## Supplementary Information

## Stochastic Ratchet Mechanisms for Replacement of Proteins Bound to DNA

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## I. TRANSITION MATRICES FOR THE DIFFERENT MODELS

We show in Table I the transition matrices $W$ for the four models of Fig. 1 in the main text. States are numbered from 1 to $2 N$ in the following way (the unbound state U is omitted as it is an absorbing state). The initially bound protein has $N$ possible bound states $\mathrm{T}_{i}$, where $i$ is the number of attached units of the protein to the DNA. The 'invading' protein has, for each state of the bound protein, only two possible configurations: on or off $(j=0,1)$, at the site corresponding to where the $(i+1)$ th unit of the bound protein would be, which we call the 'zipping' site. There are thus $2 N$ states, labelled by the index $S=2 i+j-1$, for $i=1, \ldots \ldots N, j=0,1$. The transition matrices $W$ have therefore $2 N$ lines and columns. Empty squares correspond to zero entries. For $N>3$ the boundaries (two first and two last lines and columns) are the same, and the $2 \times 2$ central blocks are repeated $N-2$ times on the band diagonal.


TABLE I: Transition matrices $W$ for the NZ (top, left), Z-NS (top, right), Z-S-NSB (bottom, left), and Z-S-SB (bottom, right) models with $N=3$ binding units. All columns sum up to zero due to probability conservation, except the first two columns as states $1\left(\mathrm{~T}_{1}\right)$ and $2\left(\mathrm{R}_{1}\right)$ may decay into the unbound state U , not represented in the matrices above.

## II. UNBINDING RATES VS. CONCENTRATION FOR DIFFERENT $\epsilon$ AND $N$

In Fig. S1, Fig. S2, and Fig. S3 we show the unbinding rate $r$ as a function of the concentration $c$ for the Z-NS, Z-S-NSB and Z-S-SB models, for $\epsilon=2,3$ and different sizes $N$.

## III. FIT OF MODEL PARAMETERS FROM EXPERIMENTAL DATA

To fit the Fis-Fis and CueR-CueR data we have proceeded as follows. We have first considered three possible elementary length $a=0.5,1$ and 2 nm . In Table II we give the elementary time $t_{0}$ and the unit of concentrations $c_{0}$ obtained in the three cases. The experiments give us access to the values of three quantities: the spontaneous dissociation rate $r(0)$, the order of magnitude of the concentration $c_{R}$ at which replacement dominates over spontaneous dissociation, and the exchange rate $R$. We have first fitted the value $N$ of the number of units for the Z-NS and Z-S models from the leading contribution to $R$ (formulas given in main text), and we have then adjusted $N$ and $\epsilon$ to fit the

| $a(\mathrm{~nm})$ | 0.5 | 1 | 5 |
| :--- | :---: | :---: | :---: |
| $t_{0}(\mathrm{~s})$ | $210^{-10}$ | $1.610^{-9}$ | $210^{-7}$ |
| $c_{0}(\mathrm{M})$ | 8 | 1 | 0.008 |

TABLE II: Values of the time constant $t_{0}$ and of the unit of concentration $c_{0}$ for three different elementary lengths $a$.
spontaneous dissociation. The order of magnitude of $c_{R}$, given the fitted $N$ and $\epsilon$ allows us to check the consistency of the model.

Interestingly we have found that the choice of $a$ does not change the values, at the leading order, of $R$ and $c_{R}$. Indeed at the leading order they both depend on the product $c_{0} \times t_{0}$, which is independent of the value of $a$. More precisely, using the experimental values for $R^{F i s}$ and $R^{C u e R}$, we have found:

$$
\begin{equation*}
N_{F i s}^{Z-N S} \approx-\log _{2}\left(c_{0} t_{0} R_{F i s}\right)=13.3 \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{C u e R}^{Z-S-N S B} \approx\left(c_{o} t_{o} R_{C u e R}\right)^{-1}-1=21 \tag{2}
\end{equation*}
$$

Given the approximate values of $N$ above the concentration at which replacement start to dominates is given by

$$
\begin{equation*}
\left(c_{R}\right)_{F i s}^{Z-N S} \approx c_{0} t_{0} r_{F i s}(0) 2^{N} \approx 1 n M \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(c_{R}\right)_{\text {CueR }}^{Z-S-N S B} \approx c_{0} t_{0} r_{\text {CueR }}(0)(N+1) \approx 17 n M \tag{4}
\end{equation*}
$$

Then we have used the spontaneous dissociation rate $r(0)$ to estimate, given $N$, the order of magnitude of the binding evergy $\epsilon$ through

$$
\begin{equation*}
\epsilon \approx-\log \left(t_{0} r(0)\right) / N \tag{5}
\end{equation*}
$$

for both Fis and CueR data. Note that the Z-NS model is not compatible with CueR data because we would obtain $N_{\text {CueR }}^{Z-N S}=4$, which would corresponds to large binding energies of $\epsilon=5.7,5.2,4 k_{B} T$, for, respectively, $a=$ $0.5,1,5 \mathrm{~nm}$. Conversely, Z-S-NSB is not compatible with Fis data because we would obtain $N_{F i s}^{Z-S-N S B}=10,000$.

We have then tuned the value of $N_{C u e R}^{Z-S-N S B}$ and $N_{F i s}^{Z-N S}$ to precisely fit $R$ when the whole expression of the exchange rate is taken into account (see Eqs. (8) and (9) of main text), in particular the multiplicative factors in $1-\rho=1-e^{-\epsilon}$. The final value for $N_{\text {CueR }}^{Z-S S B}$ ranges from 13 to 16 depending on the value of $a$ (see Fig. 3 of the main text). There is less variability in the value of $N_{\text {Fis }}^{Z-N S}$, due to the exponential dependence of $R$ upon $N$. The best fit is obtained in the three cases for $N=14$; the best value for $a$ is $a=5 \mathrm{~nm}$. Finally, once $N$ is fixed, a fine tuning of $\epsilon$ is easily done to reproduce the spontaneous dissociation rate.

We show in Fig. S4 the fits of the Fis-Fis, Fis-Hu, and CueR-CuerR experiments presented and cited in the main text. In each panel three fits are presented, corresponding to the three values of the elementary length $a=0.5,1,5 \mathrm{~nm}$. The values of $N$ and $\epsilon$ are given in the panels. The slope of the curves are the exchange constants $R$ :

- The experimental Fis-Fis exchange constant $R_{\text {Fis-Fis }}=610^{4} \pm 310^{3} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ has to be compared with $R_{\text {Fis-Fis }}^{Z-N S}=4.910^{4}, 5.110^{4}, 610^{4} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ for, respectively, $a=0.5,1,5 \mathrm{~nm}$. The replacement concentration is $\left(c_{R}\right)_{\text {Fis }}^{Z-N S}=2,2,1.6 \mathrm{nM}$ for $a=0.5,1,5 \mathrm{~nm}$.
- The experimental Fis-Hu exchange constant $R_{\text {Fis-Hu }}=2.710^{3} \pm 510^{2} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ has to be compared with $R_{\text {Fis }-H u}^{Z-N S}=2.310^{3}, 2.610^{3}, 410^{3} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ for, respectively, $a=0.5,1,5 \mathrm{~nm}$. The replacement concentration is $\left(c_{R}\right)_{F i s-H u}^{Z-N S}=400,370,180 \mathrm{nM}$ for $a=0.5,1,5 \mathrm{~nm}$.
- The experimental CueR-CueR exchange constant is $R_{C u e R-C u e R}=2.810^{7} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ has to be compared with $R_{\text {CueR-CueR }}^{Z-S-N S B}=2.810^{7}, 2.910^{7}, 3.110^{7} \mathrm{M}^{-1} \mathrm{~s}^{-1}$ for, respectively, $a=0.5,1,5 \mathrm{~nm}$. We obtain the replacement concentration $\left(c_{R}\right)_{\text {CueR-CueR }}=19,16,17 \mathrm{nM}$ for $a=0.5,1,5 \mathrm{~nm}$.


## IV. SCALING OF THE REPLACEMENT RATE $R$ WITH $N$

We start by writing the average unbinding time as

$$
\begin{equation*}
t_{u n b}(c)=\left(1-P_{R}\right) t_{u n b}(0)+P_{R} t_{u n b-R} \tag{6}
\end{equation*}
$$

where $P_{R}$ is the probability that unbinding occurs (at least partially) through the replacement pathway, $t_{u n b}(0)=$ $1 / r(0)$ is the average unbinding time through the thermal pathway only, and $t_{u n b-R}$ is the average unbinding time through the replacement pathway.

In the limit of small solution-phase protein concentrations, i.e. for small rates $c$, we expect $P_{R}$ to scale linearly with $c, P_{R} \simeq p_{R} c$, and, thus, that $P_{R} \ll 1$. In addition, as unbinding through replacement is much faster than through thermal activation alone, we have $t_{u n b-R} \ll t_{u n b}(0)$. Taking the inverse of (6) we obtain

$$
\begin{equation*}
r(c)=\frac{1}{t_{\text {unb }}(c)} \simeq r(0)+p_{R} c r(0) \tag{7}
\end{equation*}
$$

Comparing with Eq. (3) in the main text we see that the replacement rate $R$ coincides with

$$
\begin{equation*}
R=p_{R} r(0) \tag{8}
\end{equation*}
$$

To calculate $p_{R}$, which corresponds to the linear term in the expansion of $P_{R}$ in powers of $c$, we consider 'replacement' paths with only one binding event for the invader. The scaling behavior of $p_{R}$ with $N$ can be simply guessed from the most likely scenario leading to dissociation through the replacement pathway, see main text and blue configurations sketched in Fig. 1 therein. Below, we consider all contributions to $p_{R}$ to find back the exact expression for $R$ given in Eqs. (7), (8) and (9) in the main text for the three Z- kinetic models; the unbinding time for the NZ model at small $c$ is not significantly larger than the thermal rate, and the replacement rate for this model, given by Eq. (6) in the main text, cannot be exactly found back within this approach (though the asymptotic dependence with $N$ is easy to determine, see main text).

We will use the following result, true for the thermal pathway alone $(c=0)$ : the average time spent in state $T_{i}$ (before the protein eventually unbinds) is

$$
\begin{equation*}
\tau_{i}=(1-\rho) \rho^{N-i}\left(1+O\left(\rho^{N}\right)\right) / r(0) \tag{9}
\end{equation*}
$$

where $r(0)=(1-\rho)^{2} \rho^{N}$ to the leading order in powers of $\rho$, see main text. The sum of $\tau_{i}$ over all states $i=1, \ldots, N$ coincides with our expression for the thermal unbinding time, $t_{u n b}(0)=1 / r(0)$.

## A. Z-NS model

One possible scenario of unbinding due to replacement, considered in the main text, consists in reaching state $R_{N}$ from state $T_{N}$. As the average time spent in state $T_{N}$ is $\tau_{N}=(1-\rho) / r(0)$ the probability that the invader eventually binds is, to the lowest order in $c, \simeq c \tau_{N}$. Next, the probability to stay and go all the way up along the alternative replacement pathway is $2^{-N}$, because each transition $\mathrm{R}_{i} \rightarrow \mathrm{R}_{i-1}$ has the same rate as the transition $\mathrm{R}_{i} \rightarrow \mathrm{~T}_{i}$ (invader detachment). The contribution of this scenario to the replacement rate is

$$
\begin{equation*}
\tau_{N} 2^{-N} r(0)=(1-\rho) 2^{-N} \tag{10}
\end{equation*}
$$

Other replacement scenarios, with a single invader-binding event, are as follows: the system spends time in the thermal pathway, undergoes the transition $\mathrm{T}_{k} \rightarrow \mathrm{R}_{k}$ at some level (number of bound units) $k$, remains in the replacement pathway until level $\ell(\leq k)$, and goes back to the thermal pathway through $\mathrm{T}_{\ell} \rightarrow \mathrm{R}_{\ell}$, until thermal binding occurs (the scenario above corresponds to the case $k=N$, without back transition to the thermal pathway). The probability of climbing (without leaving) the replacement pathway from $\mathrm{T}_{k}$ to $\mathrm{T}_{\ell}$ is $2^{-(k-\ell)}$.

We then need to calculate the probability $q_{\ell}$ that the system reaches back the thermal pathway in state $\mathrm{T}_{\ell}$. Let us start by estimating the probability $\mu_{\ell}$ that, once in state $\ell$, the protein will unbind quickly, i.e. in a time much lower than the average time $t_{u n b}$ in the thermal pathway. The following physical picture is useful to compute $\mu_{\ell}$. A particle is undergoing a biased random random walk on discrete sites $i \geq 0$ along the 1D semi-infinite line; $i=0$ is an absorbing site. The random walk is biased by a force $F$ pushing the particle towards the right (large $i$ ), with $F=-\log \rho . \mu_{\ell}$ is the probability that, starting from state $i=\ell$, the particle with reach the absorbing state, rather than being attracted towards $i \rightarrow \infty$. We obviously have

$$
\begin{equation*}
\mu_{\ell}=\frac{1}{1+\rho} \mu_{\ell+1}+\frac{\rho}{1+\rho} \mu_{\ell-1}, \quad \forall \ell=2, \ldots, N \tag{11}
\end{equation*}
$$

with the boundary conditions $\mu_{0}=1$ and $\mu_{\infty}=0$, which gives

$$
\begin{equation*}
\mu_{\ell}=\rho^{\ell} \tag{12}
\end{equation*}
$$

for all $\ell \geq 0$. Formally speaking, $\mu_{\ell}$ is the probability that the system reaches the $U$ state in a finite time (when $N \rightarrow \infty)$, conditioned to the starting state, $\mathrm{T}_{\ell}$. The probability $q_{\ell}$ that the system reaches back the thermal pathway in state $\mathrm{T}_{\ell}$ conditioned to fast unbinding is, according to Bayes' rule,

$$
\begin{equation*}
q_{\ell}=\frac{\mu_{\ell}}{\sum_{i \geq 0} \mu_{i}}=(1-\rho) \rho^{\ell} \tag{13}
\end{equation*}
$$

The sum of the contributions corresponding to all pairs $(k, \ell)$ gives the replacement rate

$$
\begin{align*}
R^{Z-N S} & =\sum_{\ell \leq k \leq N} \tau_{k} 2^{-(\ell-k)} q_{\ell} r(0)=(1-\rho)^{2} \sum_{\ell \leq k \leq N} \rho^{N-k+\ell} 2^{-(k-\ell)}  \tag{14}\\
& =(1-\rho)^{2} \sum_{0 \leq m \leq N}(N+1-m) \rho^{N-m} 2^{-m}=\frac{(1-\rho)^{2}}{(1-2 \rho)^{2} 2^{N}} \tag{15}
\end{align*}
$$

to the leading order in $N$, in agreement with Eq. (7) in the main text.

## B. Z-S-NSB model

In the Z-S-NSB model the alternative scenario to thermal unbinding starts with a transition $\mathrm{T}_{N} \rightarrow \mathrm{R}_{N}$. Later the system changes states in the replacement pathway, until unbinding occurs. We need to estimate the probability $Q$ that this scenario occurs, rather than the systems reaches back state $\mathrm{T}_{N}$ from $\mathrm{R}_{N}$, and unbinding occurs through the thermal pathway. Let $\mu_{i, N}$, with $1 \leq i \leq N$ the probability that the system never leaves the replacement pathway (and unbinding eventually occurs), starting from state $\mathrm{T}_{i}$; we want to calculate $Q=\mu_{N, N}$. According to Fig. 1 in the main text (bottom \& left panel), the rates of the forward ( $\mathrm{R}_{i} \rightarrow \mathrm{R}_{i-1}$ ) and backward ( $\mathrm{R}_{i} \rightarrow \mathrm{R}_{i+1}$ ) transitions are identical (away from the boundaries). Hence the $\mu_{i, N} \mathrm{~S}$ fulfill the recurrence equations

$$
\begin{equation*}
\mu_{i, N}=\frac{1}{2}\left(\mu_{i-1, N}+\mu_{i+1, N}\right), \quad \forall i=2, \ldots, N \tag{16}
\end{equation*}
$$

The above equation can be extended to the states $i=N$ and $i=1$, with the prescriptions $\mu_{N+1, N} \equiv 0$ (which expresses the fact that the system cannot re-enter the replacement pathway after having left it), and $\mu_{0, N} \equiv=1$ (where the subscript 0 stand for the unbound state $U$ here). The solution to these equations is

$$
\begin{equation*}
\mu_{i, N}=\frac{N+1-i}{N+1} \tag{17}
\end{equation*}
$$

We deduce that the replacement rate is given by

$$
\begin{equation*}
R^{Z-S-N S B}=\tau_{N} \mu_{N, N} r(0)=\frac{1-\rho}{N+1} \tag{18}
\end{equation*}
$$

in perfect agreement with Eq. (8) in the main text.

## C. Z-S-SB model

The replacement rate for the Z-S-SB model is calculated in the same way as for the Z-S-NSB model. The only differences are: the replacement pathway is entered through the $\mathrm{T}_{N-1} \rightarrow \mathrm{R}_{N-1}$ transition, and the 'length' (number of states) of this replacement pathway is now $N-1$ instead of $N$. We readily obtain, using (17),

$$
\begin{equation*}
R^{Z-S-S B}=\tau_{N-1} \mu_{N-1, N-1} r(0)=\frac{\rho(1-\rho)}{N} \tag{19}
\end{equation*}
$$

in agreement with Eq. (9) in the main text.

## V. EFFECT OF SALT ON PROTEIN-DNA INTERACTION KINETICS

It should be noted that small ions (salts) could be considered to act as 'invading monomers' in DNA-ligand binding, with a net super-cooperative effect on a ligand's dissociation constant. Binding constants for positively charged ligands to nucleic acid chains have indeed been shown $[1,2]$ to have a power-law dependence on the salt concentration in solution, where the exponent is proportional to the number of counterions released by the binding. This behavior arises from the entropy associated with the release of counterions on ligand binding [1, 2]. We note that in the DNA-DNA interaction study of [2] ions are likely being exchanged from the backbones, outside the interior of the duplex where the interactions between helices occur. It would be quite interesting from the point of view of this paper to study how salt affects off-rates in cases where ions are released from a binding site upon complex formation; the theory of this paper most relevant to this case would be the NZ model.
[1] D. P. Mascot, T. Lohman. Proc. Natl. Acad. Sci. USA 87, 3142 (1990).
[2] N. F. Dupuis, E. D. Holmstrom, D. J. Nesbitt. Biophys. J. 105 , 756 (2013).


FIG. S1: Unbinding rate as a function of the concentration for the Z-NS model, and for different protein sizes $N$ ranging from 6 to 14, and monomer binding energy $\epsilon=2$ and 3 (in units of $k_{B} T$ ) in log-log scale. Full lines: numerical calculation of the unbinding rate. Dotted lines: linear approximations to the unbinding rate, see main text.


FIG. S2: Unbinding rate as a function of the concentration for the Z-S-NSB model, and for different protein sizes $N$ ranging from 6 to 14, and monomer binding energy $\epsilon=2$ and 3 (in units of $k_{B} T$ ) in $\log -\log$ scale. Full lines: numerical calculation of the unbinding rate. Dotted lines: linear approximations to the unbinding rate, see main text.


FIG. S3: Unbinding rate as a function of the concentration for the Z-S-SB model, and for different protein sizes $N$ ranging from 6 to 14, and monomer binding energy $\epsilon=2$ and 3 (in units of $k_{B} T$ ) in $\log -\log$ scale. Full lines: numerical calculation of the unbinding rate. Dotted lines: linear approximations to the unbinding rate, see main text.

Fis-Fis


Fis-Hu



FIG. S4: Fit of the experimental data on concentration dependent dissociation rates of Fis proteins bounded on DNA as a function of concentration of Fis proteins (top) and Hu proteins (middle) in solution with the Z-NS model. Bottom: fit of CueR dissociation rates as a function of CueR concentration in solution with the Z-S-NSB model. See text for details on the parameter values.

