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# A RICE METHOD PROOF OF THE NULL-SPACE PROPERTY OVER THE GRASSMANNIAN.

J.-M. AZAÏS, Y. DE CASTRO, AND S. MOURAREAU

ABSTRACT. The Null-Space Property (NSP) is a necessary and sufficient condition for the recovery of the largest coefficients of solutions to an under-determined system of linear equations. Interestingly, this property governs the success and the failure of recent developments in high-dimensional statistics, signal processing, error-correcting codes and the theory of polytopes.

Although this property is the keystone of  $\ell_1$ -minimization techniques, it is an open problem to construct deterministic matrices satisfying NSP. One can only prove that NSP holds, with an overwhelming probability, when the matrix is chosen at random, in general with a rotation invariant distribution. To the best of our knowledge, the only direct proof of NSP for rotation invariant distribution, initiated by Donoho and Tanner, is based on integral convex geometry theory.

In this article, we follow a different path and we provide a short and self-contained proof using random processes theory and the Rice method. Moreover, our analysis gives new explicit non-asymptotic bounds for NSP with respect to rotation invariant distributions.

## 1. INTRODUCTION

**1.1. Null-Space Property (NSP).** One of the simplest inverse problem is the following: given  $X \in \mathbb{R}^{n \times p}$  and  $y \in \text{Im}(X)$ , can we faithfully recover  $\beta^*$  such that the identity  $y = X\beta^*$  holds? In the ideal case where  $n \geq p$  and  $X$  is one to one (namely, the model is identifiable), this problem is trivial. However, in view of recent applications in genetics, signal processing, or medical imaging, high-dimensional statistics is governed by the opposite situation where  $n < p$ . To bypass the limitations due to the lack of identifiability, one usually assumes that  $X$  is at random and consider the  $\ell_1$ -minimization procedure [CDS98]:

$$(P_{\ell_1}) \quad \Delta_X(y) \in \arg \min_{X\beta=y} \|\beta\|_1.$$

Interestingly, Program  $(P_{\ell_1})$  can be solved efficiently using linear programming, e.g. [CT05]. Moreover, the high-dimensional models often assume that the target vector  $\beta^*$  belongs to the space  $\Sigma_s$  of  $s$ -sparse vectors:

$$\Sigma_s = \{\beta \in \mathbb{R}^p, \|\beta\|_0 \leq s\},$$

where  $\|\beta\|_0$  denotes the size of the support of  $\beta$ . Note that this framework is the baseline of the flourishing Compressed Sensing (CS), see [CRT06, Don06a, CDD09, CGLP12] and references therein. As a matter of fact, a breakthrough brought by CS states that if  $X$  is drawn at random then, with overwhelming probability, one can faithfully recover  $\beta^* \in \Sigma_s$  using  $(P_{\ell_1})$ . The interplay between randomness and  $\ell_1$ -minimization shows that with only  $\frac{n}{s} = \mathcal{O}(\log \frac{p}{s})$ , one can faithfully reconstruct any  $s$ -sparse vector  $\beta^*$  from the knowledge of  $X$  and  $y := X\beta^*$ . As a matter of

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fact, one can prove [CDD09] that  $\Delta_X$  is the identity on  $\Sigma_s$  if and only if the kernel of  $X$  satisfies NSP( $s, C$ ) for some  $C > 1$ .

**Definition** (Null-Space Property of order  $s$  and dilatation  $C$ ) — *Let  $0 < s < p$  be two integers. Let  $G$  be a sub-space of  $\mathbb{R}^p$ . We say that  $G$  satisfies NSP( $s, C$ ), the nullspace property of order  $s$  and dilatation  $C > 1$ , if and only if*

$$(1) \quad \forall \gamma \in G, \forall S \subset \{1, \dots, p\} \text{ s.t. } |S| \leq s, \quad C \|\gamma_S\|_1 \leq \|\gamma_{S^c}\|_1,$$

where  $S^c$  denotes the complement of  $S$ , the vector  $\gamma_S$  has entry equal to  $\gamma_i$  if  $i \in S$  and 0 otherwise, and  $|S|$  is the size of the set  $S$ .

We recall the following important result in CS.

**Theorem 1** ([CDD09]) — *For all  $\beta^* \in \Sigma_s$  there is a unique solution to  $(P_{\ell_1})$  and  $\Delta_X(\beta^*) = \beta^*$  if and only if the nullspace  $\ker(X)$  enjoys NSP( $s, C$ ) for some  $C > 1$ . Moreover, if  $\ker(X)$  enjoys NSP( $s, C$ ) for some  $C > 1$  then for all  $\beta^* \in \mathbb{R}^p$ ,*

$$\|\beta^* - \Delta_X(\beta^*)\|_1 \leq \frac{2(C+1)}{C-1} \min_{|S| \leq s} \|\beta^* - \beta_S^*\|_1.$$

Additionally, NSP suffices to show that any solution to  $(P_{\ell_1})$  is comparable to the  $s$ -best approximation of the target vector. Theorem 1 demonstrates that NSP is a natural property that should be required in CS. This analysis can be lead a step further considering Lasso [Tib96] or Dantzig selector [CT07]. In the frame of noisy observations,  $\ell_1$ -minimization procedures are based on hypothesis like Restricted Isometry Property (RIP) [CT07], Restricted Eigenvalue Condition (REC) [BRT09], Compatibility Condition (CC) [vdGB09], Universal Distortion Property (UDP) [dC12], or  $H_{s,1}$  condition [JN11]. Note that all of these properties imply that  $X$  satisfies NSP. While there exists pleasingly ingenious and simple proofs of RIP, see [CGLP12] for instance, a direct proof of NSP (without the help of RIP) remains a challenging issue.

**1.2. Phase transition.** In a captivating series of papers [DT05, Don06b, DT09b, DT09a], Donoho and Tanner have proved that the kernel of a matrix  $X(n, p_n) \in \mathbb{R}^{n \times p_n}$  with i.i.d. centered Gaussian entries enjoys a phase transition: there exists a function  $\rho_S: ]0, 1[ \rightarrow ]0, 1[$  such that for all  $(\rho, \delta) \in ]0, 1[^2$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{P}[\ker(X(n, p_n)) \text{ enjoys NSP}(s_n, 1)] = \begin{cases} 0 & \text{if } \rho > \rho_S(\delta), \\ 1 & \text{if } \rho < \rho_S(\delta), \end{cases}$$

where  $s_n = \lfloor \rho n \rfloor$ ,  $p_n = \lfloor \frac{n}{\delta} \rfloor$  and  $\lfloor \cdot \rfloor$  denotes the integer part. Interestingly, they have characterized implicitly and computed numerically the function  $\rho_S$  (note that the subscript  $S$  stands for ‘‘Strong’’ since  $\rho_S$  is often named the ‘‘strong threshold’’). Note that their approach is based on computation of Grassmann angles of a polytope due to Affentranger and Schneider [AS92] and Vershik and Sporyshev [VS92]. This uses exact analytical work in integral geometry of convex sets by McMullen [McM75] (nonlinear angle-sum relations), Grünbaum [Grü68] (Grassmann angles), and Ruben [Rub60] (volumes of spherical simplices). To the best of our knowledge, all the proofs of NSP are based on their approach which relies on integral convex geometry theory.

**1.3. A new proof.** In this paper, we follow a different path using the Rice method [AW05, AW09]. Interestingly, this provides a short and direct proof of NSP( $s, C$ ) for any value  $C > 1$ . In its simpler (and weaker) form, our result gives a simple and explicit lower bound  $\rho_{C,D}$  on the phase transition for NSP with dilatation  $C > 1$ .

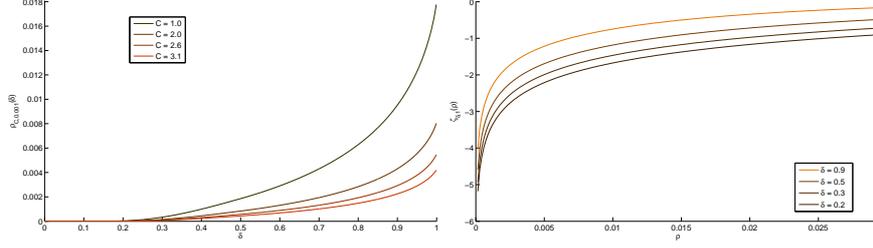


FIGURE 1. The explicit functions  $\rho_{(C,0.01)}$  (left panel) and  $\zeta_{(\delta,1)}$  (right panel) of Theorem 2.

**Theorem 2** — For all  $C > 1$  and  $D > 0$ , there exists an explicit function  $\rho_{C,D}: ]0, 1[ \rightarrow ]0, 1[$  such that for all  $(\rho, \delta) \in ]0, 1[ \times ]0, 1[$  and  $\rho < \rho_{C,D}(\delta)$ ,

$$(2) \quad \mathbb{P}[\ker(X(n, p_n)) \text{ enjoys NSP}(s_n, C)] \geq 1 - \frac{9.7}{C} \left(\frac{n}{\delta}\right)^{\frac{3}{2}} \exp(-nD),$$

where  $s_n = \lfloor \rho n \rfloor$ ,  $p_n = \lfloor \frac{n}{\delta} \rfloor$ ,  $X(n, p_n)$  is a  $(n \times p_n)$  random matrix with i.i.d. centered Gaussian entries. Moreover,  $\rho_{C,D}$  is characterized by:

$$\rho_{C,D}(\delta) = \begin{cases} \zeta_{(\delta,C)}^{-1}(-D - \frac{4-\delta}{2\delta} \log 2 - \frac{1-\delta}{\delta} \log C) & \forall \delta \in ]0, 1/2], \\ \zeta_{(\delta,C)}^{-1}(-D - \frac{2-\delta}{2\delta} \log 2 - \frac{1}{\delta} \mathbf{H}_e(\delta) - \frac{1-\delta}{\delta} \log C) & \forall \delta \in ]1/2, 1[, \end{cases}$$

where  $\mathbf{H}_e(\delta) = -\delta \log(\delta) - (1-\delta) \log(1-\delta)$  and:

$$\zeta_{(\delta,C)}(\rho) = \log\left[-\frac{\rho\delta}{C_0} \log\left[\frac{\rho\delta}{eC_0}\right]\right] \quad \text{and} \quad C_0 := \frac{\pi}{2e^2(1+C)^2}.$$

Observe that (2) depends only on the kernel of  $X$ . In particular, the result holds for any value of the variance of the entries of  $X(n, p)$ . As a matter of fact, the result holds for any kernel drawn at random uniformly on the Grassmannian  $\text{Gr}_m(\mathbb{R}^p)$  where  $m = p - n$ . Note that this is the case when the lines of  $X$  are drawn independently from (possibly distinct) rotation invariant distributions. In its more involved form, our result reads as follows.

**Theorem 3** — For all  $C > 1$  and  $0 < s < n < p$ . Denote  $X(n, p)$  a  $(n \times p)$  random matrix with i.i.d. centered Gaussian entries, it holds:

$$(3) \quad \mathbb{P}[\ker(X(n, p)) \text{ enjoys NSP}(s, C)] = 1 - \Pi,$$

where:

$$(4) \quad \Pi \leq \psi(C) \left[ \sum_{k=0}^{p-n-2} \binom{p}{k} \sqrt{\pi} (C^2 - 1 + \frac{p-k}{s})^{\frac{n+k+1-p}{2}} \frac{\Gamma(\frac{2p-2k-n-1}{2})}{\Gamma(\frac{p-k}{2})\Gamma(\frac{p-n-k}{2})} \right. \\ \left. \times C^{p-n-k-1} \left[ \frac{H_k!}{H_k^{H_k+k-p+n} (p-n-k)!} \right]^{1/2} + \binom{p}{n+1} \left[ \frac{H_{p-n-1}!}{H_{p-n-1}^{H_{p-n-1}-1}} \right]^{1/2} \right]$$

with:

$$\psi(C) = 2 \left[ \frac{s \log(eC_0 \frac{p}{s})}{C_0 p} \right]^{\frac{p}{2}}, \quad C_0 = \frac{\pi}{2e^2(1+C)^2},$$

and  $H_k = \lfloor (\frac{\pi}{2}((C^2 - 1)s + p - k) \wedge (p - n)) \rfloor$ .

Since  $\psi(C)$  assesses the concentration of one Gaussian vector, see Lemma 4, the expression into bracket in (4) can be seen as a complexity bound. Each terms of the aforementioned sum are treated in Appendix B. In particular, an overall bound

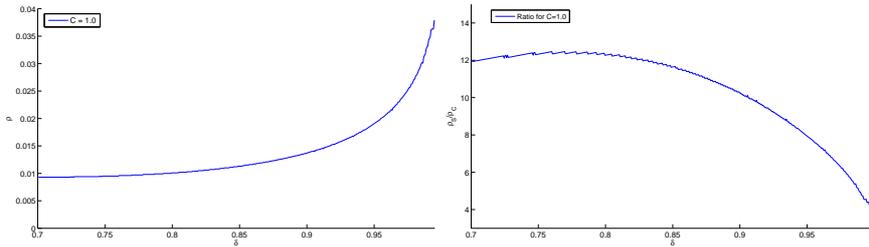


FIGURE 2. The left panel illustrates Theorem 3 showing the region  $(\rho, \delta)$  such that  $\Pi = 0$ , i.e. NSP(1) holds. The right panel shows the ration between the threshold of Theorem 3 and the strong phase transition of Donoho and Tanner [DT05].

is given in Lemma 6. From (4), we can describe (numerically, see Figure 2) a region  $(\rho, \delta)$  where NSP is satisfied, i.e.

$$\lim_{n \rightarrow +\infty} \mathbb{P}[\ker(X(n, p_n)) \text{ enjoys NSP}(s_n, 1)] = 1,$$

where  $s_n = \lfloor \rho n \rfloor$ ,  $p_n = \lfloor \frac{n}{\delta} \rfloor$ ,  $X(n, p_n)$  is a  $(n \times p_n)$  random matrix with i.i.d. centered Gaussian entries. Interestingly, our lower bound compares (up to a multiplicative factor less than 14 for  $\delta > 0.7$ ) to the phase transition of Donoho and Tanner [DT05], see Figure 2.

**1.4. Previous works.** The pioneering work of Donoho and Tanner on this subject can be found in [DT05, Don06b, DT09b, DT09a]. They provide a direct proof of NSP using integral convex geometry theory [McM75, Grü68, Rub60]. Observe that their phase transition is characterized implicitly using an equation involving inverse Mills ratio of the standard normal density. However, they have derived a nice explicit expression of the phase transition for small values of  $\delta$ , i.e. when  $\delta \rightarrow 0$ . Hence, they uncover that, in the regime  $n \ll p$ , NSP(s,1) holds when  $n \gtrsim 2s \log \frac{p}{s}$ .

In a recent work [Sto09, Sto13], Stojnic provides convenient characterizations of the exponents appearing in the expression of the “weak threshold” given by Donoho and Tanner. Note that the weak threshold governs the exact reconstruction by  $\ell_1$ -minimization of sparse vectors with prescribed support and signs, while NSP characterizes the exact reconstruction of all sparse vectors. In [ALMT13], the authors present appealing and rigorous quantitative estimates of weak thresholds appearing in convex optimization, including the location and the width of the transition region. Observe that NSP is characterized by the strong threshold. Nevertheless, the weak threshold describes a region where NSP cannot be satisfied, i.e.

$$\lim_{n \rightarrow +\infty} \mathbb{P}[\ker(X(n, p_n)) \text{ enjoys NSP}(s_n, 1)] = 0,$$

using notation of Theorem 2.

Using a similar approach to Donoho and Tanner, Xu and Hassibi have investigated [XH08, XH11] the property NSP( $s, C$ ) for values  $C > 1$ . They provide the same flavor of result as Theorem 2 using integral convex geometry theory. To the best of our knowledge, this is the only proof of NSP( $s, C$ ) for values  $C > 1$ .

Our proof relies on the Rice formula applied to locally linear processes defined on the sphere. This tool has been previously used in a different framework by Cucker and Wschebor [CW03].

## 2. PROOF OF THE MAIN RESULT

**2.1. Model and notation.** Let  $0 < s < n < p$ , let  $C > 1$  and set  $m = p - n$ . Let  $G$  be uniformly distributed on the Grassmannian  $\text{Gr}_m(\mathbb{R}^p)$ . Observe that it can be generated by  $m$  standard Gaussian vectors  $g_i \in \mathbb{R}^p$  for  $i = 1, \dots, m$ . Define  $\{Z(t); t \in \mathbb{S}^{m-1}\}$  the process with values in  $\mathbb{R}^p$  given by:

$$(5) \quad Z(t) := \sum_{i=1}^m t_i g_i,$$

Note this process spans  $G$  and it can be written as:

$$(6) \quad \text{for } j = 1, \dots, p, \quad Z_j(t) = \langle t, g^j \rangle,$$

where  $(g^j)_{j=1}^p$  are independent Gaussian random vectors of size  $m$  with standard distribution in  $\mathbb{R}^m$ . Let  $O \in \mathbb{R}^{p \times p}$  be an orthogonal matrix. Remark that, due to rotation invariance of the Gaussian distribution, it holds:

$$\forall t \in \mathbb{S}^{m-1}, \quad OZ(t) \sim Z(t).$$

Consider the order statistic of the absolute values of the coordinates of  $Z(t)$ :

$$|Z_{(1)}(t)| \geq \dots \geq |Z_{(p)}(t)|.$$

Given a sparsity  $s$ , a degree of freedom  $m$ , and a degree of constraint  $p$ , consider the real valued process  $\{X(t); t \in \mathbb{S}^{m-1}\}$ :

$$(7) \quad X(t) = C|Z_{(1)}(t)| + \dots + C|Z_{(s)}(t)| - [|Z_{(s+1)}(t)| + \dots + |Z_{(p)}(t)|].$$

NSP is equivalent to the fact that this process is always non positive. We will prove that its happen with an overwhelming probability, see Section 2.3.

**2.2. Cutting the sphere out.** Let  $A \subseteq \{1, \dots, p\}$ , define the random subsets  $\mathbb{S}_A$  and  $\dot{\mathbb{S}}_A$  of the unit sphere  $\mathbb{S}^{m-1}$  by:

$$\begin{aligned} \mathbb{S}_A &= \{t \in \mathbb{S}^{m-1}; Z_A(t) = 0\}, \\ \dot{\mathbb{S}}_A &= \{t \in \mathbb{S}^{m-1}; Z_A(t) = 0 \text{ and } \forall j \notin A, Z_j(t) \neq 0\}. \end{aligned}$$

One can check that  $\mathbb{S}_A$  is the unit sphere of the orthogonal of  $\text{Span}\{g^j; j \in A\}$ . This implies that a.s.  $\mathbb{S}_A$  is a random sphere of dimension  $m - 1 - |A|$  if  $|A| \leq m - 1$  and is almost surely empty if  $|A| > m - 1$ . It follows that the quantities  $|Z_{(1)}(t)|, \dots, |Z_{(n+1)}(t)|$  are a.s. positive and that a.s.

$$\mathbb{S}^{m-1} = \bigcup_{|A| \leq m-1} \dot{\mathbb{S}}_A,$$

giving a partition of the sphere. We define also, for later use, the random subset  $W$  by:

$$W := \{t \in \mathbb{S}^{m-1}; |Z_{(s)}(t)| = |Z_{(s+1)}(t)|\}.$$

Observe that, conditionally to  $g^j$ , the set  $W$  is closed with empty interior.

**2.3. Probability of failure.** We consider the probability:

$$\Pi := \mathbb{P}\{\mathcal{M} > 0\} \leq \sum_{|A| \leq m-1} \mathbb{P}\{\mathcal{M}_{\dot{\mathbb{S}}_A} \geq 0\},$$

where  $\mathcal{M}$  and  $\mathcal{M}_{\dot{\mathbb{S}}_A}$  are respectively the number of local maxima of  $X(t)$ , along  $\mathbb{S}^{m-1}$  satisfying  $X(t) > 0$  and that are in  $\mathbb{S}^{m-1}$ , respectively in  $\dot{\mathbb{S}}_A$ . The baseline of our proof is to upper-bound each right hand side probability, using the expectation

of the number of local maximum above zero and the Markov inequality. The first element is Lemma 4 that proves that:

$$\forall t \in \mathbb{S}^{m-1}, \quad \mathbb{P}\{X(t) > 0\} \leq \psi(C),$$

where:

$$(8) \quad \psi(C) = 2 \left[ \frac{s \log(eC_0 \frac{p}{s})}{C_0 p} \right]^{\frac{p}{2}} \quad \text{with} \quad C_0 = \frac{\pi}{2e^2(1+C)^2}.$$

The second element is that  $X(t)$  admits a density  $p_X$ . To check that, note that  $|Z_{(1)}|, \dots, |Z_{(p)}|$  are the order statistics of the absolute values of i.i.d. Gaussian variables and thus they have a known joint density on the simplex  $|Z_{(1)}| \geq \dots \geq |Z_{(p)}|$ . Formula (7) implies the existence of a density for  $X(t)$ . Moreover, this density does not depend on  $t$  due to invariance of Gaussian distribution.

**2.4. Initialization: local maxima on  $\dot{\mathbb{S}}_\emptyset$ .** By considering the symmetry properties of the sphere  $\dot{\mathbb{S}}_\emptyset$ , we have

$$\mathbb{P}\{\mathcal{M}_{\dot{\mathbb{S}}_\emptyset} \geq 0\} \leq \frac{1}{2} \mathbb{E}(\mathcal{M}_{\dot{\mathbb{S}}_\emptyset}).$$

In this part, our aim will be to give bound to the expectation using a Kac-Rice formula. One can check that if  $t$  belongs to  $\dot{\mathbb{S}}_\emptyset$  and does not belong to  $W$ ,  $X(t)$  is locally the sum of the absolute values of some  $s$  coordinates multiplied by  $C$  minus the sum of the absolute values of the other coordinates. It can be written as:

$$X(t) = C\epsilon_1 Z_{(1)}(t) + \dots + C\epsilon_s Z_{(s)}(t) + \epsilon_{s+1} Z_{(s+1)}(t) + \dots + \epsilon_p Z_{(p)}(t),$$

where  $\epsilon_1, \dots, \epsilon_p$  are random variables taking values  $\pm 1$ .

**Lemma 1** — *Let  $t \in \mathbb{S}^{m-1}$  and denote by  $P_{t^\perp}$  the orthogonal projection onto the orthogonal of  $t$ . Then the random vector*

$$(9) \quad P_{t^\perp}(C\epsilon_1 g^{(1)} + \dots + C\epsilon_s g^{(s)} + \epsilon_{s+1} g^{(s+1)} + \dots + \epsilon_p g^{(p)}),$$

*is a centered isotropic Gaussian random vector onto  $t^\perp$  with variance  $(sC^2 + (p-s))$  that is independent of  $(\epsilon_1, \dots, \epsilon_p, (1), \dots, (p))$  where  $\epsilon_i$ , resp.  $(k)$ , denotes the sign, resp. the indice, of the  $k$ -th largest entries (taking the natural order in case of ties) in absolute value of  $Z(t)$ .*

*Proof.* Remark  $Z(t)$  is a function of  $(P_t(g^1), \dots, P_t(g^p)) = (Z_1(t)t, \dots, Z_p(t)t)$  with obvious notation. It implies that the orthogonal projection  $(P_{t^\perp}(g^1), \dots, P_{t^\perp}(g^p))$  onto the orthogonal of  $t$  is independent of  $(\epsilon_1, \dots, \epsilon_p, (1), \dots, (p))$ . Moreover, observe that the random vector (9) can be written as:

$$\mathbf{A}(P_{t^\perp}(g^1), \dots, P_{t^\perp}(g^p)),$$

where  $\mathbf{A} : (t^\perp)^p \rightarrow t^\perp$  is a random linear operator which is a function of the random variables  $(\epsilon_1, \dots, \epsilon_p, (1), \dots, (p))$ . In particular,  $\mathbf{A}$  is independent of the random vector  $(P_{t^\perp}(g^1), \dots, P_{t^\perp}(g^p))$  which is a standard Gaussian vector on  $(t^\perp)^p$ . We deduce that the random vector (9) is a Gaussian random vector on  $t^\perp$  with variance operator  $\mathbf{A}^* \mathbf{A} = (sC^2 + (p-s))\text{Id}_{t^\perp}$ , where  $\text{Id}_{t^\perp}$  is the identity operator on  $t^\perp$ .  $\square$

Since the distribution of the process  $Z(t)$  is invariant by isometries of  $\mathbb{R}^p$ , it turns that all the  $2^p$  possible signs  $\epsilon_1, \dots, \epsilon_p$  and all the  $p!$  orderings  $(1), \dots, (p)$  play the same role. To its suffices consider the particular case (that we will called the condition  $\mathcal{C}$ ) where:

$$(C) \quad X(t) = CZ_1(t) + \dots + CZ_s(t) - Z_{s+1} - \dots - Z_p(t),$$

an observe that  $X(t)$  is locally linear and thus differentiable.

The next step is to prove that a.s. there is no local maximum on  $W$ . The case where there are ex-aequo among the  $|Z_i(t)|$  has to be considered though it happens with probability 0 for a fixed  $t$ . Note that in that case, the order statistics remain uniquely defined while the ordering  $(1), \dots, (p)$  is no more uniquely defined. Suppose that  $t \in W$  and that, to simplify notation and without loss of generality, similarly to condition  $\mathcal{C}$ , we have:  $Z_1(t) \geq \dots \geq Z_{s-h-1}(t) > Z_{s-h}(t) = \dots = Z_{s+k}(t) > Z_{s+k+1}(t) \geq \dots \geq Z_p(t)$  then, for  $w$  in some neighborhood  $N$ , (not included in  $W$ ) of  $t$ :

$$(10) \quad X(w) = CZ_1(w) + \dots + CZ_{s-h-1}(w) + (1+C)\text{Max}_h(Z_{s-h}(w) + \dots + Z_{s+k}(w)) - (Z_{s-h}(w) + \dots + Z_p(w)),$$

where  $\text{Max}_h$  is the sum of the  $h$  largest element of a set. As being the maximum of  $\binom{h}{(s+k)+1}$  linear forms the function  $\text{Max}_h$  is convex

Let us consider in detail the vectors  $g^{s-h}, \dots, g^{s+k}$ . With probability 1, they are pairwise different. The point  $t$  is such that their projection on  $t$  coincide. As a consequence the derivatives of the linear forms  $Z_\ell(w) = \langle g^\ell, w \rangle$ ,  $\ell = (s-h) \dots (s+k)$  on the tangent space orthogonal to  $t$  are pairwise different. This implies that the function  $\text{Max}_h$  has some direction in which it is strictly convex and as a consequence  $t$  cannot be a local maxima.

Suppose that  $t \notin W$ , conditionally the  $g^j$ 's the function  $X(t)$  is locally linear and hence differentiable. Without loss of generality we can assume we are under the condition  $\mathcal{C}$ :

$$X(t) = CZ_1(t) + \dots + CZ_s(t) - [Z_{s+1}(t) + \dots + Z_p(t)].$$

The gradient in  $\mathbb{R}^m$  of this function is given by:

$$\dot{X}(t) = Cg^1 + \dots + Cg^s - [g^{s+1} + \dots + g^p].$$

As a consequence of formula of geometry (it can be proved by local parametrization), the derivative along the sphere  $X'(t)$  is the tangent projection of the free gradient. In our case, it reads as:

$$X'(t) = P_{t^\perp} \dot{X}(t),$$

where  $P_{t^\perp}$  is the projector onto the orthogonal of  $t$ . The second derivative  $X''(t)$  along the unit sphere is given by the tangent projection of the second derivative (that vanishes) minus identity multiplied by the normal derivative (with the well chosen orientation). It holds:

$$X''(t) = -\langle \dot{X}(t), t \rangle I_{m-1}.$$

As a consequence

$$X''(t) = -X(t)I_m, \\ \text{and } X'(t) = P_{t^\perp} [Cg^1 + \dots + Cg^s - (g^{s+1} + \dots + g^p)].$$

This has several consequences:

- First, we see that  $X'(t)$  and  $(X(t), X''(t))$  are independent.
- Second, the distribution of  $P_{t^\perp} g^1, \dots, P_{t^\perp} g^p$  is not affected by the conditioning on  $P_t g^1, \dots, P_t g^p$  implied by condition  $\mathcal{C}$ . This implies that on  $t^\perp$ ,  $X'(t)$  is centered Gaussian vector with variance:

$$q_C := (C^2 - 1)s + p.$$

This is what we obtain under the condition  $\mathcal{C}$  but since every condition plays the same role and since the distribution does not depend on the condition,

it shows clearly that for every  $t$ ,  $X'(t)$  is Gaussian though  $X(t)$  is obviously not, see Lemma 1.

- Third, if we limit our attention, for example, to points  $t$  such that  $X(t) > 0$ ,  $X''(t)$  cannot be singular.

This last condition implies that we can apply Theorem 5.1.1 of [Adl81]. This lemma is a Kac type formula that shows that the zeroes of the derivative  $X'(t)$  are isolated and thus in finite number. In addition recalling that  $\mathcal{M}_{\dot{\mathbb{S}}_\theta}$  is the number of local maxima of  $X(t)$  satisfying  $X(t) > 0$  and belonging to  $\dot{\mathbb{S}}_\theta$ , this number satisfies

$$(11) \quad \mathcal{M}(\dot{\mathbb{S}}_\theta) = \lim_{\delta \rightarrow 0} \frac{1}{V(\delta)} \int_{\mathbb{S}^{m-1}} \mathbb{E}(|\det X''(t)| \mathbf{1}_{|X'(t)-0|<\delta} \mathbf{1}_{t \in \dot{\mathbb{S}}_\theta} \mathbf{1}_{X(t)>0}) \sigma(dt),$$

where  $\sigma$  is the surfacic measure on  $\mathbb{S}^{m-1}$  and  $V(\delta)$  is the volume of the ball  $B(\delta)$  with radius  $\delta$ . Passing to the limit using the Fatou lemma gives:

$$(12) \quad \begin{aligned} \mathbb{E}(\mathcal{M}(\dot{\mathbb{S}}_\theta)) &\leq \liminf_{\delta \rightarrow 0} \int_0^\infty dx \int_{\dot{\mathbb{S}}_\theta} dt p_{X(t)}(x) \\ &= \frac{1}{V(\delta)} \int_{B(\delta)} dx' p_{X'(t)}(x') \mathbb{E}(|\det(X''(t))| \mid X(t) = x, X'(t) = x'), \\ &= (2\pi q_C)^{\frac{1-m}{2}} 2 \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \int_0^\infty x^{m-1} p_{X(t)}(x) dx, \end{aligned}$$

where  $p_{X(t)}(x)$  denotes the density of  $X(t)$  at  $x$  and  $\Gamma$  denotes the Gamma function. Note that we have used:

- the fact that every point  $t$  is equivalent so we can replace the integral on the unit sphere by  $2 \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$  the value at a given point.
- $\mathbb{E}(|\det(X''(t))| \mid X(t) = x, X'(t) = x') = X^{m-1}$
- $p_{X'(t)}(x')$  is bounded by  $(2\pi q_C)^{\frac{1-m}{2}}$ .

So it remains to majorize:

$$\mathbb{E}[(X(t)^+)^{m-1}].$$

For that purpose we write  $X(t)$  as the independent product  $\|Z(t)\|_2 Y(t)$ , where the process  $Y(t)$  is constructed exactly as the process  $X(t)$  but starting now from a uniform distribution  $U$  on the unit sphere  $\mathbb{S}^{p-1}$  instead of the standard Gaussian distribution of  $Z(t)$ . Using standard results on the moments of the  $\chi^2$  distribution we have:

$$\mathbb{E}((X(t)^+)^{m-1}) = 2^{\frac{m-1}{2}} \frac{\Gamma(\frac{m-1+p}{2})}{\Gamma(\frac{p}{2})} \mathbb{E}((Y(t)^+)^{m-1}).$$

We use now the fact that  $Y(t) \leq C\sqrt{s}$  to get that:

$$\mathbb{E}((Y(t)^+)^{m-1}) \leq (C\sqrt{s})^{m-1} \mathbb{P}\{Y(t) > 0\}.$$

Moreover, Lemma 4 shows that, with probability greater than  $1 - \psi(C)$ , a standard Gaussian vector  $g$  in  $\mathbb{R}^p$  enjoys:

$$(C+1)\sqrt{s}\|g\|_2 \leq \|g\|_1.$$

This implies that:

$$(13) \quad \mathbb{P}\{Y(t) > 0\} \leq \psi(C),$$

and consequently the probability of having a local maxima above 0 on  $\dot{\mathbb{S}}_\theta$  is bounded by:

$$\begin{aligned}
& \sigma(\dot{\mathbb{S}}_\emptyset) (2\pi q_C)^{\frac{1-m}{2}} 2^{\frac{m-1}{2}} \frac{\Gamma(\frac{m-1+p}{2})}{\Gamma(\frac{p}{2})} C^{m-1} \psi(C) \\
(14) \quad & \leq 2\sqrt{\pi} \left( C^2 - 1 + \frac{p}{s} \right)^{\frac{1-m}{2}} \frac{\Gamma(\frac{m-1+p}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{m}{2})} C^{m-1} \psi(C) =: h(s, m, p).
\end{aligned}$$

**2.5. Induction: maxima on smaller spheres.** We turn now to the study of the first term that corresponds to existence of a maximum on  $\dot{\mathbb{S}}_A$ . It leads exactly to the same computations as the case  $\dot{\mathbb{S}}_\emptyset$  because conditionally to the  $g^j, j \in A$ , we can replace the space  $\mathbb{R}^m$  by the orthogonal of  $V$  and obtain the same result replacing  $m$  by  $m - k$  and  $p$  by  $p - k$ :

$$h(s, m - k, p - k) := 2\sqrt{\pi} \left( C^2 - 1 + \frac{p - k}{s} \right)^{\frac{m-1+p-k}{2}} \frac{\Gamma(\frac{m-1+p-2k}{2})}{\Gamma(\frac{p-k}{2})\Gamma(\frac{m-k}{2})} C^{m-k-1} \psi(C).$$

Observe that  $\psi(C)$  does not depend on  $k$ , see (13).

However, we have to pay attention to the value  $k = m - 1$ . In such a case  $\mathbb{S}_A$  consists of two points giving the same value of  $X(t)$  and the first term corresponds only to the probability of  $X(t)$  to be positive which is  $\psi(C)$ .

**2.6. Refinements: localization of the sub-gradient.** Let us consider now the case of a maximum on  $\dot{\mathbb{S}}_A$ ,  $A \neq \emptyset$  and set  $k = |A|$ . First we have to draw the  $g^i, i \in A$  and define the space  $V = \text{Span}\{g^i; i \in A\}$  and then conditionally to these vectors we consider  $\mathbb{S}_A$ , the random sphere orthogonal to these vectors. Note that the behavior of  $X(t)$  on  $\mathbb{S}_A$  is independent of  $\{g^i, i \in A\}$  since for all  $t \in \mathbb{S}_A$ ,  $X(t)$  depends only on the  $g^j, j \notin A$  and that for such  $j$ ,

$$Z_j(t) = \langle g^j, t \rangle = \langle \Pi_{V^\perp} g^j, t \rangle.$$

A point  $t \in \dot{\mathbb{S}}_A \setminus \mathcal{W}$  is a local maximum on  $\mathbb{S}^{m-1}$  if it is a local maximum along  $\mathbb{S}_A$  and its super-gradient in the orthogonal space contains zero. Indeed, locally around  $t$ , the behavior of  $X(w)$  along  $V$  is the sum of some linear forms (for  $j \notin A$ ) and of absolute value of linear forms (for  $j \in A$ ) thus it is locally concave and we can define its super-gradient as the opposite of the sub-gradient of  $-X$ .

As we have seen, the two conditions are independent. Let us look to the second one. Thus we have to compute the probability of the super-gradient to contain zero. More precisely for  $s$  in a neighborhood of  $t \in \dot{\mathbb{S}}_A \setminus \mathcal{W}$ ,

$$X(w) = X_A(w) + X_{A^c}(w),$$

where, because  $s \leq n$ :

$$X_A(w) = - \sum_{i \in A} |Z_i(w)|.$$

Around  $t$ ,  $X_{A^c}(s)$  is differentiable and, with a possible harmless change of sign (see Lemma 1), its gradient is given by

$$\sum_{i \in \bar{A}} C_i g^i,$$

where the coefficient  $C_i$  takes the value  $C$  for  $s$  of them and  $-1$  for the others. This gradient is distributed as an isotropic normal variable  $\xi \in V$  with variance:

$$\tilde{p}_k := (C^2 - 1)s + p - k.$$

By this we mean that the distribution of  $\xi$ , in a convenient basis, is  $\mathcal{N}(0, \tilde{p}_k I_k)$ .

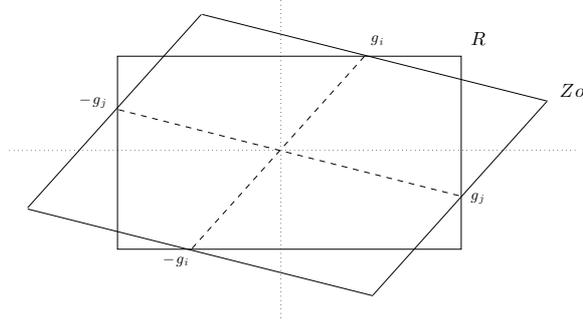


FIGURE 3. The standard Gaussian measure of the zonotope  $Zo$  is greater than the standard Gaussian measure of the rectangle  $R$ .

Let us consider now the case  $i \in A$ . Observe that the super-gradient of the concave function  $-|Z_i(t)|$  at point  $t$  is the segment  $[-g^i, g^i]$  and thus the super-gradient of  $X_A(t)$  is the zonotope:

$$(15) \quad Zo = \sum_{i \in A} [-g^i, g^i],$$

where the sum means Minkowsky addition.

In conclusion, the probability of the super-gradient to contain zero is equal to  $\mathcal{P}(k, \tilde{p}, m)$  where  $\mathcal{P}(k, \tilde{p}, m)$  is the probability of the following event:

- draw  $k$  standard Gaussian variables  $g^1, \dots, g^k$  in  $\mathbb{R}^m$  and consider the zonotope  $Zo$  given by formula (15),
- draw in the space  $V$  generated by  $g^1, \dots, g^k$  an independent isotropic normal variable  $\xi$  of variance  $\tilde{p}$ ,
- define  $\mathcal{P}(k, \tilde{p}, m)$  as the probability of  $\xi$  to be in  $Zo$ .

**Lemma 2** — Define the orthonormal basis  $e^1, \dots, e^k$  obtained by Gram-Schmidt orthogonalization of the vectors  $g^1, \dots, g^k$ . Then:

- (a)  $\mathcal{P}(k, \tilde{p}, m)$  is less than the probability  $\mathcal{Q}(k, \tilde{p}, m)$  of  $\xi$  to be in the hyper-rectangle:

$$(16) \quad R = \sum_{i \in A} [-\langle e^i, g^i \rangle, \langle e^i, g^i \rangle],$$

- (b) this last probability satisfies:

$$(\mathcal{Q}(k, \tilde{p}_k, m))^2 \leq \frac{H_k!}{H_k^{H_k + k - m} (m - k)!},$$

with  $\tilde{p}_k := (C^2 - 1)s + p - k$  and  $H_k = \lfloor (\frac{\pi}{2} \tilde{p}_k) \wedge m \rfloor$ , where  $\lfloor \cdot \rfloor$  is the integer part.

*Proof.* (a) We prove the result conditionally to the  $g^i$ 's and by induction on  $k$ . When  $k = 1$  the result is trivial since the zonotope and the rectangle are simply the same segment.

Let  $\Phi_h$  be the standard Gaussian distribution on  $\mathbb{R}^h$ ,  $\mathcal{P}(k, \tilde{p}, m)$  is equal to:

$$\Phi_k((\tilde{p})^{-1/2} \cdot Zo) =: \Phi_k(\tilde{Zo}).$$

We can compute this probability using the Fubini Theorem:

$$\mathcal{P}(k, \tilde{p}_k, m) = \int_{-\frac{\langle e^k, g^k \rangle}{\sqrt{\tilde{p}}}}^{\frac{\langle e^k, g^k \rangle}{\sqrt{\tilde{p}}}} \Phi_{k-1}(\tilde{Zo}_{k-1} + vz) \phi(z) dz = \mathcal{Q}(k, \tilde{p}_k, m),$$

where  $\phi$  is the standard Gaussian density;  $\widetilde{Z}o_{k-1}$  is the zonotope generated by  $g^1, \dots, g^{k-1}$  and normalized by  $(\tilde{p})^{-1/2}$  and  $v$  is some vector. By use of the Anderson inequality [And55], the non-centered zonotope  $(\widetilde{Z}o_{k-1} + vz)$  has a smaller standard Gaussian measure than the centered one.

$$\mathcal{P}(k, \tilde{p}, m) \leq \int_{-\frac{\langle e^k, g^k \rangle}{\sqrt{\tilde{p}}}}^{\frac{\langle e^k, g^k \rangle}{\sqrt{\tilde{p}}}} \Phi_{k-1}(\widetilde{Z}o_{k-1}) \phi(z) dz \leq \int_R \phi(z_1) \dots \phi(z_k) dz_1 \dots dz_k.$$

The last inequality is due to the induction hypothesis. It achieves the proof.

(b) We use the relation above and deconditionning on the  $g^i$ . Note the dimension of the edges of the rectangle  $R$  are independent with distribution:

$$2\chi(m), 2\chi(m-1), \dots, 2\chi(m-k+1),$$

where the law  $\chi(d)$  is defined as the square root of a  $\chi^2(d)$ . As a consequence, using the independence of the components of  $\xi$  in the basis  $e^1, \dots, e^k$  and the fact that a Student density  $T$  is uniformly bounded by  $(2\pi)^{-1/2}$ , we get that:

$$\mathcal{Q}(k, \tilde{p}, m) = \prod_{\ell=0}^{k-1} \mathbb{P} \left[ |T(m-\ell)| \leq \sqrt{\frac{m-\ell}{\tilde{p}}} \right].$$

Suppose that  $\pi\tilde{p}_k \geq 2m$ , then a convenient bound is obtained by using the fact that a Student density is uniformly bounded by  $(2\pi)^{-1/2}$ :

$$(\mathcal{Q}(k, \tilde{p}, m))^2 \leq \left(\frac{2}{\pi\tilde{p}_k}\right)^k \frac{m!}{(m-k)!}.$$

In the other case, set  $H_k = \lfloor (\pi\tilde{p}_k)/2 \rfloor$ , where  $\lfloor \cdot \rfloor$  is the integer part. Removing factors that are greater than 1 in the computation above gives

$$(\mathcal{Q}(k, \tilde{p}, m))^2 \leq \left(\frac{2}{\pi\tilde{p}_k}\right)^{H_k+k-m} \frac{H_k!}{(m-k)!}.$$

and the desired general form is obtained by setting  $H_k = m$  in the first case. Observe that  $H_k > m - k + 1$  for  $k \geq 1$ .  $\square$

**2.7. Conclusion: gathering the pieces.** Finally summing up over the  $\binom{p}{k}$  sets of size  $k$ . We have to consider apart the particular case  $k = m - 1$  to get that the probability of failure  $\Pi$  is upper bounded by:

$$\frac{1}{2} \sum_{k=0}^{m-2} \binom{p}{k} h(s, m-k, p-k) \left[ \frac{H_k!}{H_k^{H_k+k-m} (m-k)!} \right]^{\frac{1}{2}} + \binom{p}{n+1} \psi(C) \left[ \frac{H_{m-1}!}{H_{m-1}^{H_{m-1}-1}} \right]^{\frac{1}{2}}$$

Observe that if  $\delta := \frac{n}{p} > \delta_0 := \frac{2}{2+\pi}$  then for all  $0 \leq k \leq m$ ,  $H_k = m$ . It follows that the aforementioned equation simplifies to:

$$\frac{1}{2} \sum_{k=0}^{m-2} \binom{p}{k} h(s, m-k, p-k) \left[ \frac{m!}{m^k (m-k)!} \right]^{\frac{1}{2}} + \binom{p}{n+1} \psi(C) \left[ \frac{m!}{m^{m-1}} \right]^{\frac{1}{2}}.$$

This proves Theorem 3.

**2.8. Simplification: an explicit expression of a lower bound on the phase transition.** While we can compute the upper bound (4) for given  $s$ ,  $n$  and  $p$  and derive a ‘‘phase transition’’ from numerical experiments, we choose to present a more simpler (and weaker) upper bound in Theorem 2. Let  $1 \leq k \leq m - 2$  and set  $\rho := \frac{s}{n}$ ,  $\delta := \frac{n}{p}$  and  $\varepsilon := C^2 - 1$ . First, using Lemma 5, note that for all  $1 \leq u \leq v$ , it holds:

$$\binom{v}{u} \leq e^{v\mathbf{H}_e(\frac{v}{u})} \quad \text{and} \quad \frac{\Gamma(u + \frac{1}{2})}{\Gamma(u)} \leq e^{(u+1)^{\frac{1}{2}}}.$$

where  $\mathbf{H}_e(x) := -x \log(x) - (1-x) \log(1-x)$  is the Shannon entropy in basis  $e$ . Moreover, observe that:

$$\frac{H_k!}{H_k^{H_k+k-m}(m-k)!} \leq 1$$

Hence, Equation (14) gives that:

$$\binom{p}{k} h(s, m-k, p-k) \left[ \frac{H_k!}{H_k^{H_k+k-m}(m-k)!} \right]^{\frac{1}{2}} \leq \frac{9.7}{C} \left(\frac{n}{\delta}\right)^{\frac{1}{2}} \exp(n\mathbb{A}(\rho, \delta, \varepsilon, n, k)),$$

where the exponent  $\mathbb{A}$  is given by:

$$\begin{aligned} \mathbb{A}(\rho, \delta, \varepsilon, n, k) &:= \frac{1}{2} \left( \frac{k+1}{n} - \frac{1-\delta}{\delta} \right) \log \left( \varepsilon + \frac{n-k\delta}{n\rho\delta} \right) + \frac{1}{2\delta} \log \left[ -\frac{\rho\delta}{C_0} \log \left( \frac{\rho\delta}{\varepsilon C_0} \right) \right] \\ &\quad + \left( \frac{1-\delta}{2\delta} - \frac{k}{2n} \right) \log(1+\varepsilon) \\ &\quad + \left( \frac{2-\delta}{2\delta} - \frac{k}{n} \right) \mathbf{H}_e \left( \left( 2 - \frac{n}{n/\delta - k - 2} \right)^{-1} \right) + \frac{1}{\delta} \mathbf{H}_e \left( \delta \frac{k}{n} \right). \end{aligned}$$

where  $C_0$  is defined as in Lemma 4. We make the following observations:

- (1) The map  $\zeta_{(\delta, \varepsilon)} : \rho \mapsto \log \left[ -\frac{\rho\delta}{C_0} \log \left( \frac{\rho\delta}{\varepsilon C_0} \right) \right]$  is an increasing  $\mathcal{C}^1$ -diffeomorphism from  $]0, \frac{C_0}{\delta}[$  to  $] -\infty, 0[$ .
- (2) It holds:

$$\left( \frac{k+1}{n} - \frac{1-\delta}{\delta} \right) \log \left( \varepsilon + \frac{n-k\delta}{n\rho\delta} \right) \leq -\frac{2}{n} \log \left( \varepsilon + \frac{1}{\rho} \right).$$

- (3) Observe that:

$$\left( \frac{1-\delta}{2\delta} - \frac{k}{2n} \right) \log(1+\varepsilon) \leq \frac{1-\delta}{2\delta} \log(1+\varepsilon).$$

- (4) Note that for all  $1 \leq k \leq m = \frac{1-\delta}{\delta}n$ , the last term:

$$\left( \frac{2-\delta}{2\delta} - \frac{k}{n} \right) \mathbf{H}_e \left( \left( 2 - \frac{n}{n/\delta - k - 2} \right)^{-1} \right) + \frac{1}{\delta} \mathbf{H}_e \left( \delta \frac{k}{n} \right)$$

is less than:

$$\begin{cases} \left( \frac{4-\delta}{2\delta} \right) \log 2 & \text{if } \delta \in ]0, 1/2], \\ \left( \frac{2-\delta}{\delta} \right) \log 2 + \frac{1}{\delta} \mathbf{H}_e(\delta) & \text{if } \delta \in ]1/2, 1]. \end{cases}$$

It yields that:

$$\mathbb{A}(\rho, \delta, \varepsilon, n, k) \leq \zeta_{(\delta, \varepsilon)}(\rho) + \frac{4-\delta}{2\delta} \log 2 + \frac{1-\delta}{2\delta} \log(1+\varepsilon),$$

for  $\delta \in ]0, 1/2]$  and:

$$\mathbb{A}(\rho, \delta, \varepsilon, n, k) \leq \zeta_{(\delta, \varepsilon)}(\rho) + \frac{2-\delta}{\delta} \log 2 + \frac{1}{\delta} \mathbf{H}_e(\delta) + \frac{1-\delta}{2\delta} \log(1+\varepsilon),$$

for  $\delta \in ]1/2, 1]$ . We see that, if  $\rho < \rho_{C,D}(\delta)$  then  $\mathbb{A}(\rho, \delta, \varepsilon, n, k) < -D$ , where:

$$\rho_{C,D}(\delta) = \begin{cases} \zeta_{(\delta, \varepsilon)}^{-1} \left( -D - \frac{4-\delta}{2\delta} \log 2 - \frac{1-\delta}{\delta} \log C \right) & \text{if } \delta \in ]0, 1/2], \\ \zeta_{(\delta, \varepsilon)}^{-1} \left( -D - \frac{2-\delta}{\delta} \log 2 - \frac{1}{\delta} \mathbf{H}_e(\delta) - \frac{1-\delta}{\delta} \log C \right) & \text{if } \delta \in ]1/2, 1]. \end{cases}$$

Moreover, Lemma 4 shows that if  $\rho < \rho_{C,D}(\delta)$  then:

$$\binom{p}{n+1} \psi(C) \left[ \frac{H_{m-1}!}{H_{m-1}^{H_{m-1}-1}} \right]^{\frac{1}{2}} \leq 2 \exp(-Dn).$$

We conclude using (4).

## APPENDIX A. GAUSSIAN CONCENTRATION INEQUALITIES

**Lemma 3** — *Let  $g$  be a standard Gaussian vector in  $\mathbb{R}^p$  and  $t_1, t_2 > 0$  such that  $t_1^2 \geq 1 + (\pi(p-2))^{-\frac{1}{2}}$  then:*

$$(17) \quad \mathbb{P}\{\|g\|_2 > t_1\sqrt{p}\} \leq \exp\left[-\frac{p}{2}(t_1^2 - 1 - 2\log(t_1))\right],$$

$$(18) \quad \mathbb{P}\{\|g\|_1 < t_2 p\} \leq \exp\left[\left(p \log\left(e\sqrt{\frac{2}{\pi}}t_2\right)\right)\right].$$

*Proof.* • Proof of (17): By integrating in polar coordinates:

$$\begin{aligned} \mathbb{P}\{\|g\|_2 > t\sqrt{p}\} &= \int_{t\sqrt{p}}^{+\infty} v^{p-1} \sigma(\mathbb{S}^{p-1}) (2\pi)^{-p/2} e^{-v^2/2} dv, \\ &= 2^{1-\frac{p}{2}} \left(\Gamma\left(\frac{p}{2}\right)\right)^{-1} \int_{t\sqrt{p}}^{+\infty} v^{p-1} e^{-\frac{v^2}{2}} dv, \end{aligned}$$

where  $t$  stands for  $t_1$  and  $\sigma(\mathbb{S}^{p-1}) = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}$  is the  $(p-1)$ -dimensional volume of the unit sphere. Consider the case  $p$  even. By integration by parts, if  $w^2 > p$ ,

$$\begin{aligned} \int_w^{+\infty} v^{p-1} e^{-\frac{v^2}{2}} dv &= [w^{p-2} + (p-2)w^{p-4} + \dots + (p-2)(p-4)\dots 2] e^{-\frac{w^2}{2}}, \\ &\leq w^{p-2} \left(1 - \frac{p}{w^2}\right)^{-1} e^{-\frac{w^2}{2}}. \end{aligned}$$

Using this result, the Stirling formula (see Lemma 5 for instance), and the fact that  $(1-x)^{1-\frac{1}{x}} < e$  for all  $0 < x < 1$  (see See [AS65] Eq. 4.1.34), we get that:

$$\begin{aligned} \mathbb{P}\{\|g\|_2 > t\sqrt{p}\} &\leq \left(\frac{p}{2}\right)^{\frac{p-2}{2}} \frac{t^p}{t^2-1} \left(\Gamma\left(\frac{p}{2}\right)\right)^{-1} \exp\left(-t^2\frac{p}{2}\right), \\ &\leq \exp\left[-\frac{p}{2}(t^2 - 1 - 2\log t)\right]. \end{aligned}$$

The proof for the case  $p$  odd is very similar using that  $\bar{\Phi}(u) \leq \frac{\phi(u)}{u}$ .

• Proof of (18): We use the following facts:

- The density of  $g$  is bounded by  $(2\pi)^{-\frac{p}{2}}$ ,
- The volume of the  $L^1$  unit ball is  $\frac{2^p}{p!}$ ,

to get that:

$$\mathbb{P}\{\|g\|_1 \leq t_2 p\} \leq \left(\frac{2t_2 p}{\sqrt{2\pi}}\right)^p \frac{1}{p!} \leq \exp\left[\left(p \log\left(e\sqrt{\frac{2}{\pi}}t_2\right)\right)\right],$$

using the Stirling formula. □

**Lemma 4** — *Let  $C > 0$ , then, except with a probability smaller than:*

$$\psi(C) := 2\left(\frac{s \log(eC_0 \frac{p}{s})}{C_0 p}\right)^{\frac{p}{2}} \quad \text{with} \quad C_0 = \frac{\pi}{2e^2(1+C)^2},$$

*a standard Gaussian vector  $g \in \mathbb{R}^p$  enjoys for all  $S \subset \{1, \dots, p\}, |S| \leq s$*

$$(19) \quad C\|g_S\|_1 \leq \|g_{S^c}\|_1,$$

*which is equivalent to  $(C+1)\|g_S\|_1 \leq \|g\|_1$ .*

*Proof.* Define:

$$t_1 = \left(1 + \log\left(C_0 \frac{p}{s}\right)\right)^{\frac{1}{2}} \quad \text{and} \quad t_2 = t_1(C+1)\left(\frac{s}{p}\right)^{\frac{1}{2}}.$$

Lemma 3 shows that:

$$\begin{aligned} -\frac{1}{p} \log(\mathbb{P}(\|g\|_2 > t_1 \sqrt{p})) &\geq \frac{1}{2} \log(C_0 p / (s \log(e C_0 \frac{p}{s}))), \\ -\frac{1}{p} \log(\mathbb{P}(\|g\|_1 < t_2 p)) &\geq \frac{1}{2} \log(C_0 p / (s \log(e C_0 \frac{p}{s}))). \end{aligned}$$

Hence, except with a probability smaller than  $\psi(C)$ , it holds:

$$(C+1)\|g_S\|_1 \leq s^{\frac{1}{2}}(C+1)\|g\|_2 \leq (C+1)\frac{t_1}{t_2} \left[\frac{s}{p}\right]^{\frac{1}{2}} \|g\|_1 = \|g\|_1.$$

as claimed.  $\square$

## APPENDIX B. TECHNICAL LEMMAS

### B.1. Binet's formula.

**Lemma 5** — *Let  $z > 0$  then there exists  $\theta \in (0, 1)$  such that:*

$$\Gamma(z+1) = (2\pi z)^{\frac{1}{2}} \left(\frac{z}{e}\right)^z \exp\left(\frac{\theta}{12z}\right).$$

*Proof.* See [AS65] Eq. 6.1.38.  $\square$

### B.2. Overall expression.

**Lemma 6** — *Set  $\rho := \frac{s}{n}$ ,  $\delta := \frac{n}{p}$  and  $\varepsilon := C^2 - 1$ . Then for all  $1 \leq k \leq m-2$ ,*

$$\binom{p}{k} h(s, m-k, p-k) \left[ \frac{H_k!}{H_k^{H_k+k-m} (m-k)!} \right]^{1/2} \leq \exp(\mathbb{B}(\rho, \delta, \varepsilon, n, k)),$$

where:

$$\begin{aligned} \mathbb{B}(\rho, \delta, \varepsilon, n, k) &:= \frac{\delta}{12n} + \frac{1}{12\left(\left(\frac{1}{\delta} - \frac{1}{2}\right)n - k - \frac{3}{2}\right)} + \frac{1}{24H_k} + \log\left[\frac{1}{\sqrt{\pi e}}\right] \\ &+ \frac{1}{2}\left(k+1 - \frac{1-\delta}{\delta}n\right) \log\left(\varepsilon + \frac{n-k\delta}{n\rho\delta}\right) + \left(\frac{1-\delta}{\delta}n - k - 1\right) \log(C) \\ &+ \left(\frac{n}{\delta} + \frac{1}{2}\right) \log \frac{n}{\delta} + \left[\left(\frac{1}{\delta} - \frac{1}{2}\right)n - k - 1\right] \log\left(\left(\frac{1}{\delta} - \frac{1}{2}\right)n - k - \frac{3}{2}\right) \\ &- \left(\frac{n}{\delta} - k + \frac{1}{2}\right) \log\left(\frac{n}{\delta} - k\right) - \left(k + \frac{1}{2}\right) \log k \\ &- \left(\frac{n}{2\delta} - \frac{k+1}{2}\right) \log\left(\frac{n}{2\delta} - \frac{k+2}{2}\right) \\ &- \left(\frac{1-\delta}{2\delta}n - \frac{k+1}{2}\right) \log\left(\frac{1-\delta}{2\delta}n - \frac{k+2}{2}\right) \\ &+ \frac{1}{2}\left(\frac{1-\delta}{\delta}n - k + \frac{1}{2}\right) \log(H_k) - \frac{1}{2}\left(\frac{1-\delta}{\delta}n - k\right) \log\left(\frac{1-\delta}{\delta}n - k\right) \\ &+ \frac{1}{2}\left(\frac{1-\delta}{\delta}n - k - H_k\right) + \log(\psi(C)), \end{aligned}$$

with  $H_k = \lfloor \left(\frac{1-\delta}{\delta}\right)n \wedge \left[\frac{\pi}{2}(\varepsilon\rho n + \frac{n}{\delta} - k)\right] \rfloor$ .

*Proof.* Set  $P = \binom{p}{k} h(s, m-k, p-k)$ . Using (14) it holds:

$$P \leq 2\pi^{\frac{1}{2}} \left[ \varepsilon + \frac{n-k\delta}{n\delta\rho} \right]^{\frac{k}{2} + \frac{1}{2} - \frac{1-\delta}{2\delta}n} C^{\frac{1-\delta}{\delta}n - k - 1} \psi(C) R,$$

where:

$$R = \frac{\Gamma\left(\frac{n}{\delta} + 1\right) \Gamma\left(\left(\frac{1}{\delta} - \frac{1}{2}\right)n - \left(k + \frac{1}{2}\right)\right)}{\Gamma\left(\frac{n}{\delta} - k + 1\right) \Gamma(k+1) \Gamma\left(\frac{n}{2\delta} - \frac{k}{2}\right) \Gamma\left(\frac{1-\delta}{2\delta}n - \frac{k}{2}\right)}.$$

Using Lemma 5, this last expression can be written as:

$$\begin{aligned}
\log R \leq & -\log(2\pi) - \frac{1}{2} + \frac{\delta}{12n} + \frac{1}{12((\frac{1}{\delta} - \frac{1}{2})n - k - \frac{3}{2})} + (\frac{n}{\delta} + \frac{1}{2}) \log \frac{n}{\delta} \\
& + [(\frac{1}{\delta} - \frac{1}{2})n - k - 1] \log((\frac{1}{\delta} - \frac{1}{2})n - k - \frac{3}{2}) - (\frac{n}{\delta} - k + \frac{1}{2}) \log(\frac{n}{\delta} - k) \\
& - (k + \frac{1}{2}) \log k - (\frac{n}{2\delta} - \frac{k+1}{2}) \log(\frac{n}{2\delta} - \frac{k+2}{2}) \\
(20) \quad & - (\frac{1-\delta}{2\delta}n - \frac{k+1}{2}) \log(\frac{1-\delta}{2\delta}n - \frac{k+2}{2}).
\end{aligned}$$

Set:

$$T = \left[ \frac{H_k!}{H_k^{H_k+k-m}(m-k)!} \right]^{\frac{1}{2}} \leq \left[ \frac{H_k^{m-k}}{(m-k)!} e^{-H_k} \sqrt{2\pi H_k} e^{\frac{1}{12H_k}} \right]^{\frac{1}{2}}.$$

Invoking Lemma 5, it follows that:

$$\begin{aligned}
2 \log T \leq & \frac{1}{12H_k} + (\frac{1-\delta}{\delta}n - k + \frac{1}{2}) \log(H_k) - (\frac{1-\delta}{\delta}n - k) \log(\frac{1-\delta}{\delta}n - k) \\
(21) \quad & + \frac{1-\delta}{\delta}n - k - H_k.
\end{aligned}$$

From (20) and (21), we deduce that  $\log(P \times T) \leq \mathbb{B}(\rho, \delta, n, k)$ .  $\square$

*Remark.* In the case  $k = 0$ , note that:

$$p \times h(s, m, p) \leq \exp \left[ \log(\psi(C)) - \frac{1}{3} + \log p + \frac{m}{2} \log(1 + \frac{p}{m}) + \frac{p}{2} \log(1 + \frac{m}{p}) \right].$$

*Remark.* For  $C = 1$ , using that  $\binom{p}{k} \leq (\frac{ep}{k})^k / (2\pi k)^{\frac{1}{2}}$ , it holds for  $1 \leq k \leq m-1$ :

$$\begin{aligned}
\exp(\mathbb{B}(\rho, \delta, \varepsilon, n, k)) & \leq \psi(1) \frac{2\sqrt{\pi}}{\sqrt{2\pi}} \frac{e^{-5/12}}{\sqrt{2\pi}} \left(\frac{p}{k}\right)^k \frac{e^k}{\sqrt{k}} e^{\frac{m-k}{2}} \\
& \times \left(1 + \frac{p-1-k}{m-2-k}\right)^{\frac{m-1-k}{2}} \left(\frac{p}{s}\right)^{\frac{1-m+k}{2}}, \\
& \leq \psi(1) \frac{e^{-\frac{5}{12}}}{\sqrt{\pi}} \left(\frac{p}{s}\right)^{\frac{1-m}{2}} e^{\frac{m}{2}} \left(\frac{pe}{k}\right)^k e^{\frac{k}{2}} \left(\frac{p}{s}\right)^{\frac{k}{2}} \\
& \times \left(1 + \frac{p-1-k}{m-2-k}\right)^{\frac{m-1-k}{2}}, \\
& \leq \psi(1) \frac{e^{-\frac{5}{12}}}{\sqrt{\pi}} s^{\frac{m-1}{2}} e^{\frac{m}{2}} \left(\frac{pe^{\frac{3}{2}}}{k\sqrt{s}}\right)^k, \\
& \leq \psi(1) e^{-\frac{1}{6}} (se)^{\frac{m}{2}} \left(\frac{pe^{\frac{3}{2}}}{k\sqrt{s}}\right)^k, \\
(22) \quad & \leq (C_0/2) \rho \delta \log(eC_0 \rho \delta) (\rho ne)^{\frac{1-\delta}{2\delta}} \left(\frac{\sqrt{ne^{\frac{3}{2}}}}{Ck\delta\sqrt{\rho}}\right)^k,
\end{aligned}$$

where  $C_0$  is defined as in Lemma 4.

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