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Random Measurable Sets and Covariogram Realisability Problems

Bruno Galerne and Raphaël Lachièze-Rey*

May 24, 2014

Abstract

We provide a characterization of the realisable set covariograms, bringing a rigorous yet abstract solution to the S_2 problem in materials science. Our method is based on the covariogram functional for random measurable sets (RAMS) and on a result about the representation of positive operators in a locally compact space. RAMS are an alternative to the classical random closed sets in stochastic geometry and geostatistics, they provide a weaker framework allowing to manipulate more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of geometric nature. Along the way, we extend the theory of random measurable sets, and in particular the local approximation of the perimeter by local covariograms.

Keywords. Random measurable sets, realisability, S_2 problem, covariogram, perimeter, truncated moment problem.

MSC2010 subject classification. Primary 60D05; Secondary 28C05, 47B65, 60G57, 74A40, 82D30.

Introduction

An old and difficult problem in materials science is the S_2 problem, often posed in the following terms: Given a real function $S_2 : \mathbb{R}^d \rightarrow [0, 1]$, is there a stationary random set $X \subset \mathbb{R}^d$ whose standard two point correlation function is S_2 , that is, such that

$$\mathbf{P}(x - y \in X) = S_2(x - y), \quad x, y \in \mathbb{R}^d ? \quad (0.1)$$

The S_2 problem is a *realisability problem* concerned with the existence of a (translation invariant) probability measure satisfying some prescribed marginal conditions.

One can see the S_2 problem as a truncated version of the general *moment problem* that deals with the existence of a process for which all moments are prescribed. The main difficulty in only considering the moments up to some finite order is that this

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sequence of moments does not uniquely determine the possible solution. The appearance of second order realisability problems for random sets goes back to the 1950's, see for instance [McM55] in the field of telecommunications. There are applications in materials science and geostatistics, and marginal problems in general are present under different occurrences in fields as various as quantum mechanics, computer science, or game theory, see the recent work [FC13] and references therein.

Reconstruction of heterogeneous materials from a knowledge of limited microstructural information (a set of lower-order correlation functions) is a crucial stake in many applications. It is an intriguing inverse problem, and would correspond here to a constructive solution for the realisability problem. This study can serve many purposes, especially in spatial modeling, where one needs to know necessary admissibility conditions to propose new covariance models.

In estimation and reconstruction, one should test whether an estimated covariance indeed corresponds to a random structure, and propose an adapted reconstruction procedure. A series of works by Torquato and his coauthors in the field of materials science gather known necessary conditions and illustrate them for many 2D and 3D theoretical models, along with reconstruction procedures (see [JST07] and the survey [Tor02, Sec. 2.2] and references therein). This question was developed alongside in the field of geostatistics, where some authors do not tackle directly this issue, but address the realisability problem within some particular classes of models, e.g. Gaussian, mosaic, or Boolean model (see [Mas72, CD99, Lan02, Eme10]).

A related question concerns the *specific covariogram* of a stationary random set X , defined for all non empty bounded open set $U \subset \mathbb{R}^d$ by

$$\gamma_X^s(y) = \frac{\mathbf{E}\mathcal{L}^d(X \cap (y + X) \cap U)}{\mathcal{L}^d(U)} = \mathbf{E}\mathcal{L}^d(X \cap (y + X) \cap (0, 1)^d). \quad (0.2)$$

The associated realisability problem, which consists in determining whether there exists a stationary random set X whose specific covariogram is a given function, is the (*specific covariogram realisability problem*). Note that a straightforward Fubini argument gives that for any stationary random closed set X

$$\gamma_X^s(y) = \int_{(0,1)^d} \mathbf{P}(x \in X, x - y \in X) dx = S_2(-y) = S_2(y), \quad (0.3)$$

and thus the S_2 realisability problem and the specific covariogram problem are fundamentally the same.

Our main result provides an abstract and fully rigorous characterization of this problem for random measurable sets (RAMS) having locally finite mean perimeter. Furthermore, in the restrictive one-dimensional case ($d = 1$), results can be passed on to the classical framework of random closed sets. It will become clear in this paper why the covariogram approach in the framework of random measurable sets is more adapted to a rigorous mathematical study. Random measurable sets are an alternative to the classical random closed sets in stochastic geometry and geostatistics, they provide a weaker framework allowing to manipulate more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of geometric nature. Along the way, we extend the theory

of random measurable sets, and in particular the local approximation of the perimeter by local covariograms.

Our main result uses a fundamental relation between the Lipschitzness of the covariogram function of a random set, and the finiteness of its mean variational perimeter, unveiled in [Gal11]. Like in [LRM11, Th. 3.1] about point processes, we prove that the realisability of a given function $S_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ can be characterized by two independent conditions : a positivity condition, and a regularity condition, namely the Lipschitzness of S_2 . The positivity condition deals with the positivity of a linear operator extending S_2 on an appropriate space, and is of combinatorial nature. The proof of this main result relies on a theorem dealing with positive operators in a locally compact space recently derived in [LRM11] to treat realisability problems for point processes. This general method therefore proves here its versatility by being applied in the framework of random sets in a very similar manner.

Checking whether S_2 satisfies the positivity condition is completely distinct from the concerns of this paper. It is a difficult problem that has a long history. It is more or less implicit in many articles, and has been, to the best of the authors' knowledge, first addressed directly by Shepp [She63], later on by Matheron [Mat93], and more recently in [Qui08, LR13b, LR13a]. Still, a deep mathematical understanding of the problem remains out of reach.

The plan of the paper is as follows. We give in Section 1 a quick overview of the mathematical objects involved here, namely random measurable sets, positivity, perimeter, and realisability problems, and we also state the main result of the paper dealing with the specific covariogram realisability problem for stationary random measurable sets with finite specific perimeter. In section 2, we develop the theory of random measurable sets, define different notions of perimeter, and explore the relations with random closed sets, while section 3 is devoted to the local covariogram functional and its use for perimeter approximation. In section 4, we give the precise statement and the proof of the main result. We also show that our main result extends to the framework of one-dimensional stationary RACS.

1 Framework and main results

1.1 Random measurable sets and variational perimeter

Details about random measurable sets are presented in Section 2, and we give here the essential notation for stating the results. Call \mathcal{M} the class of Lebesgue measurable sets of \mathbb{R}^d . A *random measurable set* (RAMS) X is a random variable taking values in \mathcal{M} endowed with the Borel σ -algebra induced by the local convergence in measure (which corresponds to the $L^1_{\text{loc}}(\mathbb{R}^d)$ -topology for the indicator functions, see Section 2.1 for details). Remark that under this topology, one is bound to identify two sets A and A' lying within the same Lebesgue class (that is, such that their symmetric difference $A \Delta A'$ is Lebesgue-negligible), and we indeed perform this identification on \mathcal{M} . Say furthermore that a RAMS is *stationary* if its law is invariant under translations of \mathbb{R}^d .

One geometric notion that can be extended to RAMS is that of perimeter. For a deterministic measurable set A , the perimeter of A in an open set $U \subset \mathbb{R}^d$ is defined as

the variation of the indicator function 1_A in U , that is,

$$\text{Per}(A; U) = \sup \left\{ \int_U 1_A(x) \operatorname{div} \varphi(x) dx : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d), \|\varphi(x)\|_2 \leq 1 \text{ for all } x \right\}, \quad (1.1)$$

where $\mathcal{C}_c^1(U, \mathbb{R}^d)$ denotes the set of continuously differentiable functions $\varphi : U \rightarrow \mathbb{R}^d$ with compact support and $\|\cdot\|_2$ is the Euclidean norm [AFP00]. See Section 2.2 for a discussion and some properties of variational perimeters in the class of measurable sets. If X is a RAMS then for all open sets $U \subset \mathbb{R}^d$, $\text{Per}(X; U)$ is a well-defined random variable. Besides, if X is stationary then $U \mapsto \mathbf{E}(\text{Per}(X; U))$ extends into a measure invariant by translation, and thus proportional to the Lebesgue measure. One calls *specific perimeter* of X the constant of proportionality that will be denoted by $\text{Per}^s(X)$ and that is given by $\text{Per}^s(X) = \mathbf{E}\text{Per}(X; (0, 1)^d)$.

1.2 Covariogram realisability problems

For a deterministic set A , one calls *local covariogram* of A the map

$$\begin{aligned} \mathbb{R}^d \times \mathcal{W} &\rightarrow \mathbb{R} \\ (y; W) &\mapsto \delta_{y;W}(A) := \mathcal{L}^d(A \cap (y + A) \cap W) \end{aligned}$$

where \mathcal{W} denotes the set of *observation windows* defined by

$$\mathcal{W} = \{W \subset \mathbb{R}^d \text{ bounded open set such that } \mathcal{L}^d(\partial W) = 0\}.$$

Given a RAMS X , we denote by $\gamma_X(y; W) = \mathbf{E}\delta_{y;W}(X)$ the (*mean*) *local covariogram* of X . If X is stationary, then the map $W \mapsto \gamma_X(y; W)$ is translation invariant and extends into a measure proportional to the Lebesgue measure. Hence, one calls *specific covariogram* of X and denotes by $y \mapsto \gamma_X^s(y)$, the map such that $\gamma_X(y; W) = \mathbf{E}\delta_{y;W}(X) = \gamma_X^s(y)\mathcal{L}^d(W)$. Note that one simply has $\gamma_X^s(y) = \gamma_X(y, (0, 1)^d)$.

We are interested in this paper in the specific covariogram realisability problem: Given a function $S_2 : \mathbb{R}^d \rightarrow \mathbb{R}$, does there exist a stationary random measurable set $X \in \mathcal{M}$ such that $S_2(y) = \gamma_X^s(y)$ for all $y \in \mathbb{R}^d$?

The specific covariogram candidate S_2 has to verify some structural necessary condition to be realisable.

Definition 1.1 (Covariogram admissible). A function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ is said to be \mathcal{M} -*local covariogram admissible*, or just *admissible*, if for all 5-tuple $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\forall A \in \mathcal{M}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(A) \geq 0 \right] \Rightarrow c + \sum_{i=1}^q a_i \gamma(y_i; W_i) \geq 0.$$

A function $S_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be \mathcal{M} -*specific covariogram admissible*, or just *admissible*, if the function $(y; W) \mapsto S_2(y)\mathcal{L}^d(W)$ is \mathcal{M} -*local covariogram admissible*.

It is an immediate consequence of the positivity and linearity of the mathematical expectation that a realizable S_2 function is necessarily admissible. Checking whether

a given S_2 is admissible, a problem of combinatorial nature, is difficult. It will not be addressed here, but as emphasized in (0.3), it is directly related to the positivity problem for two-point covering functions, which is studied in numerous articles (see [She63, Mat93, Qui08, LR13b, LR13a], and references therein). Let us remark that being admissible is a strong constraint on S_2 that conveys the usual properties of covariogram functions, and in particular, $S_2(y) \geq 0$ for all $y \in \mathbb{R}^d$ (since for all $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ and $A \in \mathcal{M}$, $\delta_{y;W}(A) \geq 0$).

In general, the admissibility of S_2 is not sufficient for S_2 to be realizable. Consider the linear operator Φ

$$\Phi \left(c + \sum_{i=1}^q a_i \delta_{y_i; W_i} \right) = c + \sum_{i=1}^q a_i S_2(y_i) \mathcal{L}^d(W_i) \quad (1.2)$$

on the subspace of functionals on \mathcal{M} generated by the constant functions and the covariogram evaluations $A \mapsto \delta_{y;W}(A)$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$. The realisability of S_2 corresponds to the existence of a probability measure μ on \mathcal{M} representing Φ , i.e. such that $\Phi(g) = \int_{\mathcal{M}} g d\mu$ for g in the forementioned subspace. In a non-compact space such as \mathcal{M} , the positivity of Φ , i.e. the admissibility of S_2 , is not sufficient to represent it by a probability measure, as the σ -additivity is also needed.

It has been shown in [LRM11] that in such non-compact frameworks, the realisability problem should better be accompanied with an additional condition involved with the regularity of the set in some sense. This condition is carried on by a companion function, called a *regularity modulus*, depending on the functions of interest in our realisability problem. Without entering into details (see Section 4), the perimeter function fulfills this role here, mostly because it can be approximated by linear combinations of covariograms, and has compact level sets. The well-posed realisability problem with regularity condition we consider here deals with the existence of a stationary random measurable set $X \in \mathcal{M}$ such that

$$\begin{cases} S_2(y) = \gamma_X^s(y), & y \in \mathbb{R}^d, \\ \text{Per}^s(X) = \mathbf{E}\text{Per}(X; (0, 1)^d) < \infty. \end{cases}$$

The main result of this paper is the following.

Theorem 1.2. *Let $S_2 : \mathbb{R}^d \mapsto \mathbb{R}$ be a function. Then S_2 is the specific covariogram of a stationary random measurable set $X \in \mathcal{M}$ such that $\text{Per}^s(X) < \infty$ if and only if S_2 is admissible and Lipschitz in 0 along the d canonical directions.*

This result is analogous to the one obtained in [LRM11] for point processes, since the realisability condition is shown to be a positivity condition plus a regularity condition, namely the Lipschitzness of S_2 . As already discussed, a realisable function S_2 is necessarily admissible. Besides, extending results from [Gal11], we show that a stationary RAMS X has a finite specific perimeter if and only if its specific covariogram γ_X^s is Lipschitz, and we obtain an explicit relation between the Lipschitz constant of S_2 and the specific perimeter (see Proposition 3.5). Hence the direct implication of Theorem 1.2 is somewhat straightforward. The real difficulty consists in proving the converse implication. To do so we adapt the techniques of [LRM11] to our context which involves several technicalities regarding the approximation of the perimeter by linear combination of local covariogram

functional. We first establish the counterpart of Theorem 1.2 for the realisability of local covariogram function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ (see Theorem 4.1) and we then extend this result to the case of specific covariogram of stationary RAMS (see Theorem 4.9).

In addition, we study the links between RAMS and the more usual framework of random closed sets (RACS), which *in fine* enables us to obtain a result analogous to Theorem 1.2 for RACS of the real line (see Theorem 4.12).

2 Random measurable sets

2.1 Definition of random measurable sets

Random measurable sets (RAMS) are a weak framework for random sets allowing for the definition of irregular functionals. They are defined as random variable taking value in the set \mathcal{M} of Lebesgue (classes of) sets of \mathbb{R}^d endowed with the Borel σ -algebra $\mathcal{B}(\mathcal{M})$ induced by the natural topology, the so-called local convergence in measure. We recall that a sequence of measurable sets $(A_n)_{n \in \mathbb{N}}$ locally converges in measure to a measurable set A if for all bounded open set $U \subset \mathbb{R}^d$, the sequence $\mathcal{L}^d((A_n \Delta A) \cap U)$ tends to 0, where Δ denotes the symmetric difference. The local convergence in measure simply corresponds to the convergence of the indicator functions 1_{A_n} towards 1_A in the space of locally integrable functions $L^1_{\text{loc}}(\mathbb{R}^d)$, and consequently \mathcal{M} is a complete metrizable space¹.

Definition 2.1 (Random measurable sets). A *random measurable set* (RAMS) X is a measurable map

$$\begin{aligned} X : (\Omega, \mathcal{A}) &\rightarrow (\mathcal{M}, \mathcal{B}(\mathcal{M})), \\ \omega &\mapsto X(\omega) \end{aligned}$$

where $\mathcal{B}(\mathcal{M})$ denotes the Borel σ -algebra induced by the local convergence in measure.

Note that if X is a RAMS, then $\omega \mapsto 1_{X(\omega)}$ is a random integrable function. This concept of random measurable (class of) set(s) is not standard. As mentioned in [Mol05], measurable random subsets of the interval $[0, 1]$ are defined following this definition in [SS87].

In the remaining of this section, we will discuss the link between RAMS and other classical random objects, namely random Radon measures, jointly measurable subsets of $\Omega \times \mathbb{R}^d$, and random closed sets.

Random Radon measures associated with random measurable sets Following the usual construction of random objects, a random Radon measure is defined as a measurable function from a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ to the space \mathbf{M}^+ of positive Radon measures over \mathbb{R}^d equipped with the appropriate σ -algebra \mathfrak{M} defined as the smallest σ -algebra for which the evaluation maps $\mu \mapsto \mu(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$ relatively compact, are measurable (see *e.g.* [Kal86, SW08]). Any RAMS $X \subset \mathbb{R}^d$ canonically defines a random

¹ This is a consequence of the facts that $L^1_{\text{loc}}(\mathbb{R}^d)$ is a complete metrizable space and that the set of indicator functions is closed in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Radon measure that is the restriction to X of the Lebesgue measure:

$$\begin{aligned} \mathcal{B}(\mathbb{R}^d) &\rightarrow [0, +\infty], \\ B &\mapsto \mathcal{L}^d(X \cap B). \end{aligned}$$

The measurability of this restriction results from the observation that, for all $B \in \mathcal{B}(\mathbb{R}^d)$, the map $f \mapsto \int_B f(x)dx$ is measurable for the L^1_{loc} -topology.

Existence of a jointly measurable representative For a RAMS $X : \Omega \rightarrow \mathcal{M}$, one can study the measurability properties of the graph $Y = \{(\omega, x) : x \in X(\omega)\} \subset \Omega \times \mathbb{R}^d$.

Definition 2.2 (Jointly measurable representative). A subset $Y \subset \Omega \times \mathbb{R}^d$ is a *jointly measurable representative* of a RAMS X if

1. Y is a jointly measurable subset of $\Omega \times \mathbb{R}^d$ (i.e. $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$),
2. For a.a. $\omega \in \Omega$, the ω -section $Y(\omega) = \{x \in \mathbb{R}^d, (\omega, x) \in Y\}$ is equivalent in measure to $X(\omega)$ (i.e. $\mathcal{L}^d(Y(\omega) \Delta X(\omega)) = 0$).

Proposition 2.3. Any jointly measurable set $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ canonically defines a RAMS by considering the Lebesgue class of its ω -sections:

$$\omega \mapsto Y(\omega) = \{x \in \mathbb{R}^d, (\omega, x) \in Y\}.$$

Conversely, any RAMS X admits jointly measurable representatives $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. The first point is trivial. Let us prove the second point. Consider the random Radon measure μ associated to X , that is

$$\mu(\omega, B) = \mathcal{L}^d(X(\omega) \cap B) = \int_B 1_{X(\omega)}(x)dx.$$

By construction this random Radon measure is absolutely continuous with respect to the Lebesgue measure. Then according to Radon-Nikodym theorem for random measures (see Theorem A.1), there exists a jointly measurable map $g : (\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)) \rightarrow \mathbb{R}$ such that for all $\omega \in \Omega$,

$$\mu(\omega, B) = \int_B g(\omega, x)dx, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Hence for all $\omega \in \Omega$, $1_{X(\omega)}(\cdot)$ and $g(\omega, \cdot)$ are both Radon-Nikodym derivative of $\mu(\omega, \cdot)$ and thus are equal almost everywhere. In particular, for a.a. $x \in \mathbb{R}^d$, $g(\omega, x) \in \{0, 1\}$. Consequently, the function $(\omega, x) \mapsto 1(g(\omega, x) = 1)$ is also jointly measurable and is a Radon-Nikodym derivative of $\mu(\omega, \cdot)$ for all $\omega \in \Omega$, and thus the set

$$Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d, g(\omega, x) = 1\}$$

is a jointly measurable representative of X . □

Remark 2.4. Given a jointly measurable representative Y of a RAMS X , one can consider measurable events of Ω related to Y . However, only a subset of these events belongs to the sub- σ -algebra $\sigma(X)$ of $\mathcal{B}(\mathcal{M})$ induced by X . For example, for a given $x \in \mathbb{R}^d$, Fubini Theorem ensures that the x -section set $\{\omega \in \Omega, (\omega, x) \in Y\}$ is a measurable subset of Ω , but it depends on the Lebesgue representative chosen for X (e.g. $Y \setminus \{x\}$ is another measurable representative for which the section set is always empty). However, events such as $\{\omega \in \Omega, \mathcal{L}^d(Y(\omega, \cdot) \cap B) \geq a\}$, for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $a > 0$, are events of $\sigma(X)$ since they are invariant by a change of Lebesgue representative.

Random measurable sets and random closed sets Recall that $(\Omega, \mathcal{A}, \mathbf{P})$ denote our probability space. Let $\mathcal{F} = \mathcal{F}(\mathbb{R}^d)$ be the set of all closed subset of \mathbb{R}^d . Following [Mol05, p. 1] a random closed set is defined as follows.

Definition 2.5 (Random Closed Set). A map $Z : \Omega \rightarrow \mathcal{F}$ is called a *random closed set* (RACS) if for every compact set $K \subset \mathbb{R}^d$, $\{\omega : Z(\omega) \cap K \neq \emptyset\} \in \mathcal{A}$.

The framework of random closed sets is standard in stochastic geometry [Mat75, Mol05]. Let us now reproduce a result of C.J. Himmelberg that allows to link the different notions of random sets (see [Mol05, Theorem 2.3 p. 26] or the original paper [Him75] for the complete theorem),

Theorem 2.6 (Himmelberg). *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $Z : \Omega \rightarrow Z(\omega) \in \mathcal{F}$ be a map taking values into the set of closed subset of \mathbb{R}^d . Consider the two following assertions:*

- (i) $\{\omega : Z \cap F \neq \emptyset\} \in \mathcal{A}$ for every closed set $F \subset \mathbb{R}^d$,
- (ii) The graph of Z , i.e. the set $\{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in Z(\omega)\}$, belongs to the product σ -algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$,

Then the implication (i) \Rightarrow (ii) is always true, and if the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is complete (i.e. all \mathbf{P} -negligible subsets of Ω are measurable) one has the equivalence (i) \Leftrightarrow (ii).

In view of our definitions for random sets, Himmelberg's theorem can be rephrased in the following terms.

Proposition 2.7. *Any RACS Z defines a jointly measurable set $Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in Z(\omega)\}$, and thus also defines a random measurable set.*

Proof. By definition of a RACS, for every compact set $K \subset \mathbb{R}^d$, $\{\omega : Z \cap K \neq \emptyset\} \in \mathcal{A}$. Now, since we are in \mathbb{R}^d , any closed set $F \subset \mathbb{R}^d$ can be expressed as the countable union of compact sets $F = \bigcup_n K_n$ (e.g. take $K_k = F \cap (k + [0, 1]^d)$ with $k \in \mathbb{Z}^d$). But then

$$\{\omega : Z \cap F \neq \emptyset\} = \bigcup_n \{\omega : Z \cap K_n \neq \emptyset\} \in \mathcal{A}.$$

Hence for every closed set $F \subset \mathbb{R}^d$, $\{\omega : Z \cap F \neq \emptyset\} \in \mathcal{A}$, and thus by (i) \Rightarrow (ii) of Theorem 2.6, the graph Y of Z is jointly measurable, that is Y is a jointly measurable set. According to Proposition 2.3, this jointly measurable set is canonically associated with a random measurable set. \square

The converse of the above proposition is obviously false since random measurable sets take possibly non-closed set values (more precisely some Lebesgue classes do not contain any closed set, see Example 2.10). However, if a jointly measurable set only takes closed set values, then it also defines a RACS.

Proposition 2.8. *Suppose that the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is complete. Let $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ be a jointly measurable set such that for all $\omega \in \Omega$, its ω -section*

$$Y(\omega) = \{x \in \mathbb{R}^d, (\omega, x) \in Y\}$$

is a closed subset of \mathbb{R}^d . Then, the map $\omega \mapsto Y(\omega)$ defines a random closed set.

Proof. Since $\omega \mapsto Z(\omega)$ takes only closed values and $(\Omega, \mathcal{A}, \mathbf{P})$ is complete, by (ii) \Rightarrow (i) of Theorem 2.6, for every closed sets $F \subset \mathbb{R}^d$, $\{\omega : Z \cap F \neq \emptyset\} \in \mathcal{A}$. In particular this is true for every compact set $K \subset \mathbb{R}^d$ and thus, by definition, $\omega \mapsto Z(\omega)$ defines a random closed set. \square

2.2 Random measurable sets of finite perimeter

The perimeter is one of the main geometric characteristics of a set. For a closed set F , the perimeter is generally defined by the length of the topological boundary, that is $\mathcal{H}^{d-1}(\partial F)$. This definition is not valid for a measurable set $A \subset \mathbb{R}^d$, in the sense that $\mathcal{H}^{d-1}(\partial A)$ strongly depends on the representative of A within its Lebesgue class. The proper notion of perimeter for measurable sets is the variational perimeter that defines the perimeter as the variation of the indicator function of the set. An important feature of the variational perimeter is that it is lower semi-continuous for the convergence in measure, while the functional $F \mapsto \mathcal{H}^{d-1}(\partial F)$ is not lower semi-continuous on the set of closed sets \mathcal{F} endowed with the hit or miss topology. This is a key aspect for this paper since it allows to consider the variational perimeter as a regularity modulus for realisability problems in following the framework of [LRM11] (the definition of a regularity modulus will be recalled in Section 4).

Variational perimeters Let U be an open set. As already recalled in the introduction, the (*variational*) *perimeter* $\text{Per}(A; U)$ of a measurable set $A \in \mathcal{M}$ in the open set U is defined by

$$\text{Per}(A; U) = \sup \left\{ \int_U 1_A(x) \text{div} \varphi(x) dx : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d), \|\varphi(x)\|_2 \leq 1 \text{ for all } x \right\},$$

where $\mathcal{C}_c^1(U, \cdot)$ denotes the set of continuously differentiable functions with compact support and $\|\cdot\|_2$ is the Euclidean norm [AFP00]. Closely related to the perimeter, one also defines the directional variation in the direction $u \in S^{d-1}$ of A in U by [AFP00, Section 3.11]

$$V_u(A; U) = \sup \left\{ \int_U 1_A(x) \langle \nabla \varphi(x), u \rangle dx : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}), \|\varphi(x)\|_2 \leq 1 \text{ for all } x \right\}.$$

For technical reasons, we also consider the anisotropic perimeter

$$A \mapsto \text{Per}_{\mathbf{B}}(A; U) = \sum_{j=1}^d V_{e_j}(A; U)$$

which adds up the directional variations along the d directions of the canonical basis $\mathbf{B} = \{e_1, \dots, e_d\}$. In geometric measure theory, the functional $A \mapsto \text{Per}_{\mathbf{B}}(A; U)$ is described as the anisotropic perimeter associated with the anisotropy function $x \mapsto \|x\|_{\infty}$ (see e.g. [CCMN08] and the references therein). Indeed, one easily sees that

$$\text{Per}_{\mathbf{B}}(A; U) = \sup \left\{ \int_U 1_A(x) \operatorname{div} \varphi(x) dx : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d), \|\varphi(x)\|_{\infty} \leq 1 \text{ for all } x \right\}.$$

Hence the only difference between the variational definition of the isotropic perimeter $\text{Per}(A; U)$ and the one of the anisotropic perimeter $\text{Per}_{\mathbf{B}}(A; U)$ is that the test functions φ take values in the $\|\cdot\|_2$ -unit ball B_d for the former whereas they take values in the $\|\cdot\|_{\infty}$ -unit ball $[-1, 1]^d$ for the latter. The set inclusions $B_d \subset [-1, 1]^d \subset \sqrt{d}B_d$ lead to the tight inequalities

$$\text{Per}(A; U) \leq \text{Per}_{\mathbf{B}}(A; U) \leq \sqrt{d} \text{Per}(A; U). \quad (2.1)$$

Consequently a set A has a finite perimeter $\text{Per}(A; U)$ in U if and only if it has a finite anisotropic perimeter $\text{Per}_{\mathbf{B}}(A; U)$ (let us mention that this equivalence is not true when considering only one directional variation $V_u(A; U)$). One says that a measurable set $A \subset \mathbb{R}^d$ has locally finite perimeter if A has a finite perimeter $\text{Per}(A; U)$ in all bounded open set $U \subset \mathbb{R}^d$.

To finish let us mention that if X is a RAMS then $\text{Per}(X; U)$, $\text{Per}_{\mathbf{B}}(X; U)$, and $V_u(X; U)$, $u \in S^{d-1}$, are well-defined random variables since the maps $A \mapsto \text{Per}(A; U)$, $A \mapsto \text{Per}_{\mathbf{B}}(A; U)$ and $A \mapsto V_u(A; U)$ are lower semi-continuous for the convergence in measure [AFP00]. Consequently one says that a RAMS X has a.s. finite perimeter in U if the random variable $\text{Per}(X; U)$ is a.s. finite (or equivalently if $\text{Per}_{\mathbf{B}}(X; U)$ is a.s. finite). One says that a RAMS X has a.s. locally finite perimeter in U if for all bounded open set $V \subset U$ the random variable $\text{Per}(X; V)$ is a.s. finite.

Closed representative of one-dimensional sets of finite perimeter Although the general geometric structure of sets of finite perimeter is well-known (see [AFP00, Section 3.5]) it necessitates involved notions from geometric measure theory (rectifiable sets, reduced and essential boundaries, etc.). However, when restricting to the case of one-dimensional sets of finite perimeter, all the complexity vanishes since subsets of \mathbb{R} having finite perimeter all correspond to finite union of non empty and disjoint closed intervals.

More precisely, according to Proposition 3.52 p. 153 of [AFP00], if a non-negligible measurable set $A \subset \mathbb{R}$ has finite perimeter in an interval $(a, b) \subset \overline{\mathbb{R}}$, there exists an integer p and p pairwise disjoint non empty and closed intervals $J_i = [a_{2i-1}, a_{2i}] \subset \overline{\mathbb{R}}$, with $a \leq a_1 < a_2 < \dots < a_{2p} \leq b$ such that

- $A \cap (a, b)$ is equivalent in measure to the union $\bigcup_i J_i$,

- the perimeter of A in (a, b) is the number of interval endpoints belonging to (a, b)

$$\text{Per}(A; (a, b)) = \#\{a_1, a_2, \dots, a_{2p}\} \cap (a, b).$$

Let us remark that a set of the form $A = \bigcup_i [a_{2i-1}, a_{2i}]$ is closed and that such a set satisfies the identity $\text{Per}(A; (a, b)) = \mathcal{H}^0(\partial A \cap (a, b))$, where ∂A denotes the topological boundary of A and \mathcal{H}^0 is the Hausdorff measure of dimension 0 on \mathbb{R} (*i.e.* the counting measure) while in the general case one only has $\text{Per}(A; (a, b)) \leq \mathcal{H}^0(\partial A \cap (a, b))$ since A may contain isolated points.

More generally, if $A \subset \mathbb{R}$ has locally finite perimeter, then there exists a unique countable or finite family of closed and disjoint intervals $J_i = [a_{2i-1}, a_{2i}]$, $i \in I \subset \mathbb{Z}$, such that A is equivalent in measure to $\bigcup_{i \in I} J_i$ and for all bounded open interval (a, b) , $\text{Per}(A; (a, b))$ is the number of interval endpoints belonging to (a, b) .

Using both this observations and Proposition 2.8, one obtains the following proposition.

Proposition 2.9. *Suppose that the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is complete. Let X be a RAMS of \mathbb{R} that has a.s. locally finite perimeter. Then, there exists a RACS $Z \subset \mathbb{R}$ such that for \mathbf{P} -almost all $\omega \in \Omega$ and for all $a < b \in \mathbb{R}$,*

$$\mathcal{L}^1(X(\omega) \Delta Z(\omega)) = 0 \quad \text{and} \quad \text{Per}(X(\omega); (a, b)) = \mathcal{H}^0(\partial Z(\omega) \cap (a, b)).$$

In Section 4, the above proposition will be used to extend the conclusions of Theorem 1.2, that is valid for RAMS, to the case of one-dimensional RACS.

Proof of Proposition 2.9. Let us first remark that a measurable set of finite perimeter $A \subset \mathbb{R}$ equivalent in measure to $\bigcup_{i \in I} [a_{2i-1}, a_{2i}]$ for some finite or countable index set $I \subset \mathbb{Z}$ has the Lebesgue density

$$\begin{aligned} D(x, A) &= \lim_{r \rightarrow 0^+} \frac{\mathcal{L}^1(A \cap (x-r, x+r))}{2r} \\ &= \begin{cases} 1 & \text{if } x \text{ is in some open interval } (a_{2i-1}, a_{2i}), \\ \frac{1}{2} & \text{if } x \text{ is an interval endpoint } a_{2i-1} \text{ or } a_{2i} \text{ for some } i \in I, \\ 0 & \text{if } x \notin \bigcup_{i \in I} [a_{2i-1}, a_{2i}]. \end{cases} \end{aligned}$$

Let X be a RAMS of \mathbb{R} that has a.s. locally finite perimeter. Let $\Omega' \in \mathcal{A}$ be a subset of Ω of probability one such that for all $\omega \in \Omega'$, X has locally finite perimeter. For all $\omega \in \Omega'$, the Lebesgue class $X(\omega)$ admits a representative that is the union of an at most countable family of non empty and disjoint closed intervals. According to the above observation, for a fixed $\omega \in \Omega'$, the density $D(x, X(\omega))$ exists for all $x \in \mathbb{R}$ and the good representative of X is given by $\{x \in \mathbb{R}, D(x, X(\omega)) > 0\}$. Let $(r_n)_{n \in \mathbb{N}}$ be a positive sequence decreasing to 0, and let us define for all $\omega \in \Omega$,

$$g(\omega, x) = \lim_{n \rightarrow +\infty} \frac{\mathcal{L}^1(X(\omega) \cap (x-r_n, x+r_n))}{2r_n}.$$

According to the proof of Theorem A.1 and Proposition 2.3,

$$Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d, g(\omega, x) > 0\}$$

is a jointly measurable set that is a representative of X . Besides, for a given $\omega \in \Omega'$, since $D(x, X(\omega))$ exists for all $x \in \mathbb{R}$, one has the equality

$$D(x, X(\omega)) = g(\omega, x), \quad x \in \mathbb{R}.$$

Hence, for all $\omega \in \Omega'$, the ω -section $Y(\omega)$ of Y is the union of an at most countable and locally finite family of non empty and disjoint closed intervals, and in particular a closed set. Thus, by Proposition 2.8, the map $\omega \mapsto Y(\omega)$ defines a random closed set. \square

Non-closed RAMS in dimension $d > 1$ In contrast to the one-dimensional case, in dimension $d > 1$ there exists measurable sets of finite perimeter that do not have closed representative in their Lebesgue class. In [AFP00, Example 3.53 p. 154], Ambrosio *et al.* consider a set obtained as the union of an infinite family of open balls with small radii which centers form a dense subset, which yields the following result.

Example 2.10. There exists a measurable subset A of $[0, 1]^d$ with finite perimeter, finite measure $\mathcal{L}^d(A) < 1$, and such that $\mathcal{L}^d(A \cap U) > 0$ for any open subset U of $[0, 1]^d$.

Such a set clearly has no closed representative, because if it had one, say F , then F would charge every open subset of $[0, 1]^d$, and therefore it would be dense in $[0, 1]^d$. But since F is closed, one would have $F = [0, 1]^d$, which contradicts $\mathcal{L}^d(F) = \mathcal{L}^d(A) < 1$.

3 Local covariogram and perimeter approximation

In this section we establish general properties of the local covariogram of a measurable set, as well as the mean local covariogram of a RAMS. A particular emphasis is given on the relation between the local perimeter of a set and the Lipschitz constant of its local covariogram in order to adapt the results of [Gal11] to the local covariogram functional.

3.1 Definition and continuity

The local covariogram of a measurable set $A \in \mathcal{M}$ is the function which to each vector $y \in \mathbb{R}^d$ and observation window $W \in \mathcal{W}$ (recall that

$$\mathcal{W} = \{W \subset \mathbb{R}^d \text{ bounded open set such that } \mathcal{L}^d(\partial W) = 0\})$$

associates the Lebesgue measure of $A \cap (A + y) \cap W$. It will be denoted by

$$\delta_{y;W}(A) = \mathcal{L}^d(A \cap (A + y) \cap W), \quad y \in \mathbb{R}^d, W \in \mathcal{W}.$$

Remark that for all $y \in \mathbb{R}^d$, and $W \in \mathcal{W}$

$$\delta_{y;W}(A) = \delta_{y;W}(A \cap (W \cup (-y + W))), \quad A \in \mathcal{M}, \quad (3.1)$$

so that only the part of A included in the domain $W \cup (-y + W)$ has an influence on the value of $\delta_{y;W}(A)$ (hence local covariograms are indeed local). Before enunciating specific results of interest for our realisability problem, let us prove that local covariograms are continuous for the local convergence in measure.

Proposition 3.1 (Continuity of local covariograms).

(i) For all $A \in \mathcal{M}(\mathbb{R}^d)$ and $W \in \mathcal{W}$ the map $y \mapsto \delta_{y;W}(A)$ is uniformly continuous over \mathbb{R}^d .

(ii) Let $A \in \mathcal{M}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$. Then for all $U, W \in \mathcal{W}$, one has

$$|\delta_{y;U}(A) - \delta_{y;W}(A)| \leq \mathcal{L}^d(U \Delta W).$$

In particular the map $W \mapsto \delta_{y;W}(A)$ is continuous for the convergence in measure.

(iii) Let $A, B \in \mathcal{M}(\mathbb{R}^d)$ and let $W \in \mathcal{W}$. Then for all $y \in \mathbb{R}^d$,

$$|\delta_{y;W}(A) - \delta_{y;W}(B)| \leq 2\mathcal{L}^d((A \Delta B) \cap (W \cup (-y + W))).$$

In particular the map $A \mapsto \delta_{y;W}(A)$ is continuous for the local convergence in measure.

Proof. (i) Let us use the convolution interpretation for local covariograms, that is

$$\delta_{y;W}(A) = \mathcal{L}^d(A \cap (y + A) \cap W) = \int_{\mathbb{R}^d} 1_{A \cap W}(x) 1_{-A}(y - x) dx = 1_{A \cap W} * 1_{-A}(y).$$

Since $1_{A \cap W} \in L^1(\mathbb{R}^d)$ and $1_{-A} \in L^\infty(\mathbb{R}^d)$, the uniform continuity is ensured by the L^p - $L^{p'}$ -convolution theorem (see e.g. [HL99, Proposition 3.2 p. 171]).

(ii) Using the general inequality $|\mathcal{L}^d(A_1) - \mathcal{L}^d(A_2)| \leq \mathcal{L}^d(A_1 \Delta A_2)$ one gets

$$|\delta_{y;U}(A) - \delta_{y;W}(A)| \leq \mathcal{L}^d((A \cap (A + y) \cap U) \Delta (A \cap (A + y) \cap W)) \leq \mathcal{L}^d(U \Delta W).$$

(iii) Suppose first that A and B have finite Lebesgue measure. Then, one has

$$\begin{aligned} |\delta_{y;W}(A) - \delta_{y;W}(B)| &= |1_{A \cap W} * 1_{-A}(y) - 1_{B \cap W} * 1_{-B}(y)| \\ &\leq |1_{A \cap W} * 1_{-A}(y) - 1_{A \cap W} * 1_{-B}(y) + 1_{A \cap W} * 1_{-B}(y) - 1_{B \cap W} * 1_{-B}(y)| \\ &\leq |1_{A \cap W} * (1_{-A} - 1_{-B})(y)| + |(1_{A \cap W} - 1_{B \cap W}) * 1_{-B}(y)| \\ &\leq \|1_{A \cap W}\|_\infty \|1_{-A} - 1_{-B}\|_1 + \|1_{A \cap W} - 1_{B \cap W}\|_1 \|1_{-B}\|_\infty \\ &\leq \mathcal{L}^d(A \Delta B) + \mathcal{L}^d((A \cap W) \Delta (B \cap W)) \\ &\leq 2\mathcal{L}^d(A \Delta B). \end{aligned}$$

The above inequality is valid as soon as A and B have finite Lebesgue measure. The announced general inequality is obtained thanks to Equation (3.1) which ensures that one can replace A and B by $A \cap (W \cup (-y + W))$ and $B \cap (W \cup (-y + W))$ without changing the values of $\delta_{y;W}(A)$ and $\delta_{y;W}(B)$. \square

3.2 Local covariogram and anisotropic perimeter

As for the case of covariogram [Gal11], difference quotients in zero of local covariograms are related to the directional variations of the set A . This is clarified by the next results

where $V_u(f; U)$ denotes the directional variation of $f \in L^1(U)$ in U in the direction $u \in S^{d-1}$, that is

$$V_u(f; U) = \sup \left\{ \int_U f(x) \langle \nabla \varphi(x), u \rangle dx : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}), \|\varphi(x)\|_2 \leq 1 \text{ for all } x \right\}$$

(recall that for a set $A \in \mathcal{M}$, $V_u(A; U) := V_u(1_A; U)$).

For A, B two measurable sets, note

$$A \ominus B = \{x \in A : x + B \subset A\}.$$

Proposition 3.2. *Let U be an open subset of \mathbb{R}^d and $u \in S^{d-1}$. Then for all functions $f \in L^1(U)$ and $\varepsilon \in \mathbb{R}$,*

$$\int_{U \ominus [0, \varepsilon u]} \frac{|f(x + \varepsilon u) - f(x)|}{|\varepsilon|} dx \leq V_u(f; U)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{U \ominus [0, \varepsilon u]} \frac{|f(x + \varepsilon u) - f(x)|}{|\varepsilon|} dx = V_u(f; U). \quad (3.2)$$

This proposition deals with general facts from the theory of functions of bounded variation. We refer to [Gal14] for a complete proof.

The next two propositions show that when f is the indicator function of a set A , the integral

$$\int_{U \ominus [0, \varepsilon u]} \frac{|f(x + \varepsilon u) - f(x)|}{|\varepsilon|} dx$$

can be expressed as a linear combination of local covariograms $\delta_{y; W}(A)$. Since this linear combination will be central in the next results, we introduce the notation

$$\sigma_{u; W}(A) = \frac{1}{\|u\|} (\delta_{0; W \ominus [-u, 0]}(A) - \delta_{u; W \ominus [-u, 0]}(A) + \delta_{0; W \ominus [0, u]}(A) - \delta_{-u; W \ominus [0, u]}(A))$$

for any $A \in \mathcal{M}$, $u \neq 0$, and $W \in \mathcal{W}$. Let us remark that for $W \in \mathcal{W}$, $y \in \mathbb{R}^d$, $A \in \mathcal{M}(\mathbb{R}^d)$,

$$\delta_{0; W}(A) - \delta_{y; W}(A) = \mathcal{L}^d(A \cap W) - \mathcal{L}^d(A \cap (y + A) \cap W) = \mathcal{L}^d((A \setminus (y + A)) \cap W). \quad (3.3)$$

Proposition 3.3 (Local covariogram and anisotropic perimeter). *For all $A \in \mathcal{M}(\mathbb{R}^d)$, $W \in \mathcal{W}$, $\varepsilon \in \mathbb{R}$, and $u \in S^{d-1}$,*

$$0 \leq \sigma_{\varepsilon u; W}(A) \leq V_u(A; W) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon u; W}(A) = V_u(A; W). \quad (3.4)$$

When summing along the d directions of the canonical basis $\mathbf{B} = \{e_1, e_2, \dots, e_d\}$, one obtains similar results for the anisotropic perimeter, that is, for all $A \in \mathcal{M}(\mathbb{R}^d)$ and $\varepsilon \in \mathbb{R}$,

$$0 \leq \sum_{j=1}^d \sigma_{\varepsilon e_j; W}(A) \leq \text{Per}_{\mathbf{B}}(A; W) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^d \sigma_{\varepsilon e_j; W}(A) = \text{Per}_{\mathbf{B}}(A; W).$$

Proof. The announced inequalities are immediate from Proposition 3.2 and the following equality: Let $A \in \mathcal{M}(\mathbb{R}^d)$, $W \in \mathcal{W}$, $u \in S^{d-1}$, then for all $\varepsilon \in \mathbb{R}$,

$$\int_{W \ominus [0, \varepsilon u]} |1_A(x + \varepsilon u) - 1_A(x)| dx = |\varepsilon| \sigma_{\varepsilon u; W}(A).$$

The proof is elementary.

$$\begin{aligned} & \int_{W \ominus [0, \varepsilon u]} |1_A(x + \varepsilon u) - 1_A(x)| dx \\ &= \int_{W \ominus [0, \varepsilon u]} |1_{-\varepsilon u + A}(x) - 1_A(x)| dx \\ &= \mathcal{L}^d(((-\varepsilon u + A) \Delta A) \cap (W \ominus [0, \varepsilon u])) \\ &= \mathcal{L}^d(((-\varepsilon u + A) \setminus A) \cap (W \ominus [0, \varepsilon u])) + \mathcal{L}^d((A \setminus (-\varepsilon u + A)) \cap (W \ominus [0, \varepsilon u])). \end{aligned}$$

Applying the translation of vector εu one has

$$\mathcal{L}^d(((-\varepsilon u + A) \setminus A) \cap (W \ominus [0, \varepsilon u])) = \mathcal{L}^d((A \setminus (\varepsilon u + A)) \cap (\varepsilon u + (W \ominus [0, \varepsilon u]))).$$

Now remark that $\varepsilon u + (W \ominus [0, \varepsilon u]) = W \ominus [-\varepsilon u, 0]$ and thus, using (3.3)

$$\begin{aligned} & \int_{W \ominus [0, \varepsilon u]} |1_A(x + \varepsilon u) - 1_A(x)| dx \\ &= \mathcal{L}^d((A \setminus (\varepsilon u + A)) \cap (W \ominus [-\varepsilon u, 0])) + \mathcal{L}^d((A \setminus (-\varepsilon u + A)) \cap (W \ominus [0, \varepsilon u])) \\ &= \delta_{0; W \ominus [-\varepsilon u, 0]}(A) - \delta_{\varepsilon u; W \ominus [-\varepsilon u, 0]}(A) + \delta_{0; W \ominus [0, \varepsilon u]}(A) - \delta_{-\varepsilon u; W \ominus [0, \varepsilon u]}(A). \end{aligned}$$

□

We now turn to the counterpart of Proposition 3.3 for mean local covariograms of RAMS. For a RAMS X , let γ_X be the (mean) local covariogram of the RAMS X defined by

$$\gamma_X(y; W) = \mathbf{E} \delta_{y; W}(X), \quad y \in \mathbb{R}^d, W \in \mathcal{W},$$

and define similarly $\sigma_X(u; W) = \mathbf{E} \sigma_{u; W}(X)$.

Corollary 3.4. *Let X be a RAMS. Then for all $W \in \mathcal{W}$ and $u \in S^{d-1}$,*

$$\mathbf{E} V_u(X; W) = \lim_{\varepsilon \rightarrow 0} \sigma_X(\varepsilon u; W).$$

Proof. Apply Lebesgue theorem with the almost sure convergence and domination given by (3.4). □

We now turn to similar results for stationary RAMS. Let us first precise the definition of specific covariogram, specific perimeter, and specific directional variation of a stationary RAMS. Remark that for any RAMS X , the map $W \mapsto \gamma_X(y; W)$ extends into a measure for each $y \in \mathbb{R}^d$. If the law of X is invariant under translations then for all $u \in \mathbb{R}^d$,

$$\begin{aligned} \gamma_X(y; u + W) &= \mathbf{E} (\mathcal{L}^d(X \cap (y + X) \cap (u + W))) \\ &= \mathbf{E} (\mathcal{L}^d((-u + X) \cap (y - u + X) \cap W)) \\ &= \mathbf{E} (\mathcal{L}^d(X \cap (y + X) \cap W)) \\ &= \gamma_X(y; W). \end{aligned}$$

It follows that the measure $W \mapsto \gamma_X(y; W)$ is proportional to the Lebesgue measure. The constant of proportionality is called *specific covariogram* of X and is denoted by $\gamma_X^s(y)$. Hence for all $W \in \mathcal{W}$, $\gamma_X(y; W) = \gamma_X^s(y) \mathcal{L}^d(W)$. In particular, $\gamma_X^s(y) = \gamma_X(y; (0, 1)^d)$.

Similarly, if X is stationary, the measure $U \mapsto \mathbf{E}\text{Per}_{\mathbf{B}}(X; U)$ is invariant by translation and thus proportional to the Lebesgue measure. One calls *specific perimeter* the constant of proportionality that will be denoted by $\text{Per}_{\mathbf{B}}^s(X)$ and that is given by

$$\text{Per}_{\mathbf{B}}^s(X) = \mathbf{E}\text{Per}_{\mathbf{B}}(X; (0, 1)^d) \in [0, +\infty].$$

Similarly, for all $u \in S^{d-1}$, one defines the specific variation in direction u by $V_u^s(X) = \mathbf{E}V_u(X; (0, 1)^d)$.

For a function $F : \mathbb{R}^d \rightarrow \mathbb{R}$, one defines the Lipschitz constant in the j -th direction at $y \in \mathbb{R}^d$ by

$$\text{Lip}_j(F, y) = \sup_{t \in \mathbb{R}} \frac{|F(y + te_j) - F(y)|}{|t|},$$

and note $\text{Lip}_j(F) = \sup_{y \in \mathbb{R}^d} \text{Lip}_j(F, y)$. Let us note that a function F is Lipschitz in the usual sense if and only if the d constants $\text{Lip}_j(F)$, $j \in \{1, \dots, d\}$, are finite.

Proposition 3.5. *Let X be a stationary RAMS and let γ_X^s be its specific covariogram. Then γ_X^s is even and for all $y, z \in \mathbb{R}^d$,*

$$|\gamma_X^s(y) - \gamma_X^s(z)| \leq \gamma_X^s(0) - \gamma_X^s(y - z).$$

In particular, γ_X^s is Lipschitz over \mathbb{R}^d if and only if γ_X^s is Lipschitz in 0. Besides, one has for all $j \in \{1, \dots, d\}$,

$$\frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} \leq \frac{1}{2} V_{e_j}^s(X), \quad \varepsilon \neq 0,$$

and

$$\text{Lip}_j(\gamma_X^s) = \text{Lip}_j(\gamma_X^s, 0) = \lim_{\varepsilon \rightarrow 0} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} = \frac{1}{2} V_{e_j}^s(X).$$

Proof. The proof is an adaptation of similar results for covariogram functions [Gal11]. Let us first check that γ_X^s is even. For all $y \in \mathbb{R}^d$, one has

$$\gamma_X^s(-y) = \mathbf{E}(\mathcal{L}^d(X \cap (-y + X) \cap [0, 1]^d)) = \mathbf{E}(\mathcal{L}^d((y + X) \cap X \cap (y + [0, 1]^d))) = \gamma_X^s(y).$$

Let us turn to the inequality. Remark that one has for all $y, z \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $A \in \mathcal{M}(\mathbb{R}^d)$,

$$\begin{aligned} \delta_{y;W}(A) - \delta_{z;W}(A) &= \mathcal{L}^d(A \cap (y + A) \cap W) - \mathcal{L}^d(A \cap (z + A) \cap W) \\ &\leq \mathcal{L}^d((A \cap (y + A) \cap W) \setminus (A \cap (z + A) \cap W)) \\ &\leq \mathcal{L}^d(((y + A) \cap W) \setminus ((z + A) \cap W)) \\ &\leq \mathcal{L}^d((y + A) \cap W) - \mathcal{L}^d((y + A) \cap (z + A) \cap W) \\ &\leq \mathcal{L}^d(A \cap (-y + W)) - \mathcal{L}^d(A \cap (z - y + A) \cap (-y + W)) \\ &\leq \delta_{0; -y+W}(A) - \delta_{z-y; -y+W}(A). \end{aligned} \tag{3.5}$$

Consequently,

$$\gamma_X(y; W) - \gamma_X(z; W) \leq \gamma_X(0; -y + W) - \gamma_X(z - y; -y + W).$$

But since $\gamma_X(y; W) = \gamma_X^s(y) \mathcal{L}^d(W)$, one has $\gamma_X^s(y) - \gamma_X^s(z) \leq \gamma_X^s(0) - \gamma_X^s(z - y)$ and interchanging y and z yields $|\gamma_X^s(y) - \gamma_X^s(z)| \leq \gamma_X^s(0) - \gamma_X^s(y - z)$. Thanks to this inequality one has,

$$\text{Lip}_j(\gamma_X^s) = \sup_{y \in \mathbb{R}^d, \varepsilon \in \mathbb{R}} \frac{|\gamma_X^s(y + \varepsilon e_j) - \gamma_X^s(y)|}{|\varepsilon|} = \sup_{t \in \mathbb{R}} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|t|} = \text{Lip}_j(\gamma_X^s, 0).$$

Now since γ_X^s is even, for all $\varepsilon \neq 0$,

$$\begin{aligned} & \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} \\ &= \frac{1}{2} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j) + \gamma_X^s(0) - \gamma_X^s(-\varepsilon e_j)}{|\varepsilon|} \\ &= \frac{1}{2} \sup_{c > 0} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j) + \gamma_X^s(0) - \gamma_X^s(-\varepsilon e_j)}{|\varepsilon|} \frac{c^d - |\varepsilon|c}{c^d} \\ &= \frac{1}{2} \sup_{c > 0} \frac{(\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)) \mathcal{L}^d((0, c)^d \ominus [-\varepsilon e_j, 0]) + (\gamma_X^s(0) - \gamma_X^s(-\varepsilon e_j)) \mathcal{L}^d((0, c)^d \ominus [0, \varepsilon e_j])}{|\varepsilon| \mathcal{L}^d((0, c)^d)} \\ &= \frac{1}{2} \sup_{c > 0} \sigma_X(\varepsilon e_j; (0, c)^d) \frac{1}{\mathcal{L}^d((0, c)^d)} \\ &\leq \frac{1}{2} \sup_{c > 0} \mathbf{E}V_{e_j}(X; (0, c)^d) \frac{1}{\mathcal{L}^d((0, c)^d)} = \frac{1}{2} V_{e_j}^s(X), \end{aligned}$$

where we used the inequality of (3.4) for the inequality. This shows that

$$\text{Lip}_j(\gamma_X^s, 0) = \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} \leq \frac{1}{2} V_{e_j}^s(X).$$

Besides, for all $\varepsilon \neq 0$ and $c > 0$,

$$\frac{1}{2} \sigma_X(\varepsilon e_j; (0, c)^d) \frac{1}{\mathcal{L}^d((0, c)^d)} \leq \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} \leq \frac{1}{2} V_{e_j}^s(X),$$

and according to Corollary 3.4, the left-hand term tends to $\frac{1}{2} V_{e_j}^s(X)$ when ε tends to 0. Hence,

$$\lim_{\varepsilon \rightarrow 0} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|} = \frac{1}{2} V_{e_j}^s(X) = \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma_X^s(0) - \gamma_X^s(\varepsilon e_j)}{|\varepsilon|}.$$

□

3.3 Anisotropic perimeter approximation for pixelized sets

The proofs of our main result rely on several approximations involving pixelized sets and discretized covariogram. We proved in the previous section that the directional variations as well as the anisotropic perimeter can be computed from limits of difference

quotients in zero of the local covariogram. We show here that for pixelized sets, the anisotropic perimeter $\text{Per}_{\mathbf{B}}(A; W)$ can be expressed as a finite difference in zero of the local covariogram functionals. Before enunciating the formula, let us introduce some notation regarding pixelized sets.

For $n \in \mathbb{N}^*$, we consider the pixels $C_k^n = \frac{1}{n}k + [0, n^{-1}]^d$, $k \in \mathbb{Z}^d$, that are the cells of the lattice $n^{-1}\mathbb{Z}^d$. Let us denote by \mathcal{M}_n the algebra of \mathbb{R}^d induced by the sets C_k^n , $k \in \mathbb{Z}^d$, and note $\mathcal{W}_n = \mathcal{W} \cap \mathcal{M}_n$. Remark that for any set $A \in \mathcal{M}_n$, there is a unique subset I_A of \mathbb{Z}^d such that A is equivalent in measure to $\cup_{k \in I_A} C_k^n$.

Proposition 3.6. *Let $\mathbf{B} = (e_1, e_2, \dots, e_d)$ be the canonical basis of \mathbb{R}^d and let $n \in \mathbb{N}^*$. For all $A \in \mathcal{M}_n$, $W \in \mathcal{W}_n$, and $j \in \{1, \dots, d\}$,*

$$V_{e_j}(A; W) = \sigma_{n^{-1}e_j; W}(A).$$

Hence, for all $A \in \mathcal{M}_n$ and $W \in \mathcal{W}_n$, $\text{Per}_{\mathbf{B}}(A; W) = \sum_{j=1}^d \sigma_{n^{-1}e_j; W}(A)$.

Proof. Let $0 < \varepsilon \leq n^{-1}$. Let us consider the quantity

$$\begin{aligned} \delta_{\varepsilon e_j; W \ominus [-\varepsilon e_j, 0]}(A) &= \mathcal{L}^d(A \cap (\varepsilon e_j + A) \cap (W \ominus [-\varepsilon e_j, 0])) \\ &= \mathcal{L}^d \left(\left(\bigcup_{k \in I_A} C_k^n \right) \cap \left(\bigcup_{l \in I_A} (\varepsilon e_j + C_l^n) \right) \cap (W \ominus [-\varepsilon e_j, 0]) \right). \end{aligned}$$

The two unions are over sets with pairwise negligible intersection, whence

$$\delta_{\varepsilon e_j; W \ominus [-\varepsilon e_j, 0]}(A) = \sum_{k, l \in I_A} \mathcal{L}^d(C_k^n \cap (C_l^n + \varepsilon e_j) \cap (W \ominus [-\varepsilon e_j, 0])).$$

Since $0 < \varepsilon \leq n^{-1}$, for $k, l \in I_A$,

$$\mathcal{L}^d(C_k^n \cap (C_l^n + \varepsilon e_j) \cap (W \ominus [-\varepsilon e_j, 0])) = \begin{cases} n^{-(d-1)}(n^{-1} - \varepsilon) & \text{if } l = k \in I_W, \\ \varepsilon n^{-(d-1)} & \text{if } l = k - e_j \text{ and } k, l \in I_W, \\ 0 & \text{otherwise.} \end{cases}$$

These assertions are straightforward, one simply has to be cautious in the case $l = k - e_j$, $k \in I_W$, $l \notin I_W$, contribution of which is 0. Summing up those contributions and doing similar computations for the quantities $\delta_{0; W \ominus [-\varepsilon e_j, 0]}(A)$, $\delta_{-\varepsilon e_j; W \ominus [0, \varepsilon e_j]}(A)$, $\delta_{0; W \ominus [0, \varepsilon e_j]}(A)$ yields that for some real numbers α, β independent of ε , such that for all $\varepsilon \in (0, n^{-1}]$,

$$\sigma_{\varepsilon e_j; W}(A) = \alpha/\varepsilon + \beta.$$

Proposition 3.3 then implies $\alpha = 0$, $\beta = V_{e_j}(A; W)$, which yields the desired conclusion with $\varepsilon = n^{-1}$. \square

4 Realisability result

In this section we explicit some considerations related to our realisability result and give its proof.

4.1 Realisability problem and regularity modulus

Recall that the local covariogram of a RAMS X is $\gamma_X(y; W) = \mathbf{E}\delta_{y;K}(X)$.

Let us introduce a regularized realisability problem for local covariogram. Put $U_n = (-n, n)^d$. Define the *weighted anisotropic perimeter* by

$$\text{Per}_{\mathbf{B}}^\beta(A) = \sum_{n \geq 1} \beta_n \text{Per}_{\mathbf{B}}(A; U_n)$$

where the sequence (β_n) is set to $\beta_n = 2^{-n}(2n)^{-d}$ so that $\sum_{n \geq 1} \beta_n \mathcal{L}^d(U_n) = 1$.

For a given function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$, define

$$\sigma_\gamma(u; W) = \frac{1}{\|u\|} [\gamma(0; W \ominus [-u, 0]) - \gamma(u; W \ominus [-u, 0]) + \gamma(0; W \ominus [0, u]) - \gamma(-u; W \ominus [0, u])].$$

One defines for all window $W \in \mathcal{W}$ the constant $L_j(\gamma, W) \in [0, +\infty]$ by

$$L_j(\gamma, W) = \sup_{\varepsilon \in \mathbb{R}} \sigma_\gamma(\varepsilon e_j; W), \quad j \in \{1, \dots, d\}. \quad (4.1)$$

$L_j(\gamma, W)$ is related to the Lipschitzness of γ in its spatial variable. The motivation for considering this particular constant comes from Corollary 3.4 which shows that that if γ_X is the local covariogram of a RAMS X , then

$$\mathbf{E}V_{e_j}(X; W) = \sup_{\varepsilon \in \mathbb{R}} \sigma_{\gamma_X}(\varepsilon e_j; W).$$

Theorem 4.1. *Let $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ be a function and $r \geq 0$. Then γ is realizable by a RAMS X such that*

$$\mathbf{E}\text{Per}_{\mathbf{B}}^\beta(X) \leq r$$

iff γ is admissible and

$$\sum_{n \geq 1} \beta_n \left(\sum_{j=1}^d L_j(\gamma, W_n) \right) \leq r, \quad (4.2)$$

where for all $j \in \{1, \dots, d\}$ and $n \geq 1$, the constant $L_j(\gamma, W_n)$ is defined by Equation (4.1).

The stationary counterpart of the above theorem is stated and proved in Section 4.2.

Let us recall the general definition and result of [LRM11] that we use for the proof of our main result.

Definition 4.2 (Regularity modulus). Let G^* be a vector space of measurable real functions on \mathcal{M} . A G^* -regularity modulus on \mathcal{M} is a lower semi-continuous function $\chi : \mathcal{M} \mapsto [0, +\infty]$ such that for all $g \in G^*$, the level set

$$H_g = \{A \in \mathcal{M}, \chi(A) \leq g(A)\} \subset \mathcal{M}$$

is relatively compact for the convergence in measure.

We give the following result, straightforward consequence of Proposition 2.2 and Theorem 2.6 in [LRM11] for bounded continuous functions, see in particular the discussion after the proof of Theorem 2.6.

Theorem 4.3 (Lachièze-Rey - Molchanov [LRM11]). *Let G^* be a vector space of real continuous bounded functions on \mathcal{M} that comprises constant functions. Let χ be a G^* -regularity modulus, and Φ be a linear function on G^* such that $\Phi(1) = 1$. Then, for any given $r \geq 0$, there exists a random measurable set $X \in \mathcal{M}$ such that*

$$\begin{cases} \mathbf{E}g(X) = \Phi(g), & g \in G^*, \\ \mathbf{E}\chi(X) \leq r \end{cases} \quad (4.3)$$

if and only if

$$\sup_{g \in G^*} \inf_{A \in \mathcal{M}} \chi(A) - g(A) + \Phi(g) \leq r. \quad (4.4)$$

In our setting, call G the vector space generated by the constant functionals and the local covariogram functionals $A \mapsto \delta_{y;W}(A)$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$.

Proposition 4.4. $\text{Per}_{\mathbf{B}}^\beta$ is a G -regularity modulus (and therefore a G^* -regularity modulus for any subspace $G^* \subset G$).

Proof. By definition of a regularity modulus, one has to show that the $\text{Per}_{\mathbf{B}}^\beta$ -level sets are relatively compact. Consider a sequence (A_n) such that $\text{Per}_{\mathbf{B}}^\beta(A_n) \leq c$ for all $n \in \mathbb{N}$. Then for all $n, m \in \mathbb{N}$, $\text{Per}_{\mathbf{B}}(A_n; U_m) \leq \frac{c}{\beta_m} < +\infty$ and thus (A_n) is a sequence of sets of locally finite perimeter whose perimeter in any open bounded set $U \subset \mathbb{R}^d$ is uniformly bounded. According to Theorem 3.39 p. 145 of [AFP00], there exists a subsequence of (A_n) that locally converges in measure in \mathbb{R}^d . \square

For $g \in G$, denote by $\text{dom}(g)$ the smallest open set such that for every measurable set A , $g(A) = g(A \cap \text{dom}(g))$. If g is under the form

$$g = \sum_{i=1}^q a_i \delta_{y_i; W_i}, \quad (4.5)$$

we have $\text{dom}(g) \subset \cup_i (W_i \cup (-y_i + W_i))$, but there is not equality because such a decomposition is not unique.

We can now turn to the proof of Theorem 4.1. It involves several technical lemmas that are stated within the proof when needed. Their demonstrations are delayed to the end of the section in order to facilitate the reading of the proof.

Proof of Theorem 4.1. Necessity: If X is a RAMS, then the admissibility of γ_X is the consequence of the positivity of the mathematical expectation (see the discussion below Definition 1.1). According to Proposition 3.3, for all $n \geq 1$, with probability 1,

$$\sigma_{\varepsilon e_j; U_n}(X) \leq V_{e_j}(X; U_n).$$

After taking the expectation, the supremum of the left hand member over $\varepsilon > 0$ is $L_j(\gamma, W)$. Summing over j yields $\sum_{j=1}^d L_j(\gamma, U_n) \leq \text{Per}_{\mathbf{B}}(X; U_n)$, and multiplying by β_n and summing over n yields (4.2).

Sufficiency: Call $G_n \subset G$ the set of functionals $g = c + \sum_{i=1}^q a_i \delta_{y_i; W_i}$ such that for all i , $y_i \in n^{-1}\mathbb{Z}^d$, and $W_i \in \mathcal{W}_n$ (i.e. the closures of the W_i are pixelized sets). Denote by $G^* = \bigcup_{n \geq 1} G_n$. Remark that each G_n is a vector space and that $G^* \subset G$ is a vector

space as well: Indeed, if $g_1 \in G_n$ and $g_2 \in G_m$, then one has $g_1 + g_2 \in G_{mn}$. To apply Theorem 4.3 we need to show that,

$$\sup_{n \geq 1} \sup_{g \in G_n} \inf_{A \in \mathcal{M}(\mathbb{R}^d)} \text{Per}_{\mathbf{B}}^\beta(A) - g(A) + \Phi(g) \leq r,$$

where Φ is defined by

$$\Phi \left(c + \sum_{i=1}^q a_i \delta_{y_i; W_i} \right) = c + \sum_{i=1}^q \gamma(y_i, W_i).$$

First remark that Φ is a positive operator because γ is admissible (see Definition 1.1). Let $g \in G_n$. Define $M_g = \inf_{A \in \mathcal{M}(\mathbb{R}^d)} \text{Per}_{\mathbf{B}}^\beta(A) - g(A) + \Phi(g)$. Let $p \in \mathbb{N}$ large enough such that $\text{dom}(g) \subset (-p, p)^d$. For all $c > 0$ note $\mathcal{M}_n^c = \mathcal{M}_n((-c, c)^d)$. We have

$$M_g \leq \inf_{A \in \mathcal{M}_n^p} \text{Per}_{\mathbf{B}}^\beta(A) - g(A) + \Phi(g)$$

because $\mathcal{M}_n^p \subset \mathcal{M}$. The proof is based on an approximation of the perimeter by a discretized functional with compact domain, summarized by the following lemma, which is proved at the end of the section.

Lemma 4.5. *For $n, p \geq 1$, put $U_n^p = (-p - 1/n, p + 1/n)^d$. There exists $g_{n,p} \in G_n$ with $\text{dom}(g_{n,p}) \subset U_n^p$ such that*

$$g_{n,p}(A) = \text{Per}_{\mathbf{B}}^\beta(A)$$

for all $A \in \mathcal{M}_n^p$. Its explicit expression is

$$g_{n,p}(A) = \sum_{m=1}^p \beta_m \sum_{j=1}^d \sigma_{n^{-1}e_j; U_n}(A) + \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \sum_{j=1}^d \sigma_{n^{-1}e_j; U_p}(A).$$

Furthermore, for all $A \in \mathcal{M}_n$,

$$|g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| \leq E_{n,p}$$

where $E_{n,p} := 8dn2^{-p}(p+1)^{-d} \left(\left(p + \frac{1}{n} \right)^d - p^d \right)$.

Therefore $g_{n,p} = \text{Per}_{\mathbf{B}}^\beta$ on \mathcal{M}_n^p , and

$$\begin{aligned} M_g &\leq \inf_{A \in \mathcal{M}_n^p} g_{n,p}(A) - g(A) + \Phi(g) \\ &\leq \inf_{A \in \mathcal{M}_n^{p+1/n}} g_{n,p}(A) - g(A) + \Phi(g) + E_{n,p} \end{aligned}$$

because $g(A) = g(A \cap \text{dom}(g)) = g(A \cap (-p, p)^d) = g(A \cap (-p - 1/n, p + 1/n)^d)$ and where the error term $E_{n,p} \geq |g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)|$ is computed in Lemma 4.5. We now need the following lemma, also proved afterwards.

Lemma 4.6. *Any functional $g \in G_n$ reaches its infimum on a set of $\mathcal{M}_n(\text{dom}(g))$, that is*

$$\inf_{\mathcal{M}} g = \min_{\mathcal{M}_n(\text{dom}(g))} g.$$

We have $\text{dom}(g_{n,p} - g) \subset U_n^p$, whence in virtue of Lemma 4.6, $g_{n,p} - g$ reaches its infimum over \mathcal{M} on $\mathcal{M}_n^{p+1/n}$, and

$$\inf_{A \in \mathcal{M}_n^{p+1/n}} g_{n,p}(A) - g(A) = \inf_{A \in \mathcal{M}} g_{n,p}(A) - g(A) = \inf_{A \in \mathcal{M}} (g_{n,p} - g)(A) \leq \Phi(g_{n,p} - g)$$

where the last inequality is a consequence of the positivity of Φ . Therefore

$$M_g \leq \Phi(g_{n,p} - g) + \Phi(g) + E_{n,p} = \Phi(g_{n,p}) + E_{n,p}. \quad (4.6)$$

Let us bound $\Phi(g_{n,p})$. Recall that by definition $\Phi(\delta_{y;W}) = \gamma(y;W)$. Using the definition of the constants $L_j(\gamma, W)$ and the expression of $g_{n,p}$, one has

$$\Phi(g_{n,p}) \leq \sum_{m=1}^p \beta_m \left(\sum_{j=1}^d L_j(\gamma, W_m) \right) + \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \left(\sum_{j=1}^d L_j(\gamma, U_n^p) \right)$$

At this point we need to use the following lemma regarding the constants $L_j(\gamma, W)$.

Lemma 4.7. *For all admissible function γ , any $W, W' \in \mathcal{W}$, and $j \in \{1, \dots, d\}$, one has*

$$W \subset W' \quad \Rightarrow \quad L_j(\gamma; W) \leq L_j(\gamma, W').$$

By Lemma 4.7, since γ is admissible, and for all $m > p$, $U_n^p \subset W_m$,

$$\begin{aligned} \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \left(\sum_{j=1}^d L_j(\gamma, U_n^p) \right) &= \sum_{m=p+1}^{+\infty} \beta_m \left(\sum_{j=1}^d L_j(\gamma, U_n^p) \right) \\ &\leq \sum_{m=p+1}^{+\infty} \beta_m \left(\sum_{j=1}^d L_j(\gamma, W_m) \right). \end{aligned}$$

Hence,

$$\Phi(g_{n,p}) \leq \sum_{m=1}^{+\infty} \beta_m \left(\sum_{j=1}^d L_j(\gamma, W_m) \right) \leq r.$$

Coming back to (4.6) yields

$$M_g \leq r + E_{n,p}.$$

But since for all $n \geq 1$, $E_{n,p}$ tends to 0 as p tends to $+\infty$, one has $M_g \leq r$. Since $n \geq 1$ and $g \in G_n$ were arbitrarily chosen, we conclude that

$$\sup_{g \in G^*} \inf_{A \in \mathcal{M}(\mathbb{R}^d)} \text{Per}_{\mathbf{B}}^\beta(A) - g(A) + \Phi(g) \leq r.$$

Hence we can apply Theorem 4.3 that ensures that there exists a RAMS X solution of the problem (4.3). This RAMS X satisfies $\mathbf{EPer}_{\mathbf{B}}^\beta(X) \leq r$ and

$$\forall y \in \mathbb{Q}^d, W \in \mathcal{W} \cap \bigcup_{n \in \mathbb{N}^*} \mathcal{M}_n(\mathbb{R}^d), \quad \gamma_X(y; W) = \gamma(y; W).$$

It only remains to show that this equality between γ_X and γ extends to all couple $(y; W) \in \mathbb{R}^d \times \mathcal{W}$ using the density of $\mathbb{Q}^d \times \mathcal{W} \cap \bigcup_{n \in \mathbb{N}^*} \mathcal{M}_n(\mathbb{R}^d)$ and the continuity of both γ_X and γ .

First, regarding the W -variable, since γ_X and γ are both admissible and by Proposition 3.1 for all $U, W \in \mathcal{W}$, one always has $|\delta_{y;U}(A) - \delta_{y;W}(A)| \leq \mathcal{L}^d(U \Delta W)$, hence both γ_X and γ are continuous with respect to the convergence in measure. Besides, the set of pixelized sets $\mathcal{W} \cap \bigcup_{n \in \mathbb{N}^*} \mathcal{M}_n(\mathbb{R}^d)$ is dense in \mathcal{W} for the convergence in measure. Indeed, given $W \in \mathcal{W}$, one easily shows by dominated convergence that the sequence $W_n = \bigcup_{k \in \mathbb{Z}^d} \{C_k^n, C_k^n \subset W\}$ converges in measure towards W since thanks to the hypothesis $\mathcal{H}^{d-1}(\partial W) < +\infty$ for almost all $x \in \mathbb{R}^d$ either $x \in W$ or $x \in \mathbb{R}^d \setminus \overline{W}$.

Regarding the y -variable, clearly \mathbb{Q}^d is dense in \mathbb{R}^d and $y \mapsto \gamma_X(y; W) = \mathbf{E} \delta_{y;W}(X)$ is continuous since according to Proposition 3.1 for all $\omega \in \Omega$, $y \mapsto \delta_{y;W}(X(\omega))$ is continuous and bounded by $\mathcal{L}^d(W)$. To conclude the proof, let us show that $y \mapsto \gamma(y; W)$ is also continuous over \mathbb{R}^d , which is the purpose of the following lemma.

Lemma 4.8. *Let γ be an admissible function. Let $y \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $r > 0$. For all $z \in \mathbb{R}^d$ and $\rho > 0$, let us denote by $C(z, \rho)$ the hypercube of center z and half size length ρ , that is $C(z, \rho) = \{z' \in \mathbb{R}^d, \|z' - z\|_\infty \leq \rho\}$.*

Then, for all $z, z' \in C(y, r)$ one has

$$|\gamma(z; W) - \gamma(z'; W)| \leq \sum_{j=1}^d L_j(\gamma, W \oplus C(-y, 3r)) |z'_j - z_j|.$$

In particular, if γ satisfies (4.2) then for all $W \in \mathcal{W}$, the map $y \mapsto \gamma(y; W)$ is locally Lipschitz.

□

Proof of lemma 4.5. First let us remark that for all sets A of finite perimeter such that $A \subset (-p, p)^d$ for some integer $p \geq 1$ and for all $m > p$, since $A \cap (U_m \setminus (-p, p)^d) = \emptyset$,

$$\text{Per}_{\mathbf{B}}(A; U_m) = \text{Per}_{\mathbf{B}}(A; (-p - \frac{1}{n}, p + \frac{1}{n})^d) = \text{Per}_{\mathbf{B}}(A; U_n^p).$$

Consequently,

$$\text{Per}_{\mathbf{B}}^\beta(A) = \sum_{m \geq 1} \beta_m \text{Per}_{\mathbf{B}}(A; U_m) = \sum_{m=1}^p \beta_m \text{Per}_{\mathbf{B}}(A; U_m) + \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \text{Per}_{\mathbf{B}}(A; U_n^p).$$

According to Proposition 3.6, for all pixelised set $A \in \mathcal{M}_n^p$, all the perimeters $\text{Per}_{\mathbf{B}}(A; U_m)$, $1 \leq m \leq p$ and $\text{Per}_{\mathbf{B}}(A; U_n^p)$ can be expressed as some linear combination of local covariograms, which gives

$$\text{Per}_{\mathbf{B}}^\beta(A) = \sum_{m=1}^p \beta_m \sum_{j=1}^d \sigma_{n^{-1}e_j; U_m}(A) + \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \sum_{j=1}^d \sigma_{n^{-1}e_j; U_n^p}(A).$$

The linear combination on the right-hand side is an element of G that will be denoted $g_{n,p}$ in what follows. Note one has $\text{dom}(g_{n,p}) \subset U_n^p = (-p - \frac{1}{n}, p + \frac{1}{n})^d$. It remains to

show the inequality $|g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| \leq E_{n,p}$. Using Proposition 3.1, for all $A \in \mathcal{M}_n$,

$$\begin{aligned} |g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| &= |g_{n,p}(A \cap U_n^p) - g_{n,p}(A \cap (-p, p)^d)| \\ &\leq \left(\sum_{m=p+1}^{+\infty} \beta_m \right) dn 8 \mathcal{L}^d((A \cap U_n^p) \Delta (A \cap (-p, p)^d)). \end{aligned}$$

For all $m \geq p+1$, one has $\beta_m = 2^{-m}(2m)^{-d} \leq 2^{-m}(2(p+1))^{-d}$. Hence,

$$\sum_{m=p+1}^{+\infty} \beta_m \leq 2^{-d}(p+1)^{-d} \sum_{m=p+1}^{+\infty} 2^{-m} = 2^{-d-p}(p+1)^{-d}.$$

Besides, $\mathcal{L}^d((A \cap U_n^p) \Delta (A \cap (-p, p)^d)) \leq \mathcal{L}^d(U_n^p \setminus (-p, p)^d) = 2^d \left((p + \frac{1}{n})^d - p^d \right)$. Finally

$$|g_{n,p}(A \cap U_n^p) - g_{n,p}(A \cap (-p, p)^d)| \leq 8dn 2^{-p}(p+1)^{-d} \left((p + \frac{1}{n})^d - p^d \right).$$

□

Proof of Lemma 4.6. Put $W = \text{dom}(g) \in \mathcal{W}_n$. Then $g(A) = g(A \cap W)$ for any $A \in \mathcal{M}$. Now that the problem is restricted to the bounded pixelized domain W , it remains to show that the extrema of g on $\mathcal{M}(W)$ are reached by sets of $\mathcal{M}_n(W)$. Let us turn into the details.

Without loss of generality, assume that g has the form

$$g = \sum_{i=1}^q a_i \delta_{y_i; W_i},$$

for some $y_i \in n^{-1}\mathbb{Z}^d$, $W_i \in \mathcal{W}_n$, $a_i \in \mathbb{R}$. Denote by $I_n(W)$ the set of all indexes $k \in \mathbb{Z}^d$ such that the hypercube C_k^n is included in \overline{W} , we then also have $\overline{W} = \cup_{k \in I_n(W)} C_k^n$. For $A \in \mathcal{M}(W)$, $n \geq 1$, denote by $A_k^n = A \cap C_k^n$, $k \in I_n(W)$, the intersection of A with the hypercube C_k^n , and by

$$\tilde{A}_k^n = -k + nA_k^n$$

its rescaled translated version comprised in $[0, 1]^d$. Now consider the probability space $(\Omega = [0, 1]^d, \mathcal{A} = \mathcal{B}([0, 1]^d), \mathbb{P} = \mathcal{L}^d)$, on which we define the $\{0, 1\}^{I_n(W)}$ -valued random vector

$$\begin{aligned} Y^A : \Omega &\rightarrow \{0, 1\}^{I_n(W)} \\ \omega &\mapsto (1_{\tilde{A}_k^n}(\omega))_{k \in I_n(W)}. \end{aligned}$$

The measures of the pairwise intersections $\mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n)$ can thus be seen as the components of the covariance matrix $C(A) = (C_{k,l}(A))_{k,l \in I_n(W)}$ of the random vector Y^A , i.e.

$$C_{k,l}(A) = \mathbb{E}(Y_k^A Y_l^A) = \mathbb{E}\left(1_{\left(\omega \in \tilde{A}_k^n\right)} 1_{\left(\omega \in \tilde{A}_l^n\right)}\right) = \mathcal{L}^d\left(\tilde{A}_k^n \cap \tilde{A}_l^n\right), \quad k, l \in I_n(W).$$

Let us prove that $g(A)$ can be written as

$$g(A) = \sum_{k,l \in I_n(W)} \beta_{k,l} \mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n)$$

for some coefficients $\beta = (\beta_{k,l})_{k,l \in I_n(W)}$ depending solely on g . Putting $k_i = ny_i \in \mathbb{Z}^d$, one has,

$$\begin{aligned}
g(A) &= \sum_{i=1}^q a_i \delta_{n^{-1}k_i, W_i}(A) \\
&= \sum_{i=1}^q a_i \mathcal{L}^d(A \cap (n^{-1}k_i + A) \cap W_i) \\
&= \sum_{i=1}^q a_i \sum_{k \in I_n(W)} 1(C_k^n \subset \overline{W_i}) \int_{C_k^n} 1(x \in A, x \in n^{-1}k_i + A) dx \\
&= \sum_{i=1}^q a_i \sum_{k,l \in I_n(W)} 1(l = k - k_i) 1(C_k^n \subset \overline{W_i}) \int_{C_k^n} 1(x \in A, x \in n^{-1}(k - l) + A) dx \\
&= \sum_{k,l \in I_n(W)} \sum_{i=1}^q a_i 1(l = k - k_i) 1(C_k^n \subset \overline{W_i}) \int_{C_0^n} 1(x \in -n^{-1}k + A, x \in -n^{-1}l + A) dx \\
&= \sum_{k,l \in I_n(W)} n^{-d} \underbrace{\left(\sum_{i=1}^q a_i 1(l = k - k_i) 1(C_k^n \subset \overline{W_i}) \right)}_{=\beta_{k,l}} \mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n).
\end{aligned}$$

Then, one can write

$$g(A) = \sum_{k,l \in I_n(W)} \beta_{k,l} C_{k,l}(A) = \langle \beta, C(A) \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical scalar product between matrices. Denote by Γ_n the set of covariance matrices of all random vectors having values in $\{0, 1\}^{I_n(W)}$. Since for every set A one can associate some covariance matrix $C(A)$ such that $g(A) = \langle \beta, C(A) \rangle$, one can write

$$\inf_{A \in \mathcal{M}(W)} g(A) \geq \inf_{C \in \Gamma_n} \langle \beta, C \rangle.$$

The optimization problem on the right-hand side of this inequality is a linear programming problem on the bounded convex set Γ_n . Hence we are ensured that there exists an optimal solution C^* of this problem which is an extreme point of Γ_n . But, as shown in Proposition B.2 (see also [LR13b]), the extreme points of Γ_n are covariance matrices associated with deterministic random vectors. That is there exists a fixed vector $z^* \in \{0, 1\}^{I_n(W)}$ such that $C_{k,l}^* = z_k^* z_l^*$ minimizes $\langle \beta, C \rangle$. Given this vector $z^* \in \{0, 1\}^{I_n(W)}$, define the set A^* as the union of the hypercubes

$$A^* = \bigcup_{\{k, z_k^*=1\}} C_k^n \subset W.$$

Then one sees that the covariance matrix $C(A^*)$ associated with the deterministic set A^* is equal to C^* . Furthermore it is clear that A^* is measurable with respect to the σ -algebra generated by the C_k^n , meaning exactly $A^* \in \mathcal{M}_n(W)$. Hence we have shown that,

$$\inf_{A \in \mathcal{M}(K)} g(A) \geq \inf_{C \in \Gamma_n} \langle \beta, C \rangle = \min_{A \in \mathcal{M}_n(W)} g(A).$$

Since $\mathcal{M}_n(W) \subset \mathcal{M}(W)$ the reverse inequality is immediate, and thus

$$\inf_{A \in \mathcal{M}(W)} g(A) = \min_{A \in \mathcal{M}_n(W)} g(A).$$

□

Proof of Lemma 4.7. First, remark that if W and W' are two observation windows such that $W \subset W'$, then, for all $y \in \mathbb{R}^d$ and $A \in \mathcal{M}(\mathbb{R}^d)$,

$$0 \leq \delta_{0;W}(A) - \delta_{y;W}(A) \leq \delta_{0;W'}(A) - \delta_{y;W'}(A).$$

Indeed, since $W \subset W'$, one has $(A \setminus (y + A)) \cap W \subset (A \setminus (y + A)) \cap W'$ and thus, taking the Lebesgue measure and using (3.3), $0 \leq \delta_{0;W}(A) - \delta_{y;W}(A) \leq \delta_{0;W'}(A) - \delta_{y;W'}(A)$.

Let W and $W' \in \mathcal{W}$ such that $W \subset W'$, $j \in \{1, \dots, d\}$, and let us show that $L_j(\gamma; W) \leq L_j(\gamma, W')$. Suppose that $L_j(\gamma, W')$ is finite, otherwise there is nothing to show. Since $W \subset W'$ one also has $W \ominus [-\varepsilon e_j, 0] \subset W' \ominus [-\varepsilon e_j, 0]$ and $W \ominus [0, \varepsilon e_j] \subset W' \ominus [0, \varepsilon e_j]$. Hence, according to the preliminary remark, for all $A \in \mathcal{M}(\mathbb{R}^d)$, $\varepsilon \in \mathbb{R}$, $\sigma_{\varepsilon e_j; W}(A) \leq \sigma_{\varepsilon e_j; W'}(A)$. Since γ is admissible, this implies that for all $\varepsilon \in \mathbb{R}$, $\sigma_\gamma(\varepsilon e_j; W) \leq \sigma_\gamma(\varepsilon e_j; W')$. Hence, by definition of $L_j(\gamma, W')$, one has $\sigma_\gamma(\varepsilon e_j; W) \leq L_j(\gamma, W')$, and thus $L_j(\gamma; W) \leq L_j(\gamma, W')$. □

Proof of Lemma 4.8. Recall that it has been shown in the proof of Proposition 3.5 (see (3.5)) that for all $y, z \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $A \in \mathcal{M}(\mathbb{R}^d)$,

$$\delta_{y;W}(A) - \delta_{z;W}(A) \leq \delta_{0, -y+W}(A) - \delta_{z-y, -y+W}(A).$$

Let γ be an admissible function and let $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ and $r > 0$ be fixed. Let $z, z' \in C(y, r)$ be such that $z' = z + t e_j$ for some $t \in \mathbb{R}$ and $j \in \{1, \dots, d\}$. Since γ is admissible, the above inequality ensures that

$$\gamma(z; W) - \gamma(z'; W) \leq \gamma(0; -z + W) - \gamma(t e_j; -z + W).$$

As a consequence of (3.3), since γ is admissible, the difference in zero $U \mapsto \gamma(0; U) - \gamma(t e_j; U)$ is an increasing function of U . Hence, since $W \subset (W \oplus [-t e_j, 0]) \ominus [-t e_j, 0]$, one has

$$\begin{aligned} & \gamma(z; W) - \gamma(z'; W) \\ & \leq \gamma(0; -z + (W \oplus [-t e_j, 0]) \ominus [-t e_j, 0]) - \gamma(t e_j; -z + (W \oplus [-t e_j, 0]) \ominus [-t e_j, 0]) \\ & \leq |t| \frac{\gamma(0; -z + (W \oplus [-t e_j, 0]) \ominus [-t e_j, 0]) - \gamma(t e_j; -z + (W \oplus [-t e_j, 0]) \ominus [-t e_j, 0])}{|t|} \\ & \leq |t| \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma(0; -z + (W \oplus [-t e_j, 0]) \ominus [-\varepsilon e_j, 0]) - \gamma(\varepsilon e_j; -z + (W \oplus [-t e_j, 0]) \ominus [-\varepsilon e_j, 0])}{|\varepsilon|} \\ & \leq |t| L_j(\gamma, -z + W \oplus [-t e_j, 0]). \end{aligned}$$

Now according to Lemma 4.7, $W \mapsto L_j(\gamma; W)$ is increasing. Since $z \in C(y, r)$ and $|t| = \|z - z'\|_\infty \leq \|z - y\|_\infty + \|y - z'\|_\infty \leq 2r$, one has $-z + W \oplus [-t e_j, 0] \subset W \oplus C(-y, 3r)$. Hence for all $z, z' \in C(y, r)$ be such that $z' = z + t e_j$ one has

$$\gamma(z; W) - \gamma(z'; W) \leq L_j(\gamma, W \oplus C(-y, 3r)) |t|.$$

Exchanging z and z' one gets, $|\gamma(z; W) - \gamma(z'; W)| \leq L_j(\gamma, W \oplus C(-y, 3r))|t|$. To finish, let us now consider a couple of points $z, z' \in C(y, r)$ that are not necessarily aligned along an axis. Consider the finite sequence of vector $u_0 = z, u_1, \dots, u_d = z'$ defined such that the j first coordinates of u_j are the ones of z' while its $d - j$ last coordinates are the ones of z , so that $u_0 = z, u_d = z'$ and $u_j - u_{j-1} = (z'_j - z_j)e_j$. Clearly, each u_j belongs to the hypercube $C(y, r)$ and thus applying the d inequalities obtained above,

$$\begin{aligned} |\gamma(z; W) - \gamma(z'; W)| &= \left| \sum_{j=1}^d \gamma(u_j; W) - \gamma(u_{j-1}; W) \right| \\ &\leq \sum_{j=1}^d |\gamma(u_j; W) - \gamma(u_{j-1}; W)| \\ &\leq \sum_{j=1}^d L_j(\gamma, W \oplus C(-y, 3r))|z'_j - z_j|. \end{aligned}$$

If γ satisfies (4.2) then for all $n \in \mathbb{N}^*$ and $j \in \{1, \dots, d\}$ the constants $L_j(\gamma, U_n)$ are all finite. According to Lemma 4.7, this implies that the d constants $L_j(\gamma, W \oplus C(-y, 3r))$, $j \in \{1, \dots, d\}$ are all finite for any fixed $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ and $r > 0$, and thus the map $y \mapsto \gamma(y; W)$ is locally Lipschitz. \square

4.2 Stationary case

The following theorem is the main result of this paper. It is a refined version of Theorem 1.2 given in the introduction.

Theorem 4.9. *Let $S_2 : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function and $r \geq 0$. Then there is a stationary RAMS X such that*

$$\begin{cases} S_2(y) = \gamma_X^s(y), & y \in \mathbb{R}^d, \\ \text{Per}_{\mathbf{B}}^s(X) \leq r \end{cases} \quad (4.7)$$

if and only if S_2 is admissible and

$$\sum_{j=1}^d \text{Lip}_j(S_2, 0) \leq \frac{r}{2}.$$

We shall use a variant of Theorem 2.10(ii) from [LRM11], where the monotonicity assumption is replaced by a domination.

Theorem 4.10. *Let G^*, χ, Φ be like in Theorem 4.3 and assume G^* is stable under the action of a group of transformations Θ of \mathbb{R}^d : For all $\theta \in \Theta, g \in G^*, \theta g : A \mapsto g(\theta A)$ is a function of G^* . Assume furthermore that there is a sequence $(g_n)_{n \geq 1}$ of functions of G^* such that $0 \leq g_n \leq \chi$ and*

$$g_n(A) \rightarrow \chi(A), \quad A \in \mathcal{M},$$

and that χ is sub-invariant: For every $\theta \in \Theta$, there is a constant $C_\theta > 0$ such that

$$\chi(\theta A) \leq C_\theta \chi(A), \quad A \in \mathcal{M}. \quad (4.8)$$

Then if Φ is invariant under the action of Θ , that is,

$$\Phi(\theta g) = \Phi(g), g \in G^*, \theta \in \Theta,$$

for any given $r \geq 0$, Φ is realizable by a Θ -invariant RAMS X such that

$$\mathbf{E}\chi(X) \leq r$$

iff (4.4) holds.

Proof. The proof is the same as the one of Theorem 2.10 (ii) from [LRM11], itself based on Prop. 4.1 from Kuna *et al.* [KLS11]. It consists in checking hypotheses of the Markov-Kakutani fixed point theorem. Let M be the family of random elements X that realise Φ on G^* , and satisfy

$$\mathbf{E}\chi(\theta X) \leq r$$

for every $\theta \in \mathbb{R}^d$. The family M is easily seen to be convex with respect to addition of measures, it is compact by Theorem 2.8 from [LRM11], and invariant under the action of Θ thanks to the Θ -invariance of Φ . It remains to prove that M is not empty. Since (4.4) is in order, Theorem 4.3 yields the existence of a RAMS X realising Φ and such that $\mathbf{E}\chi(X) \leq r$. Now for $\theta \in \Theta$, by Lebesgue theorem,

$$\mathbf{E}\chi(\theta X) = \mathbf{E} \lim_n g_n(\theta X) = \lim_n \mathbf{E} g_n(\theta X) = \lim_n \mathbf{E} g_n(X) = \mathbf{E} \lim_n g_n(X) = \mathbf{E}\chi(X) \leq r$$

where we have used the fact that $\mathbf{E} \sup_n g_n(X) < \infty$ and

$$\mathbf{E} \sup_n g_n(\theta X) \leq \mathbf{E}\chi(\theta X) \leq C_\theta \mathbf{E}\chi(X) < \infty.$$

It follows that $M \ni X$ is non-empty, whence by Markov-Kakutani Theorem the mappings $X \mapsto \theta X, \theta \in \Theta$, admit a common fixed point X (considered here as a probability measure), which is therefore invariant under Θ . \square

Proof of Theorem 4.9. Necessity: Assume S_2 is the specific covariogram of a stationary RAMS X with $\text{Per}_{\mathbf{B}}^s(X) \leq r$. Then S_2 is admissible and by Proposition 3.5 one has

$$\sum_{j=1}^d \text{Lip}_j(S_2, 0) = \sum_{j=1}^d \frac{1}{2} V_{e_j}^s(X) = \frac{1}{2} \text{Per}_{\mathbf{B}}^s(X) \leq \frac{r}{2}.$$

Sufficiency: Define $\gamma(y; W) = \mathcal{L}^d(W) S_2(y)$. Then for all $W \in \mathcal{W}$ and $j \in \{1, \dots, d\}$,

$$\begin{aligned} & L_j(\gamma, W) \\ &= \sup_{\varepsilon \in \mathbb{R}} \frac{1}{|\varepsilon|} \left[\mathcal{L}^d(W \ominus [-\varepsilon e_j, 0]) (S_2(0) - S_2(\varepsilon e_j)) + \mathcal{L}^d(W \ominus [0, \varepsilon e_j]) (S_2(0) - S_2(-\varepsilon e_j)) \right] \\ &\leq 2 \mathcal{L}^d(W) \text{Lip}_j(S_2, 0). \end{aligned}$$

Hence, since $\sum_{j=1}^d \text{Lip}_j(S_2, 0) \leq \frac{r}{2}$, one has

$$\sum_{n \geq 1} \beta_n \left(\sum_{j=1}^d L_j(\gamma, W_n) \right) \leq \sum_{n \geq 1} \beta_n 2 \mathcal{L}^d(W_n) \sum_{j=1}^d \text{Lip}_j(S_2) \leq r.$$

Hence, according to Theorem 4.1 γ is realizable as the local covariogram of a RAMS X , and consequently, according to Theorem 4.3, γ satisfies (4.4) with $\chi = \text{Per}_{\mathbf{B}}^{\beta}$ and $G^* = G$ the space of all functionals of the form $g = \sum_{i=1}^q a_i \delta_{y_i; W_i}$. For $t \in \mathbb{R}^d, y \in \mathbb{R}^d, W \in \mathcal{W}, A \in \mathcal{M}$, we have

$$\begin{aligned} \theta_t \delta_{y; W}(A) &= \delta_{y; W}(\theta_t A) = \mathcal{L}^d((t + A) \cap (t + y + A) \cap W) = \mathcal{L}^d(A \cap (A + y) \cap (-t + W)) \\ &= \delta_{y; -t + W}(A), \end{aligned}$$

whence the space G generated by constant functions and functions $\delta_{y; W}, y \in \mathbb{R}^d, W \in \mathcal{W}$ is invariant under the action of the group $\Theta = \{\theta_t, t \in \mathbb{R}^d\}$ of translations. The linear functional defined by

$$\Phi(\delta_{y; W}) = \mathcal{L}^d(W) \gamma^s(y)$$

is invariant under the action of translations $\theta_t, t \in \mathbb{R}^d$. For $t \in \mathbb{R}^d$, let $\lceil t \rceil_{\infty} := \lceil \|t\|_{\infty} \rceil$ be the smallest integer larger than $\|t\|_{\infty}$. Then, recalling that U_n denotes the hypercube $(-n, n)^d$ one has $-t + U_n \subset (-n - \lceil t \rceil_{\infty}, n + \lceil t \rceil_{\infty})^d \subset U_{n + \lceil t \rceil_{\infty}}$. Hence, for $A \in \mathcal{M}$,

$$\text{Per}_{\mathbf{B}}^{\beta}(t + A) = \sum_{n=1}^{+\infty} \beta_n \text{Per}_{\mathbf{B}}(t + A; U_n) = \sum_{n=1}^{+\infty} \beta_n \text{Per}_{\mathbf{B}}(A; -t + U_n) \leq \sum_{n=1}^{+\infty} \beta_n \text{Per}_{\mathbf{B}}(A; U_{n + \lceil t \rceil_{\infty}}).$$

Since $\beta_n = 2^{-n}(2n)^{-d} = 2^{\lceil t \rceil_{\infty}} \left(\frac{n + \lceil t \rceil_{\infty}}{n}\right)^d \beta_{n + \lceil t \rceil_{\infty}} \leq 2^{\lceil t \rceil_{\infty}} \lceil t \rceil_{\infty}^d \beta_{n + \lceil t \rceil_{\infty}}$, one has

$$\text{Per}_{\mathbf{B}}^{\beta}(t + A) \leq 2^{\lceil t \rceil_{\infty}} \lceil t \rceil_{\infty}^d \sum_{n=1}^{+\infty} \beta_{n + \lceil t \rceil_{\infty}} \text{Per}_{\mathbf{B}}(A; U_{n + \lceil t \rceil_{\infty}}) \leq 2^{\lceil t \rceil_{\infty}} \lceil t \rceil_{\infty}^d \text{Per}_{\mathbf{B}}^{\beta}(A), \quad A \in \mathcal{M},$$

whence (4.8) is in order for $\chi = \text{Per}_{\mathbf{B}}^{\beta}$. To apply Theorem 4.10 it only remains to check that $\text{Per}_{\mathbf{B}}^{\beta}$ can be pointwise approximated from below by functions from G . According to Proposition 3.3, for $A \in \mathcal{M}$, and U a bounded open set of \mathbb{R}^d ,

$$\text{Per}_{\mathbf{B}}(A; U) = \lim_n g_n^U(A)$$

for some function $g_n^U \in G$ explicit in Proposition 3.3 that satisfy

$$0 \leq g_n^U \leq \text{Per}_{\mathbf{B}}(\cdot; U). \quad (4.9)$$

Define

$$g_n(A) := \sum_{m=1}^n \beta_m g_n^{U_m}(A).$$

Let $A \in \mathcal{M}$ with $\text{Per}_{\mathbf{B}}^{\beta}(A) < \infty$. Since for every $m \geq 1, g_n^{U_m}(A) \rightarrow \text{Per}(A; U_m)$ as $n \rightarrow \infty$, Lebesgue theorem with $0 \leq g_n(A) \leq \text{Per}_{\mathbf{B}}^{\beta}(A) < \infty$ ensures that $g_n(A) \rightarrow \text{Per}_{\mathbf{B}}^{\beta}(A)$ as $n \rightarrow \infty$.

Now if $A \in \mathcal{M}$ is such that $\text{Per}_{\mathbf{B}}^{\beta}(A) = \infty$, let $M > 0$, and n_0 be such that $\sum_{m=1}^{n_0} \beta_m \text{Per}_{\mathbf{B}}(A; U_m) \geq M + 1$. Let $n_1 \geq n_0$ be such that for $n \geq n_1, g_n^{U_m} \geq \text{Per}_{\mathbf{B}}(A; U_m) - 1$ for $1 \leq m \leq n_0$. Then for $n \geq n_1$,

$$g_n(A) \geq \sum_{m=1}^{n_0} \beta_m \text{Per}_{\mathbf{B}}(A; U_m) - \sum_{m \geq 1} \beta_m \geq M + 1 - 1 \geq M.$$

It follows that $g_n(A) \rightarrow \infty = \text{Per}_{\mathbf{B}}^\beta(A)$.

Hence according to Theorem 4.10 there exists a stationary RAMS X realizing γ , which implies that $\gamma_X^s = S_2$. But then according to Proposition 3.5,

$$\text{Per}_{\mathbf{B}}^s(X) = \sum_{j=1}^d V_{e_j^s}(X) = 2 \sum_{j=1}^d \text{Lip}_j(S_2, 0) \leq r.$$

□

4.3 Covariogram realisability problem for RACS of \mathbb{R}

The goal of this section is to establish a theorem similar to Theorem 4.9 the specific covariogram of one-dimensional stationary RACS by making use of Proposition 2.9.

First let us discuss the definition of local covariogram admissibility of functions in arbitrary dimension $d \geq 1$. According to Definition 1.1, a function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ is \mathcal{M} -local covariogram admissible if for all 5-tuple $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\forall A \in \mathcal{M}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(A) \geq 0 \right] \Rightarrow c + \sum_{i=1}^q a_i \gamma(y_i; W_i) \geq 0.$$

When considering RACS of \mathbb{R}^d , that is random variables taking values in $\mathcal{F} = \mathcal{F}(\mathbb{R}^d)$ the set of all closed subset of \mathbb{R}^d , it is natural to introduce the following definition: One says that a function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ is \mathcal{F} -local covariogram admissible if for all 5-tuple $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\forall F \in \mathcal{F}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(F) \geq 0 \right] \Rightarrow c + \sum_{i=1}^q a_i \gamma(y_i; W_i) \geq 0.$$

Besides, one says that $S_2 : \mathbb{R} \rightarrow \mathbb{R}$ is \mathcal{F} -specific covariogram admissible if $(y, W) \mapsto S_2(y)\mathcal{L}^d(W)$ is \mathcal{F} -local covariogram admissible.

However this distinction is superfluous since these two notions of admissibility are strictly equivalent.

Proposition 4.11. *A function $\gamma : \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R}$ is \mathcal{F} -local covariogram admissible if and only if it is \mathcal{M} -local covariogram admissible.*

Proof of Proposition 4.11. The proof of this equivalence relies on the continuity of local covariograms for the convergence in measure and the density of compact sets due to Lusin Theorem. It consists in showing that for all 5-tuple $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\forall F \in \mathcal{F}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(F) \geq 0 \right] \Leftrightarrow \left[\forall A \in \mathcal{M}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(A) \geq 0 \right]$$

Since $\mathcal{F} \subset \mathcal{M}$, the implication \Leftarrow is clear. To show the converse, let $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$ be such that $\forall F \in \mathcal{F}, \quad c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(F) \geq 0$, and let

us show that this inequality is valid for any $A \in \mathcal{M}$. One can suppose that A is bounded since according to Equation (3.1), one can replace A by $A \cap \bigcup_{i=1}^q W_i \cup (-y_i + W_i)$ without changing the value of $c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(A)$. Then by Lusin theorem (see e.g. [EG92]), there exists a sequence of compact sets $K_n \subset A$ that converges in measure towards A (i.e. $\mathcal{L}^d(A \Delta K_n) \rightarrow 0$). Since each K_n is closed, for all n one has $c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(K_n) \geq 0$. But since the sequence $(K_n)_{n \in \mathbb{N}}$ converges in measure towards A , thanks to the fact that the local covariogram $A \mapsto \delta_{y; W}$ is continuous for the local convergence in measure (see Proposition 3.1) for all $(y_i; W_i)$, $\delta_{y_i; W_i}(K_n)$ tends to $\delta_{y_i; W_i}(A)$, and thus letting n tends to $+\infty$ gives the inequality $c + \sum_{i=1}^q a_i \delta_{y_i; W_i}(A) \geq 0$. \square

Now that this technical point has been clarified we are in measure to formulate our result for the realisability of specific covariogram of stationary RACS of \mathbb{R} .

Theorem 4.12. *Suppose that the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is complete. Let $S_2 : \mathbb{R} \rightarrow \mathbb{R}$ be a given function and let $r > 0$. Then S_2 is the covariogram of a stationary RACS $Z \subset \mathbb{R}$ such that*

$$\mathbf{E}(\mathcal{H}^0(\partial Z) \cap (0, 1)) \leq r$$

if and only if S_2 is \mathcal{F} -specific covariogram admissible and Lipschitz with Lipschitz constant $L \leq \frac{r}{2}$.

Proof. Necessity: If there exists a stationary RACS $Z \subset \mathbb{R}$ such that $\mathbf{E}(\mathcal{H}^0(\partial Z) \cap (0, 1)) \leq r$, then S_2 is necessarily \mathcal{F} -specific covariogram admissible and, according to Proposition 3.5, S_2 is Lipschitz with Lipschitz constant $L = \frac{1}{2} \mathbf{E}(\text{Per}(Z); (0, 1))$. But since $\text{Per}(Z; (0, 1)) \leq \mathcal{H}^0(\partial Z \cap (0, 1))$, one has $L \leq \frac{1}{2} \mathbf{E}(\mathcal{H}^0(\partial Z \cap (0, 1))) \leq \frac{r}{2}$.

Sufficiency: Suppose that S_2 is \mathcal{F} -specific covariogram admissible and Lipschitz with Lipschitz constant $L \leq \frac{r}{2}$. Then by Proposition 4.11, γ is \mathcal{M} -specific covariogram positive, and thus by Theorem 4.9 there exists a RAMS $X \subset \mathbb{R}$ such that S_2 is the specific covariogram of X and $\mathbf{E}(\text{Per}(X); (0, 1)) \leq r$. By Proposition 2.9, there exists a RACS $Z \subset \mathbb{R}$ equivalent in measure to X such that $\text{Per}(X; (0, 1)) = \mathcal{H}^0(\partial Z \cap (0, 1))$ a.s. But then the specific covariogram of Z is also equal to S_2 and $\mathbf{E}(\mathcal{H}^0(\partial Z \cap (0, 1))) = \mathbf{E}(\text{Per}(X; (0, 1))) \leq r$. \square

Note that although the geometry of sets with finite perimeter on the line seems quite simplistic, a direct proof of the realisability result above is far from trivial.

A Radon-Nikodym theorem for random measures

Theorem A.1. *Let U be an open subset of \mathbb{R}^d . Let $\mu : \Omega \mapsto \mathbf{M}(U)$ be a random signed Radon measure over U such that for all $\omega \in \Omega$, the measure $\mu(\omega, \cdot)$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L}^d . Then, there exists a jointly measurable map*

$$\begin{aligned} f : (\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(U)) &\rightarrow \mathbb{R} \\ (\omega, x) &\mapsto f(\omega, x) \end{aligned}$$

such that for all $\omega \in \Omega$, $f(\omega, \cdot)$ is a Radon-Nikodym derivative of $\mu(\omega, \cdot)$ with respect to the Lebesgue measure \mathcal{L}^d , that is for all $\omega \in \Omega$, $f(\omega, \cdot) \in L^1_{\text{loc}}(U)$ and

$$\forall B \in \mathcal{B}(U), \quad \mu(\omega, B) = \int_B f(\omega, x) dx.$$

Proof. It is enough to consider the case $U = \mathbb{R}^d$, since for $U \subset \mathbb{R}^d$ one can always extend the random measure by zero over \mathbb{R}^d and take the restriction of f to U afterwards. Let us denote by $B(x, r)$ the open ball of center x and radius r , and by \varkappa_d the Lebesgue measure of the unit ball of \mathbb{R}^d , so that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mathcal{L}^d(B(x, r)) = \varkappa_d r^d$. For any $\omega \in \Omega$, according to Besicovitch derivation theorem (see e.g. [AFP00, Theorem 2.22 p. 54]), the derivative of the measure $\mu(\omega, \cdot)$ with respect to \mathcal{L}^d , that is

$$\lim_{r \rightarrow 0^+} \frac{\mu(\omega, B(x, r))}{\varkappa_d r^d}, \quad x \in \mathbb{R}^d,$$

exists for \mathcal{L}^d -almost all $x \in \mathbb{R}^d$, is in $L^1(\mathbb{R}^d)$, and is a Radon-Nikodym derivative of the measure $\mu(\omega, \cdot)$. Let $(r_n)_{n \in \mathbb{N}}$ be a positive sequence decreasing to 0. For all $\omega \in \Omega$, $x \in \mathbb{R}^d$, and $n \in \mathbb{N}$, let us define

$$f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\varkappa_d r_n^d}.$$

As a consequence of Besicovitch derivation theorem, for all $\omega \in \Omega$, the function

$$f(x, \omega) = \limsup_{n \rightarrow +\infty} f_n(\omega, x) 1 \left(\limsup_{n \rightarrow +\infty} f_n(\omega, x) = \liminf_{n \rightarrow +\infty} f_n(\omega, x) \right)$$

is a Radon-Nikodym derivative of $\mu(\omega, \cdot)$ with respect to the Lebesgue measure \mathcal{L}^d . Let us show that this function f is jointly measurable (*i.e.* $\mathcal{A} \otimes \mathcal{B}(U)$ -measurable). Given the definition of f , and since the lim sup and lim inf of a countable sequence of measurable functions is a measurable function, it is enough to show that the functions f_n are jointly measurable. Let $n \in \mathbb{N}$. For all $x \in \mathbb{R}^d$, by definition of a random Radon measure, the map $\omega \mapsto f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\varkappa_d r_n^d}$ is \mathcal{A} -measurable. Let us now show that for all $\omega \in \Omega$, the map $\omega \mapsto f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\varkappa_d r_n^d}$ is continuous over \mathbb{R}^d . Indeed let $x \in \mathbb{R}^d$ and $(x_k)_{k \in \mathbb{N}}$ a sequence of points that tends to x . Then, for all $y \in \mathbb{R}^d \setminus \partial B(x, r_n)$, $1(y \in B(x_k, r_n))$ tends to $1(y \in B(x, r_n))$. Since the sphere $\partial B(x, r_n)$ is Lebesgue negligible, by absolute continuity $\partial B(x, r_n)$ is also $\mu(\omega, \cdot)$ -negligible. Hence, $1(y \in B(x_k, r_n))$ tends to $1(y \in B(x, r_n))$ for $\mu(\omega, \cdot)$ -almost all $y \in \mathbb{R}^d$. Besides, since the sequence (x_k) tends to x it is bounded, and thus there exists $R > 0$ such that $x_k \in B(0, R)$ for all $k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$,

$$\left| \frac{1(y \in B(x_k, r_n))}{\varkappa_d r_n^d} \right| \leq \frac{1(y \in B(0, R + r_n))}{\varkappa_d r_n^d} \in L^1(\mu(\omega, \cdot)).$$

Hence by dominated convergence,

$$\lim_{k \rightarrow +\infty} \frac{\mu(\omega, B(x_k, r_n))}{\varkappa_d r_n^d} = \frac{\mu(\omega, B(x, r_n))}{\varkappa_d r_n^d},$$

that is $f_n(\omega, \cdot)$ is continuous at x . In conclusion, $\omega \mapsto f_n(\omega, x)$ is measurable and $x \mapsto f_n(\omega, x)$ is continuous, *i.e.* f_n is a Carathéodory function. Since \mathbb{R}^d is a separable metric space, one can conclude that f_n is jointly measurable [AB07, Section 4.10]. \square

Remark A.2. The above proof follows the outline of [Bog07, Exercice 6.10.72]. A more general result for the existence of a jointly measurable Radon-Nikodym derivative of a random measure absolutely continuous with respect to another random measure is established in [DM82, V.58].

B Convexity of the set of covariance matrices of random vectors in $\{0, 1\}^N$

Let N be a positive integer. We denote by Γ_n the set of all covariance matrices of random vectors $X = (X_1, \dots, X_N) \in \{0, 1\}^N$, that is $C \in \Gamma_n$ if there exists a random vector X such that $C_{k,l} = \mathbf{E}(X_k X_l)$, $k, l \in \{1, \dots, N\}$.

Proposition B.1 (Convexity of Γ_n). Γ_n is a convex set.

Proof. Let C_0 and C_1 be two covariance matrices in Γ_n , and let $\alpha \in [0, 1]$. Let X_0 and X_1 be two independent random vectors having covariance matrix C_0 and C_1 respectively, and let Y be a Bernoulli random variable of parameter α independent of (X_0, X_1) . Then, the mixing random vector

$$X = \begin{cases} X_0 & \text{if } Y = 0 \\ X_1 & \text{if } Y = 1 \end{cases}$$

has covariance matrix $C = (1 - \alpha)C_0 + \alpha C_1$. \square

Proposition B.2 (Extreme points of Γ_n). *The extreme points of Γ_n are covariance matrices of deterministic vectors (i.e. random vectors such that $X = x$ a.s. for some $x \in \{0, 1\}^N$).*

Proof. Let us denote by C^x the covariance matrix of a deterministic vector $X = x$, that is $C_{k,l}^x = x_k x_l \in \{0, 1\}$. Let X be any random vector of $\{0, 1\}^N$ and denote by C its covariance matrix. Since X takes its values in the discrete set $\{0, 1\}^N$, one can decompose its distribution \mathbf{P}_X into

$$\mathbf{P}_X = \sum_{x \in \{0,1\}^N} \mathbf{P}_X(X = x) \delta_x,$$

where δ_x is the unit mass measure in x . Then, for all $k, l \in \{1, \dots, N\}$,

$$C_{k,l} = \mathbf{E}(X_k X_l) = \sum_{x \in \{0,1\}^N} \mathbf{P}_X(X = x) x_k x_l = \sum_{x \in \{0,1\}^N} \mathbf{P}_X(X = x) C_{k,l}^x.$$

Hence each covariance of Γ_n is the convex linear combination of covariance matrices of deterministic vectors. \square

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