A nonlinear model of opinion formation on the sphere

Marco Caponigro, Anna Chiara Lai, Benedetto Piccoli

A nonlinear model of opinion formation on the sphere

Marco Caponigro[*] Anna Chiara Lai[†] Benedetto Piccoli[‡]

Abstract

In this paper we present a model for opinion dynamics on the d-dimensional sphere based on classical consensus algorithms. The choice of the model is motivated by the analysis of the comprehensive literature on the subject, both from the mathematical and the sociological point of views. The resulting dynamics is highly nonlinear and therefore presents a rich structure. Equilibria and asymptotic behaviour are then analysed and sufficient condition for consensus are established. Finally we address global stabilization and controllability.

1 Introduction

Studying the complex laws governing the evolution of opinions in social networks is a challenge that has attracted an increasing attention from researchers in the last decades. The problem is to find the interaction rules between nodes of a network, or *agents*, generating the emergence of complex patterns observed in reality. A wide literature has been developed for the mathematical description of the dynamical evolution of *opinions*, represented e.g. by colors, in a network of interacting agents. This question is an aspect of the wider challenge of modeling and analysis of emergent behaviors in multi-agent systems. Several models have been proposed in the last years, among the most widely used we cite the Sznajd model (SM), Deffuant model (DM), Hegelsmann–Krause Model (HKM), and the classical Voter model (VM). We briefly highlight the main features of the above mentioned models to gather useful informations for the design of a new model. This list is far from exhaustive but a complete review of the problem of modeling opinion formations is beyond the scope of this work.

The Sznajd model (SM), see [SWS00], is based on the Ising model for ferromagnetism in statistical mechanics. In this model opinions are discrete variables x_i taking value ± 1. The interactions are governed by two basic rules: the "ferromagnetic" interaction (that is, if $x_i = x_{i+1}$ then at the next step adjacent agents will satisfy with a given probability $x_{i-1} = x_i = x_{i+1} = x_{i+2}$) and the "antiferromagnetic" interaction (if $x_i = -x_{i+1}$ then an antisymmetric pattern forms $-x_{i-1} = x_i = -x_{i+1} = x_{i+2}$). The model has been extended to higher dimensional opinion and complex network topologies. The motivation for this model comes form the postulate that "agreement generates agreement", that is, if two agents reach a consensus then all agents directly connected to them are induced to agree. In other words, in Sznajd model, the opinion flows out from a group of agreeing agents.

In [HK02] Hegselmann and Krause present a model (HKM) of herding of opinions in a N-agents system in which the position x_i of the agent i, representing its opinion and taking values in an interval of $\mathbb{R}$, changes according to the distance from other agents $x_j, j \neq i$, rescaled by an interaction coefficient

[*]Conservatoire National des Arts et Métiers, Équipe M2N, 292 rue Saint-Martin, 75003, Paris, France. (marco.caponigro@cnam.fr)

[†]Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via Scarpa 16, Rome, Italy. (anna.lai@sbai.uniroma1.it)

[‡]Rutgers University, Department of Mathematics, Business & Science Building Room 325 Camden, NJ 08102, USA (piccoli@camden.rutgers.edu).

a_{ij} accounting for the weight given to the opinion of agent j by agent i. With these notations the opinion of agent i evolves according to

$$x_i(t+1) = a_{i1}x_1(t) + a_{i2}x_2(t) + \cdots + a_{iN}x_N(t).$$

The model has been widely studied also in its continuous-time analogue where the evolution of the opinion of agent i is governed by

$$\dot{x}_i = \sum_{|x_i - x_j| < 1} a_{ij}(x_j - x_i).$$

This model, when the interaction is zero for mutual distances above a certain threshold, is said "bounded confidence" model. The rationale for the bounded confidence is that it is unlikely for one agent to be influenced by another one whose opinion is too far from its own. This kind of interaction gives rise to clusters of opinions (see for instance [BHT10]).

The idea of opinions varying in a continuous fashion has been used also in the Deffuant Model (DM), see [DNAW00]. The interacting agents are chosen randomly and if the difference between their opinions, say x_1 and x_2, is smaller in magnitude than a certain threshold (bounded confidence) then the agents re-adjust their opinion according to the law

$$x_1(t+1) = x_1(t) + \mu(x_2(t) - x_1(t)) \quad \text{and} \quad x_2(t+1) = x_2(t) + \mu(x_1(t) - x_2(t)),$$

for a given interation parameter $\mu \geq 0$.

The Voter Model is a widely studied classical model of opinion formation with discrete opinions and stochastic interactions, that can be seen, in some cases, as a generalization of the Sznajd model ([BS03]).

All these models present very interesting features but also lack of connections with real life examples. This issue has been throughly addressed in [Sob09]. It is very hard task to describe the complex interactions between individuals with simple mathematical rules, nevertheless the models above present features that are good starting points to develop more complete models. Below, we present an analysis of strengths and weaknesses of the mentioned models divided in three topics.

Discrete vs Continuous opinions: One of the main difficulties in modeling opinion formations is the lack of good measurements of the opinions. A classical problem in sociology is to design interviews not affecting opinions, i.e. questions not influencing answers. Purely open questions do not exist and, moreover, it is very hard to collect data from open answers. On the other hand, closed questions induce quantization on the answers: opinions collapse on discrete sets representing the possible answers to a closed question. It is therefore natural to set the initial and final opinion, in an opinion formation process, on a discrete set (as in SM and VM). Nevertheless opinions do not jump instantly but follow a continuous, possibly very slow-varying, evolution. The time evolution is therefore better modeled by continuous dynamics.

Stochastic vs Deterministic evolutions: There are several models based on stochastic interactions between agents (SM, DM) and many convincing numerical simulations have been provided to validate these models. Nevertheless these models neglect the fact that the evolution of the opinion of one agent is the deterministic result of complex interactions with other agents. While the choice of the interacting agent could be random, the opinion formation should be influenced by an averaging of the opinions of the interacting agents in the spirit, for instance, of the HKM.

Interactions: One of the main challenges of the present days is represented by the study of interaction networks, or social networks. The problem is twofold: on the one hand the problem is to study

the interaction between two agents and on the other hand to study the topology of the network and, possibly, its evolution.

Interactions between agents should weighted in different way for different agents, the interacting factor between agents 1 and 2 can be different from the one between agents 2 and 3. Several models consider a weight a_{ij} in the interaction between two agents i and j (for instance HKM) and these coefficients can be exploited to model the different natures of relations. For instance, a relation could be attractive ($a_{ij} > 0$) or repulsive ($a_{ij} < 0$). From an economical point of view, we can see the attraction as cooperation and the repulsion as competition. In this framework, it could be interesting to introduce in the model concepts as "far cooperation vs local competition" or vice-versa, by including a dependence on the "position" x of the coefficients a_{ij}. Similarly, HKM and DM include the idea of "bounded confidence" (see [Lor07] for a survey).

Asymmetries on the interaction matrix $(a_{ij})_{i,j=1}^N$ could be used to model the hierarchy in the network, for example, if $a_{ij} >> a_{ji}$ then i is more likely to influence j than the converse, meaning that agent i is an opinion leader, at least from the point of view of agent j.

As mentioned, in many models (e.g. SM and DM) pairs of interacting agents are chosen randomly. Although this approach is realistic in large networks, the deterministic nature of interactions should not be neglected. Moreover, there exist also social interactions that are strong, or constant in time. This could be modeled by introducing random switching topologies also including unbreakable subnetworks.

To conclude, modeling such complex phenomena is a challenging task. Thanks to a wide literature on the problem it is possible to find interesting features with strong sociological motivations.

In this paper we propose a model including as many of these features as possible. In order to achieve this goal we design a nonlinear model on the d-dimensional sphere $\mathbb{S}^d$, the rationale for the choice of the dynamics of the state space is presented in Section 2. The nonlinearity yield a rich structures. Several new interesting features arise from the qualitative study of the model. Three kind of equilibria are present in this model: consensus, antipodal, and polygonal (see Section 3.1 for precise definitions), in contrast with the linear systems in which the only equilibrium is consensus. This allows a better representation of the complex reality of opinion formation in which (luckily) consensus is not the only possible equilibrium. In addition another interesting feature arising in our model is the presence of "dancing equilibria": steady configuration in the evolution of the mutual distances of the opinions (see Definition 3). The dynamics originating from a dancing equilibrium in some case is a rotation of the sphere with constant angular velocity. This is an equilibrium in the sense that there is no evolution of the single opinion with respect to the others, however, the system is still in constant evolution. The rest of Section 3 is devoted to the qualitative analysis of the asymptotic behavior of the model. In particular we present, in Theorem 1, a sufficient condition for consensus similar to those provided for linear systems. In Section 4 we introduce an external control to the model, accounting for the effect of mass media on public opinion. We then state three different control problems and we address the problem of stabilizing the system to a desired consensus. Finally, Section 5 contains several numerical experiments simulating the asymptotic behavior of the system to show the many interesting features of this model.

2 Building a model

We consider a system of N interacting agents. The opinion of the i-th agent is represented by a vector x_i of the sphere $\mathbb{S}^d$.

The choice of a d-dimensional vector opinion, instead of a scalar opinion, is motivated by a seek of higher fidelity in the the model. Indeed opinions on different topics are usually interconnected: economic policy attitudes and candidate choice in political elections; opinion formation and economical condition [NBL91]; opinion on research funding and religious and ideological beliefs – see for instance [Nis05] for a study on the relations between worldview and opinions on stem cell research and [SL05] for a study on opinions about nanotechnology research.

The rationale for using the sphere $\mathbb{S}^d$ instead of the Euclidean space is that, as mentioned, opinion are subjected to a quantization phenomenon when measured. We can imagine that, at the instant of measurement (elections, polls, interviews, etc.), opinions take only two values (yes/no, left/right, Demcratic/Republican, liberal/conservative, for/against, etc.), so that every component of the vector $x_i = (x_i^{(1)}, \dots, x_i^{(d+1)})$ takes a positive or a negative value. In particular x_i belongs to $\mathbb{S}^d$. The manifold $\mathbb{S}^d$ is a mathematical abstraction to describe the dynamical evolution of the opinions on a continuous (i.e. non-discrete) set.

The influence of the opinion x_i on the one of agent x_j is weighted with an interaction factor a_{ij}. The interaction could be attractive $a_{ij} > 0$, repulsive $a_{ij} < 0$, or neutral $a_{ij} = 0$. Every agent tends to agree with another one if there is an attractive interaction and to disagree in presence of repulsive interaction. In the spirit of HKM, the result of all influences is the averaging of all the distances on $\mathbb{S}^d$ weighted by a_{ij}. The time evolution of the opinion x_i of agent i is governed by

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - \langle x_i, x_j \rangle x_i), \quad x_i \in \mathbb{S}^d, i = 1, \dots, N. \tag{1}$$

The RHS is the projection of the vector x_j on the orthogonal to x_i. In this paper we will consider constant coefficient a_{ij} with no assumptions on the symmetries of the network: we will study symmetric interactions $a_{ij} = a_{ji}$, antisymmetric interactions $a_{ij} = -a_{ji}$ as well as general coefficients.

As mentioned in the introduction, a realistic model should include many different kind of interactions. It is possible to increase the complexity of the model, in order to represent opinion formation processes closer to the reality, by adapting the coefficients a_{ij} to this purpose. For example the switching topology can be represented by introducing a (stochastic or deterministic) dependence of a_{ij} on the time t. By introducing a dependence on the state x, one may model bounded confidence (e.g. $a_{ij}(x) = 0$ if $|x_i - x_j| \geq 1$), local cooperation and far competition (e.g. $a_{ij}(x) = a(|x_i - x_j|)$ with $a(0) > 0$ and a decreasing), or local competition and far cooperation (e.g. $a_{ij}(x) = a(|x_i - x_j|)$ with $a(0) < 0$ and a increasing), as well as unbreakable subnetworks (e.g. families), the arising of opinion leaders, clusters of opinions, and so on.

Other models of consensus on manifolds has been studied in the last years. We cite the Kuramuto model [Kur84] on the sphere $\mathbb{S}^1$ who attracted a wide interest of researchers over the last 30 years, motivated by its connection with the problem of syncronizing a large population of harmonic oscillators - see [Str00] for a survey. Other possible applications can be found in [VCBJ+95] and [Hop82]. Lately, adapted versions of the Kuramoto model were extensively studied in a opinion formation perspective. However, except for the circle, consensus on non-Euclidean manifolds is not widely covered by the literature. A first effort in studying consensus dynamics on more general manifolds has been made in [SS09] by Sarlette and Sepulchre who studied opinion dynamics on a wider class of manifolds including, among others, the special orthogonal group $SO(n)$, the Grassmann manifold, and $\mathbb{S}^1$ (see also [Sar09] and [Sep11] for a survey on this topic). Their models are mainly based on the projection of linear opinion dynamics on the manifold. Even if the state space is nonlinear the evolution inherits the structure of the linear case: for instance the only equilibrium is consensus and convergence results rely mainly on consensus algorithms for linear systems (as, for example, the one by Tsitsiklis [Tsi84], Jadbabaie, Lin, and Morse [JLM03], Moreau [Mor04, Mor05], Blondel, Hendrickx, Olshevsky and

Tsitsiklis [BHT10], Olfati-Saber and Murray [OSFM07], etc.). The novelty of our model is the intrinsic nonlinear nature of its dynamics which constitutes an obstruction to the direct application of these powerful tools. On the other hand, the nonlinearity yields a rich structure. Indeed our models presents many new features not present in previous works: antipodal equilibria, polygonal equilibria, dancing equilibria, etc.

3 Model Analysis

System (1) can be rewritten in matrix form as

$$\dot{x} = Lx + D(x)x \tag{2}$$

where

$$L = (l_{ij}) := \begin{cases} a_{ij} & \text{if } i \neq j \\ -\sum_{k=1}^{n} a_{ik} & \text{otherwise} \end{cases},$$

and $D(x) = (d_{ij}(x))$ is a diagonal matrix with $d_{ii}(x) = \sum_{j=1}^{N} a_{ij}(1 - \langle x_j, x_i \rangle)$.

Remark 1. *The time-continuous version of HKM reads $\dot{x} = Lx$. Then the dynamics of (1) can be seen as the classical HKM plus a non-linear diagonal term $D(x)x$, representing the projection of the velocities Lx on the tangent space of $\mathbb{S}^d$. The matrix L is called* Laplacian *of the matrix of $A = (a_{ij})_{i,j=1}^N$. The spectral properties of the Laplacian matrix are among the main topics in algebraic graph theory, expecially in the case that A is a non-negative symmetric matrix. The applications of such properties in the framework of opinion formation models and consensus algorithms were investigated, among many others, in [OSM04].*

The features of the system depends on the nature of the interactions between agents, namely on the properties of the *interaction matrix* $A = (a_{ij})_{i,j=1}^N$

Definition 1. *The matrix A is* positive *if $a_{ij} > 0$ for every $i, j = 1, \dots, N$. The matrix A is* sign-symmetric *if either $a_{ij} \cdot a_{ji} > 0$, either $a_{ij} = 0 = a_{ji}$ for every $i, j = 1, \dots, N$.*

We can also see the dynamics from the point of view of agent i. Consider the influence on agent i of all other agents, that is

$$\alpha_i = \sum_{j \neq i} a_{ij} x_j,$$

then, with this notation, system (1) reads

$$\dot{x}_i = \alpha_i - \langle \alpha_i, x_i \rangle x_i, \qquad i = 1, \dots, N. \tag{3}$$

3.1 Equilibria

System (1) presents three kind of equilibria: consensus, antipodal, and polygonal.

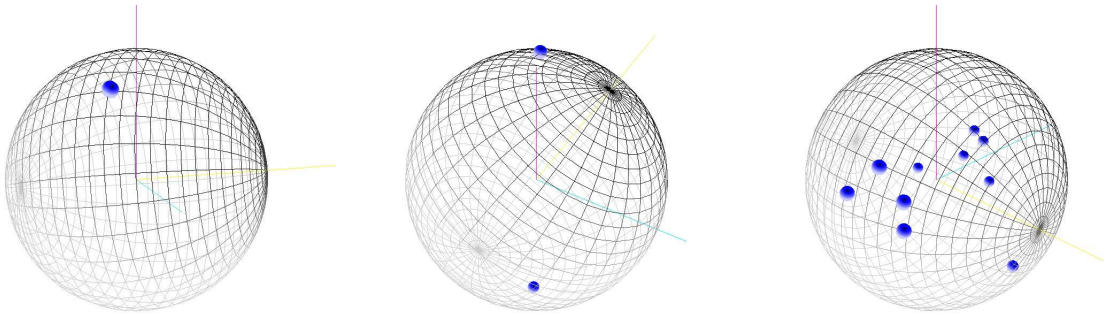
Definition 2 (Consensus, antipodal, and polygonal equilibria)**.** *The configuration*

$$x_1 = \cdots = x_N$$

is called consensus *and it is an equilibrium for system (1).*
A configuration such that, for every $j = 2, \dots, N$,

$$\text{either } x_j = x_1 \text{ or } x_j = -x_1,$$

which is not consensus is called antipodal *equilibrium.*
Every equilibrium of (1) which is not consensus nor antipodal is called polygonal.



(a) Consensus (b) Antipodal (c) Polygonal

Figure 1: Examples of equilibria on $\mathbb{S}^2$. Bigger points are closer to the viewer and smaller points are on the opposite side of the sphere.

Consensus corresponds to the case in which all opinion are equal, sometimes this configuration, in which all states coincide, is called *rendezvous*. Antipodal equilibria are configurations in which every agent is either in agreement either in disagreement with very other. Then the group splits in two subgroups, in mutual disagreement, of of agents sharing the same opinion. Unlike consensus and antipodal equilibria, *polygonal* equilibria depend, in principle on the interaction matrix A. In Figure 1 examples of the three kind of configurations on the sphere $\mathbb{S}^2$.

By definition an equilibrium for systems (1) is a point $x \in (\mathbb{S}^d)^N$ such that

$$\alpha_i - \langle \alpha_i, x_i \rangle x_i = 0 \quad \text{for every } i = 1, \dots, N.$$

In other words, an equilibrium is a vector $x \in (\mathbb{S}^d)^N$ such that the influence α_i is collinear with x_i for every $i = 1, \dots, N$. We identify three cases:

(i) $\alpha_i = c_i x_i$, for $c_i > 0$,

(ii) $\alpha_i = -c_i x_i$, for $c_i > 0$,

(iii) $\alpha_i = 0$.

Lemma 1. *Consider an equilibrium $(x_1, \dots, x_N)$ for systems (1). If*

$$\alpha_i = c_i x_i, \quad \text{for } c_i > 0, \tag{4}$$

then the equilibrium is stable with respect to x_i. *If*

$$\alpha_i = -c_i x_i, \quad \text{for } c_i > 0, \tag{5}$$

then the equilibrium is unstable with respect to x_i. *If*

$$\alpha_i = c_i x_i, \quad \text{for } c_i > 0, \text{ for every } i = 1, \dots, N,$$

the equilibrium is stable *(tout court).*

Proof. Consider the system for x_i and let us linearize the dynamics of x_i perturbing with respect to x_i only. Namely consider the linearization of system

$$\dot{x} = \alpha - \langle \alpha, x \rangle x, \quad x \in \mathbb{S}^d$$

in the equilibrium $\alpha = \pm cx$ for $c > 0$ that calculated in $v \in \mathbb{S}^d$ reads

$$-\langle \alpha, x\rangle v - \langle \alpha, v\rangle x = \mp c(v + \langle x, v\rangle x).$$

Now notice that for every $x \in \mathbb{S}^d$ the linear operator on $\mathbb{S}^d$

$$v \mapsto v + \langle v, x\rangle x$$

has positive eigenvalues. Indeed the equation

$$v + \langle v, x\rangle x = \lambda v,$$

is satisfied either by $\lambda = 1$ and $v \perp x$ either by $v = \pm x$ and $\lambda = 2$. □

One of the main differences between the present model and consensus model on Euclidean spaces (as HKM) is represented by the rich nature of equilibria. Indeed while in HKM the only equilibrium is the consensus, here the model allows also the disagreement as possible (stable) equilibrium.

Consider two agents ($N = 2$) with one opinion ($d = 1$) and with repulsive interaction $a_{12} < 0, a_{21} < 0$, in this case the antipodal position of the two opinions is a stable equilibrium. More in general we have the following proposition.

Proposition 1. *If A is positive then consensus is a stable equilibrium. If A is sign-symmetric then the antipodal configuration $(x_1, \dots, x_N)$ given by*

$$x_i = \begin{cases} x_j & \text{if } a_{ij} > 0 \\ -x_j & \text{otherwise ,} \end{cases}$$

is a stable equilibrium.

Proof. The proof is a direct consequence of Lemma 1. Indeed, in the positive case one has for every $i = 1, \dots, N$

$$\alpha_i = c_i x_i = \sum_{j=1}^{N} a_{ij} x_i,$$

and $c_i = \sum_{j=1}^{N} a_{ij} > 0$.

In the sign-symmetric case for every $i = 1, \dots, N$,

$$\alpha_i = \left(\sum_{j : a_{ij} > 0} a_{ij} - \sum_{j : a_{ij} \le 0} a_{ij} \right) x_i,$$

hence the equilibrium is stable with respect every x_i. □

For the versatility of our model it is possible to design interactions giving rise to any kind of equilibria. Indeed given any configuration of opinions it is possible to find an interaction matrix for which this configuration is an equilibrium as showed in the following.

Proposition 2. *Let $N > d+1$. Then for every $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \in (\mathbb{S}^d)^N$ there exists a square matrix $A = (a_{ij})_{i,j=1}^N$ such that $\bar{x}$ is an equilibrium of system* (1).

Proof. The configuration $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N)$ is an equilibrium if and only if

$$\frac{d}{dt}\bar{x}_i = \sum_{j=1}^{N} a_{ij}\bar{x}_j - \langle \bar{x}_i, \bar{x}_j\rangle \bar{x}_i = 0, \qquad \text{for all } i = 1, \dots, N.$$

This is a system of Nd equations in the $N^2 - N$ unknowns a_{ij} (setting the diagonal a_{ii} to 0 for every $i = 1, \dots, N$). So if $N > d + 1$ there exists a nontrivial choice of the interaction coefficients for which $\bar{x}$ is an equilibrium. □

3.2 Dancing equilibria

Beyond the complex nature of equilibria, model (1) shows another interesting kind of configuration which is the *dancing equilibrium*.

Consider the scalar products $\langle x_i, x_j\rangle, i, j = 1, \dots, N$ between the components of $x \in (\mathbb{S}^d)^N$. These quantities are, in some sense, a measure of the mutual distance between the x_i's. Indeed $\|x_i - x_j\|^2 = 2 - 2\langle x_i, x_j\rangle$. Moreover the evolution of the mutual distances is governed by the system of ODEs

$$\begin{aligned}\frac{d}{dt}\langle x_i, x_j\rangle =& \sum_{k\neq i} a_{ik}\left(\langle x_k, x_j\rangle - \langle x_k, x_i\rangle\langle x_i, x_j\rangle\right) \\ &+ \sum_{k\neq j} a_{jk}\left(\langle x_k, x_i\rangle - \langle x_k, x_j\rangle\langle x_i, x_j\rangle\right),\end{aligned} \tag{6}$$

for $i, j = 1, \dots, N$. An equilibrium of system (6) is a configuration in which all mutual distances between agents are constant.

Definition 3. *A configuration such that* (6) *always vanishes is called* dancing equilibrium *for system* (1).

The name "dancing equilibrium" is motivated by the fact that opinions are crystalized since the mutual distances are in equilibrium while the whole configuration may evolve. This can be seen as a representation of the cyclicality of opinion, for example in financial markets or in fashion design.

For the system in matrix form (2), $D(x)$ is constant in presence of a dancing equilibrium. Consequently the evolution is linear.

Definition 4. *The* kinetic energy *of system* (1) *is the quantity*

$$E(t) := \frac{1}{2}\sum_{i=1}^{n} ||\dot{x}_i(t)||^2. \tag{7}$$

A system in dancing equilibrium has constant kinetic energy.

Lemma 2. *If for every* $i, j \in \{1, \dots, N\}$,

$$\frac{d}{dt}\langle x_i(t), x_j(t)\rangle|_{t=0} = 0,$$

then $E(t)$ *is constant for every* $t \geq 0$.

Proof. First of all notice that by assumption and by (6) we have $\frac{d}{dt}\langle x_i(t), x_j(t)\rangle = 0$ for every $t \geq 0$ and for every $i, j \in \{1, \dots, N\}$. In particular $\frac{d}{dt}\langle x_j(t), \alpha_i(t)\rangle = 0$ and $\frac{d}{dt}\langle \alpha_j(t), \alpha_i(t)\rangle = 0$ for every $i, j \in \{1, \dots, N\}$.

Then, we derive the kinetic energy with respect to time to get

$$\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n} ||\dot{x}_i(t)||^2 = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\langle \alpha_i, \dot{x}_i\rangle = \frac{1}{2}\sum_{i=1}^{N}\frac{d}{dt}\left(\|\alpha_i\|^2 - \langle \alpha_i, x_i\rangle^2\right) = 0,$$

for every $t \geq 0$. □

Dancing equilibrium is a configuration not arising in other models and it can be seen an the mathematical formalization of the cyclicality of opinions.

Example 1. *Consider the* 2*-agent system with antisymmetric interaction matrix*

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

$a \neq 0$*, so that*

$$\dot{x}_1 = a(x_2 - \langle x_1, x_2 \rangle x_1)$$

and

$$\dot{x}_2 = -a(x_1 - \langle x_1, x_2 \rangle x_2).$$

A direct computation gives

$$\frac{d}{dt}\langle x_1, x_2 \rangle = \langle \dot{x}_1, x_2 \rangle + \langle x_1, \dot{x}_2 \rangle = a(1 - \langle x_1, x_2 \rangle^2) - a(1 - \langle x_1, x_2 \rangle^2) = 0.$$

Note that in Euclidean models, as HKM, an antisymmetric interaction generates the divergence to infinity of two opinions.

Example 2. *Dancing equilibria emerge also in the case in which all interactions have attractive nature (i.e.* $a_{ij} > 0$*), provided that there are at least* 3 *agents. Indeed, consider*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

with three agents along the vertices of a equilateral triangle and assume $\langle x_i(0), v \rangle = 0$ *for some* $v \in \mathbb{R}^{d+1}$ *so that*

$$\langle x_1, x_2 \rangle = \langle x_1, x_3 \rangle = \langle x_2, x_3 \rangle = -\frac{1}{2}$$

We have by direct computation

$$\frac{d}{dt}\langle x_i, x_j \rangle = 0,$$

for every $i \neq j, i, j = 1, \ldots, N$*, hence this configuration is a dancing equilibrium.*

3.3 Asymptotic behaviour

In next theorem we consider interaction coefficients depending on the positions x_i, namely $a_{ij} = a(x_i, x_j)$ and we assume the function a to be positive. In this case a sufficient condition for consensus is that all initial conditions are contained in the hemisphere as the followin theorem states.

Theorem 1. *Assume* $a_{ij} = f(x_i, x_j)$ *for a continuous positive function* $f : \mathbb{S}^d \times \mathbb{S}^d \to (0, +\infty)$*. If there exists* $w \in \mathbb{S}^d$ *such that the initial conditions satisfy*

$$\langle x_i, w \rangle > 0, \quad \text{for every } i = 1, \ldots, N.$$

then the system tends to consensus.

Proof. For every $t \geq 0$ let $i = i(t)$ be the smallest index in $\{1, \ldots, N\}$ such that $\langle x_i(t), w \rangle$ is minimal, that is $\langle x_i(t), w \rangle \leq \langle x_j(t), w \rangle$ for every $j = 1, \ldots, N$. We claim that either the system is in consensus $x_1 = x_2 = \cdots = x_N$ either $\langle \dot{x}_i(t), w \rangle > 0$ for every $t \geq 0$. Indeed if, by contradiction $\langle \dot{x}_i(t), w \rangle \leq 0$, then

$$\sum_{j \neq i} a_{ij} \left(\langle x_j, w \rangle - \langle x_j, x_i \rangle \langle x_i, w \rangle \right) \leq 0 \tag{8}$$

Then there exists $j \neq i$ such that $\langle x_j, w\rangle \leq \langle x_j, x_i\rangle\langle x_i, w\rangle < \langle x_i, w\rangle$, in contradiction with the minimality of $\langle x_i, w\rangle$.

Therefore $r(t) = \min_i \langle x_i(t), w\rangle$ is increasing for every $t \geq 0$ and bounded by 1. Let $\bar{r} = \lim_{t\to\infty} r(t)$. Note that, by assumption, $\bar{r} > 0$. By compactness there exist a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \to \infty$ and $\bar{x}_i \in \mathbb{S}^d$ for $i = 1, \dots, N$ such that $\lim_{n\to\infty} x_i(t_n) = \bar{x}_i$ for $i = 1, \dots, N$. By definition of r, $\langle \bar{x}_j, w\rangle \geq \bar{r}$ for every $j = 1, \dots, N$ and there exists $i \in \{1, \dots, N\}$ such that $\langle \bar{x}_i, w\rangle = \bar{r}$. Then

$$\begin{aligned} 0 = \lim_{n\to\infty} \langle \dot{x}_i(t_n), w\rangle &= \sum_{j\neq i} a_{ij} \left(\langle w, \bar{x}_j\rangle - \langle \bar{x}_i, \bar{x}_j\rangle\langle \bar{x}_i, w\rangle\right) \\ &\geq \bar{r} \sum_{j\neq i} a_{ij} \left(1 - \langle \bar{x}_i, \bar{x}_j\rangle\right), \end{aligned}$$

which implies

$$\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_N := \bar{x}. \tag{9}$$

To prove that the system tends to the consensus $\bar{x}$ consider the $\rho(t) = \max_j \|x_j(t) - \bar{x}\|^2$ and let i the smallest index attaining the maximum. Then note that, for every $t \geq 0$,

$$\frac{d}{dt}\rho(t) = 2\frac{d}{dt}(1 - \langle x_i(t), \bar{x}\rangle) = -\langle \dot{x}_i(t), \bar{x}\rangle = -\sum_{j\neq i} a_{ij} \left(\langle x_j, \bar{x}\rangle - \langle x_j, x_i\rangle\langle x_i, \bar{x}\rangle\right).$$

As above, we have that either the system is in consensus either ρ is decreasing for every $t \geq 0$. Then, by (9), $\lim_{t\to\infty} \rho = 0$, and the system tends to consensus. □

Example 2 shows the necessity of the condition $\langle x_i, w\rangle > 0$ for every $i = 1, \dots, N$ for some $w \in \mathbb{S}^d$, indeed we present a configuration of dancing equilibrium not verifying this assumption. Dancing equilibria represent indeed one of the main differences with linear system, like HKM, where positive interactions always yield consensus. Antipodal equilibria are another example of equilibrium configuration not contained in the hemisphere and different from consensus.

Next result establishes sufficient conditions to have $E(t) \to 0$ as $t \to \infty$. This implies that the velocities of the agents $\dot{x}_i(t)$ tend to zero and, consequently, the system tends to a steady configuration. In particular, in the symmetric case, dancing equilibria do not arise.

Theorem 2. *If the interaction matrix A is symmetric then*

$$\lim_{t\to\infty} E(t) = 0.$$

In particular the system tends to an equilibrium.

Proof. Define $F(t) := \sum_{i=1}^N \langle x_i, \alpha_i\rangle$. Using the symmetry of A, we have

$$\frac{d}{dt}F(t) = 4E(t). \tag{10}$$

Indeed

$$\begin{aligned}\frac{d}{dt}F(t) &= \sum_{i=1}^{N}\langle \dot{x}_i, \alpha_i\rangle + \langle x_i, \dot{\alpha}_i\rangle \\ &= \sum_{i=1}^{N}\left(\langle \dot{x}_i, \alpha_i\rangle + \sum_{j=1}^{N} a_{ij}\langle x_i, \dot{x}_j\rangle\right) \\ &= \sum_{i=1}^{N}\langle \dot{x}_i, \alpha_i\rangle + \sum_{j=1}^{N}\langle\sum_{i=1}^{N} a_{ji}x_i, \dot{x}_j\rangle \\ &= 2\sum_{i=1}^{N}\langle \dot{x}_i, \alpha_i\rangle \\ &= 4E(t).\end{aligned}$$

As $E(t) \geq 0$ for every $t \geq 0$, $F(t)$ is a non-decreasing function. Moreover $F(t)$ is bounded, indeed for every $t \geq 0$

$$\mid F(t) \mid = \mid \sum_{i=1}^{N}\langle x_i, \alpha_i\rangle \mid = \mid \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\langle x_i, x_j\rangle \mid \leq \sum_{i,j=1}^{N} \mid a_{ij} \mid .$$

Hence $F(t) \uparrow \ell$ as $t \to \infty$ for some $\ell \leq \sum_{i,j=1}^{N} |a_{ij}|$ and, consequently, $\frac{d}{dt}F(t) \to 0$ as $t \to \infty$. The claim hence follows by (10). □

4 Stabilization and controllability

Once sufficient conditions for consensus are established, as in Theorem 1, it is natural to study whether is possible to create or induce consensus with an external intervention. The dynamics of agent $x_i \in \mathbb{S}^d$ for $i = 1, \ldots, N$ is

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - \langle x_i, x_j\rangle x_i) + u_i \tag{11}$$

where the control $\mathbf{u}(t) = (u_1(t), \ldots, u_N(t))$ are measurable function of time $t \mapsto \mathbf{u}(t) \in U_M(x)$ where

$$U_M(x) = \left\{(u_1, \ldots, u_N) \in (\mathbb{R}^d)^N \mid \langle u_i, x_i\rangle = 0 \text{ and } \sum_{i=1}^{N} \|u_i\| \leq M, \right\}, \tag{12}$$

is the set of admissible controls and depends on the state. The control acting on agent x_i is a vector of the tangent of the sphere $\mathbb{S}^d$ at x_i.

We consider then the problem of finding a control $\mathbf{u}(t) = (u_1(t), \ldots, u_N(t))$ steering the opinion of all agents towards the leading opinion x_0. Since consensus is an equilibrium for system (11) this problem is, in fact, a stabilization problem.

Definition 5. *We say that the system is* asymptotically stabilizable *(respectively,* stabilizable in finite time*) if for every initial condition there exist a choice of the control such that the system tends to consensus (respectively, reaches consensus in finite time).*

Definition 6. *We say that the system is* asymptotically stabilizable to the consensus x_0 *(respectively,* stabilizable to the consensus x_0 in finite time*) if the stabilizing control can be chosen in such a way that the associated solution tends to the consensus x_0 as t tends to infinity (respectively, in finite T).*

Throughout this section we consider the problem of stabilizing the system to x_0. To ensure that, for the uncontrolled system, consensus is a stable equilibrium and to guarantee that the uncontrolled system tends to a steady configuration we assume that:

(*i*) $a_{ij} > 0$ for every $i, j = 1, \dots, N$ (positive interactions);

(*ii*) $a_{ij} = a_{ji}$ for every $i, j = 1, \dots, N$ (symmetric interactions);

moreover, for simplicity, we assume that $(a_{ij})_{ij}$ is a doubly stochastic matrix, namely

(*iii*) $\sum_{i=1}^N a_{ij} = \sum_{j=1}^N a_{ij} = 1$ for every $i, j = 1, \dots, N$.

We have a first stabilization result in finite time for system (15).

Proposition 3. *If $M > N$ then system* (15) *is stabilizable to the consensus x_0 in finite time.*

Proof. If $x_i = x_0$ let $u_i = 0$. If $x_i = -x_0$ then let u_i be any vector not collinear with x_0 of norm $\|u_i\| = M/N$. Otherwise consider, for every $i = 1, \dots, N$, the feedback control solution of the variational problem

$$\max\langle v, x_0\rangle \quad \text{for } v \in \mathbb{R}^d \text{ such that } \langle v, x_i\rangle = 0, \text{ and } \|v\| = \frac{M}{N}. \tag{13}$$

Then a solution u_i of (13) satisfies

$$\langle u_i, x_0\rangle = \frac{M}{N}\sqrt{1 - \langle x_i, x_0\rangle^2}.$$

Indeed for every $v \in \mathbb{R}^d$ tangent to the sphere $\mathbb{S}^d$ in x_i of norm $\|v\| = \frac{M}{N}$ one has

$$\begin{aligned}
\langle v, x_0\rangle &= \langle v - \langle v, x_i\rangle x_i, x_0\rangle \\
&= \langle v, x_0\rangle - \langle v, x_i\rangle\langle x_i, x_0\rangle \\
&= \langle v, x_0 - \langle x_i, x_0\rangle x_i\rangle \\
&\le \|v\|\|x_0 - \langle x_i, x_0\rangle x_i\| \\
&= \frac{M}{N}\sqrt{1 - \langle x_i, x_0\rangle^2},
\end{aligned}$$

and the maximum is attained when the equality in the Cauchy–Schwarz inequality holds, that is, for v linearly dependent on the projection of x_0 on the orthogonal of x_i. For every $i = 1, \dots, N$ and $t \ge 0$ we have

$$\begin{aligned}
\frac{d}{dt}(1 - \langle x_i, x_0\rangle) &= -\langle \dot{x}_i, x_0\rangle \\
&= -\langle \alpha_i, x_0\rangle + \langle \alpha_i, x_i\rangle\langle x_i, x_0\rangle - \langle u_i, x_0\rangle \\
&= -\langle \alpha_i, x_0 - \langle x_i, x_0\rangle x_i\rangle - \frac{M}{N}\sqrt{1 - \langle x_i, x_0\rangle^2} \\
&\le \left(\|\alpha_i\| - \frac{M}{N}\right)\sqrt{1 - \langle x_i, x_0\rangle^2} \\
&= \left(\|\alpha_i\| - \frac{M}{N}\right)\sqrt{1 - \langle x_i, x_0\rangle}\sqrt{1 + \langle x_i, x_0\rangle}.
\end{aligned} \tag{14}$$

Notice that

$$\left(\|\alpha_i\| - \frac{M}{N}\right) > 0$$

since $\|\alpha_i\| \leq \sum_{j\neq i} a_{ij}\|x_j\| = 1$ for Assumption (iii) on the matrix A. In particular $\langle x_i, x_0\rangle$ is non-decreasing for every $i = 1, \dots, N$. Therefore for every $t \geq 0$,

$$1 + \langle x_i(t), x_0\rangle \geq 1 + \langle x_i(0), x_0\rangle.$$

Let i be such that $x_i(0) \neq -x_0$, let $m_i = (M/N - 1)(1 + \langle x_i(0), x_0\rangle) > 0$, and $r_i(t) = (1 - \langle x_i(t), x_0\rangle)$ then (14) reads

$$\dot{r}_i(r) \leq -m_i\sqrt{r_i(t)},$$

which implies

$$r_i(t) \leq \left(r_i(0) - \frac{m_i}{2}t\right)^2.$$

In particular x_i reaches x_0 in finite time. If $x_i = -x_0$ then $(1 - \langle x_i, x_0\rangle)$ has attended the maximum and it is decreasing, since, for the choice of the control, $x_i = -x_0$ is not an equilibrium. Then we can apply the above argument. □

To ensure that the dynamics of the opinions stay on the sphere the set of admissible controls $U_M(x)$ depends on the state x, indeed the control acting on x_i must be tangent to the sphere at x_i. We can also state the control problem by considering a set of admissible controls not depending on the state. In this case we take controls in $\mathbb{R}^{d+1}$ and we consider their projection on the orthogonal of x_i. Then the dynamics of agent $x_i \in \mathbb{S}^d$ for $i = 1, \dots, N$ reads

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - \langle x_i, x_j\rangle x_i) + (u_i - \langle x_i, u_i\rangle x_i) \tag{15}$$

where the controls $u_1(t), \dots, u_N(t)$ are measurable function of time satisfying the constraint

$$\sum_{i=1}^{N} \|u_i(t)\| \leq M,$$

for every $t \geq 0$.

The two control problems (11) and (15) are equivalent. This second formulation is useful in order to find smooth stabilizers as stated in Proposition 4 below. Notice that, however, the smooth feedback ensures only asymptotic stabilization. Stabilization to x_0 in finite time, as in Proposition 3, is achieved with discontinuous feedbacks. Moreover in (15) the bound M does not represent the actual strength of a control acting on the system which is given by the projection of the control on the orthogonal of x_i.

Proposition 4. *Assume $M > N$ and let $m = M/N - 1$. Then the smooth feedback defined by*

$$u_i(x) = mx_0 - \alpha_i,$$

if $x_i \neq x_0$ and by taking any vector not collinear with x_0 of norm $\|u_i\| = M/N$ if $x_i = -x_0$ asymptotically stabilizes system (15) to the consensus x_0. More precisely,

$$\|x_i(t) - x_0\|^2 = 2\frac{e^{-2mt}}{1 + e^{-2mt}}\|x_i(0) - x_0\|^2,$$

for every $t \geq 0$, and $i = 1, \dots, N$.

Proof. Note that the feedback u_i is admissible since $\|u_i\| = \|mx_0 - \alpha_i\| \leq m+1 = M/N$. If $x_i \neq -x_0$ one has

$$\begin{aligned}\frac{d}{dt}(1 - \langle x_i, x_0\rangle) &= -\langle \dot{x}_i, x_0\rangle \\ &= -\langle \alpha_i + u_i, x_0\rangle + \langle \alpha_i + u_i, x_i\rangle\langle x_i, x_0\rangle \\ &= -m(1 - \langle x_i, x_0\rangle^2) \\ &= -m(1 - \langle x_i, x_0\rangle)(1 + \langle x_i, x_0\rangle). \end{aligned} \tag{16}$$

Now let $r_i(t) = (1 - \langle x_i(t), x_0\rangle)$ then $(1 + \langle x_i(t), x_0\rangle) = 2 - r_i(t)$. Then (16) reads

$$\dot{r}_i(t) = -mr_i(t)(2 - r_i(t))$$

which implies

$$r_i(t) = 2\frac{e^{-2mt}}{1 + e^{-2mt}} r_i(0).$$

Hence $x_i(t) \to x_0$ as $t \to \infty$. If $x_i = -x_0$ then $(1 - \langle x_i, x_0\rangle)$ has attended the maximum and it is decreasing, since, for the choice of the control, $x_i = -x_0$ is not an equilibrium. Then we apply the sequence of equalities (16). □

There are several possible ways to model the external action on agents' opinions. We consider, for instance, the problem of describing the action of a mass media, or communication enterprises, on the system. Media are opinion leaders whose opinion is not influenced by the others, at least not for small intervals of time, so that it is represented as an agent whose opinion x_0 is constant. We then consider the case in which all agents perform the same control law which is determined by the influence of the leading opinion x_0, given by the media, who can change the communication rate as a function of time. Namely we consider the case in which

$$u_1(t) = \cdots = u_N(t) = u(t)x_0,$$

for some scalar control $t \mapsto u(t) \in [0, M]$. The positive bound M represents the maximal strength of communication of the media. Therefore the dynamics of agent $x_i \in \mathbb{S}^d$ for $i = 1, \ldots, N$ is

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - \langle x_i, x_j\rangle x_i) + u(t)(x_0 - \langle x_i, x_0\rangle x_i). \tag{17}$$

We have the following partial stabilization result for system (17).

Proposition 5. *Let $m = \min_i(1 + \langle x_i(0), x_0\rangle)$. If $m > 0$ and*

$$M \geq \frac{\sqrt{2}}{m},$$

then system (17) *is asymptotically stabilizable to the consensus x_0.*

Proof. Case 1. The initial data verify $\langle x_0, x_i(0)\rangle > 0$ for every $i = 1, \ldots, N$.
Consider the control $u(t) \equiv \delta$ for $0 < \delta \leq M$. Following Theorem 1 one has that the system tends to consensus, say $\bar{x}$. Indeed for every $t \geq 0$ call $r(t) = 1 - \min_i\langle x_i(t), x_0\rangle$ and let $i \in \{1, \ldots, N\}$ be the smallest index such that $1 - \langle x_i(t), x_0\rangle = r(t)$. The quantity $r(t)$ is decreasing. Indeed if, by contradiction, $\dot{r} \geq 0$ then

$$0 \leq \dot{r} = -\langle \dot{x}_i, x_0\rangle = -\sum_{j\neq i} a_{ij}\left(\langle x_j, x_0\rangle - \langle x_j, x_i\rangle\langle x_i, x_0\rangle\right) - \delta(1 - \langle x_0, x_i\rangle^2),$$

which implies there exists $j \neq i$ such that $\langle x_j, x_0\rangle < \langle x_j, x_i\rangle\langle x_i, x_0\rangle \leq \langle x_i, x_0\rangle$, in contradiction with the minimality of $\langle x_i, x_0\rangle$. Let $\bar{r} = \lim_{t\to\infty} r(t) \geq 0$. One can prove, following the proof of Theorem 1 that $\bar{r} = 0$ and, in particular, that the system tends to a consensus $\bar{x}$. We claim that $\bar{x} = x_0$. Indeed if by contradiction $\bar{x} \neq x_0$, then

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{2}\frac{d}{dt}\|x_i - x_0\|^2 &= -\lim_{t\to\infty}\langle \dot{x}_i, x_0\rangle \\ &= \lim_{t\to\infty} -\langle \alpha_i + \delta x_0, x_0\rangle + \langle \alpha_i + \delta x_0, x_i\rangle\langle x_i, x_0\rangle \\ &= -\delta(1 - \langle \bar{x}, x_0\rangle^2) \\ &\neq 0.\end{aligned}$$

Case 2. There exists at least one index $i \in \{1, \dots, N\}$ such that $\langle x_i(0), x_0\rangle < 0$.
Consider the control $u(t) \equiv M$ for every $t \geq 0$. Let $t \geq 0$ and let $i \in \{1, \dots, N\}$ such that $\langle x_i, x_0\rangle$ is minimal. Now

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x_i - x_0\|^2 &= -\langle \dot{x}_i, x_0\rangle \\ &= -\langle x_0 - \langle x_0, x_i\rangle x_i, \alpha_i + M x_0\rangle,\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}\|x_i - x_0\|^2 < 0 &\iff \langle \alpha_i, x_0\rangle - \langle x_i, x_0\rangle\langle \alpha_i, x_i\rangle + M(1 - \langle x_i, x_0\rangle^2) > 0 \\ &\iff M(1 - \langle x_i, x_0\rangle^2) > -\langle x_0 - \langle x_0, x_i\rangle x_i, \alpha_i\rangle. \qquad (18)\end{aligned}$$

Note that, by assumption,

$$-\langle x_0 - \langle x_0, x_i(0)\rangle x_i(0), \alpha_i\rangle \leq \|\alpha_i\|\sqrt{1 - \langle x_i, x_0\rangle^2} < \sqrt{2} \leq mM. \qquad (19)$$

Therefore $\|x_i(t) - x_0\|$ is decreasing as long as

$$1 - \langle x_i, x_0\rangle^2 \geq 1 + \langle x_i, x_0\rangle \geq m.$$

Then, for the assumption on the initial data, the maximal distance from the opinion leader, namely $\max_i \|x_i - x_0\|$ is monotonically decreasing. In particular there exists i and $\bar{r} \geq 0$ such that $\lim_{t\to\infty} \|x_i - x_0\|^2 = 2\bar{r}$. We claim that $\bar{r} < 1$. Indeed if $\bar{r} \geq 1$ then $\langle x_i, x_0\rangle < 0$ and following the chain of inequalities (18)-(19) one has that

$$\lim_{t\to\infty} \frac{d}{dt}\|x_i - x_0\|^2 < 0,$$

which is a contradiction. Therefore $\lim_{t\to\infty} \max_i \|x_i - x_0\|^2 < 2$, which implies that there exists t sufficiently large such that $\langle x_i(t), x_0\rangle > 0$ for every $i = 1, \dots, N$. The statement then follows from Case 1. □

Proposition 5 provides a very simple strategy for the stabilization of system (17), given by setting $u(t) = M$ for every $t \geq 0$. If the maximal strength M of the control is sufficiently large, namely $M \geq \sqrt{2}/m$, then the action of the control is stronger than every other possible influence on each agent and the system tends to the consensus x_0. In Figure 2 we represent the results of the action of this simple strategy as a function of the number of agents N and of the maximal strength of the control M.

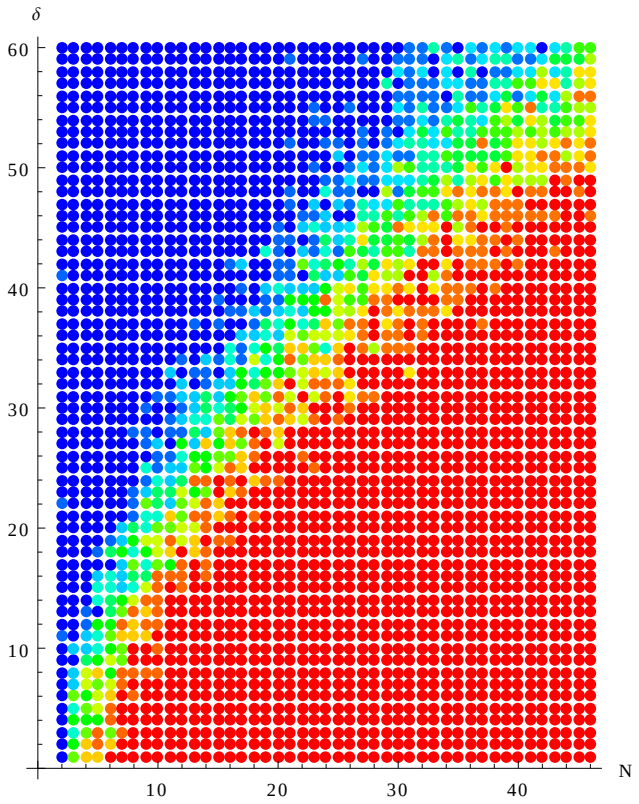


Figure 2: The result of the application of the constant control $u(t) = \delta$ as a function of the number of agents N and of the magnitude δ. Every dot represents the result of 10 trials with random initial condition and random symmetric interaction matrices (with uniformly bounded max norm and without assumptions on the sign of the interaction). The color of the dots ranges from blue to red in function of the number of trials the system reached the consensus x_0. Blue represents consensus in all 10 trials and red the non-consensus in all trials.

Remark 2. *The condition* $\min_i(1+\langle x_i(0), x_0\rangle) \neq 0$ *which implies* $x_i(0) \neq -x_0$ *for every* $i = 1, \dots, N$ *is necessary since the symmetries of the system imply that the action of the controlling agent, placed on the north pole* x_0*, is null on the south pole* $-x_0$*. As an example, consider* 3 *agents on* $\mathbb{S}^1 \subset \mathbb{C}$ *in initial positions* $x_1 = 1$*,* $x_2 = -1$*, and* $x_3 = -i$ *and consider the problem of steering the system, with interaction coefficients* $a_{ij} = 1$ *for every* $i, j = 1, 2, 3$*, to the north pole* $x_0 = i$*. Then, it is easy to see that for every choice of the control function* $t \mapsto u(t)$ *one has that* $\dot{x}_3 = 0$*.*

5 Numerical simulations

In this section we show numerical simulations of the evolution of system (1). The simulations are grouped according to the different natures the interactions.

5.1 Methodology

Interaction matrices and initial positions are chosen randomly, the associated trajectories are approximated by a Runge–Kutta scheme. This scheme ensures a 4th order convergence when considering polar coordinates in the one-dimensional cases.

In the d-dimensional case, with $d \geq 2$, we embedded the dynamics in $\mathbb{R}^{d+1}$.

This approach seems to yield numerical instability in the case of anti-symmetric matrices and for some general matrices. We did not recorded instability issues in the cases of positive, symmetric, and sign-symmetric matrices.

Instability phenomenona occur also using the Runge–Kutta scheme with adaptive discretization steps and the implicit Euler scheme.

We tried to overcome this problem by means of a renormalization of the agents coordinates, namely by projecting at every step the agents positions on the sphere, but we noticed that this approach yields a numerical dissipation of kinetic energy and, substantially, unreliable data.

In general, approximation methods based on a cartesian system of coordinates appear not to provide simulations for long times for some classes of interaction matrices. This issue seems mitigated by the fact that the system tends to stabilize within small times, possibly with an exponential rate.

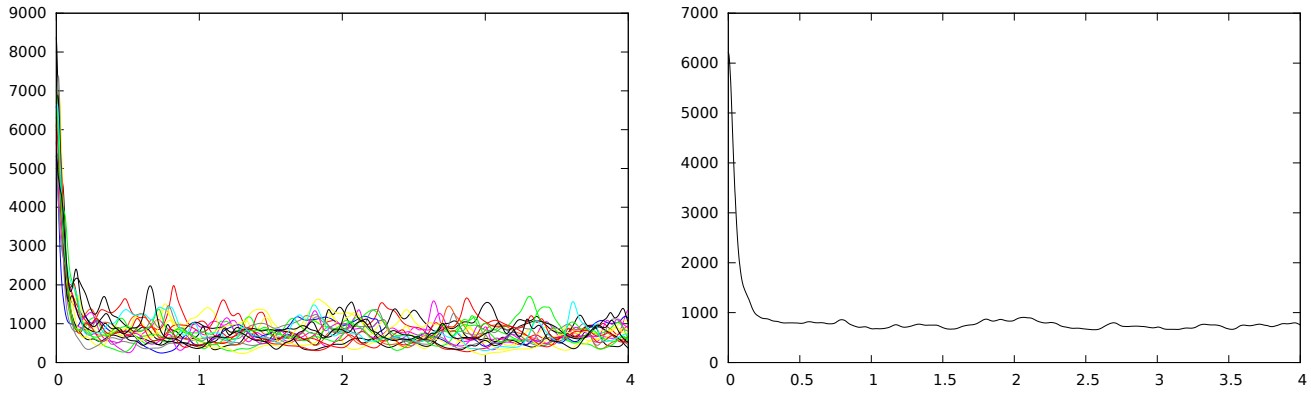
However we consider the numerical analysis of the long term behaviour of the system an important tool of investigation and we plan a further investigation (by setting the problem on appropriate internal coordinates) in a future work.

5.2 General case

We ran some tests in the general case, namely whit no symmetry or sign constrains on to the adjacency matrix. In general kinetic energy as defined in (7) has a non-monotonic behavior (see Figure 3), in contrast with the linear case (HKM in $\mathbb{R}^N$). However it is possible to observe, in simulations, a recurring initial drop of energy followed by oscillations of smaller amplitude.

Remark 3. *We notice that initial data corresponding to steady configurations (dancing equilibria) or to equilibria yield constant kinetic energy. On the other hand the set of these configurations has zero Lebesgue measure and, consequently, a randomly generated initial datum in general does not belong to this class. A suitable choice of the initial data also allows to avoid the initial drop and to observe only an oscillating behaviour of the energy.*

Another observable of the system accounting for the distance to consensus is the *centroid* (see also [SS09]).



(a) Energy evolution with 20 random initial data

(b) Mean energy evolution

Figure 3: Energy evolution for randomly a generated interaction matrix for 50 agents.

Definition 7. *The* centroid *is the unweighted barycenter of the system, that is the point*

$$C(t) := \sum_{i=1}^{N} x_i(t).$$

Note that in general the centroid does not belong to the state-space $\mathbb{S}^d$. The squared modulus of $C(t)$ reads

$$||C(t)||^2 = \sum_{i,j=1}^{N} \langle x_i(t), x_j(t) \rangle.$$

The quantity $N^2 - ||C(t)||^2$ provides an estimate of the distance from consensus. Indeed in general $||C(t)||^2 \in [0, N^2]$ and $||C(t)||^2 = N^2$ if and only if consensus is reached.

In some cases, especially when the number of agents is small, some of the equilibria described in the present paper are reached, for instance, the antipodal equilibrium and the dancing equilibrium - see Figure 4.

Periodic oscillations and dumping oscillations of kinetic energy are interesting phenomena that we observed only with general matrices, see Figure 5 and Figure 6 respectively. In the example showed in Figure 7 the dumping effect is particularly evident in the centroid evolution.

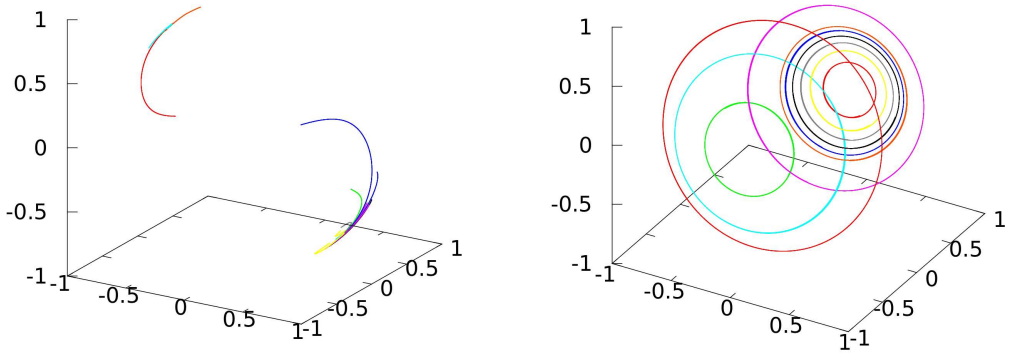
Such periodicity of energy appears to reflect, as well as dancing equilibria, the periodic (or quasi-periodic) behaviour of some social dynamics, e.g. fashion and economy cyclicality.

5.3 Symmetric case

In Theorem 2 we proved that if A is symmetric then the kinetic energy tends to zero.

Figure 8 shows the energy decay for a randomly generated symmetric interaction matrices for 50 agents. A recurring pattern in our tests in the symmetric cases is a possible initial increase of the energy and an apparently exponential convergence to zero.

We can see this pattern explicitly in the simple case of a positive symmetric matrix with 2 agents and satisfying the assumptions of Theorem 1



(a) (b)

Figure 4: An antipodal (a) and a dancing (b) equilibrium reached by two systems with general adjacency matrices and 15 and 10 agents, respectively.

Proposition 6. *Let $A = (a_{ij})$ be a positive symmetric matrix, $A_i := \sum_{j=1}^N a_{ij}$ and $\bar{A} = \min A_i$. Also assume the intial data to satisfy the assumptions of Theorem 1. Then for every constant $c \in (0, \bar{A})$ there exists $T_c \geq 0$ such that*

$$E(t) \leq M_A t + E(T_c)e^{-4c(t-T_c)} \qquad \forall t \geq T_c,$$

for some $M_A \geq 0$ depending only on A. When $N = 2$ one has $M_A = 0$ and, consequenlty, an exponential decay of $E(t)$.

Proof. First of all recall that

$$E(t) = \frac{1}{2}\sum_{i=1}^N \langle \dot{x}_i, \dot{x}_i \rangle = \frac{1}{2}\sum_{i=1}^N \langle \dot{x}_i, \alpha_i \rangle = \frac{1}{2}\sum_{i=1}^N \langle \alpha_i, \alpha_i \rangle - \langle \alpha_i, x_i \rangle^2 \qquad \forall t \geq 0.$$

Furthermore, since A is symmetric, one has

$$E(t) = \frac{1}{2}\sum_{i=1}^N \langle x_i, \dot{\alpha}_i \rangle, \qquad \forall t \geq 0.$$

Let $\theta_i(t) := \langle x_i, \alpha_i \rangle$ and $\theta(t) := \sum_{i=1}^N \theta_i(t)$. Then

$$\frac{d}{dt}\theta(t) = 4E(t) \geq 0, \qquad \forall t \geq 0. \tag{20}$$

Theorem 1 implies that for every $i, j = 1, \ldots, N$ the scalar product $\langle x_i, x_j \rangle$ tends to 1 as t tends to $+\infty$. Then for every $c \in (0, \bar{A})$ we may choose T_c such that $\theta_i(t) > c$ for every $t > T_c$. By deriving $E(t)$ we obtain:

$$\frac{d}{dt}E(t) = \sum_{i=1}^N \langle \alpha_i, \dot{\alpha}_i \rangle - \sum_{i=1}^N \theta_i(t)\frac{d}{dt}\theta_i(t) \leq \sum_{i=1}^N \langle \alpha_i, \dot{\alpha}_i \rangle - 4cE(t), \quad \forall t \geq T_c.$$

Note that when $N = 2$ we have $\langle \alpha_i, \dot{\alpha}_i \rangle = \langle x_j, \dot{x}_j \rangle = 0$ for every $i, j = 1, 2;\ i \neq j$. Now, we observe that $\sum_{i=1}^N \langle \alpha_i, \dot{\alpha}_i \rangle$ is an uniformly bounded function and we let M_A be an upper bound for it. The claim hence follows by Gronwall's Lemma. □

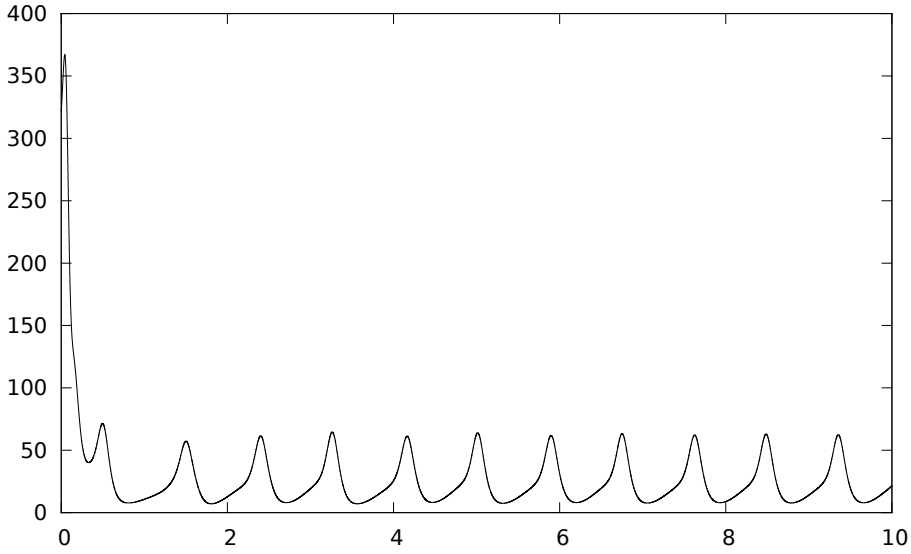
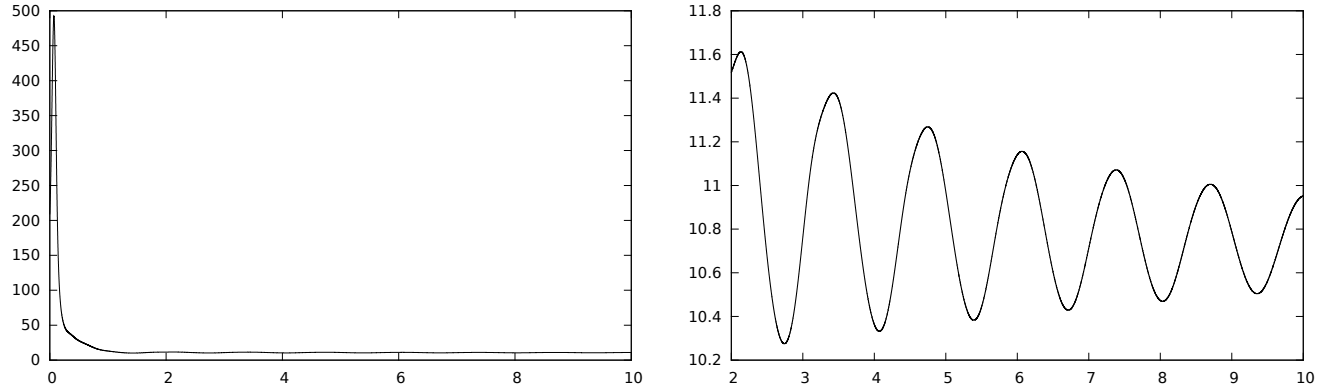


Figure 5: Energy evolution of a system with 15 agents and a general interaction matrix: after an initial drop of energy, the following oscillations suggest a possibly periodic behaviour of the energy.



(a) $t \geq 0$ (b) $t \geq 2$

Figure 6: Energy evolution of a system with 15 agents and a general interaction matrix: the initial drop of energy is followed by dumped oscillations, Figure (a). This effect is more evident when restricting the evolution of energy $E(t)$ to $t \geq 2$, Figure (b).

5.4 Sign-symmetric case

In our tests we also considered the case of sign-symmetric matrices, modeling dynamics where every pair of agents is either mutually attracted or mutually repulsed. Obviously the symmetric case is a particular case of sign-symmetric case and this reflects on the behavior of the energy: it converges as time tends to infinity, but, possibly, to a non-zero constant, see Figure 9.

In Figure 10 the energy tends to a positive constant and the system converges to periodic orbits preserving the scalar product between agents, i.e., a dancing equilibrium.

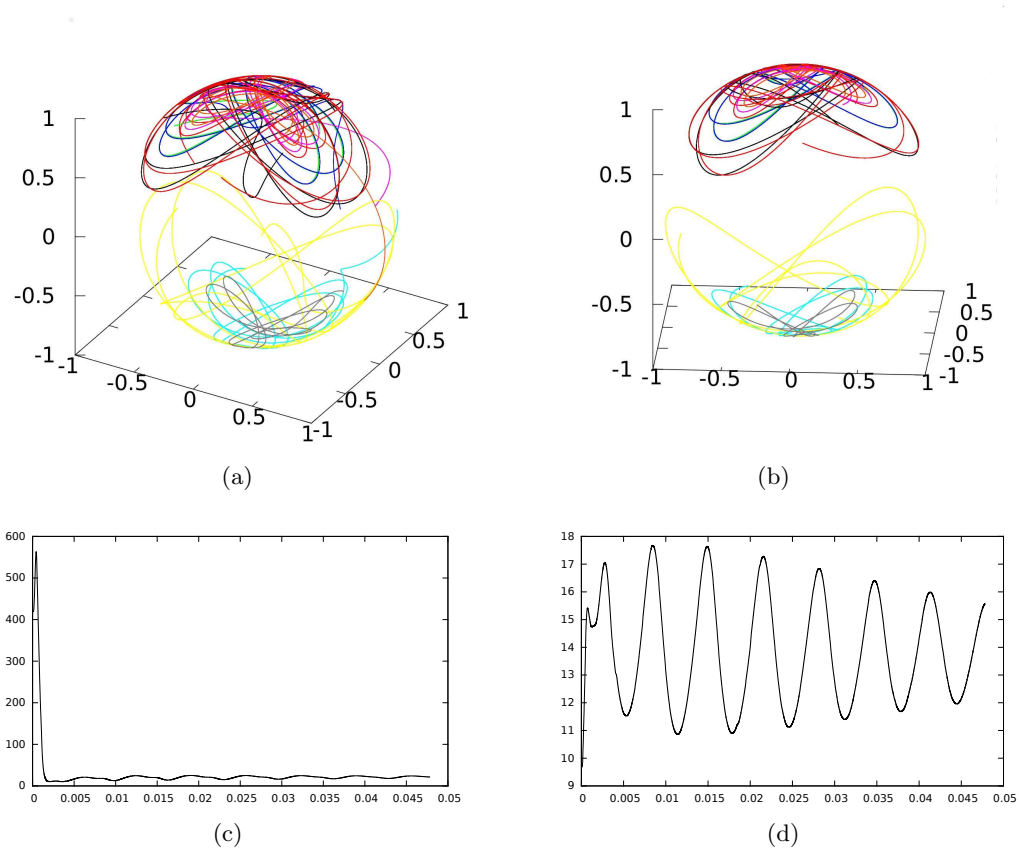
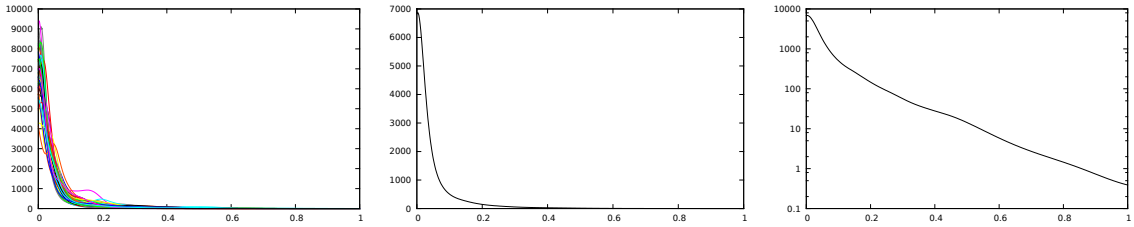


Figure 7: In (a) and (b) are represented the trajectories of 10 agents with a randomly general matrix starting from $t = 0$ and $t = 0.012$, respectively. The related evolutions of energy and of squared modulus of centroid are displayed in (c) and (d), respectively.

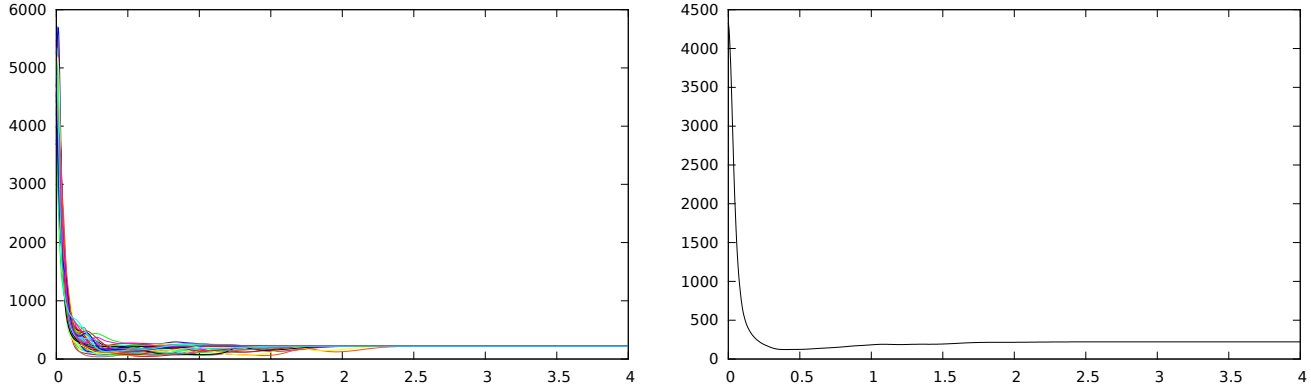


(a) Energy decay with 50 random initial data

(b) Mean energy decay

(c) Mean energy decay in logarithmic scale

Figure 8: Energy decay for a randomly generated symmetric interaction matrix for 50 agents.



(a) Energy evolution with 50 random initial data

(b) Mean energy evolution

Figure 9: Energy evolution for a randomly generated sign-symmetric interaction matrix for 50 agents.

5.5 Positive case

In the positive case, namely when $a_{ij} > 0$ for every $i, j = 1, \dots, N$, Theorem 1 ensures consensus provided that agents are in an open hemisphere. In Example 2 we present a case of dancing equilibria emerging in this kind of systems. However experimental data suggest that the evolutions not converging to consensus are particular (and unlikely) cases. In other words it seems that consensus may be reached for generic initial data.

The kinetic energy of a system with positive interaction matrix is in general non-monotone. However it seems that, in the positive case, if the system tends to consensus then the modulus of the centroid is a monotonically increasing quantity, see Figure 11.

6 Conclusions and perspectives

We presented a nonlinear model of opinion dynamics. The system is very versatile and several extensions are possible in order to describe more elaborate interactions. The nonlinearity yields a rich structure making this system suitable for modeling complex behaviors. New features arise in the analysis of the nonlinear dynamics leading to new mathematical challenges.

We highlighted the existence of three kind of equilibria. While consensus is well-known, other

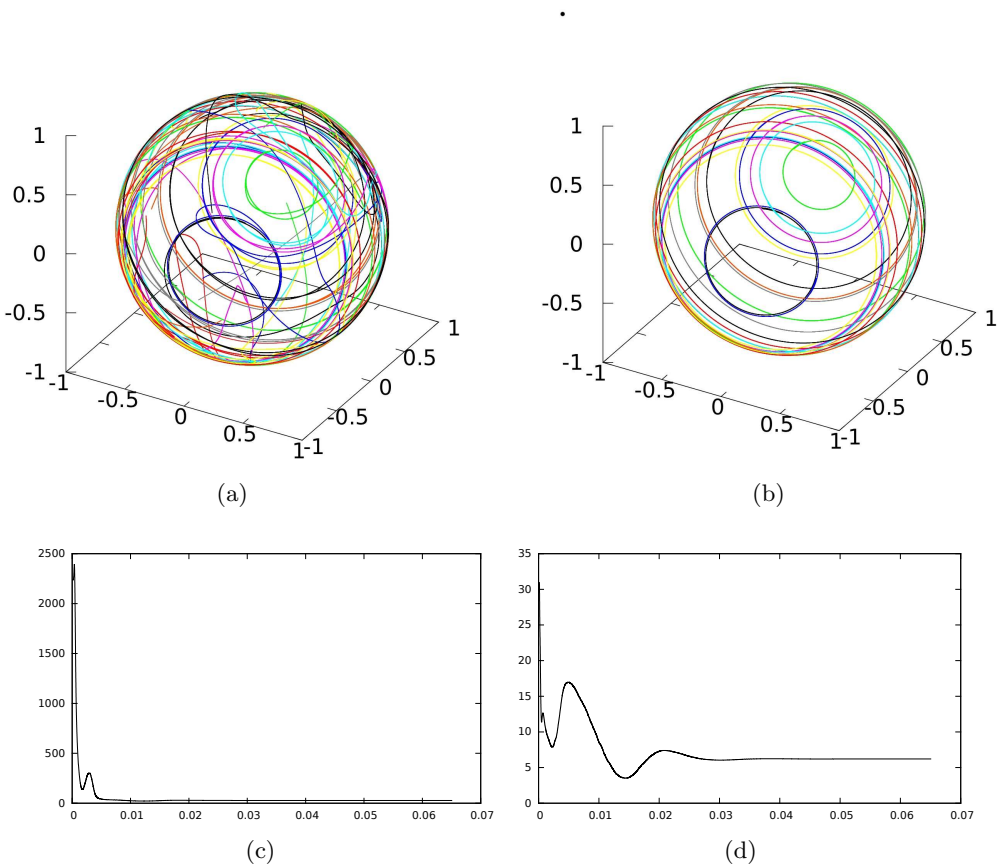


Figure 10: In (a) and (b) the trajectories of 25 agents with a randomly generated sign-symmetric matrices starting from $t = 0$ and $t = 0.03$, respectively. The evolutions of energy and of the squared modulus of centroid are displayed in (c) and (d), respectively. The system tends to a dancing equilibrium.

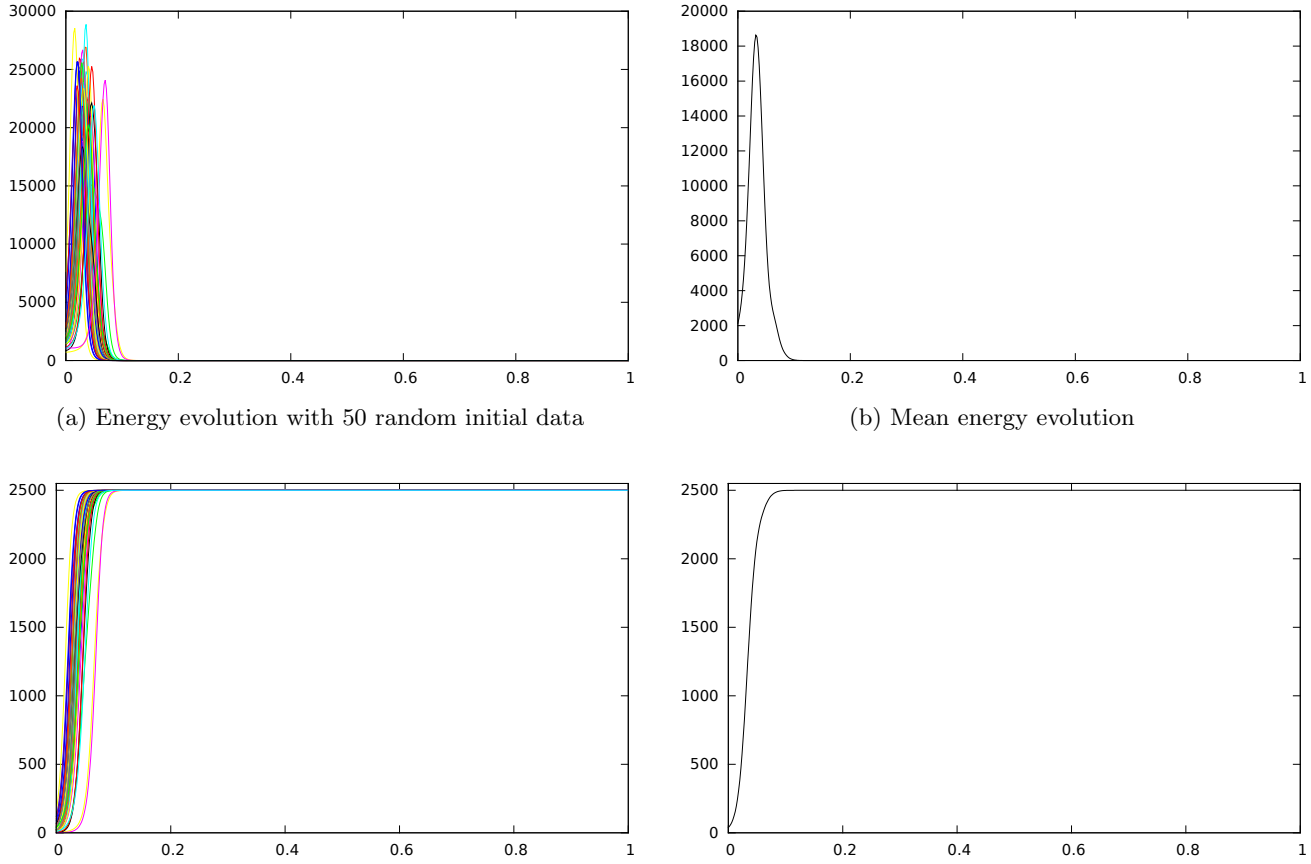
configurations like antipodal, polygonal, and dancing equilibria are peculiar of this model. We plan to characterize these configurations and to deepen the stability analysis.

We provided a sufficient condition, Theorem 1, for consensus in the positive interaction case and in the symmetric case, we showed that the kinetic energy of the system tends to zero, Theorem 2. Possible extensions of these results to more general cases can be achieved by adapting classical consensus algorithm of Moreau type to this nonlinear system.

We presented two results on the controllability of this system, i.e. the ability to force consensus using an external intervention, accounting for the influence, for instance, of mass media. We studied only distributed control but we believe that *sparse* feedback control strategies may be more effective to address the stabilization and the controllability problem.

Simulations provide several hints on the asymptotic behavior of the system: the energy, although non-monotonically, tends to stabilize. We believe that the non-monotonicity of the energy and yet its decreasing trend are the result of a trade-off between stabilizing symmetries of the systems and of non-symmetric components steering the system towards a non-steady dancing equilibrium.

In Table 1 and Table 2 below we sum up some features of the model grouped by interaction matrix type.



(a) Energy evolution with 50 random initial data

(b) Mean energy evolution

(c) Centroid evolution with 50 random initial data (the same as in (A))

(d) Mean centroid evolution

Figure 11: Energy evolution for a randomly generated sign-symmetric interaction matrix for 50 agents.

Equilibrium\Interaction	General	Positive	Symmetric	Sign-symmetric
Consensus	Yes	Yes (stable)	Yes	Yes
Antipodal	Yes (see Figure 4-(a))	Yes (unstable)	Yes (see Section 3.1)	Yes
Poygonal	Yes	Yes	Yes	Yes
Dancing	Yes (see Figure 4-(b))	Yes (see Example 2)	No (by Theorem 2)	Yes (see Section 5.4)

Table 1: Possible equilibria according to interaction matrix type. Recall that for positive matrices consensus is a stable equilibrium, while antipodal equilibrium is stable for some classes of sign-symmetric matrices.

Energy\Interaction	General	Positive	Symmetric	Sign-symmetric
Convergence to 0	Yes (see Figure 4-(a))	Yes (under the assumptions of Theorem 1)	Yes (by Theorem 2) See Section 5.3 and Propostion 6 for exponential decay	Yes
Convergence to $c > 0$	Yes (see Figure 4-(b))	No (unless there is a dancing equilbrium)	No (by Theorem 2)	Yes (Section 5.4)
Eventually monotone	Yes	Yes (see Theorem 1)	Yes	Yes (see Theorem 1)
Dumping&Periodic	Yes (see Figures 5 – 6)	No	No	No

Table 2: Observed energy behaviour according to interaction matrix type. Note that, unless specified, by writing "No" we mean that we did not observe a given phenomenon (without implying that such phenomenon is incompatible with the corresponding interaction matrix).

References

[BHT10] Vincent D Blondel, Julien M Hendrickx, and John N Tsitsiklis. Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM Journal on Control and Optimization*, 48(8):5214–5240, 2010.

[BS03] Laxmidhar Behera and Frank Schweitzer. On spatial consensus formation: Is the sznajd model different from a voter model? *International Journal of Modern Physics C*, 14(10):1331–1354, 2003.

[DNAW00] Guillaume Deffuant, David Neau, Frederic Amblard, and Gérard Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3(01n04):87–98, 2000.

[HK02] Rainer Hegselmann and Ulrich Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3), 2002.

[Hop82] John J Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the national academy of sciences*, 79(8):2554–2558, 1982.

[JLM03] Ali Jadbabaie, Jie Lin, and A Stephen Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Automatic Control, IEEE Transactions on*, 48(6):988–1001, 2003.

[Kur84] Yoshiki Kuramoto. Cooperative dynamics of oscillator community a study based on lattice of rings. *Progress of Theoretical Physics Supplement*, 79:223–240, 1984.

[Lor07] Jan Lorenz. Continuous opinion dynamics under bounded confidence: A survey. *International Journal of Modern Physics C*, 18(12):1819–1838, 2007.

[Mor04] Luc Moreau. Stability of continuous-time distributed consensus algorithms. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 4, pages 3998–4003. IEEE, 2004.

[Mor05] Luc Moreau. Stability of multiagent systems with time-dependent communication links. *Automatic Control, IEEE Transactions on*, 50(2):169–182, 2005.

[NBL91] Helmut Norpohth, Michael S Lewis Beck, and Jean Dominique Lafay. *Economics and politics: The calculus of support*. University of Michigan Press, 1991.

[Nis05] Matthew C Nisbet. The competition for worldviews: Values, information, and public support for stem cell research. *International Journal of Public Opinion Research*, 17(1):90–112, 2005.

[OSFM07] Reza Olfati-Saber, J Alex Fax, and Richard M Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[OSM04] Reza Olfati-Saber and Richard M Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[Sar09] Alain Sarlette. Geometry and symmetries in coordination control. 2009.

[Sep11] Rodolphe Sepulchre. Consensus on nonlinear spaces. *Annual reviews in control*, 35(1):56–64, 2011.

[SL05] Dietram A Scheufele and Bruce V Lewenstein. The public and nanotechnology: How citizens make sense of emerging technologies. *Journal of Nanoparticle Research*, 7(6):659–667, 2005.

[Sob09] Pawel Sobkowicz. Modelling opinion formation with physics tools: Call for closer link with reality. *Journal of Artificial Societies and Social Simulation*, 12(1):11, 2009.

[SS09] Alain Sarlette and Rodolphe Sepulchre. Consensus optimization on manifolds. *SIAM Journal on Control and Optimization*, 48(1):56–76, 2009.

[Str00] Steven H. Strogatz. From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D: Nonlinear Phenomena*, 143(14):1 – 20, 2000.

[SWS00] Katarzyna Sznajd-Weron and Józef Sznajd. Opinion evolution in closed community. *International Journal of Modern Physics C*, 11(06):1157–1165, 2000.

[Tsi84] John Nikolas Tsitsiklis. Problems in decentralized decision making and computation. Technical report, PhD Thesis, MIT, 1984.

[VCBJ+95] Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet. Novel type of phase transition in a system of self-driven particles. *Physical review letters*, 75(6):1226, 1995.