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# Gradient estimate and a Liouville theorem for a $p$ -Laplacian evolution equation with a gradient nonlinearity

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## Abstract

In this paper, we establish a local gradient estimate for a  $p$ -Laplacian equation with a fast growing gradient nonlinearity. With this estimate, we can prove a parabolic Liouville theorem for ancient solutions satisfying some growth restriction near infinity.

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*Key-words.*  $p$ -Laplacian; local gradient estimate; parabolic Liouville-type Theorem; gradient non-linearity

## 1 Introduction and main result

In this paper, we are interested in qualitative properties of solutions of the non-linear degenerate parabolic equation

$$u_t - \Delta_p u = |\nabla u|^q, \quad (1.1)$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $q > p - 1 > 1$ .

The kind of result we are going to prove are gradient estimates for local solutions in time-space, and a Liouville type theorem for ancient solutions. In the last years, gradient estimates have played a key role in geometry and PDE since at least the early work of Bernstein. Gradient a priori estimates are fundamental for elliptic and parabolic equations, leading to Harnack inequalities, Liouville theorems, and compactness theorems for both linear and nonlinear PDE. For the corresponding elliptic equation of (1.1), gradient estimates were first considered by Lions [13] for the linear diffusion case  $p = 2$ . These estimates were based upon the Bernstein

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technique. Recently for the possibly degenerate elliptic equation with  $q > p - 1 > 0$ , Bidaut-Véron, Huidobro, Véron [5] obtained a priori universal gradient estimate for equations on a domain  $\Omega$  of  $\mathbb{R}^N$  and they extended their estimates to equations on complete non compact manifolds satisfying a lower bound estimate on the Ricci curvature. These estimates allowed them to derive some Liouville type theorems.

It is natural to look also for parabolic Liouville-type-theorems. In the linear diffusion case  $p = 2$  and for  $q > 1$ , Souplet and Zhang [19] obtained local gradient estimate for locally upper bounded solution of (1.1) ( $u \leq M$ ) of the form

$$|\nabla u| \leq C(p, N, q) \left( t^{-\frac{1}{q}} + R^{-1} + R^{\frac{-1}{q-1}} \right) (M + 1 - u) \quad \text{in } B(x_0, R) \times (0, T).$$

Relying on this estimate they proved that, under some growth condition at infinity, ancient solutions in the whole of  $\mathbb{R}^N$  are constant. Motivated by their result, we generalize the gradient estimate and Liouville theorem to the case  $1 < p - 1 < q$ . We also require that the solution is locally lower bounded. Using a Bernstein method, we have the following gradient estimate.

**Theorem 1.1** *Let  $q > p - 1 > 1$ ,  $x_0 \in \mathbb{R}^N$  and  $R, T > 0$ . We set  $Q_{T,R} = B(x_0, R) \times (0, T)$ . Let  $u$  be a solution in  $L^\infty((0, T); W^{1,\infty}(B(x_0, R)))$  of*

$$\partial_t u - \Delta_p u = |\nabla u|^q \quad \text{in } Q_{T,R}.$$

*Suppose that  $|u| \leq M$  for some constant  $M \geq 1$ . Then,*

$$|\nabla u| \leq C(p, N, q) \left( t^{-\frac{1}{q}} + R^{-1} + R^{\frac{-1}{q-p+1}} \right) M \quad \text{in } Q_{T, \frac{R}{2}}. \quad (1.2)$$

For the Cauchy-Dirichlet problem associated to (1.1), a gradient estimate involving the  $W^{1,\infty}$  norm of the initial data has been obtained in [2, 19]. In Theorem 1.1 we only use the local  $L^\infty$  norm of the solution but we get a weaker estimate regarding the exponent on the distance to the boundary  $R$ .

Recently, for the singular diffusion case  $1 < p < 2$  and for  $q = p$ , F. Wang [20] established gradient estimates similar to (1.2) for smooth, upper bounded, local solutions to (1.1) on a closed manifolds or on complete noncompact Riemannian manifolds evolving under a Ricci flow. These estimates are of the form:

$$\frac{|\nabla u|}{1-u}(x, t) \leq C(N, p) \left( R^{-1} + t^{\frac{-1}{p}} + K^{\frac{2}{p}} + K \right) \quad \text{in } Q_{T, \frac{R}{2}}. \quad (1.3)$$

where  $K > 0$  is a constant related to the Ricci flow and the sectional curvature of the manifold. These estimates allowed to the author to provide some Harnack inequalities for positive solutions of the following  $p$ -Laplace heat equation

$$|z|^{p-2} z_t = \Delta_p z. \quad (1.4)$$

The estimates (1.3) have been obtained by deriving an equation for  $w = |\nabla v|^p$ ,  $v = f^{-1}(-u)$  and  $f(s) = e^{s/(p-1)} - 1$ . For  $q > p > 2$ , we take a different auxiliary function  $f$ , adapted to the degenerate diffusion case and to the fast growing gradient non-linearity.

As an application of the gradient estimate (1.2), we can state the following Liouville theorem for (1.1).

**Theorem 1.2** *Assume that  $q > p - 1 > 1$  and let  $\sigma = \min\left(1, \frac{1}{q-p+1}\right)$ . Assume that  $u \in L_{loc}^\infty((-\infty, 0); W_{loc}^{1,\infty}(\mathbb{R}^N))$  is a weak solution of*

$$u_t - \Delta_p u = |\nabla u|^q, x \in \mathbb{R}^N, -\infty < t < 0,$$

*satisfying*

$$|u(x, t)| = o(|x|^\sigma + |t|^{\frac{1}{q}}), \quad \text{as } |x|^\sigma + |t|^{\frac{1}{q}} \rightarrow \infty. \quad (1.5)$$

*Then  $u$  is constant.*

**Remark 1.1** *The growth hypothesis (1.5) is important (see the example of the function  $u(x, t) = x_1 + t$ ). However, we do not know if the exponents are sharp.*

Besides the works mentioned above, there are few other studies on gradient estimates and nonlinear Liouville theorems for a parabolic type equation on noncompact Riemannian manifolds. In this case the proof mostly relies on two types of gradient estimates or a combination of them. These estimates are known as Hamilton gradient estimate (the estimate only involves  $\nabla u$  and  $u$ ) [9] and Li-Yau's gradient estimate (the estimate involves  $\nabla u, u$  and  $u_t$ ) [12]. Let us also mention that the linear heat equation on noncompact manifolds was studied by Souplet and Zhang in [18] where they obtained a local gradient estimate related to the elliptic Cheng-Yau estimate and Hamilton's estimate for the heat equation on compact manifolds. A Liouville theorem was also proved in [18]. Hamilton-type gradient estimates were also used in [21, 14, 24]. For  $q = p > 1$ , a nonlinear analogue of Li-Yau's estimate has been established in [10] for positive solutions of (1.1) on compact manifolds with nonnegative Ricci curvature. In [10], the gradient estimate was not used to get Liouville theorems but to obtain an entropy formula. Nevertheless, Liouville theorems should be obtained as a consequence of the obtained gradient estimate.

This paper is organized as follows: In Section 2, we provide the proof of the gradient estimate (1.2) and we prove Theorem 1.2. In Sections 3 we give the proof of a technical auxiliary lemma that appears in the proof of the gradient estimate.

## 2 Bernstein-type gradient estimate

The proof of Theorem 1.1 is based on the following technical lemma which is based on a Bernstein method. The most significant difficulty being the choice of the auxiliary function  $f$  and the estimates coming from the cut-off argument. Let us mention that for different suitable choice of  $f$ , gradient bounds global in space for the Cauchy problem associated to (1.1) have been obtained in [3].

First let us make precise that by local weak solution of (1.1) we mean a function  $u \in C_{loc}(\Omega \times (0, T)) \cap L_{loc}^\infty(0, T; W_{loc}^{1,\infty}(\Omega))$  where  $\Omega$  is a smooth domain and such that the integral equality

$$\begin{aligned} \int_{\Omega} (u(x, t)\psi(x, t) - u(x, s)\psi(x, s)) dx + \int_s^t \int_{\Omega} (-u\psi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla \psi) dx d\tau \\ = \int_s^t \int_{\Omega} |\nabla u|^q \psi dx d\tau \end{aligned}$$

holds for all  $0 < s < t < T$  and for all testing function  $\psi \in C^1(\overline{\Omega} \times [0, T])$  such that  $\psi = 0$  near  $\partial\Omega \times (0, T)$ .

Now let  $\alpha \in (0, 1)$  to be chosen later on. Set  $R' = \frac{3R}{4}$ . We select a cut-off function  $\eta \in C^2(\overline{B}(x_0, R'))$ ,  $0 \leq \eta \leq 1$ , satisfying  $\eta = 0$  for  $|x - x_0| = R'$  and such that

$$\left. \begin{aligned} |\nabla\eta| &\leq CR^{-1}\eta^\alpha \\ |D^2\eta| + \eta^{-1}|\nabla\eta|^2 &\leq CR^{-2}\eta^\alpha \end{aligned} \right\} \text{ for } |x - x_0| < R', \quad (2.1)$$

for some  $C = C(\alpha) > 0$  (see [18] for the existence of such function).

**Lemma 2.1** *Assume that  $u$  is a local weak solution of (1.1) and that  $|u| \leq M$  in  $Q_{T,R}$  for some  $M > 1$ . We consider a  $C^3$  smooth increasing function  $f$  satisfying  $f'' > 0$ , the following differential equation*

$$\left(\frac{f''}{f'}\right)' + (p-1)(1+N) \left(\frac{f''}{f'}\right)^2 = 0 \quad (2.2)$$

and mapping  $[0, 3]$  onto  $[-M, M]$ . Defining  $v = f^{-1}(-u)$ , we set  $w = |\nabla v|^2$  and  $z = \eta w$ . Then at any point where  $|\nabla u| > 0$ ,  $z$  satisfies the following differential inequality

$$\begin{aligned} \mathcal{L}(z) &\leq -2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta + C(p, N)(f')^{p-2}R^{-2}\eta^\alpha w^{\frac{p}{2}} \\ &\quad + C(p, q)R^{-1}\eta^\alpha \left[ w^{\frac{p+1}{2}}(f')^{p-3}f'' + w^{\frac{q+1}{2}}(f')^{q-1} \right] \end{aligned} \quad (2.3)$$

where

$$\mathcal{L}(z) := \partial_t z - \mathcal{A}z + H \cdot \nabla z \quad (2.4)$$

with  $\mathcal{A}$  is given by (3.4)  $\mathcal{H}$  is given by (3.5).

The proof of lemma 2.1 is postponed to the the next section.

## Proof of Theorem 1.1

Let  $u \in L_{loc}^\infty((0, \infty); W_{loc}^{1, \infty}(\Omega))$  be a local weak solution of (1.1). Since  $u$  and  $\nabla u$  are locally bounded, using the result of Di Benedetto and Friedman [7, 8], we get that  $\nabla u$  is a locally Hölder continuous function. Thus  $z$  is a continuous function on  $\overline{B}(x_0, R') \times [0, T] = \overline{Q}$ , for any  $0 < T$ . Therefore, unless  $z \equiv 0$  in  $\overline{Q}$ ,  $z$  must reach a positive maximum at some point  $(\hat{x}, \hat{t}) \in \overline{B}(x_0, R') \times [t_0, T]$ . Since  $z = 0$  on  $\partial B_{R'} \times [0, T]$ , we deduce that  $\hat{x} \in B_{R'}$ . Since  $z(\hat{x}, \hat{t}) > 0$ , we have that  $|\nabla u| = f'(v)|\nabla v| > 0$  and hence we can use Lemma 2.1.

Now let us take  $f(s) = M(s+1)^\gamma - 2M$  where  $\gamma$  is given by

$$\gamma = \gamma(p, N) = \frac{(p-1)(N+1)+1}{(p-1)(N+1)} \quad (2.5)$$

It is easy to see that  $f$  satisfies the differential equation (2.2) and  $f', f'' > 0$  and  $f$  maps  $\left[0, 3^{\frac{1}{\gamma}} - 1\right]$  onto  $[-M, M]$ . Let us also note that  $\gamma \geq 1$  and  $\gamma - 1 \leq \frac{1}{p-1} \leq 1$ .

By Lemma 2.1 we get that, in a small neighbourhood  $\tilde{Q}$  of  $(\hat{x}, \hat{t})$ ,  $z$  satisfies

$$\begin{aligned}\mathcal{L}z &\leq -2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta + C(p, N, \alpha)(f')^{p-2}R^{-2}\eta^\alpha w^{\frac{p}{2}} \\ &\quad + C(p, q, \alpha)R^{-1}\eta^\alpha \left[ w^{\frac{p+1}{2}}(f')^{p-3}f'' + w^{\frac{q+1}{2}}(f')^{q-1} \right].\end{aligned}$$

Hence

$$\begin{aligned}(f')^{1-q}\mathcal{L}z &\leq -2(q-1)\frac{f''}{f'}w^{\frac{q+2}{2}}\eta + C(p, N, \alpha)(f')^{p-1-q}R^{-2}\eta^\alpha w^{\frac{p}{2}} \\ &\quad + C(p, q, \alpha)R^{-1}\eta^\alpha \left[ w^{\frac{p+1}{2}}(f')^{p-q-1}\frac{f''}{f'} + w^{\frac{q+1}{2}} \right].\end{aligned}$$

Since  $v \in \left[0, (3)^{\frac{1}{\gamma}} - 1\right]$ ,  $\gamma, M \geq 1$ , we have  $1 \leq v+1 \leq (3)^{\frac{1}{\gamma}} \leq 3$  and hence

$$\frac{1}{3(p-1)(N+1)} \leq \left(\frac{f''}{f'}\right) \leq \frac{1}{(p-1)(N+1)} \leq 1 \quad (2.6)$$

Using (2.6) together with the fact that  $1 \leq M \leq f'$  and  $p-q-1 < 0$ , we get that

$$\begin{aligned}(f')^{1-q}\mathcal{L}z &\leq -\frac{2(q-1)}{3(p-1)(N+1)}w^{\frac{q+2}{2}}\eta + C(N, p, \alpha)R^{-2}\eta^\alpha w^{\frac{p}{2}} \\ &\quad + C(p, q, \alpha)R^{-1}\eta^\alpha \left[ w^{\frac{p+1}{2}} + w^{\frac{q+1}{2}} \right].\end{aligned}$$

We take  $\alpha = \max\left(\frac{q+1}{q+2}, \frac{p+1}{q+2}\right)$ . Using the Young's inequality and recalling that  $\eta \leq 1$ , then

- for the conjugate exponents  $r_1 = \frac{q+2}{p}$ ,  $s_1 = \frac{q+2}{q-p+2}$  we have that

$$\begin{aligned}C(N, p, \alpha)R^{-2}\eta^\alpha w^{\frac{p}{2}} &= \eta^{\frac{p}{q+2}}w^{\frac{p}{2}}C(N, p, q, \alpha)\eta^{\alpha-p/(q+2)}R^{-2} \\ &\leq \varepsilon_1(N, p, q)\eta w^{\frac{q+2}{2}} + C(N, p, q, \alpha)R^{\frac{-2(q+2)}{q-p+2}},\end{aligned}$$

- for the conjugate exponents  $r_2 = \frac{q+2}{p+1}$ ,  $s_2 = \frac{q+2}{q-p+1}$  we have that

$$\begin{aligned}C(N, p, q, \alpha)R^{-1}\eta^\alpha w^{\frac{p+1}{2}} &= \eta^{\frac{p+1}{q+2}}w^{\frac{p+1}{2}}C(N, p, q, \alpha)R^{-1}\eta^{\alpha-\frac{p+1}{q+2}} \\ &\leq \varepsilon_2\eta w^{\frac{q+2}{2}} + C(N, p, q, \alpha)R^{\frac{-(q+2)}{q-p+1}}\end{aligned}$$

- and finally for the conjugate exponent  $r_3 = \frac{q+2}{q+1}$ ,  $s_3 = (q+2)$  we have that

$$\begin{aligned}C(N, p, q, \alpha)R^{-1}\eta^\alpha w^{\frac{q+1}{2}} &= \eta^{\frac{q+1}{q+2}}w^{\frac{q+1}{2}}C(N, p, q, \alpha)R^{-1}\eta^{\alpha-\frac{q+1}{q+2}} \\ &\leq \varepsilon_3\eta w^{\frac{q+2}{2}} + C(N, p, q, \alpha)R^{-(q+2)}.\end{aligned}$$

Choosing  $\varepsilon_i$  in such way that  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{4} \frac{2(q-1)}{3(p-1)(N+1)}$ , we get that

$$(f')^{1-q} \mathcal{L}z \leq -\frac{(q-1)}{2(p-1)(N+1)} w^{\frac{q+2}{2}} \eta + C(N, p, q, \alpha) R^{\frac{-2(q+2)}{q-p+2}} + C(N, p, q, \alpha) R^{\frac{-(q+2)}{q-p+1}} + C(N, p, q, \alpha) R^{-(q+2)}. \quad (2.7)$$

Using the fact that

$$\begin{cases} \frac{1}{q-p+1} \leq \frac{2}{q-p+2} \leq 1 & \text{for } q \geq p, \\ 1 \leq \frac{2}{q-p+2} \leq \frac{1}{q-p+1} & \text{for } q \leq p, \end{cases}$$

we have that

$$(f')^{1-q} \mathcal{L}z \leq -\frac{(q-1)}{2(p-1)(N+1)} w^{\frac{q+2}{2}} \eta + C(N, p, q, \alpha) \left[ R^{-\frac{q+2}{q-p+1}} + R^{-(q+2)} \right].$$

Setting

$$A = A(R, p, q, N) := C(N, p, q) \left( R^{-\frac{1}{q-p+1}} + R^{-1} \right)^2$$

and using that  $(f')^{q-1} \geq M^{q-1} \geq 1$ , it follows that

$$\mathcal{L}z \leq -\frac{(q-1)}{4(p-1)(N+1)} z^{\frac{q+2}{2}} \quad \text{in } \{(x, t) \in Q_{T, R'}; z(x, t) \geq A\}. \quad (2.8)$$

Next for  $\lambda = \lambda(q, N, p) > 0$  suitably chosen, the function  $\psi(t) = \lambda t^{\frac{-2}{q}}$  satisfies

$$\psi'(t) \geq -\frac{(q-1)}{4(p-1)(N+1)} \psi^{\frac{q+2}{2}}.$$

Now for  $t_0 \in (0, T)$  fixed, we define  $\tilde{z}(t) := z(t + t_0, x) - \psi(t)$ . It is easy to see that

$$\mathcal{L}\tilde{z} \leq 0 \quad \text{in } \{(x, t) \in Q_{T-t_0, R'}; \tilde{z}(x, t) \geq A\}.$$

Since  $\tilde{z}(t) \leq 0$  for  $t > 0$  sufficiently small, we deduce from the maximum principle that  $\tilde{z}(t) \leq A$ , i.e.  $z(x, t + t_0) \leq A + \psi(t)$  in  $Q_{T-t_0, R'}$ .

Finally using that  $z = \eta |\nabla v|^2$ , letting  $t_0$  to 0, we get that

$$|\nabla v| \leq C(N, p, q) (A + t^{\frac{-2}{q}})^{1/2}.$$

Using that

$$v + 1 = \left( 2 - \frac{u}{M} \right)^{\frac{1}{\gamma}} \quad \text{with } \left| \frac{u}{M} \right| \leq 1,$$

we get

$$\nabla v = \frac{-1}{\gamma M} \left( 2 - \frac{u}{M} \right)^{\frac{1-\gamma}{\gamma}} \nabla u.$$

It follows that

$$|\nabla u| \leq M \gamma |\nabla v| \leq C(N, p, q) (A + t^{\frac{-2}{q}})^{1/2} M \quad \text{in } Q_{T, \frac{R}{2}}. \quad (2.9)$$

Here we used the fact that  $\left( 2 - \frac{u}{M} \right)^{\frac{\gamma-1}{\gamma}} \leq 1$ .

Hence we have

$$|\nabla u| \leq C(N, p, q) \left( R^{-1} + R^{\frac{-1}{q-p+1}} + t^{\frac{-1}{q}} \right) M \quad \text{in } Q_{T, \frac{R}{2}}.$$

and the proof of Theorem 1.1 is complete.  $\square$

## Proof of Theorem 1.2

Fix  $x_0 \in \mathbb{R}^N$  and  $t_0 \in (-\infty, 0)$ . Take  $R \geq 1, T = R^{\sigma q}$  and set  $Q = B(0, R) \times (0, T)$ . Now we consider the function  $U := u(x + x_0, t + t_0 - T)$ . Using (1.5), we have that  $|U| \leq M_R$  in  $\overline{Q}$ , where

$$M_R := \sup_{B(x_0, R) \times (t_0 - T, t_0)} |u| = o(T^{\frac{1}{q}} + R^{\sigma}) = o(R^{\sigma}), \quad \text{as } R \rightarrow \infty.$$

Applying Theorem 1.1 to  $U$  in  $Q$ , we get that

$$|\nabla u(x_0, t_0)| = |\nabla U(0, T)| \leq C(N, p, q)R^{-\sigma}M_R$$

and the conclusion follows by sending  $R$  to  $+\infty$ .  $\square$

## 3 Proof of Lemma 2.1

Our proof consists of three steps.

### Step 1: computations

Let  $f$  be a  $C^3$ -function to be determined. We assume that  $f', f'' > 0$ . We put  $v = f^{-1}(-u)$  and  $w = |\nabla v|^2$ . By a straightforward computation, we have that  $v$  satisfies the following equation

$$\begin{aligned} \partial_t v &= (f')^{p-2} w^{\frac{p-2}{2}} \left[ \Delta v + (p-2) \frac{\langle D^2 v, \nabla v, \nabla v \rangle}{w} \right] + (p-1)(f')^{p-3} f'' w^{\frac{p}{2}} - (f')^{q-1} w^{\frac{q}{2}} \\ &= (f')^{p-2} w^{\frac{p-2}{2}} \left[ \Delta v + (p-2) \frac{\nabla w \cdot \nabla v}{2w} \right] + (p-1)(f')^{p-3} f'' w^{\frac{p}{2}} - (f')^{q-1} w^{\frac{q}{2}}. \end{aligned} \quad (3.1)$$

For  $i = 1, \dots, N$ , we set  $v_i = \frac{\partial v}{\partial x_i}$ . In a neighbourhood  $\tilde{Q} := \omega \times (\tau_1, \tau_2)$  of any point  $(\hat{x}, \hat{t}) \in Q_{T,R}$  for which  $|\nabla u| = f'(v)|\nabla v| > 0$ , the equation is uniformly parabolic and hence differentiating (3.1) with respect to  $x_i$ , we have

$$\begin{aligned} \partial_t v_i &= (f')^{p-2} w^{\frac{p-2}{2}} \left[ \Delta v_i + \frac{p-2}{2} \left( \frac{\nabla w_i \cdot \nabla v + \nabla w \cdot \nabla v_i}{w} - \frac{w_i \nabla w \cdot \nabla v}{w^2} \right) \right] \\ &\quad + (p-2)(f')^{p-3} f'' v_i w^{\frac{p-2}{2}} \left[ \Delta v + (p-2) \frac{\nabla w \cdot \nabla v}{2w} \right] \\ &\quad + \frac{p-2}{2} (f')^{p-2} w_i w^{\frac{p-4}{2}} \left[ \Delta v + (p-2) \frac{\nabla w \cdot \nabla v}{2w} \right] \\ &\quad + (p-1)((f')^{p-3} f'')' v_i w^{\frac{p}{2}} - (q-1)(f')^{q-2} f'' v_i w^{\frac{q}{2}} \\ &\quad + \frac{p(p-1)}{2} (f')^{p-3} f'' w_i w^{\frac{p-2}{2}} - \frac{q}{2} (f')^{q-1} w_i w^{\frac{q-2}{2}}. \end{aligned} \quad (3.2)$$

Here and in all the manuscript, the variable  $v$  is omitted in the expression of  $f', f'', \left(\frac{f''}{f'}\right)'$ , etc. The equalities are understood in a classical sense in  $\tilde{Q}$ . Multiplying (3.2) by  $2v_i$ , summing over  $i$  and using that

$$\langle D^2 v, \nabla v, \nabla v \rangle = \frac{1}{2} \nabla w \cdot \nabla v, \quad \Delta w = 2 \nabla v \cdot \nabla \Delta v + 2 |D^2 v|^2,$$



$$\sum_i 2(\nabla v_i \cdot \nabla w)v_i = |\nabla w|^2, \quad \sum_i (\nabla w_i \cdot \nabla v)v_i = \langle D^2 w, \nabla v, \nabla v \rangle,$$

we get that

$$\begin{aligned} \partial_t w &= |\nabla u|^{p-2} \Delta w + (p-2)|\nabla u|^{p-4} \langle D^2 w, \nabla u, \nabla u \rangle - 2|\nabla u|^{p-2} |D^2 v|^2 \\ &+ (p-2)(f')^{p-2} w^{\frac{p-4}{2}} \Delta v (\nabla v \cdot \nabla w) + \frac{(p-2)}{2} (f')^{p-2} w^{\frac{p-4}{2}} |\nabla w|^2 \\ &+ \frac{(p-2)(p-4)}{2} (f')^{p-2} w^{\frac{p-6}{2}} (\nabla v \cdot \nabla w)^2 \\ &- q(f')^{q-1} w^{\frac{q-2}{2}} \nabla w \cdot \nabla v + (p(p-1) + (p-2)^2)(f')^{p-3} f'' w^{\frac{p-2}{2}} \nabla w \cdot \nabla v \\ &+ 2 \left[ (p-1)((f')^{p-3} f'')' w^{\frac{p+2}{2}} - (q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} + (p-2)(f')^{p-3} f'' w^{\frac{p}{2}} \Delta v \right]. \end{aligned} \quad (3.3)$$

Here, when passing from (3.2) to (3.3), the terms have been transformed according to

$$\begin{aligned} L_{t1}^1 &\rightarrow \tilde{L}_{t1}^1 + \tilde{L}_{t3}^1, & L_{t2}^1 &\rightarrow \tilde{L}_{t2}^1, & L_{t3}^1 &\rightarrow \tilde{L}_{t2}^2, & L_{t4}^1 + L_{t2}^3 &\rightarrow \tilde{L}^3, \\ L_{t1}^2 &\rightarrow \tilde{L}_{t3}^5, & L_{t2}^2 &\rightarrow \tilde{L}_{t3}^4, \\ L_{t1}^3 &\rightarrow \tilde{L}_{t1}^2, \\ L_{t1}^4 &\rightarrow \tilde{L}_{t1}^5, & L_{t2}^4 &\rightarrow \tilde{L}_{t2}^5, \\ L_{t1}^5 &\rightarrow \tilde{L}_{t2}^4, & L_{t2}^5 &\rightarrow \tilde{L}_{t1}^4, \end{aligned}$$

(with obvious labeling).

Hence  $w$  satisfies

$$\partial_t w - \mathcal{A}(w) - \mathcal{H} \cdot \nabla w = -2|\nabla u|^{p-2} |D^2 v|^2 + \mathcal{N}(w)$$

where

$$\mathcal{A}(w) = |\nabla u|^{p-2} \Delta w + (p-2)|\nabla u|^{p-4} \langle D^2 w, \nabla u, \nabla u \rangle, \quad (3.4)$$

$$\begin{aligned} \mathcal{H} &= (p-2)(f')^{p-2} w^{\frac{p-4}{2}} \Delta v \nabla v + \frac{(p-2)}{2} (f')^{p-2} w^{\frac{p-4}{2}} \nabla w, \\ &+ \frac{(p-2)(p-4)}{2} (f')^{p-2} w^{\frac{p-6}{2}} (\nabla v \cdot \nabla w) \nabla v - q(f')^{q-1} w^{\frac{q-2}{2}} \nabla v \end{aligned} \quad (3.5)$$

$$\begin{aligned} \mathcal{N}(w) &= 2(p-1)((f')^{p-3} f'')' w^{\frac{p+2}{2}} - 2(q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} \\ &+ 2(p-2)(f')^{p-2} \frac{f''}{f'} w^{\frac{p}{2}} \Delta v. \end{aligned} \quad (3.6)$$

## Step 2: equation for $z$ and useful estimates

We set  $z = \eta w$ . Defining the operator

$$\mathcal{L}(z) := \partial_t z - \mathcal{A}(z) - \mathcal{H} \cdot \nabla z,$$

we have that

$$\begin{aligned} \mathcal{L}z &= \eta \mathcal{L}w + w \mathcal{L}\eta - 2|\nabla u|^{p-2} \nabla \eta \cdot \nabla w - 2(p-2)|\nabla u|^{p-4} (\nabla u \cdot \nabla \eta) (\nabla w \cdot \nabla u) \\ &= -2|\nabla u|^{p-2} \nabla \eta \cdot \nabla w - 2(p-2)|\nabla u|^{p-4} (\nabla u \cdot \nabla \eta) (\nabla w \cdot \nabla u) \\ &+ \eta \mathcal{N}w + w \mathcal{L}\eta - 2|\nabla u|^{p-2} |D^2 v|^2 \eta. \end{aligned}$$

## Leading estimates

Recalling that  $f$  is increasing and that  $f'' > 0$ , we get the following estimates.

1. Estimate of  $\eta\mathcal{N}w$

$$\begin{aligned}\eta\mathcal{N}w &\leq 2(p-1)((f')^{p-3}f'')'w^{\frac{p+2}{2}}\eta - 2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta \\ &\quad + \frac{(f')^{p-2}}{2}w^{\frac{p-2}{2}}|D^2v|^2\eta + 2N(p-1)^2(f')^{p-2}\left(\frac{f''}{f'}\right)^2w^{\frac{p+2}{2}}\eta.\end{aligned}\quad (3.7)$$

Here we used that

$$2(p-2)\left|\frac{f''}{f'}w\Delta v\right| \leq 2N(p-1)^2w^2\left(\frac{f''}{f'}\right)^2 + \frac{|D^2v|^2}{2}.$$

2. Estimate of  $w\mathcal{L}(\eta)$

- Estimate of  $w\mathcal{A}(\eta)$

$$|w\mathcal{A}(\eta)| \leq (f')^{p-2}w^{\frac{p}{2}}(\sqrt{N} + (p-2))|D^2\eta|. \quad (3.8)$$

- Estimate of  $|wH \cdot \nabla\eta|$

$$\begin{aligned}|wH \cdot \nabla\eta| &\leq \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}(C_1(p, N, \delta_1)\eta^{-1}|\nabla\eta|^2w + \delta_1[D^2v]^2\eta)}_1 \\ &\quad + \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}(C_2(p, N, \delta_2)\eta^{-1}|\nabla\eta|^2w + \delta_2[D^2v]^2\eta)}_2 \\ &\quad + \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}(C_3(p, N, \delta_3)\eta^{-1}|\nabla\eta|^2w + \delta_3[D^2v]^2\eta)}_3 \\ &\quad + 2(p-1)^2(f')^{p-3}f''w^{\frac{p+1}{2}}|\nabla\eta| + q(f')^{q-1}w^{\frac{q+1}{2}}|\nabla\eta|.\end{aligned}\quad (3.9)$$

(1) comes from an estimate via the Young's inequality of  $|(p-2)w\Delta v\nabla v \cdot \nabla\eta|$ .  
Recalling that  $\nabla w = (2D^2v, \nabla v)$ , (2) comes from an estimate of  $\left|\frac{(p-2)}{2}w\nabla w \cdot \nabla\eta\right|$   
and (3) come from an estimate of  $\left|\frac{(p-2)(p-4)}{2}w(\nabla v \cdot \nabla w)(\nabla v \cdot \nabla\eta)\right|$ .

3. Estimate of  $2|\nabla u|^{p-2}|\nabla\eta \cdot \nabla w|$ .

Using the Young inequality, we have

$$2|\nabla u|^{p-2}|\nabla\eta \cdot \nabla w| \leq (f')^{p-2}w^{\frac{p-2}{2}}[C_4(p, N, \delta_4)\eta^{-1}|\nabla\eta|^2w + \delta_4|D^2v|^2\eta].$$

4. Estimate of  $2(p-2)(\nabla u \cdot \nabla\eta)(\nabla w \cdot \nabla u)$

$$|2(p-2)(\nabla u \cdot \nabla\eta)(\nabla w \cdot \nabla u)| \leq (f')^2w[C_5(N, p, \delta_5)\eta^{-1}|\nabla\eta|^2w + |D^2v|^2\eta].$$

Finally recalling that  $\nabla u = f'\nabla v$  and choosing  $\delta_i$  in such way that  $-2 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = -1$  and then recalling the properties of the function  $\eta$ , we arrive at

$$\begin{aligned}\mathcal{L}(z) &\leq 2(p-1)\eta\left[\left((f')^{p-3}f''\right)'w^{\frac{p+2}{2}} + N(p-1)(f')^{p-2}\left(\frac{f''}{f'}\right)^2w^{\frac{p+2}{2}}\right] \\ &\quad - 2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta + C(p, N, \alpha)(f')^{p-2}R^{-2}w^{\frac{p}{2}}\eta^\alpha \\ &\quad + C(p, q, \alpha)\eta^\alpha R^{-1}\left[w^{\frac{p+1}{2}}(f')^{p-3}f'' + w^{\frac{q+1}{2}}(f')^{q-1}\right].\end{aligned}$$

### Step 3: suitable choice for the function $f$

To get rid of the term

$$((f')^{p-3} f'')' w^{\frac{p+2}{2}} + N(p-1)(f')^{p-2} \left(\frac{f''}{f'}\right)^2 w^{\frac{p+2}{2}} \quad (3.10)$$

$$= (f')^{p-2} w^{\frac{p+2}{2}} \left[ \left(\frac{f''}{f'}\right)' + (p-2) \left(\frac{f''}{f'}\right)^2 + (p-1)N \left(\frac{f''}{f'}\right)^2 \right] \quad (3.11)$$

$$\leq (f')^{p-2} w^{\frac{p+2}{2}} \left[ \left(\frac{f''}{f'}\right)' + (p-1)(N+1) \left(\frac{f''}{f'}\right)^2 \right]. \quad (3.12)$$

we shall take a function  $f$  satisfying the following differential equation

$$\left(\frac{f''}{f'}\right)' + (p-1)(1+N) \left(\frac{f''}{f'}\right)^2 = 0. \quad (3.13)$$

Hence we get that

$$\begin{aligned} \mathcal{L}(z) &\leq -2(q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} \eta + C(p, N, \alpha)(f')^{p-2} R^{-2} \eta^\alpha w^{\frac{p}{2}} \\ &\quad + C(p, q, \alpha) R^{-1} \eta^\alpha \left[ w^{\frac{p+1}{2}} (f')^{p-3} f'' + w^{\frac{q+1}{2}} (f')^{q-1} \right]. \end{aligned}$$

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