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SIEGEL–VEECH CONSTANTS AND VOLUMES OF STRATA OF MODULI SPACES OF QUADRATIC DIFFERENTIALS

ELISE GOUJARD

ABSTRACT. We present an explicit formula relating volumes of strata of meromorphic quadratic differentials with at most simple poles on Riemann surfaces and counting functions of the number of flat cylinders filled by closed geodesics in associated flat metric with singularities. This generalizes the result of Athreya, Eskin and Zorich in genus 0 to higher genera.

1. INTRODUCTION

1.1. Cylinders and saddle connections on half-translation surfaces. A meromorphic quadratic differential q with at most simple poles on a Riemann surface S of genus g defines a flat metric on S with conical singularities. If q is *not* the global square of a holomorphic 1-form on S , the metric has a non-trivial linear holonomy group, and in this case (S, q) is called a *half-translation* surface. In this paper we consider only quadratic differentials satisfying the previous condition. If $\alpha = \{\alpha_1, \dots, \alpha_n\} \subset \{-1\} \cup \mathbb{N}$ is a partition of $4g - 4$, $\mathcal{Q}(\alpha)$ denotes the moduli space of pairs (S, q) as above, where q has exactly n singularities of orders given by α . It is a *stratum* in the moduli space \mathcal{Q}_g of pairs (S, q) with no additional constraints on q .

In the following we will refer to a half-translation surface (S, q) simply as S .

A *saddle connection* on S is a geodesic segment on S joining a pair of conical singularities or a singularity to itself without any singularities in its interior. Note that maximal flat cylinders filled by parallel regular closed geodesics have their boundaries composed by one or several parallel saddle connections. In this paper we will evaluate the number of such cylinders on S in terms of the volumes of some strata, using the study of saddle connections by Masur and Zorich in [MZ].

1.2. Rigid collections of saddle connections. A saddle connection persists under any small deformation of S inside the stratum $\mathcal{Q}(\alpha)$. Moreover Masur and Zorich noticed in [MZ] that in some cases any small deformation which shortens a specific saddle connection shortens also some other saddle connections. More precisely, they give the following result (Proposition 1 of [MZ]):

Proposition 1 (Masur-Zorich). *Let $\{\gamma_1, \dots, \gamma_m\}$ be a collection of saddle connections on a half-translation surface S . Then any sufficiently small deformation of S inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_m|$ of the lengths of the saddle connections if and only if the saddle connections are \hat{h} omologous.*

Roughly two saddle connections are \hat{h} omologous if they define the same anti-invariant cycle in the orientation double cover. The precise definition will be recalled in § 2.1. In particular two \hat{h} omologous saddle connections are parallel with ratios of lengths equal to 1 or 2.

The geometric types of possible maximal collections of $\hat{\text{homologous}}$ saddle connections $\gamma = \{\gamma_1, \dots, \gamma_m\}$ on S are called *configurations* of saddle connections. Masur and Zorich classified all configurations of saddle connections in [MZ] in terms of combinatorial data.

We assume in the sequel that S belongs to a connected stratum (unless the non connectedness is stated explicitly), and we will speak indifferently about configurations for the surface S or for the stratum $\mathcal{Q}(\alpha)$, the second means that we look at all possible configurations on almost every surface $S \in \mathcal{Q}(\alpha)$.

We are interested in collections of $\hat{\text{homologous}}$ saddle connections, such that some of the saddle connections bound at least one cylinder filled by parallel regular closed geodesics. We refer to the geometric type of these collections as “configurations containing cylinders” or “configurations with cylinders”.

It is proved in [MZ] that such cylinders have in fact each of their two boundaries composed by exactly one or two saddle connections in the collection, and that if there are several cylinders in the configuration, the lengths of their waist curves are either the same or have the ratio 1:2. Namely, if a cylinder has at least one boundary component composed by two saddle connections, its width is twice the width of the cylinders which have each of their boundary components composed by a single saddle connection. We will refer to cylinders of the first type as “thick cylinders” and to cylinders of the second type as “thin cylinders”.

Let γ be a maximal collection of $\hat{\text{homologous}}$ saddle connections on S . Then the complimentary region of these saddle connections and the cylinders bounded by these saddle connections is the union of some surfaces with boundaries. Each of them might be obtained by a specific surgery from a flat surface belonging to a stratum $\mathcal{Q}(\alpha_i)$ or $\mathcal{H}(\beta_j)$. The union of these strata $\mathcal{Q}(\alpha') = \cup_{i,j} \mathcal{Q}(\alpha_i) \cup \mathcal{H}(\beta_j)$ is called the *boundary stratum* for the configuration \mathcal{C} . This denomination is meaningful: the boundary stratum corresponds to the degeneration of the stratum $\mathcal{Q}(\alpha)$ as the lengths of the saddle connections in the collection tend to 0.

1.3. Counting saddle connections. Let S be a half-translation surface in a connected stratum $\mathcal{Q}(\alpha)$, and \mathcal{C} a configuration with cylinders on S . It means that in some given direction, there is a collection of $\hat{\text{homologous}}$ saddle connections of type \mathcal{C} on S . Note that by results of [EM] in many other directions, one can usually find another collection of $\hat{\text{homologous}}$ saddle connections of same type \mathcal{C} .

We introduce $N(S, \mathcal{C}, L)$ the number of directions on S in which we can find a collection of saddle connections of type \mathcal{C} , with the length of the smallest saddle connection smaller than L . Since we are interested in cylinders we introduce also $N_{cyl}(S, \mathcal{C}, L)$ that counts each appearance of the configuration \mathcal{C} with weight equal to the number of the cylinders of width smaller than L , and $N_{area}(S, \mathcal{C}, L)$ that counts each appearance of the configuration \mathcal{C} with weight equal to the area of the cylinders of width smaller than L .

For each of these numbers, we introduce the corresponding Siegel–Veech constant, that gives the asymptotic of these numbers as L goes to infinity:

$$c_*(\mathcal{C}) = \lim_{L \rightarrow \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}$$

Eskin and Masur showed in [EM] that these constants do not depend on S for almost every S in the connected stratum $\mathcal{Q}(\alpha)$. Combining these results with the results of Veech ([Ve]), one concludes that all these constants are strictly positive.

1.4. Application of Siegel–Veech constants. One of the principal reasons, why the Siegel–Veech constants are more and more intensively studied during the last years (see [AEZ1], [Ba1], [Ba2], [BG], [EKZ], [Vo]) is the relation between them and the Lyapunov exponents of the Hodge bundle along the Teichmüller flow: the key formula of [EKZ] expresses the sum of the positive Lyapunov exponents for any stratum $\mathcal{Q}(\alpha)$ as a sum of a very explicit rational function in α and the Siegel–Veech constant $c_{area}(\mathcal{Q}(\alpha))$. The Lyapunov exponents are closely related to the deviation spectrum of measured foliations on individual flat surfaces, see [Fo1], [Fo2], [Zo2], [Zo3], which opens applications to billiards in polygons, interval exchanges, etc.

A recent breakthrough of A. Eskin and M. Mirzakhani provides, in particular, new tools allowing to prove that the $SL(2, \mathbb{R})$ -orbit closure of certain individual flat surfaces is an entire stratum. By the theorem of J. Chaika and A. Eskin [ChE], almost all directions for such a flat surface are Lyapunov-generic. This allows to cumulate all the technology mentioned above to compute, for example, the diffusion rate of billiards with certain periodic obstacles. The final explicit answer (as $2/3$ for the diffusion rate in the windtree model studied in [DHL]) is certain Lyapunov exponent as above. This kind of quantitative answers or estimates are often reduced to computation of the appropriate Siegel–Veech constant.

The Konsevich formula (see [K]) for the sum of the Lyapunov exponents over a Teichmüller curve and recent results of S. Filip [Fi] showing that every stratum is a quasiprojective variety suggest that an adequate intersection theory of the strata might provide algebro-geometric tools to evaluate Siegel–Veech constants (see also [KtZg] in this connection). However, such intersection theory is not developed yet, and we are limited to analytic tools in our evaluation of Siegel–Veech constants.

1.5. Principal results. Now we are ready to state the main theorem of this paper.

Theorem 1. *Let \mathcal{C} be an admissible configuration for a connected stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. Let q_1 denote the number of thin cylinders, q_2 the number of thick cylinders in the configuration \mathcal{C} , and $q = q_1 + q_2$ the total number of cylinders. Assume that the boundary stratum $\mathcal{Q}(\alpha')$ is non empty, and $q \geq 1$. Then the Siegel–Veech constants associated to \mathcal{C} are the following:*

$$\begin{aligned} (1) \quad c(\mathcal{C}) &= \frac{M}{2^{q+2}} \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)! \text{Vol } \mathcal{Q}_1(\alpha')}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2)! \text{Vol } \mathcal{Q}_1(\alpha)} \\ (2) \quad c_{cyl}(\mathcal{C}) &= \left(q_1 + \frac{1}{4} q_2 \right) c(\mathcal{C}) \\ (3) \quad c_{area}(\mathcal{C}) &= \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1} c_{cyl}(\mathcal{C}) \end{aligned}$$

where $M = \frac{M_s M_c}{M_t}$ and M_c , M_t , M_s are combinatorial constants depending only on the configuration \mathcal{C} , explicitly given by equations (8), (11) and (18).

When the boundary stratum is empty, the formulae are simpler and given in §3.3.4.

This theorem is proven in section 3.3. Note that these formulae coincide in genus 0 with formulae of [AEZ1], for the two configurations containing cylinders (named “pocket” and “dumbell” in the article).

The ratio $\frac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})} = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1}$ can be interpreted as the mean area of a cylinder in the configuration \mathcal{C} . Note that it depends only on the dimension of the ambient stratum.

For a fixed stratum $\mathcal{Q}(\alpha)$ consider all admissible configurations, and denote $q_{max}(\alpha)$ the maximal number of cylinders for all these configurations. We evaluate this number in section 5.2. The ratio $\frac{q_{max}(\alpha)}{\dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) - 1}$ represents the maximum mean total area of the cylinders in stratum $\mathcal{Q}(\alpha)$.

Proposition 2. *We have*

$$\max_{\alpha \in \Pi(4g-4+k)} \frac{q_{max}(\alpha \cup \{-1^k\})}{2g - 3 + \ell(\alpha) + k} \begin{array}{l} \xrightarrow[k \text{ fixed}]{g \rightarrow \infty} \frac{1}{3} \\ \xrightarrow[k \rightarrow \infty]{g \text{ fixed}} 0 \end{array}$$

where $\Pi(4g - 4 + k)$ denotes the set of partitions of $4g - 4 + k$ and $\ell(\alpha)$ is the length of the partition α . Furthermore for any genus g and number of poles k the bound is achieved for $\alpha \in \Pi k \sqcup \Pi_4(4g - 4)$, where $\Pi_4(4g - 4)$ denote the set of partitions of $4g - 4$ using only 4's.

1.6. Historical remarks. The Siegel–Veech constants for the strata of Abelian differentials were evaluated in the paper [EMZ]; the relations between various Siegel–Veech constants were studied in [Vo] and some further ones in a recent paper [BG]. The computation in [EMZ] involves a combination of rather involved combinatorial and geometric constructions. To test the consistence of their theoretical predictions numerically, the authors of [EMZ] compare the formulae for the Lyapunov exponents expressed in terms of the Siegel–Veech constants (reduced, in turn, to combinations of volumes of the boundary strata) with numerics provided by experiments with the Lyapunov exponents. These tests are based, in particular, on the results of A. Eskin and A. Okounkov [EOk] providing the explicit values of the volumes of all strata of Abelian differentials in small genera.

The description of combinatorial geometry of configurations of saddle connections for the strata of quadratic differentials is performed in the paper of H. Masur and A. Zorich [MZ]; for the hyperelliptic components and for strata in genus zero such description is given in the paper of C. Boissy [Bo].

The evaluation of the corresponding Siegel–Veech constants in genus zero was recently performed by J. Athreya, A. Eskin, and A. Zorich [AEZ1]; see also the related paper [AEZ2]. The results were also verified by computer experiments with Lyapunov exponents combined with the knowledge of the volumes of the strata of quadratic differentials in genus zero. (The authors prove in [AEZ1] an extremely simple explicit formula for such volumes in genus zero conjectured by M. Kontsevich.)

In the current paper we treat the strata of quadratic differentials in arbitrary genus. We should point out that we are currently very limited in numerical tests of the suggested formulae. In the contrast to the strata of Abelian differentials the analogous results of A. Eskin, A. Okounkov, and R. Pandharipande [EOPa] do not provide explicit values for the volumes of the strata of quadratic differentials. This is why we have included in this paper a straightforward evaluation of volumes of certain strata, which allows to obtain at least some exact values of Siegel–Veech

constants for the strata of quadratic differentials away from genus zero, and to show that our formulae for Siegel–Veech constants are consistent with numerics coming from Lyapunov exponents of the Hodge bundle over the Teichmüller flow.

1.7. Structure of the paper. The paper is divided into two parts. The first part, theoretical, gives the proof of Theorem 1, and develops the results on a special family of strata: $\mathcal{Q}(1^k, -1^l)$. The first part ends with the extension of some geometric results proved in [BG] for the strata of Abelian differentials to the strata of quadratic differentials.

The computations of this first part generalize the computations presented in the articles [EMZ], and [AEZ1], but in higher genus there is a huge distance between the theory and getting exact values of Siegel–Veech constants, because the techniques involve some phenomenons of higher complexity. This is why we present in a second part all pragmatismal computations.

So the second part of this paper is devoted to the computation of the values of the volumes of certain strata and of the hyperelliptic components of strata. Since the values of the corresponding Siegel–Veech constants are known, the computed values of volumes enable us to check that the formulae of the first part are coherent. This checking is primordial since the choice of the normalization for the volume and the symmetries of high complexity for the configurations affect each step of the computations. In the Abelian case this checking has been done using numerical values provided by [EOk], but for the case of quadratic differentials obtaining numerical values of volumes is still work in progress. So our hope is also that the computed values of this article will be used as test values for the future effective algorithm giving approximated values of volumes of strata of quadratic differentials.

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Part 1. Siegel–Veech constants

In this part we first recall the prerequisites and the general method of [EMZ] to compute Siegel–Veech constants. In section 3 we prove Theorem 1. Then we apply our results in § 4 to the family of strata $\mathcal{Q}(1^k, -1^l)$. In this case we can give very explicitly the constant M which appears in Theorem 1. Finally in the last section of this part we extend some geometric results proven in [BG] in the case of Abelian differentials.

2. PRELIMINARIES

2.1. $\hat{\mathbf{H}}$ omologous saddle connections. We precise here from [MZ] the notion of *homologous* saddle connections.

Recall that any flat surface (S, q) in $\mathcal{Q}(\alpha)$ admits a canonical ramified double cover $\hat{S} \xrightarrow{p} S$ such that the induced quadratic differential on \hat{S} is a global square of an Abelian differential, that is $p^*q = \omega^2$ and $(\hat{S}, \omega) \in \mathcal{H}(\hat{\alpha})$. Let $\Sigma = \{P_1, \dots, P_n\}$ denote the singular points of the quadratic differential on S , and $\hat{\Sigma} = \{\hat{P}_1, \dots, \hat{P}_N\}$ the singular points of the Abelian differential ω on \hat{S} . Note that the pre-images of poles P_i are regular points of ω so do not appear in the list $\hat{\Sigma}$. The subspace $H_-^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$ antiinvariant with respect to the action of the hyperelliptic involution provides local coordinates in the stratum $\mathcal{Q}(\alpha)$ in the neighbourhood of S .

Let γ be a saddle connection on S . We denote γ' and γ'' its two lifts on \hat{S} . If $[\gamma] = 0$ downstairs, then $[\gamma'] + [\gamma''] = 0$ upstairs, and in this case we define $[\hat{\gamma}] := [\gamma']$. In the other case we have $[\gamma'] + [\gamma''] \neq 0$ and we define $[\hat{\gamma}] := [\gamma'] - [\gamma'']$. We obtain an element of $H_-^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$.

Then two saddle connections γ_1 and γ_2 are said to be $\hat{\text{homologous}}$ if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in $H_1(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, under an appropriate choice of orientations of γ_1, γ_2 .

2.2. Configurations of saddle connections. A configuration is one of the geometric type of all possible maximal collections of $\hat{\text{homologous}}$ saddle connections. We explain here precisely which informations characterize the geometric type of a collection (Definition 3 of [MZ]). Given such a collection of saddle connections on a surface S , cutting along these saddle connections will give a union of surfaces with boundaries. These surfaces can be either flat cylinders, or surfaces obtained by a surgery from a surface of trivial or non trivial holonomy. These surfaces are called boundary surfaces. We record the genus and the order of the singularities of all these surfaces. We record also which type of surgery is applied to which singularity on each surface with the precise angles. Finally we record the way the surfaces are glued in the initial surface. All this information characterizes a configuration of homologous saddle connections.

2.3. Graphs of configurations. We recall here briefly how the graphs introduced by Masur and Zorich in [MZ] encode all combinatorial information about a configuration. Let S be a half-translation and γ a saddle connection of configuration \mathcal{C} . The graph of the configuration \mathcal{C} is given by the following procedure: associate to each of the boundary surfaces a vertex in the graph, with the following symbolic: a vertex \oplus represents a surface of trivial holonomy, a vertex \ominus a surface of non trivial holonomy, and a vertex \circ a cylinder. Then there is an edge between two vertices if the boundaries of the corresponding surfaces share a common saddle connection. At this stage we obtain a graph described by Figure 3 in [MZ].

The surgeries performed on each surface are represented by local ribbon graphs belonging to the list described in Figure 6 of [MZ]. These local graphs are decorated with numbers k_i which are the numbers of horizontal geodesic rays emerging from the zeroes on which we perform the surgery, in an angular sector delimited by two $\hat{\text{homologous}}$ saddle connections. The reunion of these local ribbon graphs forms globally a ribbon graph that can be drawn on the graph giving the organization of the surfaces. The boundary of this ribbon graph has several connected components, each of them represents a newborn zero. To compute the order of a newborn zero, one can count the number of geodesic rays emerging from this point, that is, sum all the k_i 's met when one goes along the connected component of the boundary of

the ribbon graph corresponding to the newborn zero. The cone angle around this point is then $\pi \sum_i (k_i + 1)$. See Figure 7 in [MZ] for an example.

2.4. General strategy for the computation of Siegel–Veech constants. We recall here the sketch of the general method developed in [EMZ] to evaluate Siegel–Veech constants in the Abelian case, transposed to the quadratic case in genus 0 in [AEZ1].

Let $V_{\mathcal{C}}(S)$ be the set of holonomy vectors of saddle connections on S of type \mathcal{C} . The number of configurations \mathcal{C} in S such that the length of the homologous saddle connections is bounded is then

$$N(S, \mathcal{C}, L) = \frac{1}{2} |V_{\mathcal{C}}(S) \cap B(0, L)|,$$

where the factor $\frac{1}{2}$ compensates the fact that the saddle connections are not oriented and so their holonomy vectors are defined up to a sign. If q is the number of cylinders in the configuration and q_1 the number of “thin” cylinders, we define as well

$$N_{cyl}(S, \mathcal{C}, L) = \frac{1}{2} \left(q \left| V_{\mathcal{C}}(S) \cap B\left(0, \frac{L}{2}\right) \right| + q_1 \left| V_{\mathcal{C}}(S) \cap A\left(\frac{L}{2}, L\right) \right| \right),$$

with $A\left(\frac{L}{2}, L\right) = B(0, L) \setminus B\left(0, \frac{L}{2}\right)$. Note that $N_{cyl}(S, \mathcal{C}, L)$ counts each realization of configuration \mathcal{C} with weight the number of cylinders of width smaller than L : if the width of the thin cylinders is smaller than $L/2$ then all the q cylinders have their width smaller than L , if the width of the thin cylinders is comprised between $L/2$ and L , then the thick cylinders do not count.

Simplifying the last expression we get

$$(4) \quad N_{cyl}(S, \mathcal{C}, L) = q_2 N(S, \mathcal{C}, L/2) + q_1 N(S, \mathcal{C}, L)$$

where q_2 is the number of thick cylinders ($q = q_1 + q_2$).

Finally we define

$$N_{area}(S, \mathcal{C}, L) = \frac{1}{2} \sum_{v \in V_{\mathcal{C}}(S) \cap B(0, L)} A(v)$$

where $A(v)$ is the area of the cylinders of width smaller than L among those associated to the saddle connections of type \mathcal{C} and holonomy vector $\pm v$. Note that $N_{area}(S, \mathcal{C}, L)$ weights only the cylinders which are counted by $N_{cyl}(S, \mathcal{C}, L)$.

Convention 1. Following [AEZ1] we denote $\mathcal{Q}_1(\alpha)$ the hypersurface in $\mathcal{Q}(\alpha)$ of flat surfaces of area $1/2$ such that the area of the double cover is 1.

The stratum $\mathcal{Q}(\alpha)$ is equipped with a natural $PSL(2, \mathbb{R})$ -invariant measure μ , called Masur–Veech measure, induced by the Lebesgue measure in period coordinates. We choose a normalization for μ in 3.1. This measure induces a measure μ_1 on $\mathcal{Q}_1(\alpha)$ in the following way: if E is a subset of $\mathcal{Q}_1(\alpha)$, we denote $C(E)$ the cone underneath E in the stratum $\mathcal{Q}(\alpha)$:

$$C(E) = \{S \in \mathcal{Q}(\alpha) \text{ s.t. } \exists r \in (0, +\infty), S = rS_1 \text{ with } S_1 \in E\}$$

and we define

$$\mu_1(E) = 2d \cdot \mu(C(E)),$$

with $d = \dim_{\mathbb{C}} \mathcal{Q}(\alpha)$, that is, the measure $d\mu$ disintegrates in $d\mu = r^{2d-1} dr d\mu_1$.

Eskin and Masur proved in [EM] that the asymptotics

$$\lim_{L \rightarrow \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}$$

do not depend on the surface S for almost every surface in a connected component of a stratum of Abelian differentials. Athreya Eskin and Zorich generalized their method to the quadratic case in Theorem 2.3 in [AEZ1]. Then the following Siegel–Veech constants are well defined for almost every surface S in a connected component of a stratum of quadratic differentials:

$$c_*(\mathcal{C}) = \lim_{L \rightarrow \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}.$$

Remark 1. Note that it follows directly from this formula and the definition (4) of $N_{cyl}(S, \mathcal{C}, L)$ that:

$$c_{cyl}(\mathcal{C}) = \left(q_1 + \frac{1}{4}q_2 \right) c(\mathcal{C}),$$

which is the equation (2) in Theorem 1.

Now let $\mathcal{Q}(\alpha)$ be connected stratum. The Siegel–Veech formula (cf [Ve], Theorem 0.5) gives the existence of constants $b_*(\mathcal{C})$ such that

$$\frac{1}{\text{Vol}(\mathcal{Q}_1(\alpha))} \int_{\mathcal{Q}_1(\alpha)} N_*(S, \mathcal{C}, L) d\mu_1(S) = b_*(\mathcal{C}) \pi L^2$$

so necessarily $b_*(\mathcal{C}) = 2c_*(\mathcal{C})$ and we can express the Siegel–Veech constant as

$$c_*(\mathcal{C}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi\varepsilon^2} \frac{1}{\text{Vol}(\mathcal{Q}_1(\alpha))} \int_{\mathcal{Q}_1(\alpha)} N_*(S, \mathcal{C}, \varepsilon) d\mu_1(S).$$

In fact the integral is over the subset $\mathcal{Q}_1^\varepsilon(\mathcal{C})$ of $\mathcal{Q}_1(\alpha)$ formed by the surfaces with at least one family of “short” saddle connections of type \mathcal{C} , where “short” means of length smaller than ε . We decompose this subset as $\mathcal{Q}_1^\varepsilon(\mathcal{C}) = \mathcal{Q}_1^{\varepsilon, thick}(\mathcal{C}) \cup \mathcal{Q}_1^{\varepsilon, thin}(\mathcal{C})$ where $\mathcal{Q}_1^{\varepsilon, thin}(\mathcal{C})$ is the set of surfaces having at least two distinct collections of short saddle connections of type \mathcal{C} . Eskin and Masur proved that this subset is so small that we have

$$\frac{1}{\text{Vol} \mathcal{Q}_1(\alpha)} \int_{\mathcal{Q}_1^{\varepsilon, thin}(\mathcal{C})} N_*(S, \mathcal{C}, \varepsilon) d\mu_1(S) = o(\varepsilon^2).$$

Finally we obtain

$$(5) \quad c_*(\mathcal{C}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi\varepsilon^2} \frac{\text{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C})}{\text{Vol} \mathcal{Q}_1(\alpha)}$$

where $\text{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C})$ is the weighted volume:

$$\text{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \int_{\mathcal{Q}_1^\varepsilon(\mathcal{C})} W_*(\mathcal{C}, S) d\mu_1(S)$$

with $W(\mathcal{C}, S) = 1$, $W_{cyl}(\mathcal{C}, S)$ is equal to the number of cylinders of width smaller than ε , $W_{area}(\mathcal{C}, S)$ is equal to the area of the cylinders of length smaller than ε in the configuration \mathcal{C} on S .

The last step is the computation of $\text{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C})$ in term of the volume of the boundary stratum, see § 3.3.

Counting saddle connections of type \mathcal{C} is related to a more general problem: counting saddle connections with no fixed type. Introducing the number $N(S, L)$

of distinct holonomies of saddle connections shorter than L on $S \in \mathcal{Q}(\alpha)$, the corresponding Siegel–Veech constants

$$c_*(\mathcal{Q}(\alpha)) = \lim_{L \rightarrow \infty} \frac{N(S, L) \cdot (\text{Area of } S)}{\pi L^2}$$

are also well-defined for almost every $S \in \mathcal{Q}(\alpha)$ and depend only of the stratum. Then we have naturally

$$c_*(\mathcal{Q}(\alpha)) = \sum_{\mathcal{C}} c_*(\mathcal{C}).$$

The constant $c_{area}(\mathcal{Q}(\alpha))$ is particularly important because the formula of [EKZ] relates it to the sum of Lyapunov exponents for the Teichmüller geodesic flow. So it implies a lot of applications to the dynamics in polygonal billiards. Also since there are numerical experiments on Lyapunov exponents, the Eskin-Kontsevitch-Zorich formula provides numerical approximation for the constants $c_{area}(\mathcal{Q}(\alpha))$, and that gives a way to check computations on the constants $c_{area}(\mathcal{C})$. This is the main reason why we focus on configurations containing cylinders: they are the only ones that contribute to the constant $c_{area}(\mathcal{Q}(\alpha))$.

2.5. Strata that are not connected. In the last section we explained the method to compute Siegel–Veech constants for connected strata. The classification of connected components of strata is given in [L2]. Most of the strata are connected, the only ones which are not connected are the one which have a hyperelliptic component (except some sporadic examples in genus 3 and 4), and in this case there is only one supplementary component. The three types of strata containing hyperelliptic components are recalled on § 6.1.

The general strategy for computing Siegel–Veech constants for the connected strata can be adapted for connected components. For a connected component $\mathcal{Q}^{comp}(\alpha)$ we define the Siegel–Veech constants by the means:

$$c_*(\mathcal{Q}^{comp}(\alpha), \mathcal{C}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi\varepsilon^2} \frac{1}{\text{Vol}(\mathcal{Q}_1^{comp}(\alpha))} \int_{\mathcal{Q}_1^{comp}(\alpha)} N(S, \mathcal{C}, \varepsilon) d\mu_1(S).$$

Note that the connected components of $\mathcal{Q}_1(\alpha)$ are exactly the intersection of $\mathcal{Q}_1(\alpha)$ with the connected components of $\mathcal{Q}(\alpha)$. We have also the property that

$$c_*(\mathcal{C}) = \lim_{L \rightarrow \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2},$$

for almost every S in the component $\mathcal{Q}^{comp}(\alpha)$.

So we will obtain the same evaluation:

$$(6) \quad c_*(\mathcal{Q}^{comp}(\alpha), \mathcal{C}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi\varepsilon^2} \frac{\text{Vol}_* \mathcal{Q}_1^\varepsilon(comp, \mathcal{C})}{\text{Vol} \mathcal{Q}_1(\alpha)}.$$

We apply this method in the case of hyperelliptic components in section 7.

3. COMPUTATION OF SIEGEL-VEECH CONSTANT FOR CONNECTED STRATA

In this section, $\mathcal{Q}(\alpha)$ will denote a connected stratum of quadratic differentials. We will evaluate Siegel–Veech constants $c_*(\mathcal{C})$ defined in § 2.4 using equation (5).

3.1. Choice of normalization. We have to choose a normalization for the volume element on a strata $\mathcal{Q}(\alpha)$, which is equivalent to choose a lattice in the space $H_-^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$ which gives the local model of the stratum $\mathcal{Q}(\alpha)$ around S .

Convention 2. We follow the convention of [AEZ1] and choose, as lattice in $H_-^1(\hat{S}, \hat{\Sigma}; \mathbb{C})$ of covolume 1, the subset of those linear forms which take values in $\mathbb{Z} \oplus i\mathbb{Z}$ on $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z})$, that we will denote by $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))_{\mathbb{C}}^*$.

This convention implies that the non zero cycles in $H_1(S, \Sigma, \mathbb{Z})$ (that is, those represented by saddle connections joining two distinct singularities or closed loops non homologous to zero) have half-integer holonomy, and the other ones (closed loops homologous to zero) have integer holonomy.

Convention 3. We choose to labelled all zeroes and poles. This affects the computation of volumes, but it is easy to deduce the value of volumes of strata with anonymous singularities.

3.2. Construction of a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$. In this section we recall the generic construction given in [AEZ1] of a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ from a basis of $H_1(S, \Sigma, \mathbb{Z})$, and also a specific construction for each configuration. In the following sections we will look at every configuration and use the specific basis associated to each configuration in order to have a nice expression of the measure in terms of parameters of the cylinders.

For a primitive cycle $[\gamma]$ in $H_1(S, \Sigma, \mathbb{Z})$, that is, a saddle connection joining distinct zeros or a closed cycle (absolute cycle), the lift $[\hat{\gamma}]$ is a primitive element of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$.

3.2.1. "Generic" basis. (cf [AEZ1] §3.1) Let k be the number of poles in Σ , a the number of even zeroes and b the number of odd zeroes (of order ≥ 1). Assume that the zeroes are numbered in the following way: P_1, \dots, P_a are the even zeroes, P_{a+1}, \dots, P_{a+b} are the odd zeroes and P_{a+b+1}, \dots, P_n the poles, and take a simple oriented broken line P_1, \dots, P_{n-1} . Take each saddle connexion γ_i represented by $[P_i, P_{i+1}]$ for i going from 1 to $n-2$, and a basis $\{\gamma_{n-1}, \dots, \gamma_{n+2g-2}\}$ of $H_1(S, \mathbb{Z})$.

Lemma 1. *The family $\{\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}\}$ is a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$.*

Proof. First it is clear that the elements $\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}$ are primitive elements of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ and linearly independent. Moreover they do not generate a proper sub-lattice of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$. Each of the k poles lifts to a regular point in \hat{S} so does not appear in the list $\hat{\Sigma}$. An even zero of order α_i lifts to two zeroes of degrees $\frac{\alpha_i}{2}$, and an odd zero of order α_j lifts to a zero of degree $\alpha_j + 1$. So we have $n = |\Sigma| = k + a + b$ and $N = |\hat{\Sigma}| = 2a + b$. Thus if \hat{g} is the genus of \hat{S} we have $4\hat{g} - 4 = -k + \sum_{\alpha_i \geq 1} \alpha_i$ and $2\hat{g} - 2 = \sum_{\alpha_i \geq 1} \alpha_i + b$ and so

$$\begin{aligned} \dim_{\mathbb{C}}(H_1(\hat{S}, \hat{\Sigma}, \mathbb{Z})) &= 2\hat{g} - 1 + N = (2\hat{g} - 2 + n) + (2\hat{g} - 1 + a + b) \\ &= \dim_{\mathbb{C}} H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{C}) + \dim_{\mathbb{C}} H_1^+(\hat{S}, \hat{\Sigma}, \mathbb{C}). \end{aligned}$$

This equality on dimensions shows that we can complete the family $\{\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}\}$ with $\{\gamma'_1, \dots, \gamma'_{n-k-1}, \gamma'_{n-1}, \dots, \gamma'_{n+2g-2}\}$ to form a basis of $H_1(\hat{S}, \hat{\Sigma}, \mathbb{R})$ (the linear independence is clear from the construction). The intersection matrix has integer coefficients and is of determinant 1, so that ends the proof of the lemma. \square

3.2.2. *Basis associated to a configuration.* Fix a configuration \mathcal{C} . As in [EMZ], we define an appropriate family $\{\gamma_1, \dots, \gamma_{n+2g-2}\}$ of $H_1(S, \Sigma, \mathbb{Z})$ for $S \in \mathcal{C}$, which lifts to a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, as follows:

- for each component of the principal boundary strata $\mathcal{Q}(\alpha'_i)$ take a family $\{\beta_1^i, \dots, \beta_{n_i+2g_i-2}^i\}$ of $H_1(S'_i, \Sigma_i, \mathbb{Z})$ such that $\{\hat{\beta}_1^i, \dots, \hat{\beta}_{n_i+2g_i-2}^i\}$ is a basis of $H_1^-(\hat{S}'_i, \hat{\Sigma}_i, \mathbb{Z})$ as previously,
- for each homologous cylinder take a curve δ_j joining its boundary singularities (there might be an ambiguity in the choice of such a curve, cf § 3.3.1)
- take a saddle connection or a closed curve in the homology class of γ (we denote $\pm \vec{v}$ the holonomy of γ).

Lifting this basis to $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ using the $\hat{\cdot}$ operator provides a primitive basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, as previously.

We will keep the same notations for elements in $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))_{\mathbb{C}}^*$

3.3. **Computation.** Fix a configuration \mathcal{C} containing q cylinders ($q \geq 1$). Now we give a complete description of the measure μ in terms of parameters of the configuration by disintegrating the volume element $d\mu$.

By [EM] and [MS] we have $\text{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \text{Vol}_* \mathcal{Q}_1^{\varepsilon, \text{thick}}(\mathcal{C}) + o(\varepsilon^2)$, so we will describe μ only on $\mathcal{Q}^{\varepsilon, \text{thick}}(\mathcal{C})$.

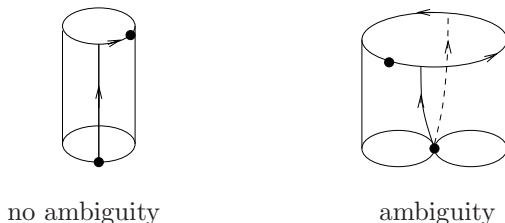
Let $S \in \mathcal{Q}^{\varepsilon, \text{thick}}(\mathcal{C})$. Local coordinates near S are given by $H_1^1(\hat{S}, \hat{\Sigma}, \mathbb{C})$, and μ is just Lebesgue measure in this coordinates. Choose now a basis associated to the configuration \mathcal{C} as above. It follows from the papers [EMZ] and [MZ] that the measure $d\mu$ in $\mathcal{Q}^{\varepsilon, \text{thick}}(\mathcal{C})$ disintegrates as the product of the measure $d\mu'$ on $\mathcal{Q}(\alpha')$ and the measure $d\nu_T$ on the space of parameters \mathcal{T} of the cylinders:

$$d\mu = M' d\mu' d\nu_T$$

where M' denotes the number of ways to get a surface S in $\mathcal{Q}^{\varepsilon, \text{thick}}(\mathcal{C})$ when the parameters of the configuration are fixed.

3.3.1. *Description of the space \mathcal{T} of the cylinders.* Roughly \mathcal{T} is described by coordinates $\pm \vec{v}, h_1, \dots, h_q, t_1, \dots, t_q$ representing the width, the heights and the twists of the cylinders, defined such that $h_i + it_i$ is the holonomy of the curve δ_i . The problem here is that there might be an ambiguity for the choice of this curve and so for the definition of the twist. In the following we assume that the cylinders are horizontal, that is $\pm \vec{v}$ represents the horizontal direction in the surface S . First note that despite the fact that the surface have a non trivial holonomy, for a given configuration \mathcal{C} it is possible to choose an orientation for each cylinder, for example by choosing an oriented path covering the graph representing the configuration. So in each cylinder we have a notion of bottom, up, left and right. Recall that thin cylinders are the one with each of their boundaries formed by a single saddle connection of holonomy $\pm \vec{v}$, and so there is only one singularity on each of their boundaries. For these cylinders we can define the twist and the height of the cylinder as usual: starting from the only one singularity on the bottom of the cylinder, draw a vertical segment going up and ending at a point P on the upper boundary of the cylinder. The length of this segment defines the height of the cylinder. Starting from the point P and following the boundary in the right horizontal direction, we meet the singularity on the upper boundary of the cylinder, which is at distance

t from P , and t defines the twist of the cylinder ($0 \leq t < |\vec{v}|$). The next picture shows a particular case where the twist is ambiguous for a thick cylinder.



For the thick cylinders, we can define their twist as follows: for such a cylinder, if one of its boundaries contains two distinct singularities (recall that the singularities are labelled), then choose the one of the smaller index. We have now in each case one distinguished singularity on each of the two boundaries. Consider the shortest geodesic segments joining these two singularities (there might be two such segments). Then their vertical coordinates coincide and define the height h of the cylinder, and their horizontal coordinate coincide modulo $\frac{2|\vec{v}|}{o_t}$, where $o_t = |\Gamma_{up}| \vee |\Gamma_{down}|$, and Γ_{up} (resp. Γ_{down}) is the group of symmetries of the upper (resp. lower) boundary. In general for cylinders appearing in a configuration the orders of these groups are 1 or 2, so o_t is equal to 1 or 2. In the example of the figure above, we have $|\Gamma_{down}| = 2$, $|\Gamma_{up}| = 1$ so $o_t = 2$. So we define the twist as the value $t \in \left[0, \frac{2|\vec{v}|}{o_t}\right)$ equal to the horizontal coordinates reduced modulo $\frac{2|\vec{v}|}{o_t}$. This definition will be interesting in the case of general cylinders, that is, cylinders which do not appear necessarily in a configuration, that are used to compute volumes of strata (cf § 8).

We have

$$d\nu_T = dhol(\hat{\gamma})dhol(\hat{\delta}_1) \dots dhol(\hat{\delta}_q).$$

Denote $n(q)$ the number of the cycles $\gamma, \delta_1, \dots, \delta_q$ in $H_1(S, \Sigma, \mathbb{Z})$ that are not homologous to 0 in $H_1(S, \Sigma, \mathbb{Z})$. Taking care of the normalization (Convention 2) we get:

$$(7) \quad d\nu_T = M_c \cdot d\vec{v} dh_1 \dots dh_q dt_1 \dots dt_q$$

with $M_c = 4^{n(q)}$.

Note that with our choice of the basis, $\delta_1, \dots, \delta_q$ are always non homologous to zero. And γ is homologous to zero if and only if the associated graph of the configuration is of type a in the classification of Masur and Zorich (Figure 3 in [MZ]): in this case a vertex corresponding to a cylinder is separating the graph, and the boundary of any cylinder in the configuration consists of a single saddle connection (homologous to γ). So we have:

$$(8) \quad M_c = \begin{cases} 4^q & \text{if } \mathcal{C} \text{ is of type } a \\ 4^{q+1} & \text{otherwise} \end{cases}$$

We choose to enumerate the cylinders such that the q_1 first cylinders have a waist curve of holonomy $\pm \vec{v}$ and the q_2 remaining cylinders have a waist curve of holonomy $2 \pm \vec{v}$.

Consider now $\mathcal{T}_1^\varepsilon$ the space of parameters of the cylinders with the additional constraint that the sum of the area of the $\hat{\gamma}$ homologous cylinders is normalized (i.e.

equal to $1/2$) and that $|v|$ is bounded by ε . Then the cone $C(\mathcal{T}_1^\varepsilon)$ underneath $\mathcal{T}_1^\varepsilon$ is given by the following equations:

$$(9) \quad |v| \left(\sum_{k=1}^{q_1} h_k + 2 \sum_{k=1}^{q_2} h_{q_1+k} \right) \leq \frac{1}{2}$$

$$(10) \quad |v| \leq \varepsilon \sqrt{2|v| \left(\sum_{k=1}^{q_1} h_k + 2 \sum_{k=1}^{q_2} h_{q_1+k} \right)}$$

3.3.2. *Computation of $c(\mathcal{C})$.* The volume of $\mathcal{T}_1^\varepsilon$ is given by:

$$\text{Vol}(\mathcal{T}_1^\varepsilon) = \dim_{\mathbb{R}}(T) \nu_T(C(\mathcal{T}_1^\varepsilon)) = 2(q+1) \nu_T(C(\mathcal{T}_1^\varepsilon))$$

with

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \int_{C(\mathcal{T}_1^\varepsilon)} d\nu_T$$

and $d\nu_T$ given by (7). Note that the measure $d\vec{v}$ on D_ε/\pm disintegrates into $w \cdot dw \cdot d\theta$ on $[0, \varepsilon] \times [0, \pi]$, and that integrating the measure of the twists $dt_1 \dots dt_q$ on $[0, w)^{q_1} \times \prod_{i=q_1+1}^q \left[0, \frac{2w}{o_{t_i}}\right)$ gives a factor $\frac{2^{q_2}}{M_t} w^q$, with

$$(11) \quad M_t = \prod_{i=q_1+1}^q o_{t_i},$$

so we get:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = M_c \pi \frac{2^{q_2}}{M_t} \int_0^{\frac{\varepsilon}{2}} w^{q+1} dw \int_{\mathbb{R}_+^q} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h \leq \frac{1}{2w} \right\} dh_1 \dots dh_q.$$

With the following changes of variables $h'_{q_1+k} = 2h_{q_1+k}$ we obtain:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \frac{M_c}{M_t} \pi \int_0^{\frac{\varepsilon}{2}} w^{q+1} dw \int_{\mathbb{R}_+^q} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h' \leq \frac{1}{2w} \right\} dh_1 \dots dh'_q.$$

with

$$h' = \sum_{i=1}^{q_1} h_i + \sum_{i=1}^{q_2} h'_{q_1+i}.$$

Using the fact that

$$\int_{\mathbb{R}_+^q} \chi \left\{ a \leq \sum_{i=1}^q h_i \leq b \right\} dh_1 \dots dh_q = \frac{1}{q!} (b^q - a^q),$$

since it is the difference of the volumes under two simplices in \mathbb{R}^q , we obtain after computation:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+1}} \frac{q}{(q+1)!}$$

Thus:

$$\text{Vol}(\mathcal{T}_1^\varepsilon) = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!}.$$

We assume now that $\mathcal{Q}(\alpha')$ is non empty, that is, the configuration \mathcal{C} is not made only by cylinders. Let $S' \in \mathcal{Q}_1(\alpha')$, then the rescaled surface $r_S S'$ where $0 < r_S \leq 1$ has area $\frac{r_S^2}{2}$. We define $\Omega(\varepsilon, r_S)$ to be the subset of \mathcal{T} formed by the

cylinders rescaled such that gluing them to $r_S S'$ after performing the appropriate surgeries gives a surface $S \in C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$. Note that the possible variations of area arising when performing the surgeries on $r_S S'$ are negligible ([EMZ] and [MZ]).

By definition $\Omega(\varepsilon, r_S)$ is exactly formed by the rescaled surfaces $r_T T$ where $0 < r_T \leq 1$, $r_T^2 + r_S^2 \leq 1$, and $T \in \mathcal{T}_1^{\tilde{\varepsilon}}$, with $\tilde{\varepsilon} = \varepsilon \sqrt{r_S^2 + r_T^2}$. So we have, denoting $Cusp(\varepsilon) = \text{Vol}(\mathcal{T}_1^\varepsilon)$,

$$\begin{aligned} \nu_T(\Omega(\varepsilon, r_S)) &= \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} Cusp\left(\frac{\tilde{\varepsilon}}{r_T}\right) dr_T \\ &= \frac{M_c \pi}{M_t 2^q (q-1)!} \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} \varepsilon^2 \frac{r_S^2 + r_T^2}{r_T^2} dr_T \end{aligned}$$

with $n_T = \dim_{\mathbb{C}}(\mathcal{T}) = q + 1$, which simplifies:

$$(12) \quad \nu_T(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!} \int_0^{\sqrt{1-r_S^2}} r_T^{2q-1} (r_S^2 + r_T^2) dr_T.$$

After computation, we obtain:

$$\nu_T(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+1} (q+1)!} (1 - r_S^2)^q (r_S^2 + q).$$

Now if M_s denote the number of ways to obtain a surface $S \in C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$ by gluing $r_T T \in \Omega(\varepsilon, r_S)$ to $r_S S' \in \mathcal{Q}(\alpha')$ (see (18)), the total measure of the cone $C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$ is:

$$(13) \quad \begin{aligned} \mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) &= M_s \text{Vol}(\mathcal{Q}_1(\alpha')) \int_0^1 r_S^{2n_S-1} \nu_T(\Omega(\varepsilon, r_S)) dr_S \\ &= \frac{M_s M_c \text{Vol}(\mathcal{Q}_1(\alpha')) \pi \varepsilon^2}{M_t 2^{q+1} (q+1)!} \underbrace{\int_0^1 r_S^{2n_S-1} (r_S^2 + q) (1 - r_S^2)^q dr_S}_I \end{aligned}$$

An easy recurrence or a change of variables gives the following lemma:

Lemma 2.

$$J(a, q) = \int_0^1 r^{2a+1} (1 - r^2)^q dr = \frac{1}{2} \frac{q! a!}{(a + q + 1)!}$$

We recognize

$$I = J(n_S, q) + qJ(n_S - 1, q).$$

After simplification we get:

$$I = \frac{(q+1)!(n_S-1)!}{2(n_S+q+1)!} (n_S + q).$$

So, denoting $M = \frac{M_s M_c}{M_t}$ we obtain:

$$\mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) = M \pi \varepsilon^2 \text{Vol}(\mathcal{Q}_1(\alpha')) \frac{(n_S-1)!(n_S+q)}{2^{q+2}(n_S+q+1)!}$$

As we have

$$\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \dim_{\mathbb{R}}(\mathcal{Q}(\alpha)) \mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C})))$$

it follows from the definition of the Siegel-Veech constant that:

$$c(\mathcal{C}) = M \dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) \frac{(n_S - 1)!(n_S + q) \text{Vol } \mathcal{Q}_1(\alpha')}{2^{q+2}(n_S + q + 1)! \text{Vol } \mathcal{Q}_1(\alpha)}.$$

Recall that $\dim_{\mathbb{C}} \mathcal{Q}(\alpha) = \dim_{\mathbb{C}} \mathcal{Q}(\alpha') + \dim_{\mathbb{C}} \mathcal{T} = n_S + q + 1$. We obtain finally the formula (1) of Theorem 1.

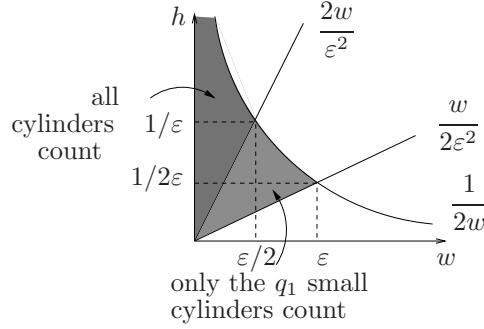
3.3.3. Computation of $c_{area}(\mathcal{C})$. Here we want to compute $c_{area}(\mathcal{C})$, so we have to count surfaces with weight the area of cylinders with waist curve smaller than ε , by definition. Note that, since there are q_1 cylinders of waist curve of length $w = |\vec{v}|$ and q_2 of waist curve of length $2w$, if $w \leq \frac{\varepsilon}{2}$ (when the area is renormalized), all cylinders count (with weight their area), and if $\frac{\varepsilon}{2} \leq w \leq \varepsilon$, only the thin cylinders count (with weight their area). Equation (10) contains two cases

$$w = |v| \leq \frac{\varepsilon}{2} \sqrt{2\text{area}}$$

and

$$\frac{\varepsilon}{2} \sqrt{2\text{area}} \leq w \leq \varepsilon \sqrt{2\text{area}}$$

of different weights. So the domain of integration of $C(\mathcal{T}_1^\varepsilon)$ splits into two parts as shown in the following picture.



This gives the following weight function:

$$W^{area}(w, h_i) = \begin{cases} \chi \left\{ \frac{2w}{\varepsilon^2} \leq h \leq \frac{1}{2w} \right\} + \frac{\sum_{i=1}^{q_1} h_i}{h} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h \leq \frac{2w}{\varepsilon^2} \right\} & \text{if } w \leq \frac{\varepsilon}{2} \\ \frac{\sum_{i=1}^{q_1} h_i}{h} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h \leq \frac{1}{2w} \right\} & \text{if } \frac{\varepsilon}{2} \leq w \leq \varepsilon \end{cases}$$

with

$$h = \sum_{k=1}^{q_1} h_k + 2 \sum_{k=1}^{q_2} h_{q_1+k}.$$

Now the weighted volume of $\mathcal{T}_1^\varepsilon$ is given by:

$$\text{Vol}^{area}(\mathcal{T}_1^\varepsilon) = \dim_{\mathbb{R}}(T) \nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) = 2(q+1) \nu_T^{area}(C(\mathcal{T}_1^\varepsilon))$$

with

$$\nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) = \int_{C(\mathcal{T}_1^\varepsilon)} W^{area}(|\vec{v}|, h_i) d\nu_T$$

and $d\nu_T$ given by (7).

Following step by step the computations of the last paragraph, using the same change of variables, we have

$$\begin{aligned} \nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) &= \frac{M_c}{M_t} \pi \left[\int_0^{\frac{\varepsilon}{2}} w^{q+1} dw \int_{\mathbb{R}_+^q} \left(\chi \left\{ \frac{2w}{\varepsilon^2} \leq h' \leq \frac{1}{2w} \right\} \right. \right. \\ &\quad \left. \left. + \frac{\sum_{i=1}^{q_1} h_i}{h'} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h' \leq \frac{2w}{\varepsilon^2} \right\} \right) dh_1 \dots dh'_q \right. \\ &\quad \left. + \int_{\frac{\varepsilon}{2}}^\varepsilon w^{q+1} dw \int_{\mathbb{R}_+^q} \frac{\sum_{i=1}^{q_1} h_i}{h'} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h' \leq \frac{1}{2w} \right\} dh_1 \dots dh'_q \right]. \end{aligned}$$

with

$$h' = \sum_{i=1}^{q_1} h_i + \sum_{i=1}^{q_2} h'_{q_1+i}.$$

Note that, since the variables h_i play symmetric roles, we have:

$$\int_{\mathbb{R}_+^q} \frac{\sum_{i=1}^{q_1} h_i}{\sum_{i=1}^q h_i} \chi \left\{ a \leq \sum_{i=1}^q h_i \leq b \right\} dh_1 \dots dh_q = \frac{q_1}{q} \int_{\mathbb{R}_+^q} \chi \left\{ a \leq \sum_{i=1}^q h_i \leq b \right\} dh_1 \dots dh_q.$$

So computations are similar to the previous ones, and we obtain:

$$\text{Vol}^{area}(\mathcal{T}_1^\varepsilon) = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+2} q!} (4q_1 + q_2).$$

Assume that $\mathcal{Q}(\alpha')$ is not empty. Now in (12) we have to multiply the integrand by the ratio of the area of the cylinders by the total area of the surface $\frac{r_T^2}{r_S^2 + r_T^2}$. We obtain:

$$\nu_T^{area}(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2 (4q_1 + q_2 - 2)}{M_t 2^{q+2} q!} \int_0^{\sqrt{1-r_S^2}} r_T^{2q+1} dr_T = \frac{M_c \pi \varepsilon^2 (4q_1 + q_2)}{M_t 2^{q+2} q!} \frac{(1-r_S^2)^{q+1}}{2(q+1)}.$$

Then:

$$\mu^{area}(C(Q_1^\varepsilon(\mathcal{C}))) = M \text{Vol } Q_1(\alpha') \frac{\pi \varepsilon^2 (4q_1 + q_2)}{2^{q+3} (q+1)!} \int_0^1 (1-r_S^2)^{q+1} r_S^{2n_S-1} dr_S.$$

Using again Lemma 2 we obtain:

$$\mu^{area}(C(Q_1^\varepsilon(\mathcal{C}))) = M \text{Vol } Q_1(\alpha') \frac{\pi \varepsilon^2 (4q_1 + q_2)}{2^{q+4}} \frac{(n_S - 1)!}{(n_S + q + 1)!}.$$

So at the end we have:

$$(14) \quad c_{area}(\mathcal{C}) = M \frac{4q_1 + q_2}{2^{q+4}} \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)! \text{Vol } Q_1(\alpha')}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)! \text{Vol } Q_1(\alpha)}.$$

Comparing to equation (1) and (2) we obtain the relation (3), which ends the proof of Theorem 1.

3.3.4. *Special case.* Assume that $\mathcal{Q}(\alpha')$ is empty that is, the configuration is made only by cylinders. This arises only on strata $\mathcal{Q}(-1^4)$, $\mathcal{Q}(2, -1^2)$ and $\mathcal{Q}(2, 2)$. Then the computations are much easier. Indeed we have in this case

$$\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \text{Vol } \mathcal{T}_1^\varepsilon = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!}$$

and

$$\text{Vol}^{area} \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \text{Vol}^{area} \mathcal{T}_1^\varepsilon = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+2} q!} (4q_1 + q_2)$$

so

$$(15) \quad c(\mathcal{C}) = \frac{M_c}{M_t 2^{q+1} (q-1)! \text{Vol } \mathcal{Q}_1(\alpha)}$$

$$(16) \quad c_{area}(\mathcal{C}) = \frac{1}{q} c_{cyl}(\mathcal{C}) = \frac{4q_1 + q_2}{4q} c(\mathcal{C})$$

since the ratio of the area of the cylinders over the total area is 1.

3.4. **Volume of the boundary strata.** Consider a strata $\mathcal{Q}(\alpha) = \prod_{i=1}^m \mathcal{Q}(\alpha_i)$ of disconnected flat surfaces. Following the notations of [AEZ1] and generalizing the result of 4.4 we obtain the following lemma:

Lemma 3.

$$\text{Vol } \mathcal{Q}_1(\alpha) = \frac{1}{2^{m-1}} \frac{\prod (\dim_{\mathbb{C}} \mathcal{Q}(\alpha_i) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)!} \prod_{i=1}^m \text{Vol } \mathcal{Q}_1(\alpha_i)$$

We also have the following relation between hyperboloids in the Abelian strata:

Lemma 4.

$$\text{Vol } \mathcal{H}_{1/2}(\alpha) = 2^{\dim_{\mathbb{C}} \mathcal{H}(\alpha)} \text{Vol } \mathcal{H}_1(\alpha)$$

So the final formula for a boundary strata $\mathcal{Q}(\alpha') = \prod \mathcal{H}(\alpha_i) \prod \mathcal{Q}(\beta_j)$ (m connected components) is:

$$(17) \quad c_{area}(\mathcal{C}) = M \frac{4q_1 + q_2}{2^{m+q+3}} \frac{\prod_i (a_i - 1)! 2^{a_i} \text{Vol } \mathcal{H}_1(\alpha_i) \prod_j (b_j - 1)! \text{Vol } \mathcal{Q}_1(\beta_j)}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)! \text{Vol } \mathcal{Q}_1(\alpha)}$$

where $a_i = \dim_{\mathbb{C}} \mathcal{H}(\alpha_i)$ and $b_j = \dim_{\mathbb{C}} \mathcal{Q}(\beta_j)$.

3.5. **Evaluation of M_s .** The general formula for M_s is given by:

$$(18) \quad M_s = \frac{K}{|\Gamma(\mathcal{C})|}$$

For each surface S_i in the principal boundary, the number of geodesic rays coming from a boundary singularity on S_i can be read on the local ribbon graph representing S_i : each boundary singularity is represented by a connected component of the local ribbon graph, summing the orders k_{i_j} along this connected component gives the number of geodesic rays emerging from this singularity. If the surface has several boundary singularities, then one has to multiply the number of geodesic rays obtained for each of them, to get the combinatorial constant responsible for the gluing of S_i in the configuration. Multiply the numbers obtained for each S_i to get the final combinatorial constant K .

$\Gamma(\mathcal{C})$ denotes the symmetries of the configuration \mathcal{C} generalising the stratum interchange and the cyclic symmetry in the Abelian case. We will explicit this group in the particular cases that we study.

4. STRATA $\mathcal{Q}(1^k, -1^l)$, WITH $k - l = 4g - 4 \geq 0$

The strata $\mathcal{Q}(1^k, -1^l)$ are particularly interesting for two reasons. First, they correspond to strata of maximal dimension at genus and number of poles fixed. Second, their boundary strata belong to the same family, so that gives recursion formulae for Siegel–Veech constants and volumes.

The strata $\mathcal{Q}(1^2, -1^2)$ and $\mathcal{Q}(1^4)$ are hyperelliptic and will be studied in § 7. In the general case there are only four types of configurations, so we give here their complete description and apply the formula for the Siegel–Veech constant $c_{area}(\mathcal{C})$ to each of them.

4.1. Configurations.

Proposition 3. *There are only four types of configurations which contain cylinders for strata $\mathcal{Q}(1^k, -1^l)$, they are described in Figure 1.*

Proof. We recall that graphs representing configurations are classified by Theorem 2 in [MZ]. Then the proof is based on the observation that there not many ways to create zeroes of order 1 or poles (see also Lemma 5 in § 5). We recall that the order of a newborn zero is given by the formula $\sum(k_i + 1) - 2$ where the k_i are the order of the boundary singularities along the boundary component of the ribbon graph that corresponds to the newborn zero (see paragraph 1.4 of [MZ] for more details), and we have $k_i \geq 0$. A boundary component admits at least one boundary singularity. So there is only one possibility for a pole: there is only one boundary singularity, which is equal to 0. For a zero of order 1 there are 3 possibilities:

- One boundary singularity of order 2
- Two boundary singularities of order 1 and 0
- Three boundary singularities of order 0

The first case is realizable when the global graph representing the configuration contain a loop with only one vertex. But in this case we can see that either there will be an other newborn zero of higher order, or there will be no cylinders in the configuration. The third case can be also be eliminated because a boundary components with exactly three boundary singularities arise only around a vertex of type +3.1 in the graph, and the parities of the boundary singularities in this case are odd.

So the only remaining possibility is the second one. We can reformulate this discussion by saying that there is only one way to get a cone angle 3π : one has to glue a cone angle π with a cone angle 2π . Looking carefully at all the ways to have boundary singularities of order 1 or 0 in the local ribbon graphs and the consequence on the boundary components in the global graph, we reduce the case to only two possibilities: the boundary singularity of order 0 arises only as cone angle around points on the boundary of a cylinder, and the one of order 1 arises either by creating a hole adjacent to a pole in a surface of non trivial holonomy (i.e. for vertices of type -1.1 and -2.2), or by breaking up a marked point on a surface of trivial holonomy (i.e. for vertices of type $+2.1$). Note that the last

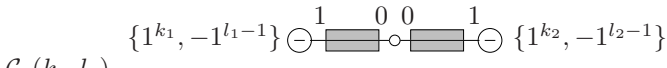
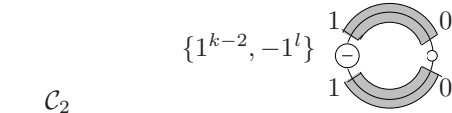
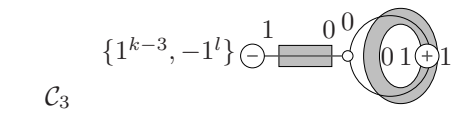
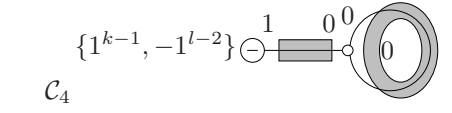
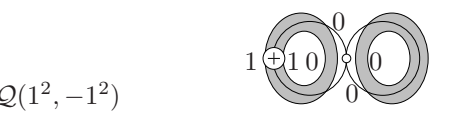
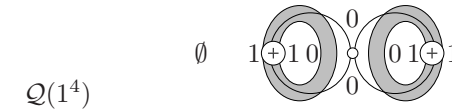
Configurations with cylinders		Boundary strata
General configurations for $g \geq 1$		
$\mathcal{C}_1(k_1, l_1)$  $\begin{cases} l_1 + l_2 - 2 = l \\ k_1 + k_2 + 2 = k \\ k_1 - l_1 = 4g_1 - 4 \\ k_2 - l_2 = 4g_2 - 4 \\ k_i \geq 0, l_i \geq 1, (k_i, l_i) \neq (1, 1) \end{cases}$		$\mathcal{Q}_{g_1}(1^{k_1}, -1^{l_1})$ and $\mathcal{Q}_{g_2}(1^{k_2}, -1^{l_2})$, for $g_1 + g_2 = g$
\mathcal{C}_2 		$\mathcal{Q}_{g-1}(1^{k-2}, -1^{l+2})$, for $k \geq 2$
\mathcal{C}_3 		$\mathcal{H}(0)$, $\mathcal{Q}_{g-1}(1^{k-3}, -1^{l+1})$ for $k \geq 3$
\mathcal{C}_4 		$\mathcal{Q}_g(1^{k-1}, -1^{l-1})$ for $l \geq 2$
Additional configurations for $g = 1, 2$		
$\mathcal{Q}(1^2, -1^2)$ 		$\mathcal{H}(0)$
$\mathcal{Q}(1^4)$ 		$\mathcal{H}(0), \mathcal{H}(0)$

FIGURE 1. Configurations containing cylinders for strata $\mathcal{Q}(1^k, -1^l)$, with $k_l = 4g - 4$ and $g \geq 1$.

surgery creates two points of cone angle π , so gluing each of them to a cylinder will create two newborn zeroes.

This situation is resumed in the following pictures (Figure 2).

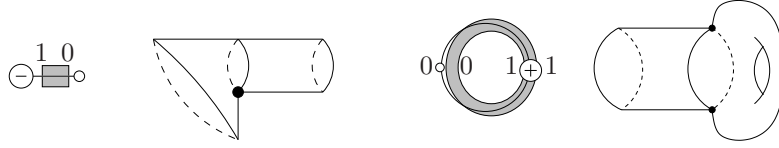


FIGURE 2. Newborn zeroes of order 1

For a pole, similar considerations give that there is only one way to get a pole (and not creating zeroes of order ≥ 2), by pinching the boundary of a cylinder (Figure 3).



FIGURE 3. Newborn poles

Note that, since the interior singularities are zeroes of order 1 or poles, the only boundary strata are $\mathcal{H}(0)$ and $\mathcal{Q}(1^K, -1^L)$.

These remarks allow us to eliminate most of the configurations, and to keep only the four possible types of configurations described on Figure 1. \square

The following tabular details the boundary strata (except $\mathcal{H}(0)$) of a stratum in genus 2.

		Number of poles						
		0	1	2	3	4	5	6
Genus	0	×	×	×	×	$\mathcal{Q}(-1^4)$	$\mathcal{Q}(1, -1^5)$	$\mathcal{Q}(1^2, -1^6)$
	1	×	×	$\mathcal{Q}(1^2, -1^2)$	$\mathcal{Q}(1^3, -1^3)$	$\mathcal{Q}(1^4, -1^4)$	$\mathcal{Q}(1^5, -1^5)$	$\mathcal{Q}(1^6, -1^6)$
	2	$\mathcal{Q}(1^4)$	$\mathcal{Q}(1^5, -1)$	$\mathcal{Q}(1^6, -1^2)$	$\mathcal{Q}(1^7, -1^3)$	$\mathcal{Q}(1^8, -1^4)$	$\mathcal{Q}(1^9, -1^5)$	$\mathcal{Q}(1^{10}, -1^6)$

Stratum Boundary strata

In general, the boundary strata of $\mathcal{Q}(1^k, -1^l)$ are those of same genus with at most $l - 1$ poles, those of lower genus with at most $l + 2$ poles, and $\mathcal{H}(0)$.

Note that, in this list, all values of volumes in genus 0 are known (cf [AEZ1]), and (25) gives the values of volumes for the first entries in genus 1 and 2 (hyperelliptic case).

4.2. Siegel–Veech constants.

Theorem 2. *Let $d = 2g - 2 + k + l = \frac{1}{2}(k + l)$ be the complex dimension of the stratum $\mathcal{Q}(1^k, -1^l)$. The Siegel–Veech constants associated to the four configurations described in Figure 1 are the following:*

If $(k_1, l_1) = (k_2, l_2)$:

$$c_{area}(\mathcal{C}_1(k_1, l_1)) = \frac{1}{8} \frac{((d_1 - 1)!)^2}{(d - 1)!} \frac{\text{Vol } \mathcal{Q}_1(1^{k_1}, -1^{l_1})^2}{\text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

Otherwise:

$$c_{area}(\mathcal{C}_1(k_1, l_1)) = \frac{1}{4} \frac{(d_1 - 1)!(d_2 - 1)! \text{Vol } \mathcal{Q}_1(1^{k_1}, -1^{l_1}) \text{Vol } \mathcal{Q}_1(1^{k_2}, -1^{l_2})}{(d - 1)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

where $d_i = \dim_{\mathbb{C}} \mathcal{Q}(1^{k_i}, -1^{l_i}) = \frac{1}{2}(3k_i + l_i)$.

$$c_{area}(\mathcal{C}_2) = 2 \frac{(d - 3)! \text{Vol } \mathcal{Q}_1(1^{k-2}, -1^{l+2})}{(d - 1)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

$$c_{area}(\mathcal{C}_3) = \frac{\pi^2}{3} \frac{(d - 5)! \text{Vol } \mathcal{Q}_1(1^{k-3}, -1^{l+1})}{(d - 1)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

$$c_{area}(\mathcal{C}_4) = \frac{1}{2} \frac{(d - 3)! \text{Vol } \mathcal{Q}_1(1^{k-1}, -1^{l-1})}{(d - 1)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

If all the four configurations appear in a stratum $\mathcal{Q}(1^k, -1^l)$, then the Siegel-Veech constant for the whole stratum is given by:

$$\begin{aligned} c_{area}(\mathcal{Q}(1^k, -1^l)) &= \sum_{\text{admissible } (k_1, l_1)} k(k-1) \binom{k-2}{k_1} \binom{l}{l_1-1} c_{area}(\mathcal{C}_1(k_1, l_1)) \\ &+ \binom{k}{2} c_{area}(\mathcal{C}_2) + \frac{1}{2} k(k-1)(k-2) c_{area}(\mathcal{C}_3) + \binom{k}{1} \binom{l}{2} c_{area}(\mathcal{C}_4) \end{aligned}$$

For the additional configurations in genera 1 and 2, see § 7.2.1.

Proof. For each configuration we compute the combinatorial data and apply equation (17).

(1) Configuration 1 (Figure 4):

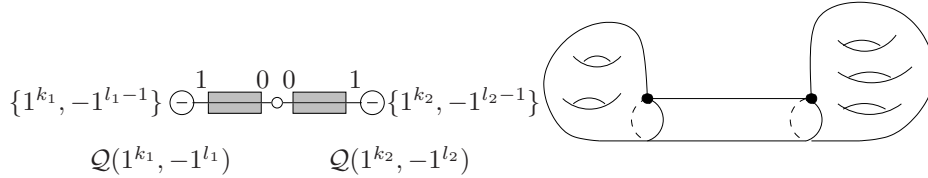


FIGURE 4. Configurations $\mathcal{C}_1(k_1, l_1)$ for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$

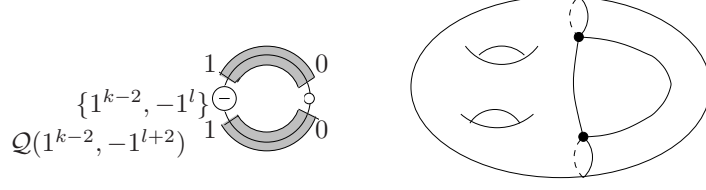
We have the following combinatorial data for this configuration:

- $M_c = 4^1$, $M_t = 1$
- $M_s = \frac{1}{|\Gamma|}$ with $|\Gamma| = 2$ if $(k_1, l_1) = (k_2, l_2)$, $|\Gamma| = 1$ otherwise
- $q_1 = 1, q_2 = 0$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k_i}, -1^{l_i}) = 2g_i - 2 + k_i + l_i = \frac{1}{2}(3k_i + l_i)$

Applying Theorem 1 we get:

$$c_{area}(\mathcal{C}_1(k_1, l_1)) = \frac{4}{|\Gamma|} \frac{4 \left(\frac{3k_1+l_1}{2} - 1\right)! \left(\frac{3k_2+l_2}{2} - 1\right)! \text{Vol } \mathcal{Q}_1(1^{k_1}, -1^{l_1}) \text{Vol } \mathcal{Q}_1(1^{k_2}, -1^{l_2})}{2^6 \left(\frac{3k+l}{2} - 1\right)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{k-1}{1} \times \binom{k-2}{k_1} \times \binom{l}{l_1-1} = k(k-1) \binom{k-2}{k_1} \times \binom{l}{l_1-1}$ configurations $\mathcal{C}_1(k_1, l_1)$.

FIGURE 5. Configuration \mathcal{C}_2 for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $k \geq 2$

(2) Configuration 2 (Figure 5):

The combinatorial data are:

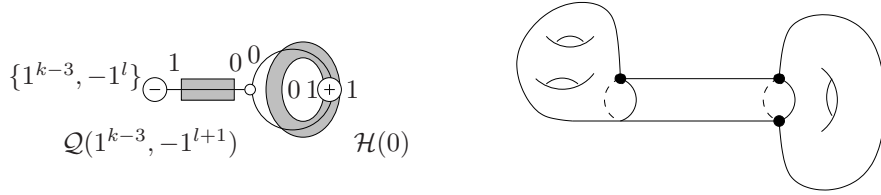
- $M_c = 4^2$, $M_t = 1$
- $M_s = 1/|\Gamma| = 1$
- $q_1 = 1, q_2 = 0$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-2}, -1^{l+2}) = 2g + k + l - 4$

We get:

$$c_{area}(\mathcal{C}_2) = 4^2 \frac{4}{2^5} \frac{(2g + k + l - 5)! \text{Vol } \mathcal{Q}_1(1^{k-2}, -1^{l+2})}{(2g + k + l - 3)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{2}$ configurations \mathcal{C}_2 .

(3) Configuration 3 (Figure 6):

FIGURE 6. Configuration \mathcal{C}_3 for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $k \geq 3$

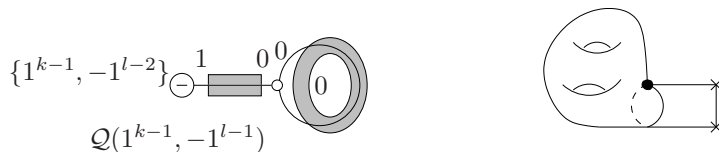
The combinatorial data are:

- $M_c = 4^2$, $M_t = 1$
- $M_s = 2/|\Gamma| = 1$ because of the involution of $\mathcal{H}(0)$.
- $q_1 = 0, q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-3}, -1^{l+1}) = 2g + k + l - 6$
- $\text{Vol } \mathcal{H}_{1/2}(0) = \frac{4\pi^2}{3}$

We obtain:

$$\begin{aligned} c_{area}(\mathcal{C}_3) &= 4^2 \frac{1}{2^6} \frac{(2g + k + l - 7)! \text{Vol } \mathcal{Q}_1(1^{k-3}, -1^{l+1})(2-1)! \text{Vol } \mathcal{H}_{1/2}(0)}{(2g + k + l - 3)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)} \\ &= \frac{\pi^2}{3} \frac{(2g + k + l - 7)! \text{Vol } \mathcal{Q}_1(1^{k-2}, -1^{l+2})}{(2g + k + l - 3)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)} \end{aligned}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{k-1}{2} = \frac{1}{2}k(k-1)(k-2)$ configurations \mathcal{C}_3 .

FIGURE 7. Configuration \mathcal{C}_4 for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $l \geq 2$

(4) Configuration 4 (Figure 7):

The combinatorial data are:

- $M_c = 4^2$, $M_t = 1$, $M_s = 1$
- $q_1 = 0, q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-1}, -1^{l-1}) = 2g + k + l - 4$

Theorem 1 gives:

$$c_{area}(\mathcal{C}_4) = 4^2 \frac{1}{2^5} \frac{(2g + k + l - 5)! \text{Vol } \mathcal{Q}_1(1^{k-1}, -1^{l-1})}{(2g + k + l - 3)! \text{Vol } \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{l}{2}$ configurations \mathcal{C}_4 .

After simplification of the formulae we obtain the results of Theorem 2. \square

5. GEOMETRY OF CONFIGURATIONS CONTAINING CYLINDERS

This section develops the quadratic version of some geometric results on configurations, proven in the Abelian case in [BG].

5.1. Variants of Siegel–Veech constants. The result (3) of Theorem 1 can be interpreted as follows: the ratio $\frac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})}$ represents the mean area of a cylinder in configuration \mathcal{C} . It does not depend on the configuration, but only on the dimension of the stratum. Summing on all configurations in a stratum we get a result of Vorobets (Theorem 1.6 in [Vo]).

We introduce variants of Siegel–Veech constants whose ratios admit a geometric interpretation. Some of them were introduced by Vorobets.

We define $N_{A_1 \geq p}(S, \mathcal{C}, L)$ (resp. $N_{A \geq p}(S, \mathcal{C}, L)$) that counts configurations \mathcal{C} on S only if the area of a fixed cylinder (resp. all cylinders) filled at least proportion p of the area of the entire surface. As before we denote

$$c_*(\mathcal{C}) = \lim_{L \rightarrow \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}$$

the associated Siegel–Veech constants.

We give the analogue of Theorems 4 and 5 of [BG]. Proofs are very similar to the Abelian case so we keep them short.

We introduce the *incomplete Beta function*

$$B(x; n, q) = \int_0^x u^{n-1} (1-u)^{q-1} du$$

and the *Beta function* $B(n, q) = B(1; n, q)$. It is a standard fact that

$$B(x; n, q) = B(n, q) \sum_{k=n}^{n+q-1} \binom{n+q-1}{k} x^k (1-x)^{n+q-1-k}.$$

Theorem 3. *Let \mathcal{C} be an admissible configuration for a connected stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. Let q denote the total number of cylinders. Assume that the boundary stratum $\mathcal{Q}(\alpha')$ is non empty, and $q \geq 1$. Then the ratios of Siegel–Veech constants associated to \mathcal{C} are the following:*

$$(19) \quad \frac{c_{A>p}(\mathcal{C})}{c(\mathcal{C})} = \frac{B(1-p; n_S, q)}{B(n_S, q)}$$

$$(20) \quad \frac{c_{A_1>p}(\mathcal{C})}{c(\mathcal{C})} = (1-p)^{\dim \mathcal{Q}(\alpha)-2}$$

The first ratio can be interpreted as the probability for the cylinders to fill a large part of the area of the surface, and the second ratio the probability for a distinguished cylinder to fill a large part of the area of the surface. Note that the first ratio depends on the number of cylinders q in the configuration, as the second ratio depends only of the dimension of the stratum.

Proof. We begin with the proof of (19). We follow step by step the computations of § 3.3.2. The value of $Cusp(\varepsilon)$ does not change. The only adjustment to made is that the area of the surface $r_T T$ which we glue to $r_S S'$ has to satisfy $r_T^2 > p(r_T^2 + r_S^2)$, which is equivalent to $r_T > \sqrt{\frac{p}{1-p}} r_S$. So (12) becomes

$$\nu_T^{A>p}(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!} \int_{\sqrt{\frac{p}{1-p}} r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1} (r_S^2 + r_T^2) dr_T.$$

and using the constraint $r_T^2 + r_S^2 \leq 1$ we obtain the following bound on r_S : $r_S \leq \sqrt{1-p}$, so (13) becomes

$$\mu^{A>p}(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) = \frac{M \text{Vol}(\mathcal{Q}_1(\alpha')) \pi \varepsilon^2}{2^{q+1} (q+1)!} \underbrace{\int_0^{\sqrt{1-p}} r_S^{2n_S-1} \int_{\sqrt{\frac{p}{1-p}} r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1} (r_S^2 + r_T^2) dr_T dr_S}_{I_p}$$

Using an appropriate change of variables as the Abelian case, we recognize

$$I_p = \frac{B(1-p; n_S, q)}{4(n_S + q + 1)}$$

where $B(1-p; n_S, q)$ is the incomplete Beta function. Comparing the result to (1) we get (19).

Now we compute $c_{A_1>p}(\mathcal{C})$: we have the same constraints as before, plus the additional constraint that the first cylinder fills at least part p of the area of the surface. This affects the calculus of the cusp. Note that a cylinder in $S \in \mathcal{Q}_1(\alpha)$ fills at least part p of the surface if it fills at least part $a = p \cdot \frac{r_S^2 + r_T^2}{r_T^2}$ in the space

of the cylinders \mathcal{T}_1 . So we have to replace $Cusp(\varepsilon)$ by

$$\begin{aligned} Cusp^{A_1 > a}(\varepsilon) &= 2(q+1)\nu_T^{A_1 > p}(C(\mathcal{T}_1^\varepsilon)) \\ &= 2(q+1)\frac{M_c}{M_t}\pi \int_0^{\frac{\varepsilon}{2}} w^{q+1}dw \int_{\mathbb{R}_+^q} \chi \left\{ \frac{w}{2\varepsilon^2} \leq h' \leq \frac{1}{2w} \right\} \chi\{h_1 \geq ah'\} dh_1 \dots dh'_q. \end{aligned}$$

Using the change of variables $h'_1 = h_1 - ah$ we get:

$$Cusp^{A_1 > a}(\varepsilon) = Cusp(\varepsilon) \cdot (1-a)^{q-1}$$

Note that if we choose another cylinder, the computations are exactly the same, even if it is a thick cylinder. Finally (13) becomes

$$\begin{aligned} \mu^{A_1 > p}(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) &= \frac{M \text{Vol}(\mathcal{Q}_1(\alpha'))\pi\varepsilon^2}{2^{q+1}(q+1)!} \\ &\cdot \underbrace{\int_0^{\sqrt{1-p}} r_S^{2n_S-1} \int_{\sqrt{\frac{p}{1-p}}r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1}(r_S^2 + r_T^2) \left(1 - p \cdot \frac{r_S^2 + r_T^2}{r_T^2}\right) dr_T dr_S}_{I'_p} \end{aligned}$$

and we get

$$I'_p = \frac{(1-p)^{n_S+q-1}}{4(n_S+q+1)} \cdot B(n, q).$$

Comparing the result to (1) we get (20). \square

5.2. Maximal number of cylinders. Configurations of quadratic differentials in genus 0 are detailed in [AEZ1]. They contain at most one cylinder. The following proposition gives the maximal number of cylinders in a configuration in higher genus.

Proposition 4. *Consider a stratum $Q(\alpha)$ in genus $g \geq 1$, with $\alpha = (4l_1, \dots, 4l_m, 4k_1 + 2, \dots, 4k_n + 2, b_1, \dots, b_p, -1^k)$, and $l_i \geq 0$, $k_i \geq 0$, b_i odd. First assume that $2n + \sum_{i=1}^p b_i - k + 4 \geq 0$, then the maximal number of homologous cylinders satisfies:*

$$q_{max}(\alpha) = \lfloor \frac{n}{2} \rfloor + m + \varepsilon_\alpha,$$

where $\varepsilon_\alpha \in \{0, 1, 2\}$.

Without this assumption, the maximal number of homologous cylinders is given by:

$$\begin{aligned} q_{max}(\alpha) &= \max\left\{\text{card } I + \frac{\text{card } J}{2}; I \subset \{1, \dots, m\}, J \subset \{1, \dots, n\}, \text{card } J \text{ even}, \right. \\ &\quad \left. 4 \sum_{i \in I} l_i + 4 \sum_{j \in J} k_j + 2n + \sum_{k=1}^p b_k + 4 - k \geq 0\right\} + \varepsilon_\alpha \end{aligned}$$

To prove this proposition we will need the following lemma:

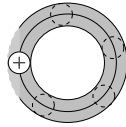
Lemma 5. *Odd zeroes are created by surfaces of non trivial holonomy \ominus or by loops in the graphs of configurations. At most four newborn odd zeroes can be created in a configuration.*

Proof. Since the zeroes on which we perform surgeries on surfaces of non trivial holonomy \ominus are of any order (even or odd), it is easy to see that we can obtain any parity order for newborns zeroes created by surfaces \ominus .

This is not the case of surfaces \oplus . In fact, a newborn zero represented in the graph by a boundary of a ribbon graph which frames a chain of surfaces \oplus (as in the picture below) surrounded by surfaces \oplus or cylinders has always an even order. This is due to the fact that we perform surgeries such as creating a hole on surfaces of trivial holonomy, so on singularities of cone angle $2k\pi$. If we glue all these surfaces identifying all boundary singularities, then the new cone angle is also multiple of 2π , so the newborn zero is of even order. Boundary types involved in these chains are $\circ 2.2$, $+2.1$, $+2.2$, $+3.2a$, $+3.2b$, $+3.3$, $+4.2a$, $+4.3a$, $+4.4$.



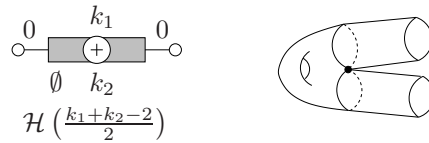
Then we just have to look at the remaining cases, namely, graphs containing surfaces of boundary type $\circ 3.2$ $\circ 4.2$, $+3.1$, $+4.1a$, $+4.1b$, $+4.2b$, $+4.2c$, $+4.3b$. Then one can see case by case that if the ribbon graph is locally as on the picture above, one or two odd zeroes are created (one can replace the surface \oplus by a cylinder \circ).



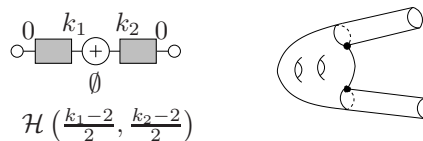
As an example, Figure 3 represents how poles are created by loops in the graph of the configuration. □

Proof of Proposition 4. This result is a corollary of the classification of configurations of homologous cylinders by Masur and Zorich (Figure 3 in [MZ]). Each configuration is represented by a graph with one, two or three chains of surfaces \oplus (with trivial linear holonomy) and cylinders \circ (see also § 2.3 for more details about these graphs). Then there are some remarks:

- A surface \oplus of type $+2.1$ (cf Figure 6 in [MZ]) in a chain is surrounded by at most two cylinders. In that case if there is no interior singularity it creates a newborn zero of order $4g = k_1 + k_2 + 2$, where g is the genus of the boundary strata $\mathcal{H}(\frac{k_1+k_2-2}{2})$ (k_1 and k_2 are odd).



- A surface \oplus of type $+2.2$ in a chain is surrounded by at most two cylinders and in that case if there is no interior singularity it creates two newborn zeroes of order k_1 and k_2 (even) with $k_1 + k_2 = 4g$ where g is the genus of the boundary strata $\mathcal{H}(\frac{k_1-2}{2}, \frac{k_2-2}{2})$
- By Lemma 5, at most 4 zeros of odd order can be realized as newborn zeroes (created by loops in the graph of the configuration or by surfaces \ominus), the others are necessarily interior singularities (of surfaces \ominus).



- Realizing zeroes as newborn zeroes instead of interior singularities increases the number of cylinders.

First we assume that $2n + \sum_{i=1}^p b_i - k + 4 \geq 0$. One procedure to construct the configuration containing the most cylinders is the following: we consider all zeroes of order $4l$ and realize them as newborn zeroes with a surface of type +2.1 as described above. Then we consider the other even zeroes and realize them by pairs as newborn zeroes with surfaces of type +2.2 as described above. At this stage we obtain a chain of $m + \lfloor \frac{n}{2} \rfloor$ surfaces with a cylinder between each surface \oplus . We consider the remaining zeroes (at most one even zero and all the odd zeroes). If there are at least 5 odd zeroes, we have to choose graph *a*), *b*) or *c*) following notations of Figure 6 in [MZ] to complete your configuration. If not, we can choose graph *c*), *d*) or *e*). In all cases we will get at most 2 additional cylinders, by looking carefully at all possible configurations depending on the number of odd/even zeroes and poles.

In the general case, we have to choose carefully the even zeroes that we realize as newborn zeroes. Indeed all remaining zeroes should be produced by another surface of non-negative genus in a boundary strata. This condition implies that we can choose to realize zeroes of orders $4l_i$ or pairs of zeroes $4k_{j_1} + 2, 4k_{j_2} + 2$ with $i \in I$ and $j_1, j_2 \in J$ while $4 \sum_{i \in I} l_i + 4 \sum_{j \in J} k_j + 2n + \sum_{k=1}^p b_k + 4 - k \geq 0$. This explains the general formula for the maximal number of cylinders. \square

We are interested in the asymptotic geometry of configurations, in particular when the genus or the number of zeroes tends to infinity, so we will consider $\tilde{q}_{max}(\alpha) = q_{max}(\alpha) - \varepsilon_\alpha$ instead of $q_{max}(\alpha)$, to simplify the computations.

As a corollary of Proposition 4 we obtain that the strata maximizing the number of cylinders at genus fixed are the ones with the most even zeroes:

Corollary 1. *Fix the genus $g \geq 1$ and the number of poles k . Denote $\Pi(4g - 4 + k)$ the set of partitions α of $4g - 4 + k$, and $l = \lfloor \frac{k}{4} \rfloor$. Then:*

$$\max_{\alpha \in \Pi(4g-4+k)} \tilde{q}_{max}(\alpha \cup \{-1^k\}) = g - l$$

and the maximum is realized for $\alpha \in \Pi k \sqcup \Pi_{4,2}(4g - 4)$, where $\Pi_{4,2}(4g - 4)$ denotes the set of partitions of $4g - 4$ using only 4 and 2.

Note that the maximal number of cylinders does not depend on k anymore.

Recall that the mean area of a cylinder is given by $\frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1}$ (cf Theorem 1), so $\frac{q_{max}(\alpha)}{\dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) - 1}$ represents the maximum mean total area of the cylinders in stratum $\mathcal{Q}(\alpha)$. As another corollary of Proposition 4, we obtain Proposition 2.

5.3. Configurations with simple surfaces. This section provides an answer in the quadratic case to the following question of Alex Eskin and Alex Wright: for a given stratum or a connected component of a stratum is it possible to find an admissible configuration whose boundary surfaces are only tori ?

Lemma 5 gives the main obstruction to solve this problem in the quadratic case: odd zeroes are created by surfaces of non trivial holonomy \ominus or by loops in graphs of configurations, and there are at most two surfaces of non trivial holonomy or two loops in a configuration. That means that a strata with enough odd zeroes will never have a configuration with only tori as boundary surfaces.

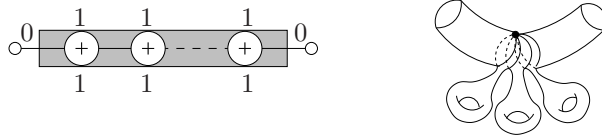
The second obstruction is that, as in the case of abelian differentials, there is no way to have a decomposition into simple surfaces in hyperelliptic components of strata, since they are made from at most two surfaces and cylinders (cf [Bo] and § 7).

Considering these two obstructions (odd zeroes and hyperelliptic components), we can formulate the following result, which is very similar to the case of Abelian differentials (cf [BG]).

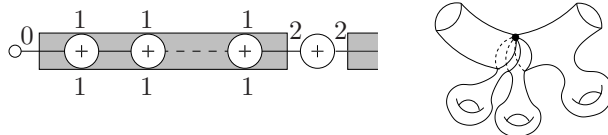
Proposition 5. *Let $\mathcal{Q}^{comp}(\alpha_1, \dots, \alpha_s)$ be a connected component of a stratum of quadratic differentials, which is not hyperelliptic. If all the α_i are even then there exists a configuration in this component containing only tori and cylinders.*

Proof. Denote n the number of zeroes of order $4k + 2$ and m the number of zeroes of order $4k$. As in the case of abelian differentials, we just look at what type of zeroes can be created by chains of tori and cylinders. We obtain the same type of zeroes as in the case of Abelian differentials.

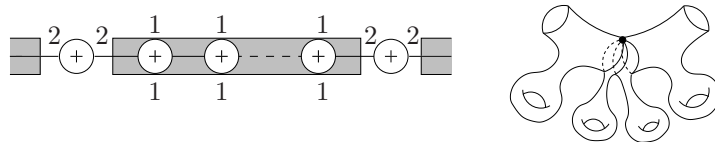
For the first type represented in the picture above, the cone angle around the singularity is also $2(2k + 1)\pi$ so we obtain a zero of order $4k$.



Zeroes of the second type represented are of order $4k + 2$ since the cone angle is $(4k + 4)\pi$.



Finally zeroes of the third type are of order $4k + 4$.



With these chains we can easily construct a bigger chain that realizes all zeroes. It remains to embed this chain in a graph of configuration. We can see that if there is at least two zeroes of order greater than 4, or if there is at least one zero of order greater than 8, then we can embed this chain in the graph e) with local ribbon graph of type $+4.2a$.

Since $\mathcal{Q}(4)$ is empty, it remains only strata $\mathcal{Q}(2, 2, \dots, 2)$, which is realizable with a graph of type $e)$ and a local ribbon graph of type $\circ 4.2$, by example.

□

Part 2. Volumes of strata and hyperelliptic components of strata

This part details certain cases where we can find a closed formula for the volume of a stratum or a connected component of a stratum. In genus 0 the volumes of strata are computed by Athreya, Eskin and Zorich (Theorem 1.6. of [AEZ1]). In higher genus, no closed formula is currently known, even though some calculations are possible using [EOPa].

The two cases where we can find explicit formulae for the volumes are the hyperelliptic components, using the known volumes in genus 0, and the strata of complex dimension smaller than 5, using techniques developed by Zorich in [Z1], and by Athreya, Eskin and Zorich in [AEZ2].

Note that in all these cases, we obtain a formula:

$$\text{Vol } \mathcal{Q}_1(\alpha) = r \cdot \pi^{2g_{\text{eff}}}, \quad r \in \mathbb{Q},$$

which is expected to be general. Here $g_{\text{eff}} = \hat{g} - g$ where \hat{g} is the genus of the double cover \hat{S} .

In these two cases, we apply our previous results and compute Siegel–Veech constants for the entire components or for the entire strata, using values of Siegel–Veech constants for configurations. We choose in purpose examples where we know the values of Siegel–Veech constants in order to show that the results are coherent.

6. VOLUMES OF HYPERELLIPTIC COMPONENTS

We begin with hyperelliptic components of strata: the values of their volumes are easier to compute since they are related to values of volumes in genus 0, which are computed in [AEZ1].

6.1. Volumes of hyperelliptic components of strata of quadratic differentials. The strata of the moduli spaces of quadratic differentials have one or two connected components: for genus $g \geq 5$ there are two components when the stratum contains a hyperelliptic component (cf [L2]). For genus $g \leq 4$ some strata are hyperelliptic and connected (cf [L1]): namely $\mathcal{Q}(1^2, -1^2)$ and $\mathcal{Q}(2, -1^2)$ in genus 1, $\mathcal{Q}(1^4)$, $\mathcal{Q}(2, 1^2)$, and $\mathcal{Q}(2, 2)$ in genus 2. For these strata and for hyperelliptic components of strata in higher genus the volume is easier to compute. We explain here the general strategy to compute these volumes, that we apply in section 7.

Proposition 6. *The volumes of hyperelliptic components of strata of quadratic differentials are given by the following formulae:*

- *First type ($k_1 \geq -1$ odd, $k_2 \geq -1$ odd, $(k_1, k_2) \neq (-1, -1)$):*
If $k_1 \neq k_2$:

$$(21) \quad \text{Vol}^{\text{numb}} \mathcal{Q}_1^{\text{hyp}}(k_1^2, k_2^2) = \frac{2^d}{(d)!} \pi^d \frac{k_1!!}{(k_1 + 1)!!} \frac{k_2!!}{(k_2 + 1)!!}$$

Otherwise:

$$(22) \quad \text{Vol}^{\text{numb}} \mathcal{Q}_1^{\text{hyp}}((g-1)^4) = \frac{3 \cdot 2^{2g+2}}{(2g+2)!} \pi^{2g+2} \left(\frac{(g-1)!!}{g!!} \right)^2$$

- *Second type ($k_1 \geq -1$ odd, $k_2 \geq 0$ even):*

$$(23) \quad \text{Vol}^{\text{numb}} \mathcal{Q}_1^{\text{hyp}}(k_1^2, 2k_2 + 2) = \frac{2^d}{(d)!} \pi^{d-1} \frac{k_1!!}{(k_1 + 1)!!} \frac{k_2!!}{(k_2 + 1)!!}$$

- *Third type* (k_1, k_2 even):

$$(24) \quad \text{Vol}^{numb} \mathcal{Q}_1^{hyp}(2k_1 + 2, 2k_2 + 2) = \frac{2^{d+1}}{(d)!} \pi^{d-2} \frac{k_1!!}{(k_1 + 1)!!} \frac{k_2!!}{(k_2 + 1)!!}$$

where $d = k_1 + k_2 + 4$ is the complex dimension of the strata.

Example 1. For the five strata that are connected and hyperelliptic we obtain:

$$(25) \quad \text{Vol} \mathcal{Q}_1(1^2, -1^2) = \frac{\pi^4}{3} = 30\zeta(4) \quad \text{Vol} \mathcal{Q}_1(1^4) = \frac{\pi^6}{15} = 63\zeta(6)$$

$$(26) \quad \text{Vol} \mathcal{Q}_1(2, -1^2) = \frac{4\pi^2}{3} = 8\zeta(2) \quad \text{Vol} \mathcal{Q}_1(2, 1^2) = \frac{2\pi^4}{15} = 12\zeta(4)$$

$$(27) \quad \text{Vol} \mathcal{Q}_1(2, 2) = \frac{4\pi^2}{3} = 8\zeta(2)$$

For an alternative computation of some of these volumes using graphs, see Appendix B.

Proof. By Convention 3 we compute volumes of strata with numbered zeroes. We denote $\text{Vol}^{numb} \mathcal{Q}(\alpha)$ the volume of the strata $\mathcal{Q}(\alpha)$ when the zeroes are numbered and $\text{Vol}^{unnumb} \mathcal{Q}(\alpha)$ when they are not. We have the following relation:

$$\text{Vol}^{numb} \mathcal{Q}_1(d_1^{\alpha_1}, d_2^{\alpha_2}, \dots, d_m^{\alpha_m}) = \frac{\alpha_1! \alpha_2! \dots \alpha_m!}{|\Gamma(\alpha)|} \text{Vol}^{unnumb} \mathcal{Q}_1(d_1^{\alpha_1}, d_2^{\alpha_2}, \dots, d_m^{\alpha_m})$$

where $\Gamma(\alpha)$ denotes the group of possible symmetries of all surfaces in the stratum $\mathcal{Q}(\alpha)$.

We recall here the three types of strata that contain hyperelliptic components (cf [L1]):

- First type:

$$\mathcal{Q}_g^{hyp}(k_1^2, k_2^2) \xrightarrow{\pi} \mathcal{Q}_0(k_1, k_2, -1^{2g+2})$$

for $k_1 \geq -1$ odd, $k_2 \geq -1$ odd, $(k_1, k_2) \neq (-1, -1)$, $g = \frac{1}{2}(k_1 + k_2) + 1$. The ramification points are $2g + 2$ poles. Note that for $k_i = -1$ there are $2g + 3$ poles and $\binom{2g+3}{1}$ choices for the cover, so in that case π is $(2g + 3) : 1$.

- Second type:

$$\mathcal{Q}_g^{hyp}(k_1^2, 2k_2 + 2) \xrightarrow{\pi} \mathcal{Q}_0(k_1, k_2, -1^{2g+1})$$

for $k_1 \geq -1$ odd, $k_2 \geq 0$ even, $g = \frac{1}{2}(k_1 + k_2 + 3)$. The ramification points are $2g + 1$ poles and the zero of order k_2 . Note that for $k_1 = -1$ there are $2g + 2$ poles and $\binom{2g+2}{1}$ choices for the cover, so in that case π is $(2g + 2) : 1$.

- Third type:

$$\mathcal{Q}_g^{hyp}(2k_1 + 2, 2k_2 + 2) \xrightarrow{\pi} \mathcal{Q}_0(2k_1, k_2, -1^{2g})$$

for k_1, k_2 even, $g = \frac{1}{2}(k_1 + k_2) + 2$. The ramification points are over all the singularities.

Except the special cases, π is always $1 : 1$.

We introduce the following notation for the general case:

$$\mathcal{Q}^{hyp}(\alpha) \xrightarrow[I:1]{\pi} \mathcal{Q}(\beta)$$

with $\alpha = (d_1^{\alpha_1}, \dots, d_n^{\alpha_n})$ and $\beta = (\tilde{d}_1^{\beta_1}, \dots, \tilde{d}_m^{\beta_m})$.

Let $d = \dim_{\mathbb{C}} \mathcal{Q}(\beta)$ be the complex dimension of the stratum that we consider.

Recall that, by definition, the volume of the hyperboloid of surfaces of area equal to $1/2$ is given by the volume of the cone underneath times the real dimension of the stratum:

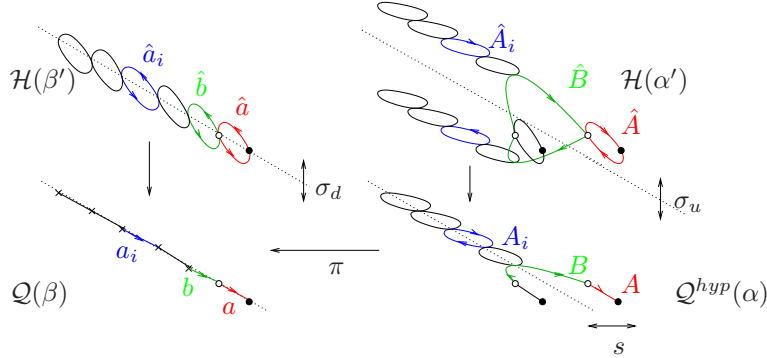
$$\text{Vol } \mathcal{Q}_1(\beta) = 2d \cdot \text{Vol}\{S \in \mathcal{Q}(\beta), \text{area}(S) \leq 1/2\}$$

Let S be a point in $\mathcal{Q}_1(\beta)$, and let S' be one of the I possible lifts $\pi^*(S)$. As S is of area $1/2$, S' is of area 1 so belongs to $\mathcal{Q}_2^{hyp}(\alpha)$. So the cone underneath $\mathcal{Q}_1(\beta)$ is in $1 : I$ correspondence with the cone underneath $\mathcal{Q}_2^{hyp}(\alpha)$. Now we want to compare the volume elements of $\mathcal{Q}^{hyp}(\alpha)$ and $\mathcal{Q}(\beta)$. So we have to understand how the lattice $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))_{\mathbb{C}}^*$ is lifted by π and compare it with the lattice $(H_1^-(\hat{S}', \hat{\Sigma}'; \mathbb{Z}))_{\mathbb{C}}^*$.

For the first type we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_{g+1}(k_1 + 1, k_2 + 1) & \xleftarrow{\dots\dots\dots} & \mathcal{H}_{2g+1}((2k_1 + 1)^2, (k_2 + 1)^2) \\ \downarrow & & \downarrow \\ \mathcal{Q}_0(k_1, k_2, -1^{2g+2}) & \xleftarrow[\substack{\pi \\ I:1}]{} & \mathcal{Q}_g^{hyp}(k_1^2, k_2^2) \end{array}$$

On $S \in \mathcal{Q}_0(k_1, k_2, -1^{2g+2})$ we consider the saddle connections defined by taking a broken line joining all the singularities except one pole, as in the picture above, such that a joins the two zeroes, b joins a zero to a pole, and a_i, b_i join the remaining poles except the last one, for i going from 1 to g . Then $\hat{a}, \hat{b}, \hat{a}_1, \dots, \hat{b}_g$ is a primitive basis of $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z})$ (cf Lemma 1). On the other hand consider the saddle connections on $\mathcal{Q}_g^{hyp}(k_1^2, k_2^2)$ constructed using a, b, a_1, \dots, b_g in the following way: for all a_i and b_i and for b , take the combination of the two lifts by π to obtain primitive cycles A_i, B_i , and B in $H_1(S', \Sigma', \mathbb{Z})$. Take only one of the two preimages of a to get a primitive cycle A . Then $\hat{A}, \hat{B}, \hat{A}_1, \dots, \hat{B}_g$ define a primitive basis of $H_1^-(\hat{S}', \hat{\Sigma}'; \mathbb{Z})$ (same arguments as in Lemma 1).



On the picture σ_u and σ_d are the involutions of the double covers and s is the hyperelliptic involution.

In this local coordinates volume elements are given by:

$$d\nu_{down} = d\hat{a} d\hat{b} d\hat{a}_1 \dots d\hat{b}_g = 2^{2d} da db da_1 \dots db_g$$

and

$$d\nu_{up} = d\hat{A} d\hat{B} d\hat{A}_1 \dots d\hat{B}_g = 2^{2d} dA dB dA_1 \dots dB_g$$

with $dA = \pi^*(da)$, $dB = 4\pi^*(db)$, $dA_i = 4\pi^*(da_i)$ and $dB_i = 4\pi^*(db_i)$.

So we obtain the following relation between the volume elements:

$$d\nu_{up} = 2^{2d-2} \pi^*(d\nu_{down})$$

Same considerations for the two other types give the same result.

So now we have all the elements to compute the relation between $\text{Vol}^{numb} \mathcal{Q}_1(\beta)$ and $\text{Vol}^{numb} \mathcal{Q}_1^{hyp}(\alpha)$:

$$\begin{aligned} \text{Vol}^{numb} \mathcal{Q}_1^{hyp}(\alpha) &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \text{Vol}^{unnumb} \mathcal{Q}_1^{hyp}(\alpha) \\ &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \cdot 2d \text{Vol}^{unnumb} \{S' \in \mathcal{Q}^{hyp}(\alpha), \text{area}(S') \leq 1/2\} \\ &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \cdot 2d \cdot \frac{1}{2^d} \text{Vol}^{unnumb} \{S \in \mathcal{Q}^{hyp}(\alpha), \text{area}(S) \leq 1\} \\ &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \cdot 2d \cdot \frac{1}{2^d} \cdot I \cdot 2^{2d-2} \text{Vol}^{unnumb} \{S \in \mathcal{Q}(\beta), \text{area}(S) \leq 1/2\} \\ &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \cdot I \cdot 2^{d-2} \text{Vol}^{unnumb} \mathcal{Q}_1(\beta) \\ &= \frac{\alpha_1! \dots \alpha_n!}{|\Gamma^{hyp}(\alpha)|} \cdot I \cdot 2^{d-2} \cdot \frac{|\Gamma(\beta)|}{\beta_1! \dots \beta_m!} \text{Vol}^{numb} \mathcal{Q}_1(\beta) \end{aligned}$$

Note that, for the first two types, the hyperelliptic involution exchanges the zeroes which are preimages of the same zero downstairs. So for these types $|\Gamma^{hyp}(\alpha)| = 2$. For the third type $|\Gamma^{hyp}(\alpha)| = 1$. Downstairs there is no symmetry for each stratum that we consider so $|\Gamma(\beta)| = 1$ for each β .

The values of the volumes of strata of quadratic differentials in genus 0 are given in [AEZ1], Theorem 1.6:

$$(28) \quad \text{Vol} \mathcal{Q}_1(d_1, \dots, d_n) = 2\pi^2 \prod_{i=1}^n v(d_i),$$

with

$$v(n) = \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$$

for $n \in \{-1, 0\} \cup \mathbb{N}$ and with

$$n!! = n(n-2)(n-4) \dots,$$

by convention $(-1)!! = 0!! = 1$.

In particular we have:

- for the first type ($k_1 \geq -1$ odd, $k_2 \geq -1$ odd, $(k_1, k_2) \neq (-1, -1)$, $d = 2g + 2$):

$$\text{Vol}^{numb} \mathcal{Q}_1(k_1, k_2, -1^d) = 2\pi^d \frac{k_1!!}{(k_1+1)!!} \cdot \frac{k_2!!}{(k_2+1)!!},$$

- for the second type ($k_1 \geq -1$ odd, $k_2 \geq 0$ even, $d = 2g + 1$):

$$\text{Vol}^{numb} \mathcal{Q}_1(k_1, k_2, -1^d) = 4\pi^{d-1} \frac{k_1!!}{(k_1+1)!!} \cdot \frac{k_2!!}{(k_2+1)!!},$$

- for the third type (k_1, k_2 even, $d = 2g$):

$$\text{Vol}^{numb} \mathcal{Q}_1(k_1, k_2, -1^d) = 8\pi^{d-2} \frac{k_1!!}{(k_1+1)!!} \cdot \frac{k_2!!}{(k_2+1)!!}.$$

So we obtain the result. \square

6.2. Volumes of hyperelliptic components of strata of Abelian differentials.

Proposition 7. *The volumes of hyperelliptic components of strata of Abelian differentials with area $1/2$ are given by the following formulae:*

$$(29) \quad \text{Vol}^{numb} \mathcal{H}_{1/2}^{hyp}(k-1) = \frac{2^{k+2}}{(k+2)!} \cdot \frac{(k-2)!!}{(k-1)!!} \cdot \pi^{k+1}$$

$$(30) \quad \text{Vol}^{numb} \mathcal{H}_{1/2}^{hyp} \left(\left(\frac{k}{2} - 1 \right)^2 \right) = \frac{2^{k+3}}{(k+2)!} \cdot \frac{(k-2)!!}{(k-1)!!} \cdot \pi^k$$

Proof. We recall here the two types of strata of Abelian differentials that contain hyperelliptic components (cf [KZ]):

- First type ($g \geq 2$):

$$\mathcal{H}^{hyp}(2g-2) \xrightarrow{\pi} \mathcal{Q}(2g-3, -1^{2g+1})$$

- Second type ($g \geq 2$):

$$\mathcal{H}^{hyp}((g-1)^2) \xrightarrow{\pi} \mathcal{Q}(2g-2, -1^{2g+2})$$

In both cases, π is an isomorphism. By conventions 2 and 1, the volume elements are chosen to be invariant under this isomorphism, so we have:

$$\text{Vol}^{unnumb} \mathcal{H}_1^{hyp}(2g-2) = \text{Vol}^{unnumb} \mathcal{Q}_1(2g-3, -1^{2g+1})$$

$$\text{Vol}^{unnumb} \mathcal{H}_1^{hyp}((g-1)^2) = \text{Vol}^{unnumb} \mathcal{Q}_1(2g-2, -1^{2g+2})$$

So considering the naming of the singularities we obtain:

$$\begin{aligned} \text{Vol}^{numb} \mathcal{H}_1^{hyp}(2g-2) &= \frac{1}{(2g+1)!} \text{Vol}^{numb} \mathcal{Q}_1(2g-3, -1^{2g+1}) \\ &= \frac{2}{(2g+1)!} \cdot \frac{(2g-3)!!}{(2g-2)!!} \cdot \pi^{2g} \\ \text{Vol}^{numb} \mathcal{H}_1^{hyp}((g-1)^2) &= \frac{2!}{2} \text{Vol}^{unnumb} \mathcal{H}_1^{hyp}((g-1)^2) \\ &= \frac{2}{(2g+2)!} \text{Vol}^{numb} \mathcal{Q}_1(2g-2, -1^{2g+2}) \\ &= \frac{8}{(2g+2)!} \cdot \frac{(2g-2)!!}{(2g-1)!!} \cdot \pi^{2g} \end{aligned}$$

by plugging values of volumes given in (28). For the first type, for $k = 2g - 1$ we have $\dim_{\mathbb{C}} \mathcal{H}(k-1) = 2g = k + 1$. For the second type, for $k = 2g$ we have $\dim_{\mathbb{C}} \mathcal{H} \left(\left(\frac{k}{2} - 1 \right)^2 \right) = 2g + 1 = k + 1$. Finally, using Lemma 4 we obtain the result. \square

7. COHERENCE OF THE FORMULAE FOR THE HYPERELLIPTIC COMPONENTS OF STRATA

7.1. Configurations containing cylinders in hyperelliptic components. The complete list of all configurations of $\hat{\text{homologous}}$ saddle connexions is described by C. Boissy in [Bo]. We extract from this list the configurations containing cylinders, and recall them on Figure 8.

The following proposition precise the boundary of the hyperelliptic components of strata.

Proposition 8. *Let S be a flat surface in a hyperelliptic component of a stratum of quadratic differentials $\mathcal{Q}^{\text{hyp}}(\alpha)$. Les γ be a collection of $\hat{\text{homologous}}$ saddle connexions realizing a configuration \mathcal{C} on the previous list (Figure 8). Then the two possible boundary components $S_1, S_2 \in \mathcal{Q}(\alpha'_1), \mathcal{Q}(\alpha'_2)$ of S are hyperelliptic.*

For every surfaces $S_1 \in \mathcal{Q}^{\text{hyp}}(\alpha'_1), S_2 \in \mathcal{Q}^{\text{hyp}}(\alpha'_2)$, there is at least one way to assemble S_1 and eventually S_2 following configuration \mathcal{C} to obtain a hyperelliptic surface S .

Proof. If $S \in \mathcal{Q}^{\text{hyp}}(\alpha)$, following Lemma 10.3 of [EMZ], we may assume that the hyperelliptic involution fixes each boundary component. So it implies that S_1 and S_2 are also hyperelliptic.

If $S_1 \in \mathcal{Q}^{\text{hyp}}(\alpha'_1)$ and $S_2 \in \mathcal{Q}^{\text{hyp}}(\alpha'_2)$, we can make the surgeries on the boundary surfaces in such a way that the new surfaces stay invariant under the hyperelliptic involution (cf § 14 in [EMZ]). Then we construct an application on S that acts on each boundary component as the hyperelliptic involution for the corresponding stratum and on the cylinder either by fixing its boundaries and rotating or by exchanging its two boundaries depending on the configuration \mathcal{C} , in such a way that the global application is an involution of S . The action of the hyperelliptic involution on the configurations is detailed in [Bo]. \square

7.2. Siegel–Veech constants for configurations in hyperelliptic components. Note that the complex dimension of any hyperelliptic component is given by: $\dim_{\mathbb{C}} \mathcal{Q}^{\text{hyp}}(k_1^2, 2k_2 + 2) = \dim_{\mathbb{C}} \mathcal{Q}^{\text{hyp}}(k_1^2, k_2^2) = \dim_{\mathbb{C}} \mathcal{Q}^{\text{hyp}}(2k_1 + 2, 2k_2 + 2) = k_1 + k_2 + 4 =: d$.

First recall that the constants for the entire components are known ([EKZ]):

Lemma 6.

$$(31) \quad c_{\text{area}}(\mathcal{Q}^{\text{hyp}}(\alpha)) = \frac{k_1 + k_2 + 4}{4\pi^2} \left(2 + \frac{1}{(k_1 + 2)(k_2 + 2)} \right)$$

for $\alpha = (k_1^2, k_2^2)$, $\alpha = (k_1^2, 2k_2 + 2)$ or $\alpha = (2k_1 + 2, 2k_2 + 2)$.

Proof. It is a direct consequence of Corollary 3 in [EKZ]. We L^- denote the sum of the Lyapunov exponents $\lambda_1^-, \dots, \lambda_{g_{\text{eff}}}^-$ for the hyperelliptic component $\mathcal{Q}^{\text{hyp}}(\alpha)$. Recall that by Theorem 1 of [EKZ], we have:

$$c_{\text{area}}(\mathcal{Q}^{\text{hyp}}(\alpha)) = \frac{3}{\pi^2} (L^- - I - K)$$

where

$$I = \frac{1}{4} \sum_{d_j \text{ odd}} \frac{1}{d_j + 2}, \quad K = \frac{1}{24} \sum_j \frac{d_j(d_j + 4)}{d_j + 2}.$$

Configurations with cylinders		Boundary strata
$\mathcal{Q}^{hyp}(k_1^2, k_2^2), k_1, k_2 \text{ odd}, (k_1, k_2) \neq (-1, -1)$		
\mathcal{C}_1		$\mathcal{H}^{hyp}(k_1 - 1), \mathcal{H}^{hyp}(k_2 - 1)$
$\mathcal{C}_2(k_i)$	$\{k_j^2\}$	$\mathcal{Q}_{q-1}^{hyp}(k_j^2, (k_i - 2)^2)$
$\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2), k_1 \text{ odd and } k_2 \text{ even}$		
\mathcal{C}_1		$\mathcal{H}^{hyp}(k_1 - 1), \mathcal{H}^{hyp}((\frac{k_2}{2} - 1)^2)$
\mathcal{C}_2	$\{k_1^2\}$	$\mathcal{Q}_{q-1}^{hyp}(k_1^2, 2k_2 - 2)$
\mathcal{C}_3	$\{2k_2 + 2\}$	$\mathcal{Q}_{q-1}^{hyp}(2k_2 + 2, (k_1 - 2)^2)$
$\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2), k_1, k_2 \text{ even}$		
\mathcal{C}_1		$\mathcal{H}^{hyp}((\frac{k_1}{2} - 1)^2), \mathcal{H}^{hyp}((\frac{k_2}{2} - 1)^2)$
$\mathcal{C}_2(k_i)$	$\{2k_j + 2\}$	$\mathcal{Q}_{q-1}^{hyp}(2k_j + 2, 2k_i - 2)$
Additional configurations		
$\mathcal{Q}^{hyp}(k_i^2, 2)$		$\mathcal{H}^{hyp}(k_i - 1)$
$\mathcal{Q}^{hyp}(2k_i + 2, 2)$		$\mathcal{H}^{hyp}((\frac{k_i}{2} - 1)^2)$
$\mathcal{Q}(2, 2)$		\emptyset

FIGURE 8. Configurations containing cylinders for hyperelliptic components of strata of quadratic differentials

Corollary 3 in [EKZ] gives the values of L^- for hyperelliptic components, that we recall here:

$$\begin{aligned} L^- &= \frac{k_1 + k_2 + 4}{4} \left(1 + \frac{1}{(k_1 + 2)(k_2 + 2)} \right) && \text{for } \mathcal{Q}^{hyp}(k_1^2, k_2^2) \\ L^- &= \frac{k_1 + k_2 + 4}{4} \left(1 + \frac{1}{k_1 + 2} \right) && \text{for } \mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2) \\ L^- &= \frac{k_1 + k_2 + 4}{4} && \text{for } \mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2) \end{aligned}$$

□

Now to compute these constants for each configuration, we use the method described in § 2.5, we follow step by step the computations of § 3 and make only a few adjustments.

First the boundary of $\mathcal{Q}^{hyp}(\alpha)$ is described by Proposition 8 and consists of hyperelliptic components of the boundary strata of $\mathcal{Q}(\alpha)$, so $\text{Vol}_* \mathcal{Q}_1^{\bar{c}}(\text{comp}, \mathcal{C})$ will be expressed in terms of $\prod_i \text{Vol } \mathcal{Q}^{hyp}(\alpha'_i)$. Second we will have to take care of the symmetries induced by the hyperelliptic involution.

Finally we obtain the following variation of formula (17):

$$(32) \quad c_{area}(\mathcal{C}) = M \frac{4q_1 + q_2 \prod_i (a_i - 1)! 2^{a_i} \text{Vol } \mathcal{H}_1^{hyp}(\alpha_i) \prod_j (b_j - 1)! \text{Vol } \mathcal{Q}_1^{hyp}(\beta_j)}{2^{m+q+3} (\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)! \text{Vol } \mathcal{Q}_1^{hyp}(\alpha)}$$

where $M = \frac{M_s M_c}{M_t}$ and M_c , M_t are given by (8) and (11), and M_s takes care of the hyperelliptic involution. It will be detailed for each configuration in the following paragraphs.

7.2.1. $\mathcal{Q}^{hyp}(k_1^2, k_2^2)$, with k_1 and k_2 odd and $(k_1, k_2) \neq (-1, -1)$.

Proposition 9. *For the first type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 8 are given by the following formulae:*

$$(33) \quad c_{area}(\mathcal{C}_1) = \frac{k_1 + k_2 + 4}{4(k_1 + 2)(k_2 + 2)\pi^2}$$

$$(34) \quad c_{area}(\mathcal{C}_2(k_i)) = \frac{(k_1 + k_2 + 4)(k_i + 1)}{(k_1 + k_2 + 2)2\pi^2}$$

Moreover:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2, k_2^2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_2(k_2))$$

Example 2.

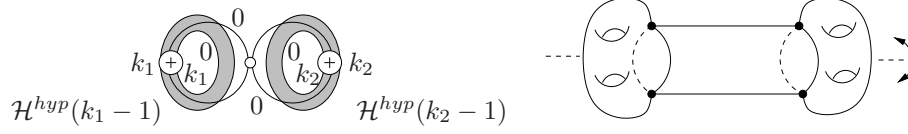
$$c_{area}(\mathcal{Q}(1^2, (-1)^2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(1)) = \frac{1}{3\pi^2} + \frac{2}{\pi^2} = \frac{7}{3\pi^2}$$

$$c_{area}(\mathcal{Q}(1^4)) = c_{area}(\mathcal{C}_1) + 2c_{area}(\mathcal{C}_2(1)) = \frac{1}{6\pi^2} + \frac{3}{\pi^2} = \frac{19}{6\pi^2}$$

Proof. For each configuration on the list (Figure 8) we apply the formula (32).

(1) Configuration \mathcal{C}_1 for $k_1 \geq 1, k_2 \geq 1$:

- $M_c = 4^2, M_t = 1$

FIGURE 9. Configuration \mathcal{C}_1 for $\mathcal{Q}^{hyp}(k_1^2, k_2^2)$

- $M_s = \frac{4k_1k_2}{|\Gamma|}$, $|\Gamma| = 2 \cdot 2$ because of the action of the hyperelliptic involutions on the two boundary components.
- $q_1 = 0, q_2 = 1, m = 2$
- $\dim_{\mathbb{C}} \mathcal{H}(k_i - 1) = k_i + 1$

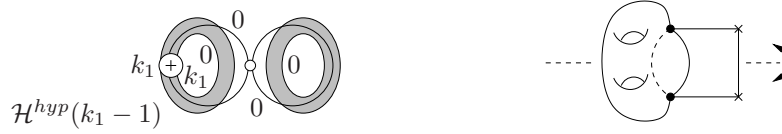
Applying formula (32) we get:

$$c_{area}(\mathcal{C}_1) = 4^2 k_1 k_2 \frac{1}{2^6} \frac{k_1! k_2! \text{Vol } \mathcal{H}_{1/2}^{hyp}(k_1 - 1) \text{Vol } \mathcal{H}_{1/2}^{hyp}(k_2 - 1)}{(k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, k_2^2)}$$

Plugging values (29) and (21) of volumes we obtain (33).

Taking care of the numbering of the zeroes, there is only one such configuration.

- (2) Configuration \mathcal{C}_1 for $k_1 \geq 1, k_2 = -1$: it is a degeneration of the first configuration:

FIGURE 10. Configuration \mathcal{C}_1^{deg} for $\mathcal{Q}^{hyp}(k_1^2, (-1)^2)$

- $M_c = 4^2, M_t = 1$
- $M_s = \frac{2k_1}{|\Gamma|}$, $|\Gamma| = 2$ (hyperelliptic involution)
- $q_1 = 0, q_2 = 1, m = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(k_i - 1) = k_i + 1$

Applying formula (32) we get:

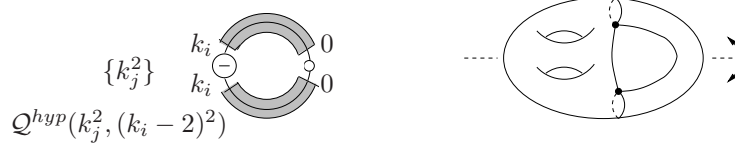
$$c_{area}(\mathcal{C}_1^{deg}) = 4^2 \frac{2k_1}{2} \frac{1}{2^5} \frac{k_1! \text{Vol } \mathcal{H}_{1/2}^{hyp}(k_1 - 1)}{(k_1 + 2)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, (-1)^2)}$$

Plugging values (29) and (21) of volumes we obtain

$$c_{area}(\mathcal{C}_1^{deg}) = \frac{k_1 + 3}{(k_1 + 2)4\pi^2}$$

which is equivalent to (33).

- (3) Configuration $\mathcal{C}_2(k_i)$ for $k_i \geq 1$:
- $M_c = 4^2, M_t = 1$

FIGURE 11. Configuration \mathcal{C}_2 for $\mathcal{Q}^{hyp}(k_1^2, k_2^2)$

- $M_s = k_i$. Here once we have chosen a direction in which we make the surgery for one of the two zeroes of order $k_i - 2$ of the boundary surface, the direction for the other zero is determined by the hyperelliptic involution.
- $q_1 = 1, q_2 = 0, m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(k_j^2, (k_i - 2)^2) = k_1 + k_2 + 2$

Applying formula (32) we get:

$$c_{area}(\mathcal{C}_2(k_i)) = 4^2 k_i \frac{4 (k_1 + k_2 + 1)! \text{Vol } \mathcal{Q}_1^{hyp}(k_j^2, (k_i - 2)^2)}{2^5 (k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, k_2^2)}$$

Plugging values (21) of volumes we obtain (34).

For each k_i there is only one such configuration.

Summing on all configurations we find the known value (6) for the entire hyperelliptic component. \square

7.2.2. $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$ with k_1 odd and k_2 even.

Proposition 10. *For the second type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 8 are given by the following formulae:*

$$(35) \quad c_{area}(\mathcal{C}_1) = \frac{(k_1 + k_2 + 4)}{(k_1 + 2)(k_2 + 2)4\pi^2}$$

$$(36) \quad c_{area}(\mathcal{C}_2) = \frac{(k_2 + 1)(k_1 + k_2 + 4)}{2(k_1 + k_2 + 2)\pi^2}$$

$$(37) \quad c_{area}(\mathcal{C}_3) = \frac{(k_1 + 1)(k_1 + k_2 + 4)}{2(k_1 + k_2 + 2)\pi^2}$$

Moreover if $k_2 \neq 0$:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2) + c_{area}(\mathcal{C}_3).$$

If $k_2 = 0$ and $k_1 \neq -1$, for the additional configuration we have:

$$(38) \quad c_{area}(\mathcal{C}_{add}) = \frac{5(k_1 + 4)}{8(k_1 + 2)\pi^2}$$

and we have:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2, 2)) = c_{area}(\mathcal{C}_{add}) + c_{area}(\mathcal{C}_3).$$

For $\mathcal{Q}(2, -1^2)$, cf § 7.3.1.

Example 3.

$$c_{area}(\mathcal{Q}(2, 1^2)) = c_{area}(\mathcal{C}_{add}) + c_{area}(\mathcal{C}_3) = \frac{25}{24\pi^2} + \frac{5}{3\pi^2} = \frac{65}{24\pi^2}$$

Proof. For each configuration on the list (Figure 8) we apply the formula (32).

(1) Configuration \mathcal{C}_1 :

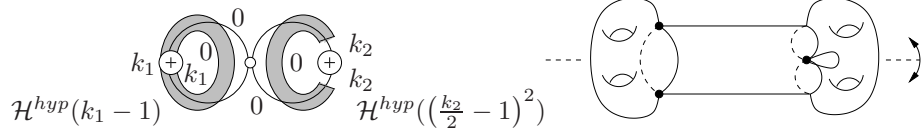


FIGURE 12. Configuration \mathcal{C}_1 for $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$

- $M_c = 4^2$, $M_t = 2$
 - $M_s = k_1 k_2$, same reasons as for the first type.
 - $q_1 = 0$, $q_2 = 1$, $m = 2$
 - $\dim_{\mathbb{C}} \mathcal{H}(k_1 - 1) = k_1 + 1$, $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_2}{2} - 1)^2) = k_2 + 1$
- Applying formula (32) we get:

$$c_{area}(\mathcal{C}_1) = 4^2 \frac{k_1 k_2}{2} \frac{1}{2^6} \frac{k_1! k_2! \text{Vol } \mathcal{H}_{1/2}^{hyp}(k_1 - 1) \text{Vol } \mathcal{H}_{1/2}^{hyp}((\frac{k_2}{2} - 1)^2)}{(k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, 2k_2 + 2)}$$

Plugging values (29), (30) and (23) of volumes we obtain (35).

There is only one such configuration.

(2) Configuration \mathcal{C}_2 :



FIGURE 13. Configuration \mathcal{C}_2 for $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$

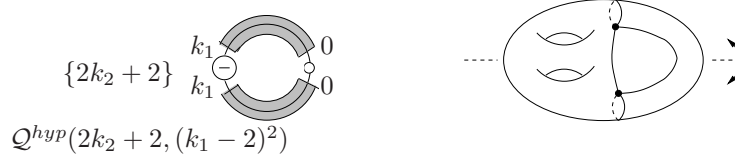
- $M_c = 4^2$, $M_t = 1$
 - $M_s = \frac{2k_2}{|\Gamma|}$, $|\Gamma| = 2$.
 - $q_1 = 1$, $q_2 = 0$, $m = 1$
 - $\dim_{\mathbb{C}} \mathcal{Q}(k_1^2, 2k_2 - 2) = k_1 + k_2 + 2$
- Applying formula (32) we get:

$$c_{area}(\mathcal{C}_2) = 4^2 \frac{2k_2}{2} \frac{4}{2^5} \frac{(k_1 + k_2 + 1)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, 2k_2 - 2)}{(k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1^{hyp}(k_1^2, 2k_2 + 2)}$$

Plugging values (23) of volumes we obtain (36). There is only one configuration of this type.

(3) Configuration \mathcal{C}_3 :

- $M_c = 4^2$, $M_t = 1$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 0$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(2k_2 + 2, (k_1 - 2)^2) = k_1 + k_2 + 2$

FIGURE 14. Configuration \mathcal{C}_3 for $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$

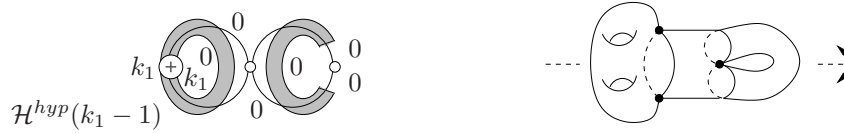
Applying formula (32) we get:

$$c_{area}(\mathcal{C}_3) = 4^2 k_1 \frac{4 (k_1 + k_2 + 1)! \text{Vol } \mathcal{Q}_1(2k_2 + 2, (k_1 - 2)^2)}{2^5 (k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1(k_1^2, 2k_2 + 2)}$$

Plugging values (23) of volumes we obtain (37). There is only one configuration of this type.

Summing on all the three configurations we find the known value (6) for the entire component.

(4) For $\mathcal{Q}^{hyp}(k_1^2, 2)$ there is an additional configuration:

FIGURE 15. Additional configuration for $\mathcal{Q}^{hyp}(k_1^2, 2)$

- $M_c = 4^3$, $M_t = 2$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 1$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(k_1 - 1) = k_1 + 1$

Applying formula (32) we get:

$$c_{area}(\mathcal{C}_{add}) = 4^3 \frac{k_1}{2} \frac{4 + 1}{2^6} \frac{k_1! \text{Vol } \mathcal{H}_{1/2}(k_1 - 1)}{(k_1 + 3)! \text{Vol } \mathcal{Q}_1(k_1^2, 2)}$$

Plugging values (29) and (21) of volumes we obtain (10). □

7.2.3. $\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2)$ with k_1 and k_2 even.

Proposition 11. *For the third type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 8 are given by the following formulae:*

$$(39) \quad c_{area}(\mathcal{C}_1) = \frac{k_1 + k_2 + 4}{4(k_1 + 2)(k_2 + 2)\pi^2}$$

$$(40) \quad c_{area}(\mathcal{C}_2(k_i)) = \frac{(k_1 + k_2 + 4)(k_i + 1)}{2(k_1 + k_2 + 2)\pi^2}$$

Moreover:

$$c_{area}(\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_2(k_2))$$

If $k_2 = 0$ and $k_1 \neq 0$:

$$(41) \quad c_{area}(\mathcal{C}_{add}) = \frac{5(k_1 + 4)}{8(k_1 + 2)\pi^2}$$

and:

$$c_{area}(\mathcal{Q}^{hyp}(2k_1 + 2, 2)) = c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_{add}).$$

For $\mathcal{Q}(2, 2)$, cf § 7.3.2.

Proof. For each configuration on the list (Figure 8) we apply the formula (32).

(1) Configuration \mathcal{C}_1 :

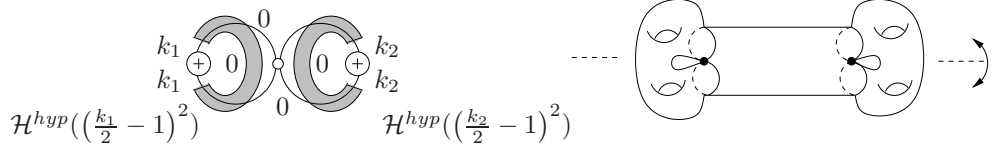


FIGURE 16. Configuration \mathcal{C}_1 for $\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2)$

- $M_c = 4^2$, $M_t = 1$, $M_s = k_1 k_2$
 - $q_1 = 0$, $q_2 = 1$, $m = 2$
 - $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_i}{2} - 1)) = k_i + 1$
- Applying formula (32) we get:

$$c_{area}(\mathcal{C}_1) = 4^2 \frac{k_1 k_2}{1} \frac{1}{2^6} \frac{k_1! k_2! \text{Vol } \mathcal{H}_{1/2}((\frac{k_1}{2} - 1)^2) \text{Vol } \mathcal{H}_{1/2}((\frac{k_2}{2} - 1)^2)}{(k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1(2k_1 + 2, 2k_2 + 2)}$$

Plugging values (30) and (24) of volumes we obtain (39).

(2) Configuration $\mathcal{C}_2(k_i)$:



FIGURE 17. Configuration \mathcal{C}_2 for $\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2)$

- $M_c = 4^2$, $M_t = 1$, $M_s = k_i$
 - $q_1 = 1$, $q_2 = 0$, $m = 1$
 - $\dim_{\mathbb{C}} \mathcal{Q}(2k_j + 2, 2k_i - 2) = k_1 + k_2 + 2$
- Applying formula (32) we get:

$$c_{area}(\mathcal{C}_2) = 4^2 k_i \frac{4}{2^5} \frac{(k_1 + k_2 + 1)! \text{Vol } \mathcal{Q}_1(2k_j + 2, 2k_i - 2)}{(k_1 + k_2 + 3)! \text{Vol } \mathcal{Q}_1(2k_1 + 2, 2k_2 + 2)}$$

Plugging values (24) of volumes we obtain (40).

(3) For $\mathcal{Q}^{hyp}(2k_i + 2, 2)$ there is an additional configuration:

- $M = 4^3$, $M_t = 2$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 1$, $m = 1$

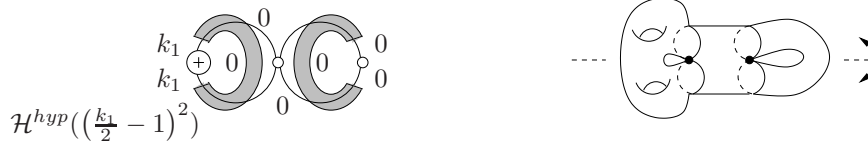


FIGURE 18. Additional configuration for $\mathcal{Q}^{hyp}(2k_1 + 2, 2)$

- $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_1}{2} - 1)^2) = k_1 + 1$
Applying formula (32) we get:

$$c_{area}(\mathcal{C}_{add}) = 4^3 \frac{k_1}{2} \frac{5}{2^5} \frac{k_1! \text{Vol } \mathcal{H}_{1/2}((\frac{k_1}{2} - 1)^2)}{(k_1 + 3)! \text{Vol } \mathcal{Q}_1(2k_1 + 2, 2)}$$

Plugging values (29) and (21) of volumes we obtain (11). □

7.3. Special cases: empty boundary stratum. The strata $\mathcal{Q}(2, -1^2)$ and $\mathcal{Q}(2, 2)$ have no boundary stratum (cf §3.3.4), so their configurations are degenerations of the configurations presented on the previous paragraph. With $\mathcal{Q}(-1^4)$ they are the only strata with an empty boundary stratum.

Note that furthermore they are connected.

7.3.1. Stratum $\mathcal{Q}(2, -1^2)$. This stratum is hyperelliptic of second type with $k_2 = 0$ and $k_1 = -1$.

Recall the value of the volume computed in (26):

$$\text{Vol } \mathcal{Q}_1^{num}(2, -1^2) = \frac{4}{3}\pi^2.$$

For an alternative computation of this volume using graphs, see Appendix B.1.

The only one configuration is a degeneration of the configuration of Figure 15, shown in Figure 19.

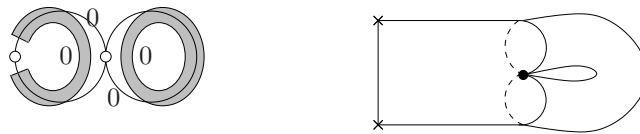


FIGURE 19. The only configuration of $\mathcal{Q}(2, -1, -1)$ containing cylinders, and its topological picture

We have the following configuration data:

- $M_c = 4^3$, $M_t = 2$
- $q_1 = 1 = q_2$

In this case we have to use a variation of the main formula, given by equation (16), we obtain:

$$\begin{aligned} c_{area}(\mathcal{C}_2) &= \frac{1}{2} 4^3 \frac{4+1}{2^5(2)! \text{Vol } \mathcal{Q}_1(2, -1^2)} \\ &= \frac{5}{2 \text{Vol } \mathcal{Q}_1(2, -1^2)} \end{aligned}$$

The configuration counts only once, so with the computed value of the volume it gives:

$$c_{area}(\mathcal{Q}_1(2, -1^2)) = \frac{15}{8\pi^2}$$

which is the known value of $c_{area}(\mathcal{Q}(2, -1^2))$

7.3.2. *Stratum* $\mathcal{Q}(2, 2)$. This stratum is hyperelliptic of the third type with $k_1 = k_2 = 0$.

Recall the value of the volume computed in (27):

$$\text{Vol } \mathcal{Q}_1^{num}(2, 2) = \frac{4}{3}\pi^2.$$

There is only one configuration shown on Figure 20.

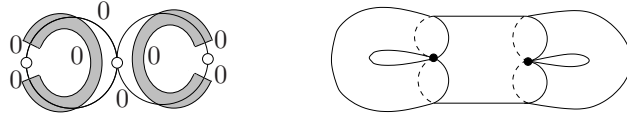


FIGURE 20. Configuration \mathcal{C} for $\mathcal{Q}(2, 2)$

Data:

- $M_c = 4^4$, $M_t = 2$
- $q_1 = 2$, $q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(2, 2) = 4$

Equation (16) gives:

$$\begin{aligned} c_{area}(\mathcal{C}) &= \frac{1}{2} 4^4 \frac{4 \cdot 2 + 1}{2^6(3)! \text{Vol } \mathcal{Q}(2, 2)} \\ &= \frac{3}{\text{Vol } \mathcal{Q}_1(2, 2)} \end{aligned}$$

The configuration counts only once, so with the computed value of the volume it gives:

$$c_{area}(\mathcal{Q}(2, 2)) = \frac{9}{4\pi^2}$$

which is the known value for the stratum.

8. VOLUMES OF STRATA OF COMPLEX DIMENSION SMALLER THAN 5

For strata of complex dimension $d \leq 5$, we can use the ideas developed by Eskin and Okounkov in [EOk], Zorich in [Z1], Athreya Eskin and Zorich in [AEZ2], and compute volumes by counting “integer points” in the stratum.

The relation between volume and number of “integer points” is given in § 2.3 of [AEZ2]:

Proposition 12 (Athreya-Eskin-Zorich).

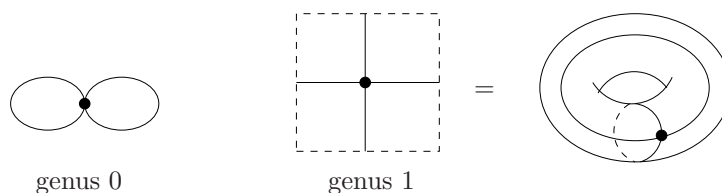
$$(42) \quad \text{Vol } \mathcal{Q}_1(\alpha) = 2d \cdot \lim_{N \rightarrow \infty} N^{-d} \cdot (\text{Number of “integer points” of area at most } N/2 \text{ in } \mathcal{Q}(\alpha))$$

Here we recall briefly the techniques of Athreya, Eskin and Zorich to count integer points (or square-tiles surfaces, or pillowcase covers) in genus 0, and explain how generalize them to genus $g > 0$.

A flat surface (S, ω) corresponding to an integer point, i.e. a point in the lattice $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))_{\mathbb{C}}^*$ in local coordinates, can be decomposed into horizontal cylinders with half-integer or integer widths, with zeroes and poles lying on the boundaries of these cylinders, that are called singular layers in [AEZ2]. Each layer defines a ribbon graph (graph with a tubular neighbourhood inside the surface), called *map* in combinatorics. A zero of order d_i belonging to a layer corresponds to a vertex of valency $d_i + 2$ in the associated graph, and edges of the graph emerging from this vertex correspond to horizontal rays emerging from the zero in the surface. The graph is metric: edges have half-integer lengths. A ribbon graph or a map carries naturally a genus: it is the minimal genus of the surface in which it can be embedded. So a ribbon graph associated to a singular layer in S has a genus lower or equal to the genus g of S . Also a ribbon graph has some faces corresponding to the connected components of its complementary in the minimal surface in which it can be embedded. In our case faces correspond to cylinders emerging from the layer. In genus 0 each face corresponds to a distinct cylinder, in higher genus some cylinders may have both of their boundaries glued to the same layer. For a ribbon graph Γ we have the Euler relation:

$$\chi_{\Gamma} = 2 - 2g_{\Gamma} = V_{\Gamma} - E_{\Gamma} + F_{\Gamma}$$

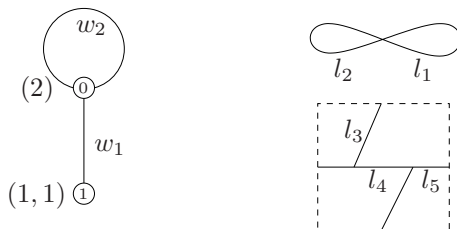
where g_{Γ} is the genus of Γ , V_{Γ} , E_{Γ} and F_{Γ} the number of respectively vertices, edges and faces of Γ . In the figure below we represent the two maps with one 4-valent vertex: one is of genus 0 and has 3 faces, the other is of genus 1 and has 1 face.



We encode the decomposition of the surface S into horizontal cylinders in a supplementary graph T , by representing each singular layer by a point in this graph and each cylinder emerging from a layer by an edge emerging from the corresponding vertex. So a layer with k faces corresponds to a k -valent vertex in T . We record also the information on the order of the zeroes lying in each layer, and on the genus

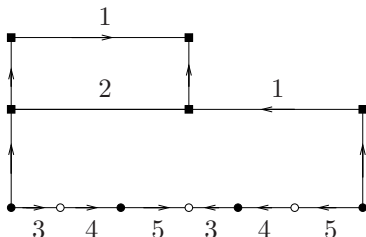
of the ribbon graph: that gives a decoration of the graph T . For surfaces S of genus 0 this graph is a tree.

As an example we consider a surface in $\mathcal{Q}(2, 1^2)$ represented by the following graph:



On the left we figure the graph T . The lower vertex represents a ribbon graph of genus 1 with two zeroes of order 1 (two 3-valent vertices): the corresponding layer is drawn on the right. The higher vertex corresponds to the ribbon graph of genus 0 with one 4-valent vertex (zero of order 2) drawn on the right. The width w_i of the cylinders and the lengths l_i of the edges of the ribbon graphs are also recorded.

Below is a flat representation of a surface corresponding to the configuration described above.



Note that the genus of S is the sum of the genera of the vertices of T , and the genus created by loops in the graph T : namely, the dimension of the homology of the graph T . In the example, the surface is of genus 2.

Note also that horizontal cylinders in S which are homologous to 0 correspond to separating edges on the graph T . It will be useful because with Convention 2, the width w of a cylinder is an integer if its waist curve is homologous to 0, and half-integer otherwise. In the example w_1 is integer and w_2 half-integer (furthermore here w_1 is necessarily equal to $2w_2$).

We have to choose the l_i such that the length of the boundaries of the faces of the ribbon graphs Γ_j correspond to the w_k . In the example we have necessarily $w_2 = l_1 = l_2$ and $w_1 = 2l_1 = 2w_2 = 2(l_3 + l_4 + l_5)$. So we have only one choice for l_1 and l_2 and exactly $\sum_{i=1}^{w_1-2} (i-1) = \frac{(w_1-2)^2}{2}$ choices for (l_3, l_4, l_5) (see also Lemma 7), because with the convention 2, w_2 is an integer and the l_i are half-integer.

To count surfaces of area lower than $N/2$ corresponding to lattice points, we have to sum on the possible graphs T and on the possible corresponding layers Γ , the number of distinct flat surfaces of this combinatorial type. So for a fixed graph T and fixed layers Γ_i we have to count the number of twists t_j , widths w_i , heights h_i and lengths of saddle connexions l_i satisfying the combinatorial configuration, and such that the area $w \cdot h = \sum_i w_i h_i$ is lower or equal to $N/2$. More precisely by (42) we have to get the asymptotic of this number as N goes to infinity. In the

example all the l_i are half-integer, h_1, t_1, h_2, t_2 also because they are coordinates of saddle connexions that are non homologous to zero, w_2 is half-integer and w_1 is integer. Twists t_1 and t_2 take respectively $2w_1$ and $2w_2$ half-integer values. We have already seen that the l_i take $\frac{(w_1-2)^2}{2}$ values (with the condition $w_1 = 2w_2$). So we want to find the asymptotic of

$$\sum_{\substack{w_1 h_1 + w_2 h_2 \leq N/2 \\ w_1 \in \mathbb{N}, \\ w_2, h_1, h_2 \in \mathbb{N}/2}} 2w_1 2w_2 \frac{(w_1 - 2)^2}{2} \mathbb{1}_{\{w_1 = 2w_2\}} = \sum_{\substack{w(h_1 + 2h_2) \leq N/2 \\ w \in \mathbb{N}, h_1, \\ h_2 \in \mathbb{N}/2}} 8w^2 \frac{(2w - 2)^2}{2}$$

Remark that since we want only the term of higher order in N we just need to take the term of higher order in w_i , so we can replace $\frac{(2w-2)^2}{2}$ by $\frac{(2w)^2}{2} = 2w^2$. In general the asymptotic for such sums is given by Lemma 3.7 of [AEZ2]. For this particular case, it is given by Lemma 9, and we obtain $\frac{N^5}{10}(32\zeta(4) - 33\zeta(5))$.

This approach is somehow limited because we need to know all the ribbon graphs of a certain type and the number of these ribbon graphs increases fast as the dimension of the stratum grows. So we apply this method to strata of complex dimension $d \leq 5$, using the complete description of ribbon graphs with at most 5 edges given in [JV]: recall that a zero of order d_i corresponds in the ribbon graph to a vertex with $d_i + 2$ adjacent edges, so the maximal number of edges of a ribbon graph in a strata $\mathcal{Q}(d_1, \dots, d_n)$ is

$$\frac{\sum_{i=1}^n d_i}{2} = 2g - 2 + n = \dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_n).$$

In genus 0, Athreya, Eskin and Zorich were able to compute the volumes of all strata of type $\mathcal{Q}(1^K, -1^{K+4})$ with this method because they used a formula which gives directly the number of ways the cylinders of widths w_i can be glued at a vertex j of a tree T . This formula was deduced from a formula of M. Kontsevich by a recurrence on the number of poles. The formula of Kontsevich works also for higher genus, but for distinct widths w_i , and since cylinders can form some loops in the surface, it is not obvious to get a general formula for the higher genus case, even for the strata $\mathcal{Q}(1^k, -1^l)$.

Convention 4. In the following we write the half-integers in lower case and the integers in capitals.

9. FIRST EXAMPLE: $\mathcal{Q}(5, -1)$

For this stratum, the Siegel–Veech constant $c_{area}(\mathcal{Q}(5, -1))$ is known, cf Theorem 9.1. in [CM]. We compute the volume of this stratum following the method described in section 8. We evaluate also all combinatorial parameters appearing in (1), and so we check formula (1) in this case.

9.1. Volume of $\mathcal{Q}(5, -1)$. We use here the method described in § 8 to compute by hands the volume of $\mathcal{Q}(5, -1)$. In this case, there are only two possible graphs T , and for each graph, only two possible layers. This gives four configurations (note that here we do not speak about configurations of homologous cylinders, but about configurations of horizontal cylinders for integer surfaces in the stratum). The computations of the asymptotics are detailed in the appendix C.

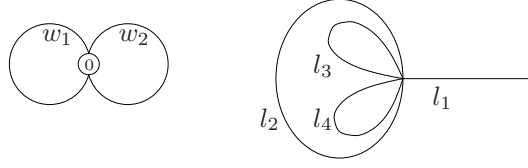


FIGURE 21. Configuration 1

- Configuration 1:

Convention 2 implies that all parameters w_i, h_i, t_i, l_i are half-integers. The possible lengths of the waist curves of the cylinders are $l_3, l_4, l_2 + 2l_1$ and $l_2 + l_3 + l_4$. Since $l_2 + l_3 + l_4 > l_3$ and $l_2 + l_3 + l_4 > l_4$ we should have $l_3 = l_4$ and $l_2 + 2l_1 = l_2 + 2l_3$:

$$\begin{cases} w_1 = l_3 = l_4 \\ w_2 = l_2 + 2l_1 = l_2 + 2l_3 \end{cases}$$

There is one way to find such (l_1, l_2, l_3, l_4) , if $2w_1 < w_2$. The contribution to the counting for this configuration is:

$$\sum_{(w_1 h_1 + w_2 h_2) \leq N/2} 4w_1 w_2 (\mathbb{1}_{\{2w_1 < w_2\}}) = \sum_{(W_1 H_1 + W_2 H_2) \leq 2N} W_1 W_2 (\mathbb{1}_{\{2W_1 < W_2\}})$$

- Configuration 2:

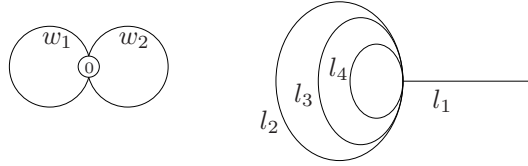


FIGURE 22. Configuration 2

All parameters are half-integers. The possible lengths for the waist curves of the cylinders are $l_4, l_3 + l_4, l_2 + l_3$ and $l_2 + 2l_1$. Since $l_3 + l_4 > l_4$ and the situation

$$\begin{cases} l_4 = l_2 + 2l_1 \\ l_3 + l_4 = l_2 + l_3 \end{cases}$$

is impossible, the only remaining case is:

$$\begin{cases} w_1 = l_4 = l_2 + l_3 \\ w_2 = l_3 + l_4 = l_2 + 2l_1 \end{cases} .$$

This implies that $l_3 = l_1$ and there is only one way to find such l_i , but only if $w_1 < w_2 < 2w_1$. The contribution to the counting is:

$$\sum_{(w_1 h_1 + w_2 h_2) \leq N/2} 4w_1 w_2 (\mathbb{1}_{\{w_1 < w_2 < 2w_1\}}) = \sum_{(W_1 H_1 + W_2 H_2) \leq 2N} W_1 W_2 (\mathbb{1}_{\{W_1 < W_2 < 2W_1\}})$$

Summing the contributions of the 2 first configurations gives:

$$\begin{aligned} \sum_{(W \cdot H) \leq 2N} W_1 W_2 (\mathbf{1}_{\{2W_1 < W_2\}} + \mathbf{1}_{\{W_1 < W_2 < 2W_1\}}) &= \sum_{W \cdot H \leq 2N} W_1 W_2 \mathbf{1}_{\{W_1 < W_2\}} \\ &\sim \frac{1}{2} \frac{(2N)^4}{4!} (\zeta(2))^2 = \frac{N^4 (\zeta(2))^2}{3} \end{aligned}$$

- Configuration 3:

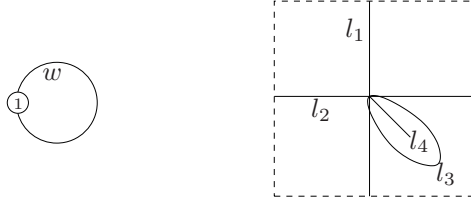


FIGURE 23. Configuration 3

All parameters are half-integers. The two lengths are $2l_1 + 2l_2 + l_3$ and $l_3 + 2l_4$ so we should have $l_4 = l_2 + l_1$ in order that the two are equal. Then we search the number of (l_1, l_2, l_3) such that $w = l_3 + 2(l_1 + l_2)$. It is a polynomial of w with leading term $\frac{1}{4} \frac{(2w)^2}{2} = \frac{w^2}{2}$.

The contribution to the counting is:

$$\sum_{wh \leq N/2} 2 \frac{w^3}{2} = \sum_{WH \leq 2N} \left(\frac{W}{2} \right)^3 \sim \frac{1}{8} \frac{(2N)^4}{4} \zeta(4) = \frac{\zeta(4)}{2}$$

- Configuration 4:

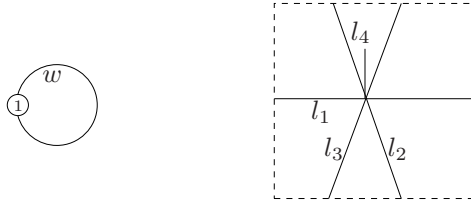


FIGURE 24. Configuration 4

All parameters are half integers. The lengths for the waist curves are $2l_1 + l_2 + l_3$ and $2l_4 + l_2 + l_3$, so we have $l_1 = l_4$. The number of solutions of $w = 2l_1 + l_2 + l_3$ is approximately $\frac{1}{2} \frac{(2w)^2}{2} = w^2$.

The contribution to the counting for this configuration:

$$\sum_{wh \leq N/2} 2w^3 = \sum_{WH \leq 2N} 2 \left(\frac{W}{2} \right)^3 = 2 \frac{1}{8} \frac{(2N)^4}{4} \zeta(4) = \zeta(4).$$

- Total:

The sum of the 4 contributions is:

$$N^4 \left(\frac{(\zeta(2))^2}{3} + \frac{3}{2}\zeta(4) \right) = \frac{7\pi^4 N^4}{2 \cdot 3^3 \cdot 5}$$

We obtain:

$$\text{Vol } Q(5, -1) = \dim_{\mathbb{R}} Q(5, -1) \frac{7}{2 \cdot 3^3 \cdot 5} \pi^4 = \frac{2^2 \cdot 7}{3^3 \cdot 5} \pi^4$$

9.2. Siegel–Veech constant. For this strata, the value of the Siegel–Veech constant is known:

$$c_{area} Q(5, -1) = \frac{15}{7\pi^2}.$$

There is only one configuration \mathcal{C} containing cylinders, cf Figure 25. To understand how to read the graph corresponding to the configuration, see § 2.3.

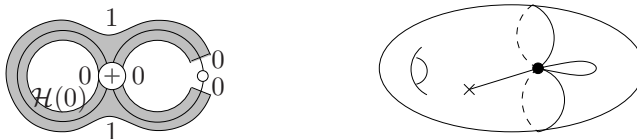


FIGURE 25. The only configuration of $Q(5, -1)$ containing cylinders, and its topological picture

For this configuration, we take the family of curves as in § 3.2.2, cf Figure 26. We evaluate the combinatorial constants that appear in (17).

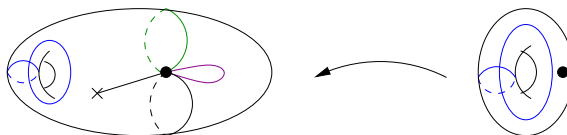


FIGURE 26. Family of curves associated to the configuration

First we can see that all cycles in the family $\{\gamma, \delta\}$ are not homologous to zero in $H_1(S, \{P_i\}, \mathbb{Z})$. Thus $M_c = 4^2$.

There is only one thin cylinder so $M_t = 1$.

The surgery applied to the marked point on the torus belonging to the principal boundary stratum $\mathcal{H}(0)$ is of type +4.1a (local construction). There are 2 geodesics rays coming from this point. But choosing either of the geodesic rays do not change the configuration because the involution of the torus exchanges these two rays, so $M_s = \frac{2}{|\Gamma|} = 1$.

We have the following combinatorial data:

- $\dim_{\mathbb{C}} \mathcal{H}(0) = 2$,
- $\dim_{\mathbb{C}} Q(5, -1) = 4$,
- $\text{Vol } \mathcal{H}_1(0) = \frac{\pi^2}{3}$,
- $q_1 = 1, q_2 = 0$.

Applying formula (17) we get :

$$c_{area}(\mathcal{C}) = 4^2 \frac{4}{2^5} \frac{(2-1)!2^2\pi^2/3}{(4-1)! \text{Vol } \mathcal{Q}(5, -1)}.$$

So:

$$c_{area}(\mathcal{C}) = \frac{2^2\pi^2}{3^2 \text{Vol } \mathcal{Q}_1(5, -1)}$$

The value of the volume computed in the previous section is $\text{Vol } \mathcal{Q}_1(5, -1) = \frac{2^2 \cdot 7}{3^3 \cdot 5} \pi^4$, it gives:

$$c_{area}(\mathcal{C}) = \frac{15}{7\pi^2},$$

which coincides with value of Theorem 9.1. in [CM].

10. SECOND EXAMPLE $\mathcal{Q}(3, -1^3)$

This stratum is also non-varying so the Siegel-Veech constant for the entire stratum is known (cf [CM]), and it is of complex dimension 4, which allows a computation of the volumes by counting graphs.

10.1. **Volume.** As previously we compute the volume of this stratum using the method described in § 8.

- Configuration 1

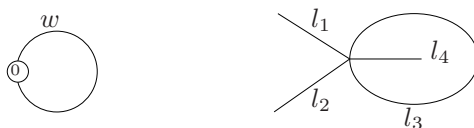


FIGURE 27. Configuration 1

All parameters are half-integers. The constraints are given by: $w = l_3 + 2l_4 = l_3 + 2l_1 + 2l_2$. There are $\sim \frac{1}{4} \frac{(2w)^2}{2} = \frac{w^2}{2}$ choices for the l_i . There are 6 ways to give name to the poles. The contribution to the counting is

$$6 \sum_{w,h \leq N/2} 2w \frac{w^2}{2} = 6 \sum_{WH \leq 2N} \left(\frac{W}{2}\right)^3 \sim \frac{3}{4} \frac{(2N)^4}{4} \zeta(4) = 3\zeta(4)$$

- Configuration 2

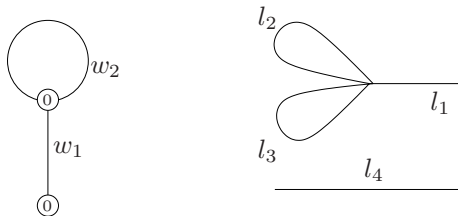


FIGURE 28. Configuration 2

The parameter $w_1 = W_1$ is an integer and all remaining parameters are half-integers. Note that here there are 3 ways to give names to the poles. The equations

$$\begin{cases} w_2 = l_2 = l_3 \\ W_1 = 2l_1 + l_2 + l_3 = 2l_4 \end{cases}$$

have one solution if $W_1 > 2w_2$.

The contribution of this configuration is:

$$3 \sum_{W_1 h_1 + w_2 h_2 \leq N/2} 2w_1 2w_2 \mathbb{1}_{\{W_1 > 2w_2\}} = 6 \sum_{2W_1 H_1 + W_2 H_2 \leq 2N} W_1 W_2 \mathbb{1}_{\{W_1 > W_2\}}$$

- Configuration 3

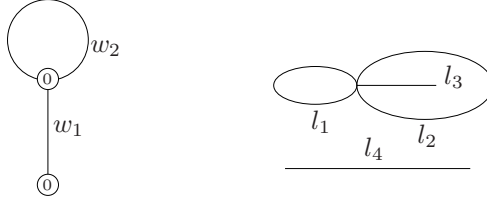


FIGURE 29. Configuration 3

The parameter $w_1 = W_1$ is an integer and all remaining parameters are half-integers. Note that here there are 3 ways to give names to the poles.

Two ribbon graphs are possible for the second layer:



For the first ribbon graph, the equations

$$\begin{cases} W_1 = 2l_4 = l_1 + l_2 \\ w_2 = l_1 = l_2 + 2l_3 \end{cases}$$

have one solution if $w_2 < W_1 < 2w_2$.

For the second ribbon graph, the equations

$$\begin{cases} W_1 = 2l_4 = l_1 \\ w_2 = l_2 + 2l_3 = l_2 + l_1 \end{cases}$$

have one solution if $W_1 < w_2$.

The total number of solutions is then:

$$\mathbb{1}_{\{w_2 < W_1 < 2w_2\}} + \mathbb{1}_{\{W_1 < w_2\}} = \mathbb{1}_{\{W_1 < 2w_2\}} - \underbrace{\mathbb{1}_{\{W_1 = w_2\}}}_{\text{negligible}}$$

This gives a contribution:

$$3 \sum_{W_1 h_1 + w_2 h_2 \leq N/2} 2w_1 2w_2 \mathbb{1}_{\{W_1 < 2w_2\}} = 6 \sum_{2W_1 H_1 + W_2 H_2 \leq 2N} W_1 W_2 \mathbb{1}_{\{W_1 < W_2\}}$$

Summing the contributions of configurations 2 and 3 we get:

$$6 \sum_{2W_1 H_1 + W_2 H_2 \leq 2N} W_1 W_2 = \frac{1}{4} 6 \sum_{W \cdot H \leq 2N} W_1 W_2 \sim \frac{3}{2} \frac{(2N)^4}{4!} (\zeta(2))^2 = \frac{5N^4}{2} \zeta(4)$$

- Configuration 4

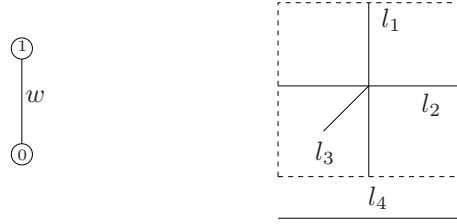


FIGURE 30. Configuration 4

The parameter $w = W$ is an integer and all remaining parameters are half-integers. Note that here also there are 3 ways to give name to the poles.

The constraints are:

$$W = 2l_4 = 2(l_1 + l_2 + l_3)$$

So there are $\sim \frac{W^2}{2}$ ways to choose (l_1, \dots, l_4) .

The contribution of this configuration is:

$$3 \sum_{W, h \leq N/2} 2W \frac{W^2}{2} = 3 \sum_{WH \leq N} W^3 \sim \frac{3N^4}{4} \zeta(4)$$

- The sum of all contributions is $\frac{25N^4}{4} \zeta(4)$ so it gives

$$\text{Vol}^{comp} \mathcal{Q}(3, -1^3) = 50\zeta(4) = \frac{5\pi^4}{9}$$

10.2. Siegel-Veech constant. The stratum $\mathcal{Q}(3, -1^3)$ is non-varying, so the value of the sum of the Lyapunov exponents λ^+ of the invariant subbundle H_+^1 of the Hodge bundle over the stratum along the Teichmüller flow is given in [CM]: $L^+ = \frac{2}{5}$. By the Eskin-Kontsevich-Zorich formula ([EKZ]) we obtain

$$c_{area}(\mathcal{Q}(3, -1^3)) = \frac{9}{5\pi^2}.$$

There are two configurations for this stratum.

- (1) Configuration \mathcal{C}_1 :

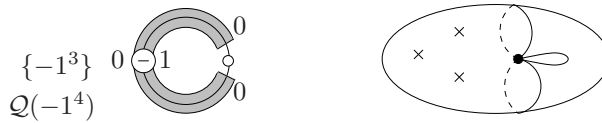


FIGURE 31. Configuration \mathcal{C}_1 of $\mathcal{Q}(3, -1, -1, -1)$ containing cylinders, and its topological picture

The combinatorial data for this configuration are:

- $M_c = 4^2, M_s = 1, M_t = 1$
- $q_1 = 1, q_2 = 0$
- $\dim_{\mathbb{C}} \mathcal{Q}(3, -1^3) = 4$

- $\dim_{\mathbb{C}} \mathcal{Q}(-1^4) = 2$.
 - $\text{Vol } \mathcal{Q}_1(-1^4) = 2\pi^2$ ([AEZ1])
- Applying formulae of Theorem 1 gives:

$$c_{area}(\mathcal{C}_1) = \frac{2\pi^2}{3 \text{Vol } \mathcal{Q}_1(3, -1^3)}$$

There is only one configuration of type \mathcal{C}_1 .

(2) Configuration \mathcal{C}_2 :

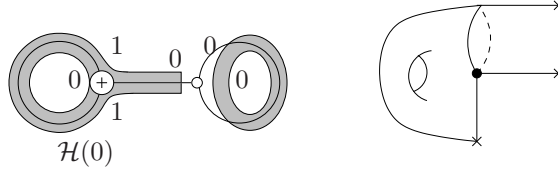


FIGURE 32. Configuration \mathcal{C}_2 of $\mathcal{Q}(3, -1, -1, -1)$ containing cylinders, and its topological picture

The combinatorial data are:

- $M_c = 4^2$, $M_s = \frac{2}{2}$ (hyperelliptic involution), $M_t = 1$.
- $q_1 = 0$, $q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(0) = 2$.
- $\text{Vol } \mathcal{H}_{1/2}(0) = \frac{4\pi^2}{3}$

Applying formulae of Theorem 1 gives:

$$c_{area}(\mathcal{C}) = \frac{\pi^2}{9 \text{Vol } \mathcal{Q}_1(3, -1^3)}$$

Due to the numbering of the poles there are 3 configurations of type \mathcal{C}_2 .

Substituting the computed value of the volume and summing on all configurations we find

$$c_{area}(\mathcal{Q}(3, -1^3)) = c_{area}(\mathcal{C}_1) + 3c_{area}(\mathcal{C}_2) = \frac{6}{5\pi^2} + 3 \cdot \frac{1}{5\pi^2} = \frac{9}{5\pi^2}$$

which is coherent with the formula of Theorem 8.1. in [CM].

APPENDIX A. SUMMARY

In the following tabular we gather the data about the strata studied in this paper. We denote L^+ (resp. L^-) the sum of the Lyapunov exponents λ^+ (resp. λ^-) of the invariant subbundle H_+^1 (resp. H_-^1) of the Hodge bundle over the stratum $\mathcal{Q}(d_1, \dots, d_n)$ along the Teichmüller flow.

Denoting

$$K = \frac{1}{24} \sum_{j_1}^n \frac{d_j(d_j + 4)}{d_j + 2} \quad \text{and} \quad I = \frac{1}{4} \sum_{d_j \text{ odd}} \frac{1}{d_j + 2},$$

Theorem 2 of [EKZ] gives

$$\begin{aligned} L^+ &= K + \frac{\pi^2}{3} c_{area}(\mathcal{Q}(\alpha)) \\ L^- - L^+ &= I. \end{aligned}$$

So c_{area} is given by the formula

$$c_{area}(\mathcal{Q}(d_1, \dots, d_n)) = \frac{3}{\pi^2}(L^+ - K) = \frac{3}{\pi^2}(L^- - I - K).$$

We denote also $g_{eff} = \hat{g} - g$ where \hat{g} is the genus of the double cover surface \hat{S} for $S \in \mathcal{Q}(\alpha)$. Recall that $H^1_-(\hat{S})$ has dimension $2g_{eff}$. In comparison to the Abelian case we expect that the values of the volumes of the strata of quadratic differentials are given by $r\pi^{2g_{eff}}$ with $r \in \mathbb{Q}$. This is true for the examples cited below.

strata	g	g_{eff}	$\dim_{\mathbb{C}}$	K	I	L^+	L^-	c_{area}	boundary comp.	hyp	Vol
$\mathcal{Q}(2, -1^2)$	1	1	3	$-\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{15}{8\pi^2}$	\emptyset	$\mathcal{Q}(0, -1^4)$	$\frac{4}{3}\pi^2$
$\mathcal{Q}(1^2, -1^2)$	1	2	4	$-\frac{1}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{7}{3\pi^2}$	$\mathcal{H}(0), \mathcal{Q}(-1^4)$	$\mathcal{Q}(1, -1^5)$	$\frac{1}{3}\pi^4$
$\mathcal{Q}(2, 2)$	2	1	4	$\frac{1}{4}$	0	1	1	$\frac{9}{4\pi^2}$	\emptyset	$\mathcal{Q}(-1^4, 0^2)$	$\frac{4}{3}\pi^2$
$\mathcal{Q}(2, 1^2)$	2	2	5	$\frac{19}{72}$	$\frac{1}{6}$	$\frac{7}{6}$	$\frac{4}{3}$	$\frac{65}{24\pi^2}$	$\mathcal{Q}(2, -1^2), \mathcal{H}(0)$	$\mathcal{Q}(1, -1^5, 0)$	$\frac{2}{15}\pi^4$
$\mathcal{Q}(1^4)$	2	3	6	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{19}{6\pi^2}$	$\mathcal{Q}(1^2, -1^2), \mathcal{H}(0)$	$\mathcal{Q}(1^2, -1^6)$	$\frac{1}{15}\pi^6$
$\mathcal{Q}(5, -1)$	2	2	4	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$\frac{15}{7\pi^2}$	$\mathcal{H}(0)$	no	$\frac{28}{135}\pi^4$
$\mathcal{Q}(3, -1^3)$	1	2	4	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{9}{5\pi^2}$	$\mathcal{Q}(-1^4), \mathcal{H}(0)$	no	$\frac{5}{9}\pi^4$

APPENDIX B. ALTERNATIVE COMPUTATIONS OF VOLUMES

Here we use the method of § 8 and the lemmas of appendix C to compute some volumes of hyperelliptic strata already computed in § 6. This allows us to check one more time that our choices of normalization for the volumes are consistent.

B.1. $\mathcal{Q}(2, -1^2)$.

- Configuration 1:



FIGURE 33. Configuration 1

All parameters are half-integers. We have $w = 2l_1 + l_3 = 2l_2 + l_3$, which has $\simeq \frac{2w}{2} = w$ solutions. The contribution is therefore:

$$\sum_{wh \leq N/2} 2w^2 = \sum_{WH \leq 2N} \frac{W^2}{2} \sim \frac{1}{2} \frac{(2N)^3}{3} \zeta(3) = \frac{4N^3}{3} \zeta(3)$$

- Configuration 2:

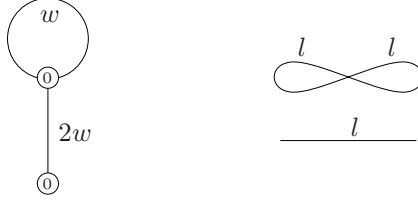


FIGURE 34. Configuration 2

All parameters are half-integers. Equation $w = l$ has 1 solution. We have an additional factor $\frac{1}{\sigma_t} = \frac{1}{2}$ which comes from the definition of the twist (cf § 3.3.1). By Lemma 9 we obtain the following contribution:

$$\frac{1}{2} \sum_{w(2h_1+h_2) \leq N/2} 2w(4w) = \sum_{W(2H_1+H_2) \leq 2N} W^2 \sim \frac{N^3}{6}(8\zeta(2) - 9\zeta(3))$$

- Configuration 3:



FIGURE 35. Configuration 3

$w = W$ is an integer and l_1, l_2, h, t are half-integers. Equation $W = 2(l_1 + l_2)$ has approximately W solutions. There is an additional factor $\frac{1}{2}$ for the twist, and another factor $\frac{1}{2}$ because (l_2, l_1) and (l_1, l_2) give the same surfaces. The contribution of this configuration is then:

$$\frac{1}{4} \sum_{Wh \leq N/2} 2W \cdot W = \frac{1}{2} \sum_{WH \leq N} W^2 \sim \frac{N^3}{6}\zeta(3)$$

Summing all the contributions we get $\frac{4N^3}{3}\zeta(2)$ so by (42), we obtain:

$$\text{Vol}\mathcal{Q}(2, -1^2) = 8\zeta(2) = \frac{4\pi^2}{3},$$

which coincides with the value found in (26).

B.2. $\mathcal{Q}(1^2, -1^2)$.

- Configuration 1:

$w = W$ is an integer and all remaining parameters are half-integers.

Equation $W = 2(l_1 + l_2 + l_3) = 2l_4$ has approximately $\frac{W^2}{2}$ solutions.



FIGURE 36. Configuration 1

Here $o_t = 3$ so we have an extra factor $\frac{1}{3}$. So the contribution of this configuration is:

$$\frac{1}{3} \sum_{Wh \leq N/2} 2W \cdot \frac{W^2}{2} = \frac{1}{3} \sum_{WH \leq N} W^3 \sim \frac{N^4}{12} \zeta(4)$$

- Configuration 2:



FIGURE 37. Configuration 2

All parameters are half-integers. Equation $w = 2l_1 + l_2 + l_3 = 2l_4 + l_2 + l_3$ has $\sim \frac{1(2w)^2}{2} = w^2$ solutions. There are 2 ways to give name to the zeroes. Therefore the contribution of this configuration is:

$$2 \sum_{wh \leq N/2} 2w \cdot w^2 = 2 \sum_{WH \leq 2N} W \left(\frac{W}{2}\right)^2 \sim \frac{1(2N)^4}{2 \cdot 4} \zeta(4) = 2N^4 \zeta(4)$$

- Configuration 3:



FIGURE 38. Configuration 3

All parameters are half-integers. There are two ways to give name to the zeroes. There are 2 ways to numbered the faces of the ribbon graphs: the first gives:

$$\begin{cases} w_1 = l_1 \\ w_2 = l_1 + 2l_2 \end{cases}$$

so there is $\mathbb{1}_{\{w_1 < w_2\}} \mathbb{1}_{\{w_2 - w_1 \in \mathbb{N}\}}$ solution. The second intertwines w_1 and w_2 , but w_1 and w_2 play symmetric roles in the graph T . We get the

following contribution:

$$2 \sum_{w \cdot h \leq N/2} 2w_1 2w_2 \mathbb{1}_{\{w_1 < w_2\}} \mathbb{1}_{\{w_2 - w_1 \in \mathbb{N}\}} = \frac{1}{2} \sum_{W \cdot H \leq 2N} W_1 W_2 \sim \frac{(2N)^4}{4!} (\zeta(2))^2 = \frac{5N^4}{6} \zeta(4)$$

- Configuration 4:

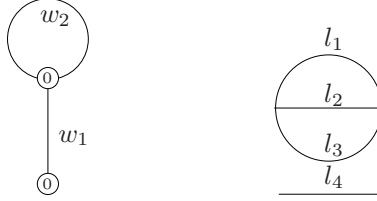


FIGURE 39. Configuration 4

$w_1 = W_1$ is an integer and all remaining parameters are half-integers. There is $1 = \frac{1}{3} \binom{3}{1}$ way to give name to the faces, say

$$\begin{cases} W_1 = 2l_4 = l_1 + l_3 \\ w_2 = l_1 + l_2 = l_2 + l_3 \end{cases} .$$

So there is $\mathbb{1}_{\{2w_2 > W_1\}}$ solution.

- Configuration 5:

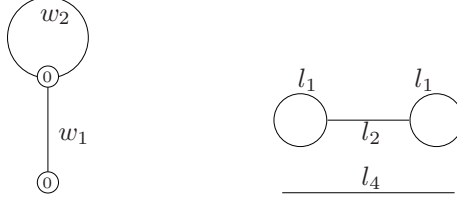


FIGURE 40. Configuration 5

$w_1 = W_1$ is an integer and all remaining parameters are half-integers. There is $1 = \frac{1}{3} \binom{3}{1}$ way to give name to the faces, say

$$\begin{cases} W_1 = 2l_4 = 2(l_2 + l_1) \\ w_2 = l_1 \end{cases} .$$

So there is $\mathbb{1}_{\{2w_2 < W_1\}}$ solution.

Summing the contributions of configurations 4 and 5 we obtain:

$$\sum_{W_1 h_1 + w_2 h_2 \leq N/2} 2W_1 2w_2 = \sum_{2W_1 H_1 + W_2 H_2 \leq 2N} 2W_1 W_2 \sim 2 \frac{1}{4} \frac{(2N)^4}{4!} (\zeta(2))^2 = \frac{5N^4}{6} \zeta(4)$$

(To understand the factor $1/4$, take $x_1 = \frac{2W_1 H_1}{2N}$ in the proof of Lemma 3.7 of [AEZ2]).

- Total Summing all the contributions, we obtain $\frac{15N^4}{4}\zeta(4)$. So by (42):

$$\text{Vol } \mathcal{Q}(1^2, -1^2) = 30\zeta(4) = \frac{\pi^4}{3},$$

which coincides with the value found in (25).

APPENDIX C. TOOLBOX

Recall that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90} \quad \text{so } (\zeta(2))^2 = \frac{5}{2}\zeta(4).$$

Lemma 7.

$$(43) \quad \forall m \geq 2, \quad \sum_{k \geq 0} \frac{1}{(2k+1)^m} = \frac{2^m - 1}{2^m} \zeta(m)$$

$$(44) \quad \forall m \geq 1, \quad \sum_{i=1}^N i^m \underset{N \rightarrow \infty}{\sim} \frac{N^{m+1}}{m+1}$$

$$(45) \quad \forall m \geq 1, \quad \text{card}\{(l_1, \dots, l_m) \in \mathbb{N}^m \mid N = 2l_1 + \dots + 2l_j + l_{j+1} + \dots + l_m\} \underset{N \rightarrow \infty}{\sim} \frac{N^{m-1}}{2^j(m-1)!}$$

We recall the following standard fact (Lemma 3.7 of [AEZ2]):

Lemma 8 (Athreya-Eskin-Zorich).

$$\sum_{\substack{H \cdot W \leq N \\ W \in \mathbb{N}^k, \tilde{W} \in \mathbb{N}^k}} W_1^{a_1+1} \dots W_k^{a_k+1} \sim \frac{N^{a+2k}}{(a+2k)!} \cdot \prod_{i=1}^k (a_i+1) \zeta(a_i+2)$$

We will need the following variation of the previous lemma:

Lemma 9.

$$(46) \quad \sum_{W(H_1+2H_2) \leq 2N} W^m \sim \frac{N^{m+1}}{2(m+1)} (2^{m+1}\zeta(m) - (2^{m+1}+1)\zeta(m+1))$$

Proof.

$$A = \sum_{W(H_1+2H_2) \leq 2N} W^m = \sum_{WH \leq 2N} W^m \text{card}\{(H_1, H_2) \in \mathbb{N}^2 \text{ s.t. } H = H_1 + 2H_2\}$$

Since $2H_2$ is even and goes from 2 to $H-1$ or $H-2$ depending on the parity of H , we have :

$$\text{card}\{(H_1, H_2) \text{ s.t. } H = H_1 + 2H_2\} = \lfloor \frac{H-1}{2} \rfloor.$$

$$\begin{aligned} A &\sim \sum_{WH \leq 2N} W^m \lfloor \frac{H-1}{2} \rfloor = \sum_{W(2K+1) \leq 2N} W^m K + \sum_{W(2K+2) \leq 2N} W^m K \\ &\sim \sum_{K \geq 1} K \left(\frac{1}{m+1} \left(\frac{2N}{2K+1} \right)^{m+1} + \frac{1}{m+1} \left(\frac{2N}{2K+2} \right)^{m+1} \right) \end{aligned}$$

using (44). So

$$A = \frac{N^{m+1}}{m+1} \left(\underbrace{2^{m+1} \sum_{K \geq 0} \frac{K}{(2K+1)^{m+1}}}_{S_1(m)} + \underbrace{\sum_{K \geq 0} \frac{K}{(K+1)^{m+1}}}_{S_2(m)} \right)$$

$$2S_1(m) + \sum_{K \geq 0} \frac{1}{(2K+1)^{m+1}} = \sum_{K \geq 0} \frac{1}{(2K+1)^m}$$

So using (43) we obtain:

$$S_1(m) = \frac{1}{2^{m+2}}((2^{m+1} - 2)\zeta(m) - (2^{m+1} - 1)\zeta(m+1))$$

Similarly,

$$S_2(m) = \zeta(m) - \zeta(m+1),$$

which gives the result. □

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