Blocks of the Grothendieck ring of equivariant bundles on a finite group
Cédric Bonnafé

To cite this version:
Cédric Bonnafé. Blocks of the Grothendieck ring of equivariant bundles on a finite group. New Zealand Journal of Mathematics, 2018, 48, pp.157-163. hal-00995158v2

HAL Id: hal-00995158
https://hal.science/hal-00995158v2
Submitted on 11 Sep 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract. — If $G$ is a finite group, the Grothendieck group $K_G(G)$ of the category of $G$-equivariant $\mathbb{C}$-vector bundles on $G$ (for the action of $G$ on itself by conjugation) is endowed with a structure of (commutative) ring. If $K$ is a sufficiently large extension of $\mathbb{Q}_p$ and $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in $K$, the $K$-algebra $K\mathcal{O}_K(G) = K \otimes_\mathbb{Z} K_G(G)$ is split semisimple. The aim of this paper is to describe the $\mathcal{O}$-blocks of the $\mathcal{O}$-algebra $K\mathcal{O}_K(G)$.

1. Notation, introduction

1.A. Groups. — We fix in this paper a finite group $G$, a prime number $p$ and a finite extension $K$ of the $p$-adic field $\mathbb{Q}_p$ such that $K H$ is split for all subgroups $H$ of $G$. We denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}_p$ in $K$, by $p$ the maximal ideal of $\mathcal{O}$, by $k$ the residue field of $\mathcal{O}$ (i.e. $k = \mathcal{O}/p$). We denote by $\mathrm{Irr}(K G)$ the set of irreducible characters of $G$ (over $K$).

A $p$-element (respectively $p'$-element) of $G$ is an element whose order is a power of $p$ (respectively prime to $p$). If $g \in G$, we denote by $g_p$ and $g_{p'}$ the unique elements of $G$ such that $g = g_p g_{p'} = g_{p'} g_p$. $g_p$ is a $p$-element and $g_{p'}$ is a $p'$-element. The set of $p$-elements (respectively $p'$-elements) of $G$ is denoted by $G_p$ (respectively $G_{p'}$).

If $X$ is a $G$-set (i.e. a set endowed with a left $G$-action), we denote by $[G \setminus X]$ a set of representatives of $G$-orbits in $X$. The reader can check that we will use formulas like

$$\sum_{x \in [G \setminus X]} f(x)$$

(or families like $(f(x))_{x \in [G \setminus X]}$) only whenever $f(x)$ does not depend on the choice of the representative $x$ in its $G$-orbit. If $X$ is a set-$G$ (i.e. a set endowed with a right $G$-action), we define similarly $[X/G]$ and will use it according to the same principles.

The author is partly supported by the ANR (Project No ANR-12-JS01-0003-01 ACORT).
1.B. Blocks. — A block idempotent of \( kG \) (respectively \( \partial G \)) is a primitive idempotent of the center \( Z(kG) \) (respectively \( Z(\partial G) \)) of \( \partial G \). We denote by Blocks\((kG)\) (respectively Blocks\((\partial G)\)) the set of block idempotents of \( kG \) (respectively \( \partial G \)). Reduction modulo \( \mathfrak{p} \) induces a bijection Blocks\((\partial G)\) \( \sim \) Blocks\((kG)\), \( e \mapsto \bar{e} \) (whose inverse is denoted by \( e \mapsto \bar{e} \)).

A \( p \)-block of \( G \) is a subset \( \mathcal{B} \) of Irr\((G)\) such that \( \mathcal{B} = \text{Irr}(kGe) \), for some block idempotent \( e \) of \( \partial G \).

1.C. Fourier coefficients. — Let

\[
\text{IrrPairs}(G) = \{(g, \gamma) \mid g \in G \text{ and } \gamma \in \text{Irr}(kC_G(g))\}
\]

and
\[
\text{BlPairs}_p(G) = \{(s, e) \mid s \in G_{p'} \text{ and } e \in \text{Blocks}(\partial C_G(s))\}.
\]

The group \( G \) acts (on the left) on these two sets by conjugation. We set

\[
\mathcal{M}(G) = [G \setminus \text{IrrPairs}(G)] \quad \text{and} \quad \mathcal{M}_p(G) = [G \setminus \text{BlPairs}(G)].
\]

If \((g, \gamma), (h, \eta) \in \text{IrrPairs}(G)\), we define, following Lusztig [Lu, 2.5(a)],

\[
\{(g, \gamma), (h, \eta)\} = \frac{1}{|C_G(g)|} \sum_{x \in G} \gamma(xh^{-1})\eta(x^{-1}g^{-1}x).
\]

Note that \( \{(g, \gamma), (h, \eta)\} \) depends only on the \( G \)-orbit of \( (g, \gamma) \) and on the \( G \)-orbit of \( (h, \eta) \).

1.D. Vector bundles. — Except from Proposition 2.3 below, all the definitions, all the results in this subsection can be found in [Lu, §2]. We denote by \( \mathcal{B} \text{un}_G(G) \) the category of \( G \)-equivariant finite dimensional \( K \)-vector bundles on \( G \) (for the action of \( G \) by conjugation). Its Grothendieck group \( K_G(G) \) is endowed with a ring structure. For each \((g, \gamma) \in \mathcal{M}(G)\), let \( V_{g, \gamma} \) be the isomorphism class (in \( K_G(G) \)) of the simple object in \( \mathcal{B} \text{un}_G(G) \) associated with \((g, \gamma)\), as in [Lu, §2.5] (it is denoted \( U_{g, \gamma} \) there). Then

\[
K_G(G) = \bigoplus_{(g, \gamma) \in \mathcal{M}(G)} \mathbb{Z} V_{g, \gamma}.
\]

The \( K \)-algebra \( K K_G(G) = K \otimes_{\mathbb{Z}} K_G(G) \) is split semisimple and commutative. Its simple modules (which have dimension one) are also parametrized by \( \mathcal{M}(G) \): if \((g, \gamma) \in \mathcal{M}(G)\), the \( K \)-linear map

\[
\Psi_{g, \gamma} : K K_G(G) \longrightarrow K
\]

defined by

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \frac{|C_G(g)|}{\gamma(1)} \cdot \{(h^{-1}, \eta), (g, \gamma)\}
\]
is a morphism of $K$-algebras and all morphisms of $K$-algebras $K\mathcal{K}_G(G) \to K$ are obtained in this way.

We define similarly block idempotents of $k\mathcal{K}_G(G)$ and $\mathcal{O}\mathcal{K}_G(G)$, as well as $p$-blocks of $\mathcal{M}(G) \to \text{Irr}(K\mathcal{K}_G(G))$.

1.E. Brauer maps. — Let $\Lambda$ denote one of the two rings $\mathcal{O}$ or $k$. If $g \in G$ (and if we set $s = g_{p'}$), we denote by $\text{Br}^\Lambda_g$ the $\Lambda$-linear map

$$\text{Br}^\Lambda_g : \Lambda C_G(s) \to \Lambda C_G(g)$$

such that

$$\text{Br}^\Lambda_g(h) = \begin{cases} h & \text{if } h \in C_G(g), \\ 0 & \text{if } h \notin C_G(g), \end{cases}$$

for all $h \in C_G(s)$. Recall [Is, Lemma 15.32] that

$$\text{Br}^k_g \text{ induces a morphism of algebras } Z(k C_G(s)) \to Z(k C_G(g)).$$

Therefore, if $e \in \text{Blocks}(\mathcal{O} C_G(s))$, then $\text{Br}^k_g(e)$ is an idempotent of $Z(k C_G(g))$ (possibly equal to zero) and we can write it a sum $\text{Br}^k_g(e) = e_1 + \cdots + e_n$, where $e_1, \ldots, e_n$ are pairwise distinct block idempotents of $k C_G(g)$. We then set

$$\beta^g_e = \sum_{i=1}^n \tilde{e}_i,$$

It is an idempotent (possibly equal to zero, possibly non-primitive) of $Z(\mathcal{O} C_G(g))$.

1.F. The main result. — In order to state more easily our main result, it will be more convenient (though it is not strictly necessary) to fix a particular set of representatives of conjugacy classes of $G$.

---

**Hypothesis and notation.** From now on, and until the end of this paper, we denote by:

- $[G_{p'}/\sim]$ a set of representatives of conjugacy classes of $p'$-elements in $G$.
- $[G/\sim]$ a set of representatives of conjugacy classes of elements of $G$ such that, for all $g \in [G/\sim]$, $g_{p'} \in [G_{p'}/\sim]$.

We also assume that, if $(g, \gamma) \in \mathcal{M}(G)$ or $(s, e) \in \mathcal{M}^p(G)$, then $g \in [G/\sim]$ and $s \in [G_{p'}/\sim]$.

If $(s, e) \in \mathcal{M}^p(G)$, we define $\mathcal{B}_G(s, e)$ to be the set of pairs $(g, \gamma) \in \mathcal{M}(G)$ such that:

1. $g_{p'} = s$.
2. $\gamma \in \text{Irr}(K C_G(g)\beta^g_e)$.
**Theorem 1.2.** — The map \((s, e) \mapsto \mathcal{B}_G(s, e)\) induces a bijection between \(\mathcal{M}^p(G)\) to the set of \(p\)-blocks of \(\mathcal{M}(G)\).

**2. Proof of Theorem 1.2**

2.A. Central characters and congruences. — If \((g, \gamma) \in \text{IrrPairs}(G)\), we denote by \(\omega_{g, \gamma} : Z(KC_G(g)) \rightarrow K\) the central character associated with \(\gamma\) (if \(z \in Z(KC_G(g))\), then \(\omega_{g, \gamma}(z)\) is the scalar through which \(z\) acts on an irreducible \(KC_G(g)\)-module affording the character \(\gamma\)). It is a morphism of algebras: when restricted to \(Z(\partial C_G(g))\), it has values in \(\partial\).

If \(h \in C_G(g)\), we denote by \(\Sigma_g(h)\) conjugacy class of \(h\) in \(C_G(g)\) and we set

\[
\hat{\Sigma}_g(h) = \sum_{v \in \Sigma_g(h)} v \in Z(\partial C_G(g)).
\]

We have

\[
(2.1) \quad \omega_{g, \gamma}(\hat{\Sigma}_g(h)) = \frac{|\Sigma_g(h)| \cdot \gamma(h)}{\gamma(1)}.
\]

We also recall the following classical results:

**Proposition 2.2.** — If \(g \in G\) and \(\gamma, \gamma'\) are two irreducible characters of \(C_G(g)\), then \(\gamma\) and \(\gamma'\) lie in the same \(p\)-block of \(C_G(g)\) if and only if

\[
\omega_{g, \gamma}(\hat{\Sigma}_g(h)) \equiv \omega_{g, \gamma'}(\hat{\Sigma}_g(h)) \mod p
\]

for all \(h \in C_G(g)\).

**Proposition 2.3.** — Let \((g, \gamma)\) and \((g', \gamma')\) be two elements of \(\mathcal{M}(G)\). Then \((g, \gamma)\) and \((g', \gamma')\) belong to the same \(p\)-block of \(\mathcal{M}(G)\) if and only if

\[
\Psi_{g, \gamma}(V_{h, \eta}) \equiv \Psi_{g', \gamma'}(V_{h, \eta}) \mod p
\]

for all \((h, \eta) \in \mathcal{M}(G)\).

2.B. Around the Brauer map. — As Brauer maps are morphisms of algebras, we have

\[
\sum_{e \in \text{Blocks}(KC_G(g_p))} \text{Br}^p_G(e) = 1,
\]

and so

\[
(2.4) \quad \text{The family } \{\mathcal{B}_G(g, e)\}_{(g,e) \in \mathcal{M}^p(G)} \text{ is a partition of } \mathcal{M}(G).
\]
Now, let \((g, \gamma) \in \mathcal{M}(G)\) and let \(s = g_p\). If \(e \in \text{Blocks}(\mathcal{O} C_G(s))\) is such that \(\gamma \in \text{Irr}(K C_G(g)\beta_g^p(e))\), and if \(\sigma \in \text{Irr}(K C_G(s)e)\), then [Is, Lemma 15.44]

\[
\omega_{s, \sigma}(z) \equiv \omega_{g, \gamma}(\text{Br}_g^\sigma(z)) \mod p
\]

for all \(z \in \mathbb{Z}(\mathcal{O} C_G(s))\).

2.C. Rearranging the formula for \(\Psi_{g, \gamma}\). — If \((g, \gamma), (h, \eta) \in \text{IrrPairs}(g)\) then

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(g) \cap G/C_G(h)]} \eta(x^{-1}g x)\omega_{g, \gamma}(\hat{\Sigma}_g(x h x^{-1})).
\]

Proof. — By definition,

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \frac{1}{\gamma(1)} \sum_{x \in [C_G(g) \cap G/C_G(h)]} \eta(x^{-1}g x)\gamma(x h x^{-1}) = \frac{1}{\gamma(1)} \sum_{x \in [G/C_G(h)]} \eta(x^{-1}g x)\gamma(x h x^{-1}).
\]

Now, if \(x \in G\) is such that \(x h x^{-1} \in C_G(g)\) and if \(u \in C_G(g)\), then

\[
\eta((ux)^{-1}g(ux))\gamma((ux)h(ux)x^{-1}) = \eta(x^{-1}g x)\gamma(x h x^{-1}).
\]

So we can gather the terms in the last sum according to their \(C_G(g)\)-orbit. We get

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(g) \cap G/C_G(h)]} \eta(x^{-1}g x)\frac{|C_G(g)|}{|C_G(g) \cap x C_G(h)x^{-1}|} \frac{\gamma(x h x^{-1})}{\gamma(1)}.
\]

But, for \(x \in G\) such that \(x h x^{-1} \in C_G(g)\),

\[
\frac{|C_G(g)|}{|C_G(g) \cap x C_G(h)x^{-1}|} = |\Sigma_g(x h x^{-1})|,
\]

so the result follows from 2.1.

Corollary 2.7. — Let \(g \in [G/\sim]\) and let \(\gamma, \gamma' \in \text{Irr}(K C_G(g))\) lying in the same \(p\)-block of \(C_G(g)\). Then \((g, \gamma)\) and \((g, \gamma')\) lie in the same \(p\)-block of \(\mathcal{M}(G)\).

Proof. — This follows from 2.6 and Proposition 2.3.
2.D. \( p' \)-part. — Fix \((g, \gamma) \in \mathcal{M}(G)\). Then it follows from 2.6 that, for all \( \chi \in \text{Irr}(KG) \),

\[
(2.8) \quad \Psi_{g, \gamma}(V_{1, \chi}) = \chi(g).
\]

**Proposition 2.9.** — Let \((g, \gamma)\) and \((h, \eta)\) be two elements in \( \mathcal{M}(G) \) which lie in the same \( p \)-block. Then \( g_{p'} = h_{p'} \).

**Proof.** — By Proposition 2.3 and Equality 2.8, it follows from the hypothesis that \( \chi(g) \equiv \chi(h) \mod p \) for all \( \chi \in \text{Irr}(KG) \). Hence \( g_{p'} \) and \( h_{p'} \) are conjugate in \( G \) (see [Bo, Proposition 2.14]), so they are equal according to our conventions explained in §1.F.

**Proposition 2.10.** — Let \( s \in G_{p'} \) and let \( \sigma, \sigma' \in \text{Irr}(KC_G(s)) \). Then \( (s, \sigma) \) and \( (s, \sigma') \) lie in the same \( p \)-block if and only if \( \sigma \) and \( \sigma' \) lie in the same \( p \)-block of \( C_G(s) \).

**Proof.** — The if part has been proved in Corollary 2.7. Conversely, assume that \( (s, \sigma) \) and \( (s, \sigma') \) lie in the same \( p \)-block. Fix \( h \in C_G(s) \). Then \( s \in C_G(h) \). Let \( \eta_{s, h} : C_G(h) \to K \) be the class function on \( C_G(h) \) defined by:

\[
\eta_{s, h}(g) = \begin{cases} 
1 & \text{if } g_{p'} \text{ and } s \text{ are conjugate in } C_G(h), \\
0 & \text{otherwise.}
\end{cases}
\]

It follows from [Bo, Proposition 2.20] that \( \eta_{s, h} \in \mathcal{O}(\text{Irr}(KC_G(h))) \). Therefore, by 2.6 and Proposition 2.3,

\[
(\#) \quad \sum_{x \in C_G(s) \setminus G/C_G(h)} \eta_{s, h}(x^{-1}sx) \left[ \omega_{s, \sigma}(\Sigma_{g}(xhx^{-1})) - \omega_{s, \sigma'}(\Sigma_{g}(xhx^{-1})) \right] \equiv 0 \mod p.
\]

Now, let \( x \in G \) be such that \( xhx^{-1} \in C_G(s) \). Since \( x^{-1}sx \) is also a \( p' \)-element, \( \eta_{s, h}(x^{-1}sx) = 1 \) if and only if \( s \) and \( x^{-1}sx \) are conjugate in \( C_G(h) \) that is, if and only if \( x \in C_G(s)C_G(h) \). So it follows from (\#) that

\[
\omega_{s, \sigma}(\Sigma_{g}(h)) \equiv \omega_{s, \sigma'}(\Sigma_{g}(h)) \mod p
\]

for all \( h \in C_G(s) \). This shows that \( \sigma \) and \( \sigma' \) lie in the same \( p \)-block of \( C_G(s) \).
2.E. Last step. — We shall prove here the last intermediate result:

**Proposition 2.11.** — Let \((s, e) \in \mathcal{M}^p(G)\) and let \((g, \gamma), (g', \gamma') \in \mathcal{B}_G(s, e)\). Then \((g, \gamma)\) and \((g', \gamma')\) are in the same \(p\)-block of \(\mathcal{M}(G)\).

**Proof.** — We fix \(\sigma \in \text{Irr}(KC_G(s)e)\). It is sufficient to show that \((g, \gamma)\) and \((s, \sigma)\) are in the same \(p\)-block of \(\mathcal{M}(G)\). For this, let \((h, \eta) \in \mathcal{M}(G)\). By Proposition 2.9, we have \(g_{\nu} = s\), so \(C_G(g) \subset C_G(s)\). So 2.6 can be rewritten:

\[
\Psi_{g, \gamma}(V_{h, \eta}) = \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \sum_{y \in [C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)]} \eta(y^{-1} g y) \omega_{g, \gamma}(g y y^{-1}).
\]

Now, let \(x \in [C_G(s) \setminus G/C_G(h)]\) and \(y \in [C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)]\) be such that \(yhy^{-1} \in C_G(g)\). Then \(yhy^{-1} \in C_G(s)\) and so \(xhx^{-1} \in C_G(s)\). Moreover, \(y^{-1}sy\) is conjugate to \(x^{-1}sx\) in \(C_G(h)\). Finally, it is well-known (and easy to prove) that \(\eta(y^{-1}h) \equiv \eta(y^{-1}sy) \mod p\) (see for instance [Bo, Proposition 2.14]). Therefore:

\[
(\diamondsuit) \quad \Psi_{g, \gamma}(V_{h, \eta}) \equiv \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \sum_{y \in [C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)]} \eta(x^{-1}s x) \omega_{g, \gamma}(\hat{\Sigma}_g(y y^{-1})) \mod p.
\]

Now, let \(x \in [C_G(s) \setminus G/C_G(h)]\) be such that \(xhx^{-1} \in C_G(s)\). Then, by definition of the Brauer map,

\[
(\heartsuit) \quad \text{Br}^G_s(xhx^{-1}) = \sum_{z \in [C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)]} \hat{\Sigma}_g((z x) h(z x)^{-1}).
\]

But \(\{zx \in [C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)] \mid xhx^{-1} \in C_G(s)\}\) is a set of representatives of double classes in \(C_G(g) \setminus C_G(s) \times C_G(h)/C_G(h)\). So it follows from \((\diamondsuit)\) and \((\heartsuit)\) that

\[
\Psi_{g, n}(V_{h, \eta}) \equiv \sum_{x \in [C_G(s) \setminus G/C_G(h)]} \sum_{xhx^{-1} \in C_G(s)} \eta(x^{-1}s x) \omega_{g, \gamma}(\text{Br}^G_s(xhx^{-1})).
\]

Using now 2.5 and 2.6, we obtain that

\[
\Psi_{g, \eta}(V_{h, \eta}) \equiv \Psi_{s, \sigma}(V_{h, \eta}) \mod p,
\]

as desired. \(\square\)

**Proof of Theorem 1.2.** — Let \((s, e)\) and \((s', e')\) be two elements of \(\mathcal{M}^p(G)\) such that \(\mathcal{B}_G(s, e)\) and \(\mathcal{B}_G(s', e')\) are contained in the same \(p\)-block of \(\mathcal{M}(G)\) (see Proposition 2.11). Let \(\sigma \in \text{Irr}(KC_G(s)e)\) and \(\sigma' \in \text{Irr}(KC_G(s')e')\).

Then \((s, \sigma)\) and \((s, \sigma')\) are in the same \(p\)-block, so it follows from Proposition 2.9 that \(s = s'\) and it follows from Proposition 2.10 that \(\gamma\) and \(\gamma'\) are in the same \(p\)-block of \(C_G(s)\), that is \(e = e'\). This completes the proof of Theorem 1.2. \(\square\)
References


September 11, 2015

CÉDRIC BONNAFÉ, Institut Montpelliérain Alexander Grothendieck (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE • E-mail: cedric.bonnafe@univ-montp2.fr