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# A nonlinear model of opinion formation on the sphere

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## Abstract

In this paper we present a model for opinion dynamics on the  $d$ -dimensional sphere based on classical consensus algorithms. The choice of the model is motivated by the analysis of the comprehensive literature on the subject, both from the mathematical and the sociological point of views. The resulting dynamics is highly nonlinear and therefore presents a rich structure. Equilibria and asymptotic behaviour are then analysed and sufficient condition for consensus are established.

## 1 Introduction

Studying the complex laws governing the evolution of opinions in social networks is a challenge that has attracted an increasing attention from researchers in the last decades. The problem is to find the interaction rules between nodes of a network, or *agents*, generating the emergence of complex patterns observed in reality. A wide literature has been developed for the mathematical description of the dynamical evolution of *opinions*, represented e.g. by colors, in a network of interacting agents. This question is an aspect of the wider challenge of modeling and analysis of emergent behaviors in multi-agents systems. Several models have been proposed in the last years, among the most widely used we cite the Sznajd model (SM), Deffuant model (DM), Hegselmann–Krause Model (HKM), the classical Voter model (VM). We briefly highlight the main features of the above mentioned models to gather useful informations for the design of a new model. This list is far from exhaustive but a complete review of the problem of modeling opinion formations is beyond the scope of this work.

The Sznajd model (SM), see [SWS00], is based on the Ising model for ferromagnetism in statistical mechanics. In this model opinions are discrete variables  $x_i$  taking value  $\pm 1$ . The interactions are governed by two basic rules: the “ferromagnetic” interaction (that is, if  $x_i = x_{i+1}$  then at the next step adjacent agents will satisfy with a given probability  $x_{i-1} = x_i = x_{i+1} = x_{i+2}$ ) and the “antiferromagnetic” interaction (if  $x_i = -x_{i+1}$  then an antisymmetric pattern forms  $-x_{i-1} = x_i = -x_{i+1} = x_{i+2}$ ). The model has been extended to higher dimensional opinion and complex network topologies. The motivation for this model comes from the postulate that “agreement generates agreement”, that is, if two agents reach a consensus then all agents directly connected to them are induced to agree. In other words, in Sznajd model, the opinion flows out from a group of agreeing agents.

In [HK02] Hegselmann and Krause presented a model (HKM) of herding of opinions in a  $N$ -agents system in which the position  $x_i$  of the agent  $i$ , representing its opinion and taking values in an interval of  $\mathbb{R}$ , changes according to the distance from other agents  $x_j, j \neq i$  rescaled by an interaction coefficient

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$a_{ij}$  accounting for the weight given to the opinion of agent  $j$  by agent  $i$ . With these notations the opinion formation of agent  $i$  evolves according to

$$x_i(t+1) = a_{i1}x_1(t) + a_{i2}x_2(t) + \dots + a_{iN}x_N(t).$$

The model has been widely studied also in its continuous-time analogue where the evolution of the opinion of agent  $i$  is governed by

$$\dot{x}_i = \sum_{|x_i - x_j| < 1} a_{ij}(x_j - x_i).$$

Such a model with an interaction that is zero when mutual distances are above a certain threshold is called a model with “bounded confidence”. The rationale for the bounded confidence is that it is unlikely that one agent would be influenced by another agent whose opinion is too far from its own. This kind of interaction gives rise to clusters of opinions (see for instance [BHT10]).

The idea of opinions varying in a continuous fashion has been used also in the Deffuant Model (DM), see [DNAW00]. The interacting agents are chosen randomly and if the difference between their opinions, say  $x_1$  and  $x_2$ , is smaller in magnitude than a certain threshold (bounded confidence) then the agents re-adjust their opinion according to the law

$$x_1(t+1) = x_1(t) + \mu(x_2(t) - x_1(t)) \quad \text{and} \quad x_2(t+1) = x_2(t) + \mu(x_1(t) - x_2(t)),$$

for a given interaction parameter  $\mu \geq 0$ .

The Voter Model is a widely studied classical model of opinion formation with discrete opinions and stochastic interactions, that can be seen, in some cases, as a generalization of the Sznajd model ([BS03]).

All these models present very interesting features but also lack of connections with real life examples. This issue has been thoroughly addressed in [Sob09]. It is very difficult to describe the complex interactions between individuals with simple mathematical rules. Nevertheless the models above show features that are good starting points to develop more complete models. Below, we present an analysis of strengths and weaknesses of the mentioned models divided in three topics.

**Discrete vs Continuous opinions:** One of the main difficulties in modeling opinion formations is the lack of good measurements of the opinions. A classical problem in sociology is to design interviews not affecting opinions, i.e. questions not influencing answers. Purely open questions do not exist and, moreover, it is very hard to collect data from open answers. On the other hand, closed questions induce quantization on the answers: opinions collapse on discrete sets representing the possible answers to a closed question. It is therefore natural to set the initial and final opinion, in an opinion formation process, on a discrete set (as in SM and VM). Nevertheless opinions do not jump instantly but follow a continuous, possibly very slow-varying, evolution. The time evolution is therefore better modeled by continuous dynamics.

**Stochastic vs Deterministic evolutions:** There are several models based on stochastic interactions between agents (SM, DM) and many convincing numerical simulations have been provided to validate these models. Nevertheless these models neglect the fact that the evolution of the opinion of one agent is the deterministic result of complex interactions with other agents. While the choice of the interacting agent could be random, the opinion formation should be influenced by an averaging of the opinions of the interacting agents in the spirit, for instance, of the HKM.

**Interactions:** One of the main challenges of the present days is represented by the study the interaction network, or social network. The problem is twofold: on the one hand the problem

is to study the interaction between two agents, on the other hand the problem is to study the topology of the network and, possibly, its evolution.

Interactions between agents should be weighted in different ways for different agents, the interacting factor between agents 1 and 2 can be different from the one between agents 2 and 3. Several models consider a weight  $a_{ij}$  in the interaction between two agents  $i$  and  $j$  (for instance HKM) and these coefficients can be exploited to model the different natures of relations. For instance, a relation could be attractive ( $a_{ij} > 0$ ) or repulsive ( $a_{ij} < 0$ ). From an economical point of view, we can see the attraction as cooperation and the repulsion as competition. In this framework, it could be interesting to introduce in the model concepts as “far cooperation vs local competition” or vice-versa, by including a dependence on the “position”  $x$  of the coefficients  $a_{ij}$ . Similarly, HKM and DM include the idea of “bounded confidence” (see [Lor07] for a survey).

Asymmetries on the interaction matrix  $(a_{ij})_{i,j=1}^N$  could be used to model the hierarchy in the network, for example, if  $a_{ij} \gg a_{ji}$  then  $i$  is more likely to influence  $j$  than the converse, meaning that agent  $i$  is an opinion leader, at least from the point of view of agent  $j$ .

As mentioned, in many models (e.g. SM and DM) pairs of interacting agents are chosen randomly. Although this approach is realistic in large networks, the deterministic nature of interactions should not be neglected. Moreover, there also exist social interactions that are strong, or constant in time. This could be modeled by introducing random switching topologies also including unbreakable subnetworks.

To conclude, modeling such complex phenomena is a challenging task. Thanks to a wide literature on the problem it is possible to find interesting features with strong sociological motivations. Here we propose a preliminary model including as many of these features as possible.

The paper is organized as follows. In Section 2 we introduce a new model of opinion formation taking into account above analysis. Section 3 is devoted to a qualitative study of the model: we investigate the equilibria of the associated dynamical system and its asymptotic behaviour. In Section 4 a distributed control, modeling the effect of mass media on public opinion, is introduced. Section 5 contains several numerical experiments simulating the asymptotic behaviour of the system and of related quantities.

## 2 Building a model

We consider a system of  $N$  interacting agents. The opinion of the  $i$ -th agent is represented by the vector  $x_i$  of the sphere  $\mathbb{S}^d$ .

The choice of a  $d$ -dimensional vector opinion, instead of a scalar opinion, is motivated by a seek of higher fidelity in the model. Indeed opinions on different topics are usually interconnected: economic policy attitudes and candidate choice in political elections; opinion formation and economical condition [NBL91]; opinion on research funding and religious and ideological beliefs – see for instance [Nis05] for a study on the relations between worldview and opinions on stem cell research and [SL05] for a study on opinions about nanotechnology research.

The rationale for using the sphere  $\mathbb{S}^d$  instead of the Euclidean space is that, as mentioned, opinions are subjected to a quantization phenomenon when measured. We can imagine that at the moment of measurement (elections, polls, interviews, etc.) opinions take only two values (yes/no, left/right, Democratic/Republican, liberal/conservative, for/against, etc.) so every component of the vector  $x_i = (x_i^{(1)}, \dots, x_i^{(d+1)})$  takes a positive or a negative value. In particular  $x_i$  belongs to  $\mathbb{S}^d$ . The manifold  $\mathbb{S}^d$  is a mathematical abstraction to describe the dynamical evolution of the opinions on a continuous (i.e. non-discrete) set.

The influence of the opinion  $x_i$  on the one of agent  $x_j$  is weighted with an interaction factor  $a_{ij}$ . The interaction could be attractive  $a_{ij} > 0$ , repulsive  $a_{ij} < 0$ , or neutral  $a_{ij} = 0$ . Every agent tends to agree with another one if there is an attractive interaction and to disagree in presence of repulsive interaction. In the spirit of HKM, the result of all influences is the averaging of all the distances on  $\mathbb{S}^d$  weighted by  $a_{ij}$ . The time evolution of the opinion  $x_i$  of agent  $i$  is governed by

$$\dot{x}_i = \sum_{j=1}^N a_{ij}(x_j - \langle x_i, x_j \rangle x_i), \quad x_i \in \mathbb{S}^d, i = 1, \dots, N. \quad (1)$$

The RHS is the projection of the vector  $x_j$  on the orthogonal to  $x_i$ . We can write

$$x_j - \langle x_i, x_j \rangle x_i.$$

In this paper we will consider constant coefficient  $a_{ij}$  with no assumptions on the symmetries of the network: we will study symmetric interactions  $a_{ij} = a_{ji}$ , antisymmetric interactions  $a_{ij} = -a_{ji}$  as well as general coefficients.

As mentioned in the introduction, a realistic model should include many different kind of interactions. It is possible to increase the complexity of the model, in order to represent opinion formation processes closer to the reality, by adapting the coefficients  $a_{ij}$  to this purpose. For example the switching topology can be represented by introducing a (stochastic or deterministic) dependence of  $a_{ij}$  on the time  $t$ . By introducing a dependence on the state  $x$ , one may model bounded confidence (e.g.  $a_{ij}(x) = 0$  if  $|x_i - x_j| \geq 1$ ), local cooperation and far competition (e.g.  $a_{ij}(x) = a(|x_i - x_j|)$  with  $a(0) > 0$  and  $a$  decreasing), or local competition and far cooperation (e.g.  $a_{ij}(x) = a(|x_i - x_j|)$  with  $a(0) < 0$  and  $a$  increasing), as well as unbreakable subnetworks (e.g. families), the arising of opinion leaders, clusters of opinions, and so on.

Models of consensus on manifolds has been thoroughly studied in the last years. We cite the Kuramoto model [Kur84] on the sphere  $\mathbb{S}^1$  who attracted a wide interest of researchers over the last 30 years, motivated by its connection with the problem of synchronizing a large population of harmonic oscillators - see [Str00] for a survey. Other possible applications can be found in [VCBJ<sup>+</sup>95] and [Hop82]. Lately, adapted versions of the Kuramoto model were extensively studied in a opinion formation perspective. In [SS09] Sarlette and Sepulchre studied opinion dynamics on a wider class of manifolds including, among others, the special orthogonal group  $SO(n)$ , the Grassmann manifold, and  $\mathbb{S}^1$ , see also [Sar09]. We refer to [Sep11] for a survey on this topic. These results rely on consensus algorithms for linear systems who have been developed by several authors, including Tsitsiklis [Tsi84], Jadbabaie, Lin, and Morse [JLM03], Moreau [Mor04, Mor05], Blondel, Hendrickx, Olshevsky and Tsitsiklis [BHT10], Olfati-Saber and Murray [OSFM07].

The novelty of our model relies in the intrinsic nonlinear nature of its dynamics which constitutes an obstruction to the direct application of these powerful tools and, on the other hand, yields a rich structure.

### 3 Model Analysis

System (1) can be rewritten in matrix form as

$$\dot{x} = Lx + D(x)x \quad (2)$$

where

$$L = (l_{ij}) := \begin{cases} a_{ij} & \text{if } i \neq j \\ -\sum_{k=1}^n a_{ik} & \text{otherwise} \end{cases},$$

and  $D(x) = (d_{ij}(x))$  is a diagonal matrix with  $d_{ii}(x) = \sum_{j=1}^N a_{ij}(1 - \langle x_j, x_i \rangle)$ .

**Remark 1.** *The time-continuous version of HKM reads  $\dot{x} = Lx$ . Then the dynamics of (1) can be seen as the classical HKM plus a non-linear diagonal term  $D(x)x$ , representing the projection of the velocities  $Lx$  on the tangent space of  $\mathbb{S}^d$ . The matrix  $L$  is called Laplacian of the matrix of  $A = (a_{ij})_{i,j=1}^N$ . The spectral properties of the Laplacian matrix are among the main topics in algebraic graph theory, especially in the case that  $A$  is a non-negative symmetric matrix. The applications of such properties in the framework of opinion formation models and consensus algorithms were investigated, among many others, in [OSM04].*

We can also see the dynamics from the point of view of agent  $i$ . Consider the influence on agent  $i$  of all other agents, that is

$$\alpha_i = \sum_{j \neq i} a_{ij} x_j,$$

then, with this notation, system (1) reads

$$\dot{x}_i = \alpha_i - \langle \alpha_i, x_i \rangle x_i, \quad i = 1, \dots, N. \quad (3)$$

### 3.1 Equilibria

System (1) presents three kind of equilibria: consensus, antipodal, and polygonal. Consensus corresponds to the case in which all opinion are equal. Antipodal equilibria are the ones in which for every  $j = 2, \dots, N$ , either  $x_j = x_1$  or  $x_j = -x_1$ . Every other equilibrium is said *polygonal* and it depends in principle on the interacting matrix  $A$ , unlike consensus and antipodal equilibria.

One of the main differences between the present model and consensus model on Euclidean spaces (as HKM) is represented by the rich nature of equilibria. Indeed while in HKM the only equilibrium is the consensus, here the model allows also the disagreement as possible (stable) equilibrium. Consider two agents ( $N = 2$ ) with one opinion ( $d = 1$ ) and with repulsive interaction  $a_{12} < 0, a_{21} < 0$ , in this case the antipodal position of the two opinions is a stable equilibrium.

Another interesting feature of our model is that given any configuration of opinions it is possible to find an interaction matrix for which this configuration is an equilibrium. Indeed consider any configuration of opinions  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N)$  then  $\bar{x}$  is an equilibrium if and only if

$$\frac{d}{dt} \bar{x}_i = \sum_{j=1}^N a_{ij} \bar{x}_j - \langle \bar{x}_i, \bar{x}_j \rangle \bar{x}_i = 0, \quad \text{for all } i = 1, \dots, N,$$

which is a system of  $Nd$  equations in the  $N^2 - N$  unknowns  $a_{ij}$  (being the diagonal  $a_{ii}$  set to 0 for every  $i = 1, \dots, N$ ). So if  $N > d + 1$  there exists a nontrivial choice of the interaction coefficients for which  $\bar{x}$  is an equilibrium.

By definition an equilibrium for systems (1) is a point  $x \in (\mathbb{S}^d)^N$  such that

$$\alpha_i - \langle \alpha_i, x_i \rangle x_i = 0 \quad \text{for every } i = 1, \dots, N.$$

In other words, an equilibrium is a vector  $x \in (\mathbb{S}^d)^N$  such that the influence  $\alpha_i$  is collinear with  $x_i$  for every  $i = 1, \dots, N$ . We identify three cases:

- (i)  $\alpha_i = c_i x_i$ , for  $c_i > 0$ ,
- (ii)  $\alpha_i = -c_i x_i$ , for  $c_i > 0$ ,
- (iii)  $\alpha_i = 0$ .

In case (i) the equilibrium is *stable with respect to  $x_i$* . Indeed consider the system for  $x_i$  and let us linearize the dynamics of  $x_i$  perturbing with respect to  $x_i$  only, namely consider the linearization of system

$$\dot{x} = \alpha - \langle \alpha, x \rangle x, \quad x \in \mathbb{S}^d$$

in the equilibrium  $\alpha = cx$  for  $c > 0$ , which calculated in  $v \in \mathbb{S}^d$  reads

$$-\langle \alpha, x \rangle v - \langle \alpha, v \rangle x = -c(v + \langle x, v \rangle x).$$

Now for every  $x \in \mathbb{S}^d$  the linear operator on  $\mathbb{S}^d$

$$v \mapsto v + \langle v, x \rangle x$$

has positive eigenvalues. Indeed the equation

$$v + \langle v, x \rangle x = \lambda v$$

is satisfied either by  $\lambda = 1$  and  $v \perp x$  either by  $v = \pm x$  and  $\lambda = 2$ .

Similarly it is possible to prove that in case (ii) the equilibrium is *unstable with respect to  $x_i$* .

The equilibrium is *stable* (tout court), in the classical sense, if  $\alpha_i = c_i x_i$  with  $c_i > 0$  for every  $i = 1, \dots, N$ , that is, if it is stable with respect every  $x_i$ ,  $i = 1, \dots, N$ .

### 3.2 Dancing equilibria

Beyond the complex nature of equilibria, model (1) shows another interesting kind of configuration which is the *dancing equilibrium*.

Consider the scalar products  $\langle x_i, x_j \rangle$ ,  $i, j = 1, \dots, N$  between the components of  $x \in (\mathbb{S}^d)^N$ . These quantities are, in some sense, a measure of the mutual distance between the  $x_i$ 's. Indeed  $\|x_i - x_j\|^2 = 1 - \langle x_i, x_j \rangle$ . Moreover the evolution of the mutual distances is governed by the system of ODEs

$$\begin{aligned} \frac{d}{dt} \langle x_i, x_j \rangle &= \sum_{k \neq i} a_{ik} (\langle x_k, x_j \rangle - \langle x_k, x_i \rangle \langle x_i, x_j \rangle) \\ &\quad + \sum_{k \neq j} a_{jk} (\langle x_k, x_i \rangle - \langle x_k, x_j \rangle \langle x_i, x_j \rangle), \end{aligned} \tag{4}$$

for  $i, j = 1, \dots, N$ . An equilibrium of system (4) is a configuration in which all mutual distances between agents are constant.

**Definition 1.** *A steady configuration of system (4) is called dancing equilibrium for system (1).*

The name “dancing equilibrium” is motivated by the fact that opinions are crystalized since the mutual distances are in equilibrium while the whole configuration may evolve.

For the system in matrix form (2),  $D(x)$  is constant in presence of a dancing equilibrium. Consequently the evolution is linear.

We define the *kinetic energy* of the system as the quantity

$$E(t) := \frac{1}{2} \sum_{i=1}^n \|\dot{x}_i(t)\|^2. \tag{5}$$

A system in dancing equilibrium has constant kinetic energy.

**Lemma 1.** *If for every  $i, j \in \{1, \dots, N\}$ ,*

$$\frac{d}{dt} \langle x_i(t), x_j(t) \rangle|_{t=0} = 0,$$

*then  $E(t)$  is constant for every  $t \geq 0$ .*

*Proof.* First of all notice that by assumption and by (4) we have  $\frac{d}{dt} \langle x_i(t), x_j(t) \rangle = 0$  for every  $t \geq 0$  and for every  $i, j \in \{1, \dots, N\}$ . In particular  $\frac{d}{dt} \langle x_j(t), \alpha_i(t) \rangle = 0$  and  $\frac{d}{dt} \langle \alpha_j(t), \alpha_i(t) \rangle = 0$  for every  $i, j \in \{1, \dots, N\}$ .

Then, we derive the kinetic energy with respect to time to get

$$\frac{d}{dt} E(t) = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^n \|\dot{x}_i(t)\|^2 = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \langle \alpha_i, \dot{x}_i \rangle = \frac{1}{2} \sum_{i=1}^N \frac{d}{dt} (\|\alpha_i\|^2 - \langle \alpha_i, x_i \rangle^2) = 0,$$

for every  $t \geq 0$ . □

Dancing equilibrium is a configuration not arising in other models and it can be seen as the mathematical formalization of the cyclicity of opinions.

**Example 1.** *Consider the 2-agent system with antisymmetric interaction matrix*

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

$a \neq 0$ , so that

$$\dot{x}_1 = a(x_2 - \langle x_1, x_2 \rangle x_1)$$

and

$$\dot{x}_2 = -a(x_1 - \langle x_1, x_2 \rangle x_2).$$

A direct computation gives

$$\frac{d}{dt} \langle x_1, x_2 \rangle = \langle \dot{x}_1, x_2 \rangle + \langle x_1, \dot{x}_2 \rangle = a(1 - \langle x_1, x_2 \rangle^2) - a(1 - \langle x_1, x_2 \rangle^2) = 0.$$

Note that in Euclidean models, as HKM, an antisymmetric interaction generates the divergence to infinity of two opinions.

**Example 2.** *Dancing equilibria emerge also in the case in which all interactions have attractive nature (i.e.  $a_{ij} > 0$ ), provided that there are at least 3 agents. Indeed, consider*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

*with three agents along the vertices of an equilateral triangle and assume  $\langle x_i(0), v \rangle = 0$  for some  $v \in \mathbb{R}^{d+1}$  so that*

$$\langle x_1, x_2 \rangle = \langle x_1, x_3 \rangle = \langle x_2, x_3 \rangle = -\frac{1}{2}$$

*We have by direct computation*

$$\frac{d}{dt} \langle x_i, x_j \rangle = 0,$$

*for every  $i \neq j, i, j = 1, \dots, N$ , hence this configuration is a dancing equilibrium.*



### 3.3 Asymptotic behaviour

**Theorem 1.** *If the interaction matrix  $A$  is positive, namely,  $a_{ij} > 0$  for every  $i, j = 1, \dots, N$  and if there exists  $w \in \mathbb{S}^d$  such that the initial condition satisfies*

$$\langle x_i, w \rangle > 0, \quad \text{for every } i = 1, \dots, N.$$

*then the system tends to consensus.*

*Proof.* For every  $t \geq 0$  and  $y \in \mathbb{S}^d$  let

$$\rho(y, t) = \max_{i=1, \dots, N} 1 - \langle x_i, y \rangle$$

and

$$r(t) = \min \left\{ \rho(y, t) \mid y \in \mathbb{S}^d \right\}.$$

Notice that, by assumption,  $r(0) < 1$ . Now let  $v(t)$  be such that  $1 - \langle x_i(t), v(t) \rangle \leq r(t)$  for every  $i = 1, \dots, N$ .

Let  $v_0 = v(0)$ . Let  $\Lambda_0$  be a set of indexes such that  $\langle x_i(0), v_0 \rangle$  is minimal for every  $i \in \Lambda_0$ , that is  $\langle x_i(0), v_0 \rangle < \langle x_j(0), v_0 \rangle$  for every  $j \notin \Lambda_0$ . Let  $i_0$  be the smallest index among the latest to loose minimality, namely, let  $s_i$  for  $i \in \Lambda_0$  be

$$s_i^0 = \inf \{ t \geq 0 \mid \exists j \neq i : \langle x_i(t), v_0 \rangle > \langle x_j(t), v_0 \rangle \}$$

let  $t_1 = \max_{i \in \Lambda_0} s_i^0$  and  $i_0 = \min \{ i \in \Lambda_0 : s_i^0 = t_1 \}$ . Note that  $t_1$  can be infinity. If this is not the case consider  $v_1 = v(t_1)$  and construct

$$\Lambda_1 = \{ i \in \{1, \dots, N\} \mid \langle x_i(t_1), v_1 \rangle \leq \langle x_j(t_1), v_1 \rangle, \forall j \in \{1, \dots, N\} \},$$

again we set for every  $i \in \Lambda_1$ ,

$$s_i^1 = \inf \{ t \geq t_1 \mid \exists j \neq i : \langle x_i(t), v_1 \rangle > \langle x_j(t), v_1 \rangle \},$$

$t_2 = \max_{i \in \Lambda_1} s_i^1$ , and  $i_1 = \min \{ i \in \Lambda_1 : s_i^1 = t_2 \}$ .

Iterating we find an increasing sequence of times  $0 = t_0 < t_1 < t_2 < \dots < t_k$  and a sequence of indexes  $i_0, i_1, \dots, i_k \in \{1, \dots, N\}$ . Let  $T \in (0, +\infty]$  be the limit of  $t_k$  as  $k \rightarrow \infty$ . For every  $t \in [0, T)$  there exists  $k \geq 0$  such that  $t_k \leq t < t_{k+1}$  and  $i_k$  such that  $\langle x_{i_k}(t), v_k \rangle \geq \langle x_j(t), v_k \rangle$  for every  $j$ , where  $v_k = v(t_k)$ . Therefore for every  $t \in [t_k, t_{k+1})$ ,

$$\frac{d}{dt} (1 - \langle x_{i_k}(t), v_k \rangle) = -\langle \dot{x}_{i_k}(t), v_k \rangle < 0$$

indeed, by contradiction, if  $\langle \dot{x}_{i_k}(t), v_k \rangle \leq 0$  then

$$\sum_{j \neq i_k} a_{i_k j} (\langle x_j, v_k \rangle - \langle x_j, x_{i_k} \rangle \langle x_{i_k}, v_k \rangle) \leq 0$$

which implies the existence of  $j \neq i_k$  such that  $\langle x_j, v_k \rangle \leq \langle x_j, x_{i_k} \rangle \langle x_{i_k}, v_k \rangle < \langle x_{i_k}, v_k \rangle$ , in contradiction with the minimality of  $\langle x_{i_k}, v_k \rangle$ .

We have that  $r(t)$  is decreasing for every  $t \in [0, T)$  and it is bounded from below by 0, hence, by compactness there exists a sequence  $(t_n)_{n \in \mathbb{N}}$ ,  $t_n \rightarrow T$  such that there exists  $\lim_{n \rightarrow \infty} r(t_n) =: \bar{r} \geq 0$ ,  $\lim_{n \rightarrow \infty} v(t_n) = \bar{v}$ , and  $\lim_{n \rightarrow \infty} x_i(t_n) = \bar{x}_i$  for  $i = 1, \dots, N$ . Let us prove that  $\bar{r} = 0$ . Notice that

$\bar{r} \leq r(0) < 1$ . By definition of  $r$ ,  $\langle \bar{x}_j, \bar{v} \rangle \geq 1 - \bar{r}$  for every  $j = 1, \dots, N$  and there exists  $i \in \{1, \dots, N\}$  such that  $\langle \bar{x}_i, \bar{v} \rangle = 1 - \bar{r}$ . Then

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{d}{dt} \langle x_i(t_n), \bar{v} \rangle = \sum_{j \neq i} a_{ij} (\langle \bar{v}, \bar{x}_j \rangle - \langle \bar{x}_i, \bar{x}_j \rangle \langle \bar{x}_i, \bar{v} \rangle) \\ &\geq (1 - \bar{r}) \sum_{j \neq i} a_{ij} (1 - \langle \bar{x}_i, \bar{x}_j \rangle), \end{aligned}$$

which implies

$$\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_N. \quad (6)$$

Therefore  $\bar{r} = 0$ . Note that (6) also implies that an equilibrium is reached in time  $T$  and, consequently,  $T = \infty$  by uniqueness of solutions. Thus the arguments above are valid for every  $t \geq 0$ .

Finally the system tends to consensus since for every  $i, j = 1, \dots, N$ ,

$$\begin{aligned} 1 - \lim_{t \rightarrow \infty} \langle x_i(t), x_j(t) \rangle &= \lim_{t \rightarrow \infty} \frac{1}{2} \|x_i(t) - x_j(t)\|^2 \\ &\leq \lim_{t \rightarrow \infty} \frac{1}{2} \|x_i(t) - v(t)\|^2 + \lim_{t \rightarrow \infty} \frac{1}{2} \|v(t) - x_j(t)\|^2 \\ &\leq \lim_{t \rightarrow \infty} 1 - \langle x_i(t), v(t) \rangle + \lim_{t \rightarrow \infty} 1 - \langle v(t), x_j(t) \rangle \\ &\leq \lim_{t \rightarrow \infty} 2r(t) = 0, \end{aligned}$$

this concludes the proof.  $\square$

For a numerical example of the application of Theorem 1 see Figure 1. Example 2 shows the necessity of the condition  $\langle x_i, w \rangle > 0$  for every  $i = 1, \dots, N$  for some  $w \in \mathbb{S}^d$ , indeed we present a configuration of dancing equilibrium not verifying this assumption.

Next result establishes sufficient conditions to have  $E(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Theorem 2.** *If the interaction matrix  $A$  is symmetric then*

$$\lim_{t \rightarrow \infty} E(t) = 0.$$

*Proof.* Define  $F(t) := \sum_{i=1}^N \langle x_i, \alpha_i \rangle$ . Using the symmetry of  $A$ , we have

$$\frac{d}{dt} F(t) = 4E(t). \quad (7)$$

Indeed

$$\begin{aligned} \frac{d}{dt} F(t) &= \sum_{i=1}^N \langle \dot{x}_i, \alpha_i \rangle + \langle x_i, \dot{\alpha}_i \rangle \\ &= \sum_{i=1}^N \left( \langle \dot{x}_i, \alpha_i \rangle + \sum_{j=1}^N a_{ij} \langle x_i, \dot{x}_j \rangle \right) \\ &= \sum_{i=1}^N \langle \dot{x}_i, \alpha_i \rangle + \sum_{j=1}^N \left\langle \sum_{i=1}^N a_{ji} x_i, \dot{x}_j \right\rangle \\ &= 2 \sum_{i=1}^N \langle \dot{x}_i, \alpha_i \rangle \\ &= 4E(t). \end{aligned}$$

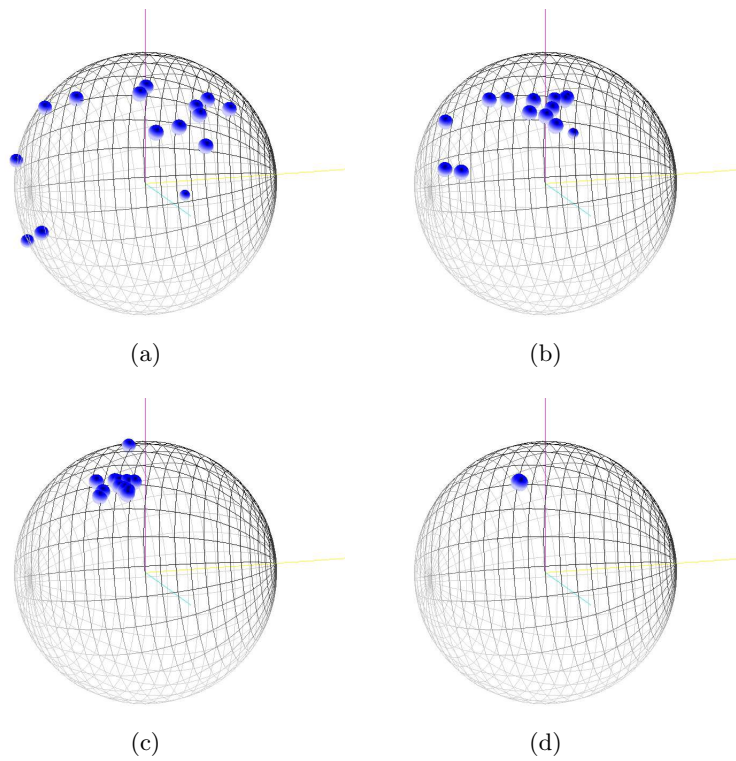


Figure 1: 15 agents tending to consensus in dynamics with positive adjacency matrix, as in Theorem 1.

As  $E(t) \geq 0$  for every  $t \geq 0$ ,  $F(t)$  is a non-decreasing function. Moreover  $F(t)$  is bounded, indeed for every  $t \geq 0$

$$|F(t)| = \left| \sum_{i=1}^N \langle x_i, \alpha_i \rangle \right| = \left| \sum_{i=1}^N \sum_{j=1}^N a_{ij} \langle x_i, x_j \rangle \right| \leq \sum_{i,j=1}^N |a_{ij}|.$$

Hence  $F(t) \uparrow \ell$  as  $t \rightarrow \infty$  for some  $\ell \leq \sum_{i,j=1}^N |a_{ij}|$  and, consequently,  $\frac{d}{dt}F(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The claim hence follows by (7).  $\square$

## 4 Distributed Control

Once sufficient conditions for consensus are established, as in Theorem 1, it is natural to study whether is possible to *create* or induce consensus with an external intervention. There are several ways to model the external action of an agent. Here we address the problem of describing the action of a mass media, or communication enterprises, on the system. Media are opinion leaders whose opinion is not influenced by the others, at least not for small periods of times. We can represent media by an agent whose opinion  $x_M$  is constant. Therefore the dynamics of agent  $x_i \in \mathbb{S}^d$  for  $i = 1, \dots, N$  is

$$\dot{x}_i = \sum_{j=1}^N a_{ij} (x_j - \langle x_i, x_j \rangle x_i) + u(t) (x_M - \langle x_i, x_M \rangle x_i) \quad (8)$$

where the scalar function of time  $u(t)$  with value in  $[0, \delta]$  represents the strength of communication of media. The positive bound  $\delta$  represents the maximal strength of communication of the media. We can consider then the problem of steering the opinion of all other agents to the media's opinion  $x_M$ . Since consensus to  $x_M$  is an equilibrium for system (8) this problem can be seen as a asymptotic stabilization problem.

A very simple strategy to stabilize the system is given by setting  $u(t) = \delta$  for every  $t \geq 0$ , possibly perturbing  $u$  in order to break the symmetries and avoid polygonal equilibria. If the maximal strength  $\delta$  of the control is sufficiently large, say  $\delta \geq N|a_{ij}|$  for every  $i, j = 1, \dots, N$ , then the action of the control is stronger than every other possible influence on each agent and the system tends to the consensus  $x_M$ .

The estimate on the minimal  $\delta$  for which this simple control strategy lead to consensus can be refined. Empirical observations lead to the conjecture that  $\delta \sim \sqrt{N}$  in the symmetric case - see Figure 2.

The strategy itself is very rough. We believe that feedback control strategies, in which opinion  $x_M$  changes according to the opinion of other agents, may ensure consensus under milder assumptions on the size of  $\delta$ . This kind of analysis is however beyond the scope of this paper and we plan to address the controllability problem in future works.

## 5 Numerical simulations

In this section we show numerical simulations of the evolution of system (1). Interaction matrices and initial positions are chosen randomly, the associated trajectories are approximated by the Runge–Kutta scheme. We present the simulations arranged according to the different natures the interactions.

Note that in general kinetic energy as defined in (5) has a non-monotonic behavior (see Figure 3), in contrast with the linear case (HKM in  $\mathbb{R}^N$ ). However it is possible to observe, in simulations, a decreasing trend of the energy.

Another observable of the system accounting for the distance to consensus is the *centroid* (see also [SS09]).

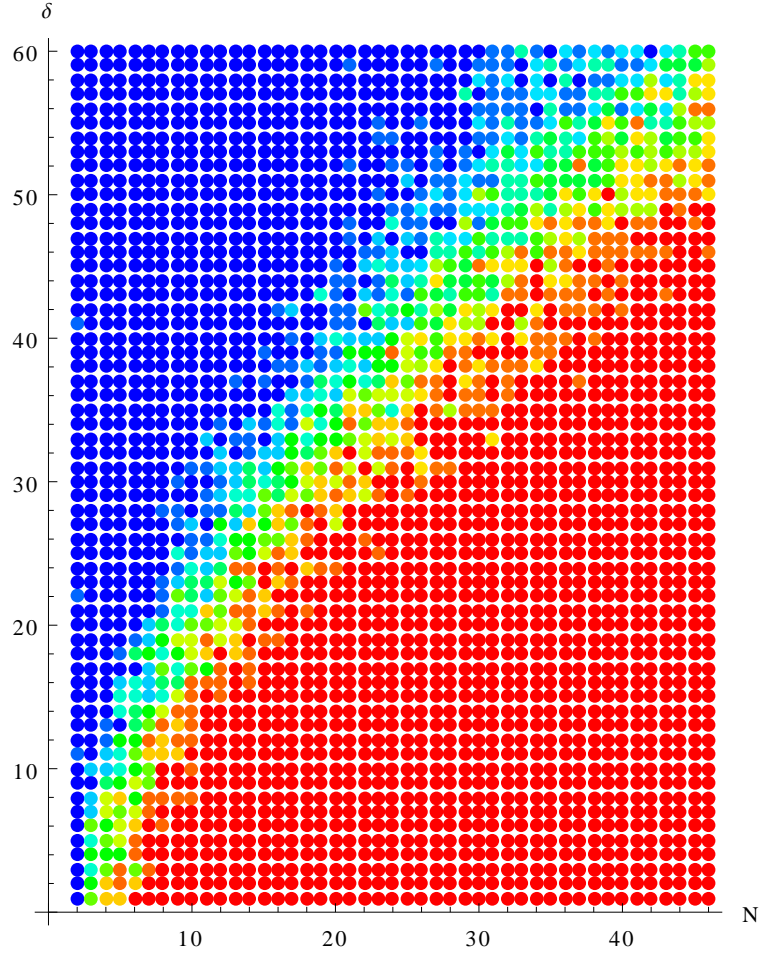


Figure 2: The result of the application of the constant control  $u(t) = \delta$  as a function of the number of agents  $N$  and of the magnitude  $\delta$ . Every dot represents the result of 10 trials with random initial condition and random symmetric interaction matrices (with uniformly bounded max norm). The color of the dots ranges from blue to red in function of the number of trials the system reached the consensus  $x_M$ . Blue represents consensus in all 10 trials and red the non-consensus in all trials.

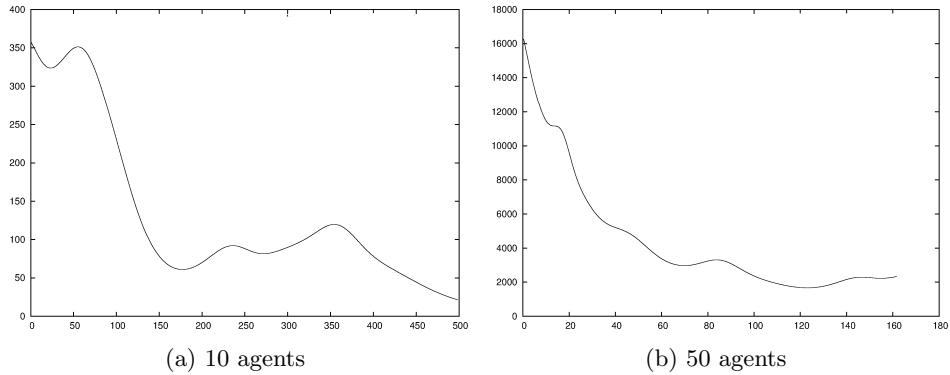


Figure 3: Energy evolution for randomly generated interaction matrices.

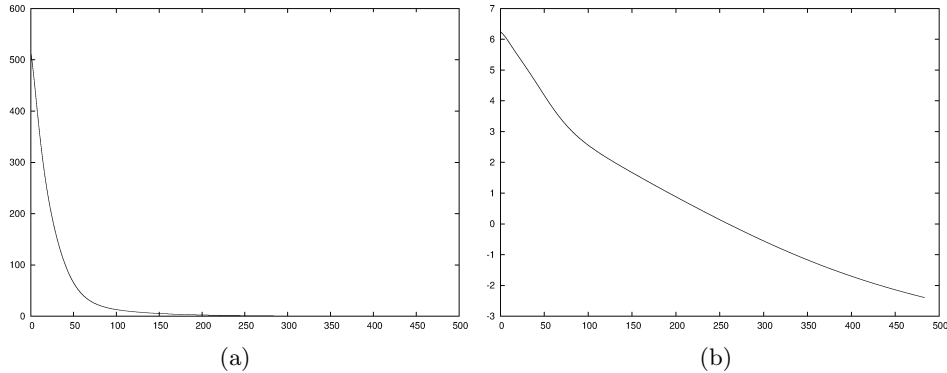


Figure 4: Energy decay for a randomly generated symmetric interaction matrices for 10 agents. In (b) data are reported in logarithmic scale

**Definition 2.** *The centroid is the unweighted barycenter of the system, that is the point*

$$C(t) := \sum_{i=1}^N x_i(t).$$

Note that in general the centroid does not belong to the state-space  $\mathbb{S}^d$ . The squared modulus of  $C(t)$  reads

$$\|C(t)\|^2 = \sum_{i,j=1}^N \langle x_i(t), x_j(t) \rangle,$$

The quantity  $N^2 - \|C(t)\|^2$  provides an estimate of the distance from consensus. Indeed in general  $\|C(t)\|^2 \in [0, N^2]$  and  $\|C(t)\|^2 = N^2$  if and only if consensus is reached.

## 5.1 Symmetric case

In Theorem 2 we proved that if  $A$  is symmetric then the kinetic energy tends to zero.

Figure 4 shows the energy decay for a randomly generated symmetric interaction matrices for 10 agents. A recurring pattern in our tests in the symmetric cases is a possible initial increase of the energy and an apparently exponential convergence to zero.

We can see this pattern explicitly in the simple case of 2 agents with positive interaction coefficient  $a$ . Indeed if the initial state is not an (antipodal) equilibrium, then conditions of Theorem 1 are satisfied and the system tends to consensus. In particular the function  $\theta(t) = \langle x_1(t), x_2(t) \rangle$  tends to 1. Moreover the convergence is monotone since

$$\frac{d}{dt}\theta(t) = 2\frac{E(t)}{a} > 0.$$

Thus, for every  $0 < c < 1$  there exists  $T_c > 0$  such that  $\theta(t) \geq c$  for every  $t \geq T_c$ . The time derivative of the energy reads

$$\frac{d}{dt}E(t) = -4\theta(t)E(t) \leq -4cE(t),$$

for every  $t > T_c$ . The exponential decay of  $E(t)$  hence follows by Gronwall's Lemma.

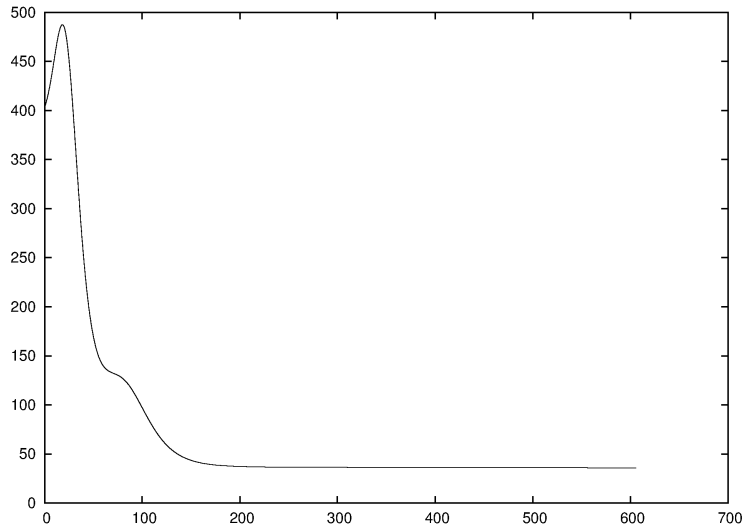


Figure 5: Energy decay for a randomly generated sign-symmetric interaction matrix for 10 agents.

## 5.2 Sign-symmetric case

In our tests we also considered the case of sign-symmetric matrices, modeling dynamics where every pair of agents is either mutually attracted or mutually repulsed. Obviously the symmetric case is a particular case of sign-symmetric case and this reflects on the behavior of the energy: it yet converges as time tends to infinity, but possibly to a non-zero constant, see Figure 5.

In Figure 6 and Figure 7 the energy tends to a positive constant and the system converges to periodic orbits preserving the scalar product between agents, i.e., a dancing equilibrium.

## 5.3 Positive case

In the positive case, namely when  $a_{ij} > 0$  for every  $i, j = 1, \dots, N$ , Theorem 1 ensures consensus provided that agents are in an open semi-sphere. In Example 2 we present a case of dancing equilibria emerging in this kind of systems. However experimental data suggest that the evolutions not converging to consensus are particular (and unlikely) cases. In other words it seems that consensus may be reached for generic initial data.

The kinetic energy of a system with positive interaction matrix is in general non-monotone. However it seems that, in the positive case, if the system tends to consensus then the modulus of the centroid is a monotonically increasing quantity, see Figure 8.

## 5.4 General case

We ran some tests in the general case, namely when no symmetry or sign constraints are applied to the adjacency matrix. In some cases, especially when the number of agents is small, some of the equilibria described in the present paper are reached, for instance the antipodal equilibrium and the dancing equilibrium - see Figure 9. Another interesting phenomenon that we found only in the general case is dumping oscillations in kinetic energy. In the example showed in Figure 10 this effect is particularly evident in the centroid evolution.

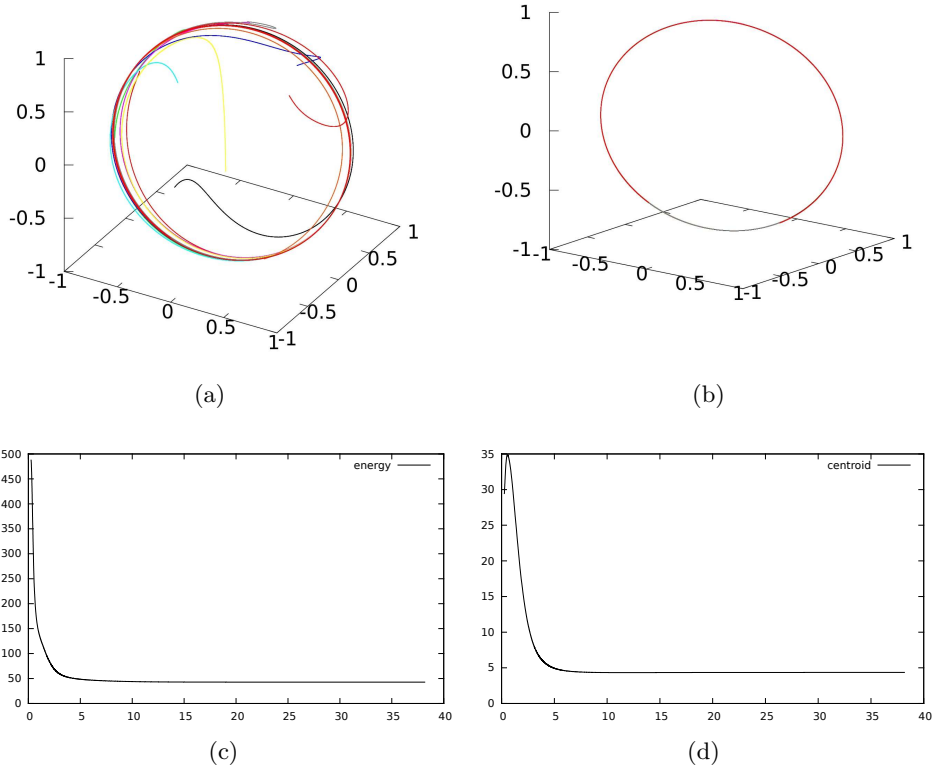


Figure 6: In (a) and (b) the trajectories of 10 agents with a randomly generated sign-symmetric matrices starting from  $t = 0$  and  $t = 20$ , respectively. The energy and the squared modulus of centroid are displayed in (c) and (d), respectively. The system tends to a dancing equilibrium on the equator of the sphere.



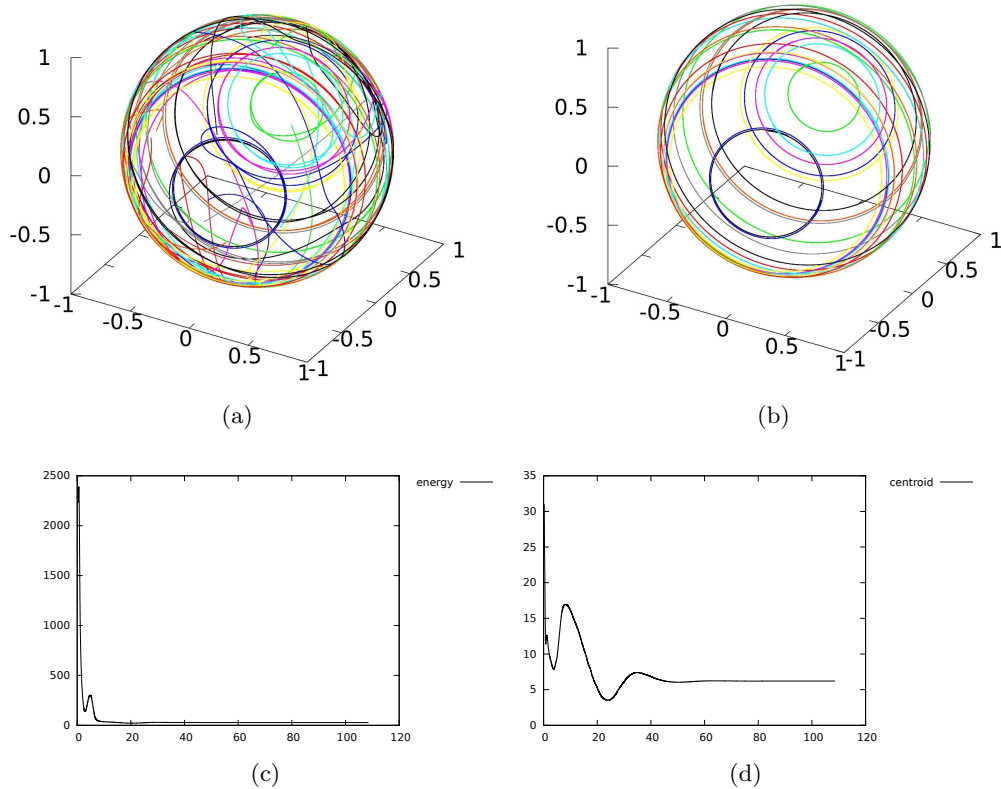


Figure 7: In (a) and (b) the trajectories of 25 agents with a randomly generated sign-symmetric matrices starting from  $t = 0$  and  $t = 50$ , respectively. The evolutions of energy and of the squared modulus of centroid are displayed in (c) and (d), respectively. The system tends to a dancing equilibrium.

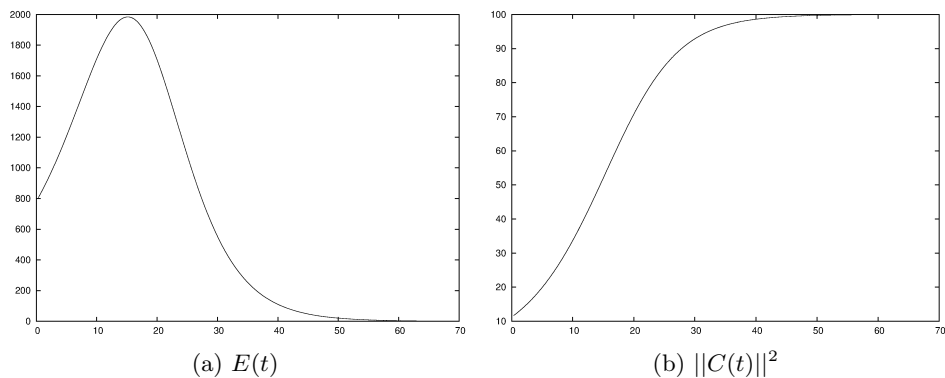


Figure 8: Energy decay and evolution of the squared modulus of centroid for a randomly generated positive interaction matrix for 10 agents.

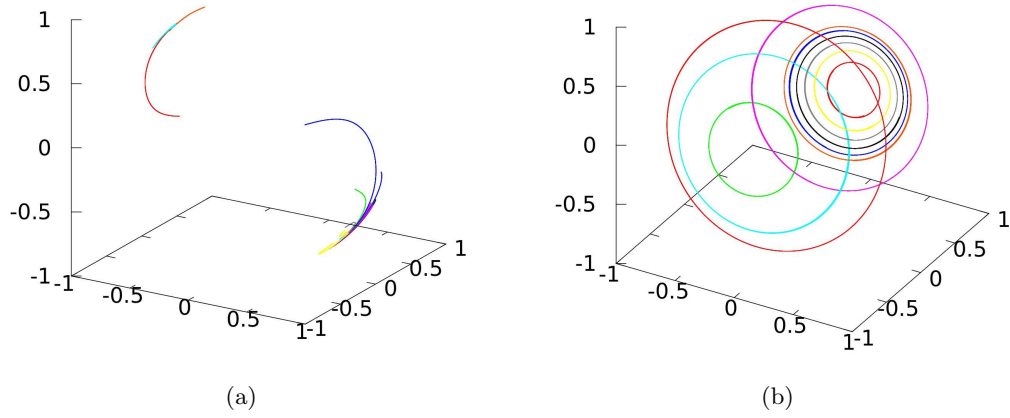


Figure 9: An antipodal (a) and a dancing (b) equilibrium reached by two systems with general adjacency matrices and 15 and 10 agents, respectively.

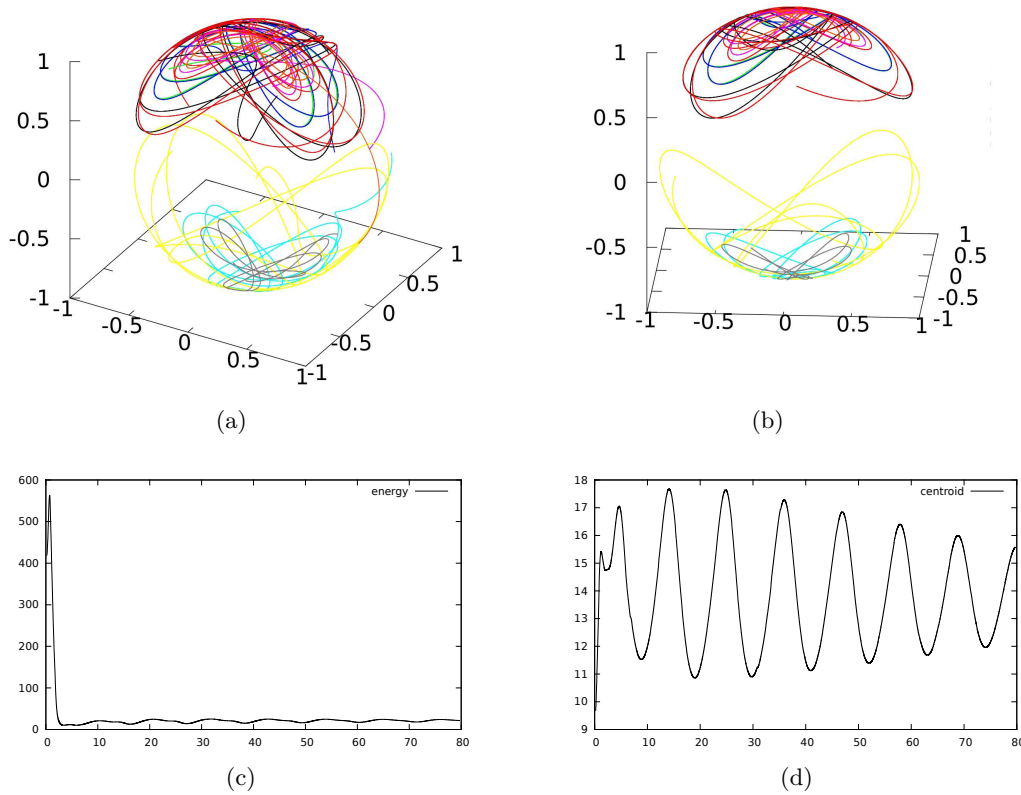


Figure 10: In (a) and (b) are represented the trajectories of 10 agents with a randomly general matrix starting from  $t = 0$  and  $t = 20$ , respectively. The related evolutions of energy and of squared modulus of centroid are displayed in (c) and (d), respectively.

## 6 Conclusions and perspectives

We presented a nonlinear model of opinion dynamics. The system is very versatile and several extensions are possible in order to describe more elaborate interactions. The nonlinearity yields a rich structure making this system suitable for modeling complex behaviors. New features arise in the analysis of the nonlinear dynamics leading to new mathematical challenges.

We highlighted the existence of three kind of equilibria. While consensus is well-known, other configurations like antipodal, polygonal, and dancing equilibria are peculiar of this model. We plan to characterize these configurations and to deepen the stability analysis.

We provided a sufficient condition, Theorem 1, for consensus in the positive interaction case and in the symmetric case, we showed that the kinetic energy of the system tends to zero, Theorem 2. Possible extensions of these results to more general cases can be achieved by adapting classical consensus algorithm of Moreau type to this nonlinear system.

We presented a first step in the study of the controllability of this system, i.e. the ability to force consensus using an external intervention, accounting for the influence, for instance, of mass media. We believe that feedback control strategies may be more effective to address the controllability problem.

Simulations provide several hints on the asymptotic behavior of the system: the energy, although non-monotonically, tends to stabilize. We believe that the non-monotonicity of the energy and yet its decreasing trend are the result of a trade-off between stabilizing symmetries of the systems and of non-symmetric components steering the system towards a non-steady dancing equilibrium.

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