le e^{\mathrm{sign}(u)\sqrt{\frac{3}{2}}(\alpha+1-x)}u(t,\alpha). \tag{1.12}$$

From (1.12) one immediately deduces the corresponding estimates for global solutions to the Degasperis-Procesi equation with dispersion.

2. Comparison with some earlier results

In [8, Theorem 3.8], Escher, Liu and Yin established the blowup for equation (1.1) assuming that $u_0\in H^s(\mathbb{S})$, $u_0\not\equiv0$, and that the corresponding solution $u(t,x)$ *vanishes in at least one point $x_t\in\mathbb{S}$ for all $t\in[0,T^*)$*. Theorem 1.1 improves their result (and the corresponding corollaries) by providing the same conclusion $T^*<\infty$ with a shorter proof, and under a condition that is easier to check.

Applying Proposition 1.3 with $\kappa=0$ and $a=0$ improves Yin's blowup criterion [11, Theorem 3.2], establishing the blowup for *odd* initial data with negative derivative at the origin.

In the particular case $\kappa=0$ Proposition 1.3 improves and simplifies the wave-breaking criterion of [8, Theorem 4.3] (and its corollaries), that established the blowup under a condition of the form $v_0'(a)<-\big(c_0\|v_0\|_{L^\infty}+c_1\|v_0\|_{L^2}\big)$, with suitable $c_0,c_1>0$. In fact, Proposition 1.3 shows that one can take $c_1=0$, and more importantly, one only needs to check the behavior of u_0 in a neighborhood of a single point to get the blowup condition.

For general $\kappa \in \mathbb{R}$, Proposition 1.3 extends and considerably simplifies the blowup condition $v_0'(a) < -M$ established in [9, Theorem 4.1], where $M = M(\kappa, \|v_0\|_{L^2}, \|v_0\|_{L^\infty})$ was given by a quite involved expression. In the same way, the pointwise estimates (1.12) allow to improve results like [9, Theorem 4.2] and its corollaries.

A Liouville-type theorem in the same spirit as Theorem 1.1 has been established for periodic solutions of the hyperelastic rod equation in [2], when the physical parameter γ of the model belongs to a suitable range (including $\gamma = 1$, that corresponds to the dispersionless Camassa–Holm equation). The simpler structure of the nonlocal term of the Degasperis-Procesi equation makes possible the much more concise proof presented here.

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