Nonlocal anisotropic damage model and related computational aspects for quasi-brittle materials

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Abstract

A three dimensional damage model with induced damage anisotropy is proposed for quasi-brittle materials such as concrete. The thermodynamics framework is used, considering then a single 2nd order tensorial damage variable whatever the intensity and the sign of the loading. The quasi-unilateral conditions of microcracks closure are written on the hydrostatic stress only. Altogether with the consideration of damage laws ensuring a damage rate proportional to the positive part of the strain tensor this is sufficient to model a strongly different behavior due to damage in tension and in compression. A proof of the positivity of the intrinsic dissipation due to such an induced anisotropic damage is given.

An efficient scheme for the implementation of the damage model in commercial Finite Element codes is then detailed and numerical examples of structural failures are given. Plain concrete, reinforced and pre-stressed concrete structures are computed up to high damage level inducing yielding of the reinforcement steels. Local and nonlocal computations are performed.

A procedure for the control of rupture is proposed. It is a key point making the computations with anisotropic damage truly efficient.

Key words: concrete, damage, induced anisotropy, nonlocal, Finite Element computations

Introduction

To extend existing isotropic damage models to induced anisotropy is not an easy task as difficulties and questions specifically related to anisotropy arise. How to write the coupling damage/elasticity? What becomes then the effective stress concept associated

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with the principle of strain equivalence [1, 2] ? Which tensorial representation of damage shall be used ? If general damage anisotropy can be represented by a fourth order damage tensor [3, 4, 5], this formally simple choice is difficult to work with, usually because of the high number of material parameters introduced. And how to model properly and efficiently (when numerical computations are in mind) the stiffness recovery due to the micro-defects closure effect [6, 7, 8, 9, 10, 11]? the damage growth higher in tension than in compression? In order to give a practical answers to these questions, the choice to represent the damage state by a second order damage tensor has been made by many authors [12, 2, 13, 14, 15, 16].

Concerning physical damage anisotropy, quasi-brittle materials such as concrete exhibit a micro-cracking pattern different in tensile and in compressive loadings [17, 18]: the micro-cracks are mainly orthogonal to the loading direction in tension and parallel to it in compression. This induced anisotropy is responsible for the large dissymmetry tension/compression of concrete behavior and must be introduced to do so in constitutive modeling. For thermodynamics consistency and according to this last remark only one damage variable must be considered. As a state variable a damage variable represents the micro-cracks pattern, whatever the sign of the loading [19, 20], and cannot be related to either tension or compression. For the sake of relative simplicity anisotropic damage is next represented by the second order tensor $\boldsymbol{D}$ of components D_{ij}. If a tensile loading is applied in direction 1, induced anisotropic (diagonal) damage shall act as its component $D_1 = D_{11}$ instead of a damage variable "for tension". If the loading is compressive, damage shall act as its component $D_2 = D_{22} = D_{33}$ instead of a damage variable "for compression". Written in the thermodynamics framework the model presented next follows these guidelines.

1 Coupling damage/elasticity using a 2nd order damage tensor

The thermodynamics framework proposed by Ladevèze leading to 3D continuous stress-strain responses is used [6, 21, 22]. The damage state is represented by the 2nd order tensor $\boldsymbol{D}$ and there is one known thermodynamics potential $\rho\psi_0^\star$ from which derives a symmetric effective stress $\tilde{\boldsymbol{\sigma}}$ independent from the elasticity parameters [23]:

$$\rho\psi_0^\star = \frac{1+\nu}{2E} tr\left(\boldsymbol{H}\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D\right) + \frac{1-2\nu}{6E}\frac{(tr\,\boldsymbol{\sigma})^2}{1-\eta D_H} \tag{1}$$

with E, ν the Young modulus and Poisson ratio of initially isotropic elasticity, η the hydrostatic sensitivity parameter ($\eta \approx 3$ for most materials [22]), ρ the density, where $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{3} tr\,\boldsymbol{\sigma}\,\mathbf{1}$ is the deviatoric stress and where $\boldsymbol{H}$ is the effective damage tensor, D_H the hydrostatic damage,

$$\boldsymbol{H} = (\mathbf{1} - \boldsymbol{D})^{-1/2} \quad \text{and} \quad D_H = \frac{1}{3} tr\,\boldsymbol{D} \tag{2}$$

Quasi-brittle materials such as concrete exhibit a strong difference of behavior in tension and in compression due to damage. This micro-defects closure effect usually leads to complex models when damage anisotropy is considered [24, 9, 25, 26] and the purpose here is to show that for monotonic applications it is sufficient to consider damage anisotropy with a quasi-unilateral effect acting on the hydrostatic stress only, with a thermodynamics potential rewritten

$$\rho\psi^{\star} = \frac{1+\nu}{2E} tr \left(\boldsymbol{H}\boldsymbol{\sigma}^{D}\boldsymbol{H}\boldsymbol{\sigma}^{D} \right) + \frac{1-2\nu}{6E} \left[\frac{\langle tr\ \boldsymbol{\sigma} \rangle^{2}}{1 - tr\ \boldsymbol{D}} + \langle -tr\ \boldsymbol{\sigma} \rangle^{2} \right] \tag{3}$$

so that the elasticity law reads

$$\boldsymbol{\epsilon} = \rho \frac{\partial \psi^{\star}}{\partial \boldsymbol{\sigma}} = \frac{1+\nu}{E} \tilde{\boldsymbol{\sigma}} - \frac{\nu}{E} tr\ \tilde{\boldsymbol{\sigma}}\ \mathbf{1} \tag{4}$$

and defines the symmetric effective stress $\tilde{\boldsymbol{\sigma}}$ independent from the elasticity parameters,

$$\tilde{\boldsymbol{\sigma}} = (\boldsymbol{H}\boldsymbol{\sigma}^{D}\boldsymbol{H})^{D} + \frac{1}{3} \left[\frac{\langle tr\ \boldsymbol{\sigma} \rangle}{1 - tr\ \boldsymbol{D}} - \langle -tr\ \boldsymbol{\sigma} \rangle \right] \mathbf{1} \tag{5}$$

with $(.)^{D} = (.) - \frac{1}{3} tr\ (.)\ \mathbf{1}$ the deviatoric part. The notation $\langle . \rangle$ stands for the positive part of a scalar, $\langle x \rangle = \max(x, 0)$.

Because of the splitting between deviatoric and hydrostatic contributions in the thermodynamics potential, the bulk modulus of the damaged material $\tilde{K}$ remains constant for negative hydrostatic stresses, equal to the undamaged modulus K ($\tilde{K} = K = \frac{E}{3(1-2\nu)}$ if $tr\ \boldsymbol{\sigma} < 0$). In other words no volumetric damage is supposed to be associated with negative strains, and the collapse of micropores in the cement past is neglected under high hydrostatic compression. This simple and therefore first stage modeling choice has already been made by several authors [17, 27, 21]. It is consistent with further consideration of Mazars damage criterion, an open criterion for the tricompression states. To consider a constant bulk modulus in compression is nevertheless a better choice for concrete-like materials than the feature of a constant (damage independent) apparent Poisson ratio encountered in most isotropic damage models. As often as long as the permanent strains due to micro-cracking and internal friction are not taken into account [28, 29, 30, 21, 10, 26, 31], dilatancy is not reproduced.

Once the coupling damage/elasticity is defined, it can be used to make quantitative damage measurements. For an uniaxial state of stress, the elasticity law reads

$$\boldsymbol{\epsilon} = \frac{\sigma}{E}\ \boldsymbol{B}(D_1, D_2) \tag{6}$$

with $\boldsymbol{B}$ a dimensionless tensor, diagonal, function of the principal damages D_1 and $D_3 = D_2$ only (the loading direction is denoted 1),

- in tension,

$$
\begin{aligned}
B_{11} &= (1+\nu)\left[\frac{4}{9(1-D_1)}+\frac{2}{9(1-D_2)}\right]+\frac{1-2\nu}{3(1-D_1-2D_2)}\\
B_{22} = B_{33} &= -(1+\nu)\left[\frac{2}{9(1-D_1)}+\frac{1}{9(1-D_2)}\right]+\frac{1-2\nu}{3(1-D_1-2D_2)}
\end{aligned}
\tag{7}
$$

- in compression,

$$
\begin{aligned}
B_{11} &= (1+\nu)\left[\frac{4}{9(1-D_1)}+\frac{2}{9(1-D_2)}\right]+\frac{1-2\nu}{3}\\
B_{22} = B_{33} &= -(1+\nu)\left[\frac{2}{9(1-D_1)}+\frac{1}{9(1-D_2)}\right]+\frac{1-2\nu}{3}
\end{aligned}
\tag{8}
$$

so that the measurement of the secant modulus $E_S = \tilde{E} = \sigma/\epsilon_{11}$ gives a possibility to measure the damage if either one of the principal damages D_1 or D_2 is zero (or constant) or if the ratio D_2/D_1 is known for example from the damage evolution law (see section 3),

$$B_{11}(D_1, D_2) = \frac{E}{\tilde{E}} \quad \longrightarrow \quad D_1 \text{ or } D_2 \tag{9}$$

Last, the state potential (3) defines the energy density release rate tensor $\boldsymbol{Y}$ as the thermodynamics variable associated with $\boldsymbol{D}$, $\boldsymbol{Y} = \rho\frac{\partial\psi^\star}{\partial\boldsymbol{D}}$ [22].

2 Local and nonlocal damage criterion functions

As for plasticity, the elasticity domain can be defined through a criterion function f such as the domain $f < 0$ corresponds to elastic loading or unloading. Many criterion can be used, written in terms of stresses such as plasticity criteria [32, 24, 33, 34], strains [17, 28, 35, 36], or strain energy release rate density [37, 5, 29] The purpose here is to built a nonlocal constitutive model with a restricted number of material parameters, robust and easy to implement in Finite Element computer codes. Dilatancy will not be taken into account and one will accept an open criterion for the tricompression states.

These remarks lead us for the present work to the simple choice of Mazars criterion, function of the positive extensions $\langle\epsilon_I\rangle$ of the I^{th} principal strain ϵ_I,

$$f = \hat{\epsilon} - \kappa \qquad \hat{\epsilon} = \sqrt{\sum_{I=1}^{3}\langle\epsilon_I\rangle^2} = \sqrt{\langle\boldsymbol{\epsilon}\rangle_+ : \langle\boldsymbol{\epsilon}\rangle_+} \tag{10}$$

where $\hat{\epsilon}$ is the equivalent strain for quasi-brittle materials and κ is the elastic strain limit in tension. The notation $\langle\boldsymbol{\epsilon}\rangle_+$ stands for the positive part of the strain tensor in terms of principal values. Note that the choice of Mazars equivalent strain is the most efficient in terms of number of material parameters introduced (none!) but it leads to a tension/compression dissymmetry of the elasticity limit usually not large enough in the uniaxial case (case adjusted next by an adequate set of damage threshold and of damage parameters), feature to be even more emphasized for the bitension/bicompression dissymmetry case. One uses next Mazars criterion with these drawbacks in mind, i.e. with in mind that further studies on the confined multiaxial loading paths will need improvements concerning the definition of the equivalent strain $\hat{\epsilon}$ as in [38]. The proposed framework will nevertheless strictly apply with just such a change of definition. Note that Mazars strain present choice already allows to face 3D Finite Element computations with anisotropic damage of plain but also reinforced and pre-stressed concrete structures, as illustrated in section 8.

To build a nonlocal damage model, it will be sufficient to consider nonlocal Mazars criterion,

$$f = \hat{\epsilon}^{nl} - \kappa \tag{11}$$

introducing the nonlocal equivalent strain $\hat{\epsilon}^{nl}$ which can be defined using an integral form ($\mathcal{W}$ is the nonlocal weight function, [39],

$$\hat{\epsilon}^{nl} = \hat{\epsilon}^{nl}(\mathbf{x}) = \frac{1}{V_r}\int_\Omega \mathcal{W}(\mathbf{x}-\mathbf{s})\ \hat{\epsilon}(\mathbf{s})\ ds \qquad V_r = V_r(\mathbf{x}) = \int_\Omega \mathcal{W}(\mathbf{x}-\mathbf{s})ds \tag{12}$$

or using a second gradient form [40, 41],

$$\hat{\epsilon}^{nl} - c\ \nabla^2 \hat{\epsilon}^{nl} = \hat{\epsilon} \tag{13}$$

Both the integral form (through $\mathcal{W}$) and the gradient form (through c) introduce a characteristic length l_c. Even if induced anisotropy is considered next, the introduction of a single (isotropic) internal length will prove sufficient for practical applications.

3 Damage evolution laws for induced anisotropy

As already mentionned, a single (tensorial) damage variable is considered. This choice is thermodynamically consistent with the representation by an internal variable of the degradation mechanisms. It solves the problem of the artificial introduction of two scalar damage variables – one for tension, one for compression – in case of isotropic damage, as a damage variable represents a state of microcraking which remains the same at constant internal variables whatever the microcracks are positively or negatively loaded. Damage neverthess acts differently in tension and in compression and the induced anisotropy is then naturally responsible for the tension/compression dissymmetry. Damage is generally larger in the tensile direction denoted 1 in further developments with D_1, D_2, D_3 as principal damages.

For metals loaded in tension in direction 1 the damages D_2 and D_3 are of the order of $D_1/2$ [23]. This feature is modelled by a damage evolution law ensuring a damage rate proportional to the absolute value (in terms of principal components) of the plastic strain rate tensor:

$$\dot{\boldsymbol{D}} \propto |\dot{\boldsymbol{\epsilon}}^p| \tag{14}$$

For concrete, the damage pattern is different in tension and in compression [17, 30, 26]. For a tension loading applied in direction 1:

$$\boldsymbol{D} \approx \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

when for a compression loading applied in direction 1:

$$\boldsymbol{D} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \tag{16}$$

These two damage states can be modelled by damage evolution laws ensuring a damage tensor rate proportional to the positive part (in terms of principal components) of the total strain tensor:

$$\dot{\boldsymbol{D}} \propto \langle \boldsymbol{\epsilon} \rangle_+ \tag{17}$$

but also by any rate law of the form

$$\dot{\boldsymbol{D}} \propto \langle \boldsymbol{\epsilon} \rangle_+^\alpha \tag{18}$$

with α a damage exponent.

3.1 Non associated thermodynamics model

To propose a damage model written in the thermodynamics framework, consider as damage pseudo-potentiel :

$$F = F(\boldsymbol{Y}; \boldsymbol{\epsilon}) = \boldsymbol{Y} : \langle \boldsymbol{\epsilon} \rangle_+^\alpha \tag{19}$$

where $\boldsymbol{\epsilon}$ acts as a parameter so that the damage evolution law is derived from the normality rule as

$$\dot{\boldsymbol{D}} = \dot{\lambda} \frac{\partial F}{\partial \boldsymbol{Y}} = \dot{\lambda} \langle \boldsymbol{\epsilon} \rangle_+^\alpha \tag{20}$$

The damage multiplier $\dot{\lambda}$ is determined from the consistency condition $f = 0$, $\dot{f} = 0$. Making the simple choice $\kappa = \kappa(tr\, \boldsymbol{D})$ [35] gives for the local model

$$\dot{\lambda} = \frac{d\kappa^{-1}}{d\hat{\epsilon}} \frac{\dot{\hat{\epsilon}}}{tr \langle \boldsymbol{\epsilon} \rangle_+^\alpha} \tag{21}$$

The anisotropic damage law takes then the general form:

$$\dot{\boldsymbol{D}} = \frac{d\kappa^{-1}}{d\hat{\epsilon}} \frac{\langle \boldsymbol{\epsilon} \rangle_+^\alpha}{tr\langle \boldsymbol{\epsilon} \rangle_+^\alpha} \dot{\hat{\epsilon}} \tag{22}$$

where the exponent α mainly plays a role in multiaxial states of stresses. Setting $\alpha = 2$ (so that $tr\langle \boldsymbol{\epsilon} \rangle_+^\alpha = \hat{\epsilon}^2$) makes once more the equivalent strain $\hat{\epsilon}$ appear. This choice is then consistent with Mazars criterion, the damage law simplifying as

$$\dot{\boldsymbol{D}} = \frac{d\kappa^{-1}}{d\hat{\epsilon}} \frac{\langle \boldsymbol{\epsilon} \rangle_+^2}{\hat{\epsilon}^2} \dot{\hat{\epsilon}} \tag{23}$$

In tension performed in direction 1, $\hat{\epsilon} = \epsilon_1 > 0$ and :

$$\langle \boldsymbol{\epsilon} \rangle_+ = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \langle \boldsymbol{\epsilon} \rangle_+^\alpha = \begin{bmatrix} \epsilon_1^\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

so that the form (15) of the damage tensor in tension is recovered.

In compression performed in direction 1, $\hat{\epsilon} = \sqrt{2}\epsilon_2 > 0$,

$$\langle \boldsymbol{\epsilon} \rangle_+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix} \qquad \langle \boldsymbol{\epsilon} \rangle_+^\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_2^\alpha & 0 \\ 0 & 0 & \epsilon_2^\alpha \end{bmatrix} \tag{25}$$

so that the form (16) of the damage tensor in compression is recovered.

3.2 Nonlocal damage law

For the nonlocal model, the equivalent strain $\hat{\epsilon}$ is replaced by $\hat{\epsilon}^{nl}$ in the criterion function. The only change is then in the expression for the damage multiplier with now:

$$\dot{\lambda} = \frac{d\kappa^{-1}}{d\hat{\epsilon}^{nl}} \frac{\dot{\hat{\epsilon}}^{nl}}{tr\langle \boldsymbol{\epsilon} \rangle_+^\alpha} \tag{26}$$

so that the damage evolution law becomes non local, i.e. $\dot{\boldsymbol{D}}(\mathbf{x})$ at point $\mathbf{x}$ is (through $\hat{\epsilon}^{nl}$) either function of the values of the surrounding points $\mathbf{s}$ or function of the strains gradient,

$$\dot{\boldsymbol{D}} = \frac{d\kappa^{-1}}{d\hat{\epsilon}^{nl}} \frac{\langle \boldsymbol{\epsilon} \rangle_+^\alpha}{tr\langle \boldsymbol{\epsilon} \rangle_+^\alpha} \dot{\hat{\epsilon}}^{nl} \qquad \text{or} \qquad \dot{\boldsymbol{D}} = \frac{d\kappa^{-1}}{d\hat{\epsilon}^{nl}} \frac{\langle \boldsymbol{\epsilon} \rangle_+^2}{\hat{\epsilon}^2} \dot{\hat{\epsilon}}^{nl} \qquad (\alpha = 2) \tag{27}$$

Of course, on the level of a single material point the nonlocal model will give the same response than the local one.

4 Positivity of the intrinsic dissipation

The thermodynamics framework has been used to derive both the state law and the damage evolution law but, due to the choices made for the potentials $\rho\psi^\star$ and F, the model is not standard generalized [42]. The positivity of the intrinsic dissipation, here $\boldsymbol{Y} : \dot{\boldsymbol{D}}$, is then not guaranteed and must be checked.

Even if the state potential (3) is not convex with respect to $\boldsymbol{D}$, it can be differentiated as $\rho d\psi^\star = \boldsymbol{\epsilon} : d\boldsymbol{\sigma} + \boldsymbol{Y} : d\boldsymbol{D}$ and also as

$$\begin{aligned}\rho d\psi^\star =& \frac{1+\nu}{E}(\boldsymbol{H}\boldsymbol{\sigma}^D\boldsymbol{H})^D : d\boldsymbol{\sigma} + \frac{1-2\nu}{3E}\left[\frac{\langle tr\,\boldsymbol{\sigma}\rangle}{1-tr\,\boldsymbol{D}} - \langle -tr\,\boldsymbol{\sigma}\rangle\right] tr\,d\boldsymbol{\sigma} \\ &+ \frac{1+\nu}{E}(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D) : d\boldsymbol{H} + \frac{1-2\nu}{6E}\frac{\langle tr\,\boldsymbol{\sigma}\rangle^2}{(1-tr\,\boldsymbol{D})^2}\,tr\,d\boldsymbol{D}\end{aligned} \tag{28}$$

so that:

$$\boldsymbol{Y} : \dot{\boldsymbol{D}} = \frac{1+\nu}{E}(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D) : \dot{\boldsymbol{H}} + \frac{1-2\nu}{6E}\frac{\langle tr\,\boldsymbol{\sigma}\rangle^2}{(1-tr\,\boldsymbol{D})^2}\,tr\,\dot{\boldsymbol{D}} \tag{29}$$

The term in $tr\,\dot{\boldsymbol{D}}$ will be obviously positive for any monotonic increasing κ^{-1}-function so one has to concentrate on the term $(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D) : \dot{\boldsymbol{H}}$.

Any damage law of the form (18) gives positive increasing eigenvalues D_J for the damage tensor $\boldsymbol{D}$. With the relation (5), the eigenvalues for $\boldsymbol{H}$ are

$$H_J = \frac{1}{\sqrt{1-D_J}} \tag{30}$$

and are therefore also positive and increasing during any damage process. The positivity of the matrix $(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D)$ is last gained by checking the sign of its eigenvalues, denoted χ, solution of $(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D)\,\vec{g} = \chi\,\vec{g}$ (with $\vec{g}$ standing for the corresponding eigenvectors), or equivalently solution of

$$(\boldsymbol{H}\boldsymbol{\sigma}^D)^2\vec{g} = \chi\,\boldsymbol{H}\vec{g} \tag{31}$$

with $(\boldsymbol{H}\boldsymbol{\sigma}^D)^2$ obviously a positive matrix. The eigenvalues χ read then:

$$\chi = \frac{\vec{g}^T(\boldsymbol{H}\boldsymbol{\sigma}^D)^2\vec{g}}{\vec{g}^T\boldsymbol{H}\vec{g}} \tag{32}$$

As ratios of positive terms they are positive as is then the second order tensor $(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D)$. The tensorial product two positive tensors, $(\boldsymbol{\sigma}^D\boldsymbol{H}\boldsymbol{\sigma}^D)$ and $\dot{\boldsymbol{H}}$ being positive, one can conclude that the positivity of the intrinsic dissipation is satisfied for any loading, eventually 3D, complex and/or non proportional.

5 Local and nonlocal anisotropic damage models

In order to derive and to implement the full set of constitutive equations for the anisotropic damage model one has last to define the function $\kappa = \kappa(tr\ \boldsymbol{D})$. The simplest choice is to consider a linear function introducing two parameters only: the damage threshold $\kappa_0 = \kappa(0)$ and a damage parameter A as [35, 26, 43]

$$\kappa(tr\ \boldsymbol{D}) = \frac{1}{A} tr\ \boldsymbol{D} + \kappa_0 \tag{33}$$

This leads to

$$\kappa^{-1}(\hat{\epsilon}) = A\left(\hat{\epsilon} - \kappa_0\right) \tag{34}$$

and to the damage law ($\alpha = 2$, nonlocal model):

$$\dot{\boldsymbol{D}} = A \frac{\langle \boldsymbol{\epsilon} \rangle_+^2}{\hat{\epsilon}^2}\ \dot{\hat{\epsilon}}^{nl} \qquad \text{(damage law 1)} \tag{35}$$

with for the local damage model the rate $\dot{\hat{\epsilon}}^{nl}$ replaced by local Mazars strain rate $\dot{\hat{\epsilon}}$.

Damage anisotropy is different in tension and in compression. It affects differently the elasticity law and a strong difference in tension and in compression is finally obtained with the quite simple damage evolution law 1 (figure 2). Important point, this feature is gained with the consideration of one (tensorial) damage variable only, in accordance with the thermodynamics definition of a state variable: if one degradation mechanism is observed, only one damage variable shall represent the micro-cracks or micro-defects pattern, whatever the material is in tension or is in compression. The dissymmetry is nevertheless not sufficient with the linear κ-function with a too high damage rate in compression leading to a non physical snapback. One prefers then to consider as damage evolution law:

$$\dot{\boldsymbol{D}} = A \left[1 + \left(\frac{\hat{\epsilon}^{nl}}{a}\right)^2\right]^{-1} \frac{\langle \boldsymbol{\epsilon} \rangle_+^2}{\hat{\epsilon}^2}\ \dot{\hat{\epsilon}}^{nl} \qquad \text{(damage law 2)} \tag{36}$$

with a a material parameter of the order of magnitude the value of the strain reached in compression. In tension the term $(\hat{\epsilon}/a)^2 << 1$ is negligible so that the damage law 1 (35) is recovered.

Comparing equations (27) and (36) allows to determine the κ-function,

$$\frac{d\kappa^{-1}}{d\hat{\epsilon}^{nl}} = A \left[1 + \left(\frac{\hat{\epsilon}^{nl}}{a}\right)^2\right]^{-1} \tag{37}$$

This defines κ^{-1} and κ as

$$\kappa^{-1}(\hat{\epsilon}^{nl}) = aA \left[\arctan\left(\frac{\hat{\epsilon}^{nl}}{a}\right) - \arctan\left(\frac{\kappa_0}{a}\right)\right] \tag{38}$$

$$\kappa(tr\ \boldsymbol{D}) = a \cdot \tan\left[\frac{tr\ \boldsymbol{D}}{aA} + \arctan\left(\frac{\kappa_0}{a}\right)\right] \tag{39}$$

which have the same expression for the local damage model with also $\hat{\epsilon}^{nl}$ (and $\dot{\hat{\epsilon}}^{nl}$ for the damage law) replaced by $\hat{\epsilon}$.

For accuracy reasons, the damage parameters A, a as well as the damage threshold must be identified by curve fitting on both the tension and compression curves of the material. Nevertheless, the damage measurement by eq. (9) with $D_2 = 0$ in tension and $D_1 = 0$ in compression allows to plot the trace of the damage tensor versus the equivalent strain curve. In tension, $tr\ \boldsymbol{D} = D_1$, $\hat{\epsilon} = \epsilon_{11}$; in compression, $tr\ \boldsymbol{D} = 2D_2$, $\hat{\epsilon} = \sqrt{2}\epsilon_{22} = \sqrt{2}\frac{B_{22}(D_2)}{B_{11}(D_2)}\epsilon_{11}$. An example is given in figure 1 for concrete from which one can then justify to consider the function κ as a function of the trace of $\boldsymbol{D}$: the experimental points from tension and from compression determine the same curve. Note also the characteristic shape of the damage curve obtained for models with no permanent strains: the damage seems to reach an asymptotic value. The concavity of the damage evolution will probably change if the permanent strains are modeled and if the damage is measured from the elastic unloading slope instead of from the secant modulus. In any case the principal damages D_1, D_2, D_3 are each bounded by 1 or most often by a critical damage D_c. The trace $tr\ \boldsymbol{D}$ can then take values larger than 1 (and nevertheless bounded by 3 or $3D_c$) but only in cases of negative states of stresses as $tr\ \boldsymbol{D}$ does not affect the compressibility modulus $\tilde{K}$ kept equal to the initial modulus K in such cases.

Fig. 1. Trace of $\boldsymbol{D}$ function of Mazars equivalent strain (marks: experiments, continuous line: model)

The full set of constitutive equations finally reads:

- Elasticity,

$$\boldsymbol{\epsilon} = \frac{1+\nu}{E}\tilde{\boldsymbol{\sigma}} - \frac{\nu}{E} tr\ \tilde{\boldsymbol{\sigma}}\ \mathbf{1} \qquad \text{or} \qquad \boldsymbol{\epsilon} = \underline{\boldsymbol{E}}^{-1} : \tilde{\boldsymbol{\sigma}} \tag{40}$$

- Effective stress,

$$\tilde{\boldsymbol{\sigma}} = \left[(\mathbf{1}-\boldsymbol{D})^{-1/2}\ \boldsymbol{\sigma}^D\ (\mathbf{1}-\boldsymbol{D})^{-1/2}\right]^D + \frac{1}{3}\left[\frac{\langle tr\ \boldsymbol{\sigma}\rangle}{1 - tr\ \boldsymbol{D}} - \langle -tr\ \boldsymbol{\sigma}\rangle\right]\mathbf{1} \tag{41}$$

- Mazars equivalent strain,

$$\hat{\epsilon} = \sqrt{\langle\boldsymbol{\epsilon}\rangle_+ : \langle\boldsymbol{\epsilon}\rangle_+} \tag{42}$$

- Nonlocal Mazars strain, defined either in gradient form (13) or in integral form (12). With a Gaussian weight function,

$$\hat{\epsilon}^{nl} = \frac{1}{V_r}\int_\Omega \exp\left(-\frac{\|\mathbf{x}-\mathbf{s}\|^2}{l_c^2}\right)\ \hat{\epsilon}(\mathbf{s})\ ds \qquad V_r = \int_\Omega \exp\left(-\frac{\|\mathbf{x}-\mathbf{s}\|^2}{l_c^2}\right) ds \tag{43}$$

- Damage criterion, local $f = \hat{\epsilon} - \kappa(tr\ \boldsymbol{D})$ or nonlocal or $f = \hat{\epsilon}^{nl} - \kappa(tr\ \boldsymbol{D})$,

$$\begin{aligned} &f < 0 \text{ or } \dot{f} \neq 0 \longrightarrow \text{elastic loading or unloading} \\ &f = 0 \text{ and } \dot{f} = 0 \longrightarrow \text{damage growth} \end{aligned} \tag{44}$$

- Damage evolution law, local (if $\hat{\epsilon}^{nl} = \hat{\epsilon}$ is set) or nonlocal,

$$\dot{\boldsymbol{D}} = A \left[1 + \left(\frac{\hat{\epsilon}^{nl}}{a} \right)^2 \right]^{-1} \frac{\langle \boldsymbol{\epsilon} \rangle_+^2}{\hat{\epsilon}^2} \, \dot{\hat{\epsilon}}^{nl} \tag{45}$$

  where $\alpha = 2$ is set.

It is a first order differential set of equations, time independent as no viscosity is introduced. The use of a damage criterion function f written in terms of strains will allow for a quite simple implementation in a Finite Elements computer code. Note that the elasticity law will then need to be inverted. This can be done in a closed form as:

$$\boldsymbol{\sigma} = (\mathbf{1} - \boldsymbol{D})^{1/2} \, \tilde{\boldsymbol{\sigma}} \, (\mathbf{1} - \boldsymbol{D})^{1/2} - \frac{(\mathbf{1} - \boldsymbol{D}) : \tilde{\boldsymbol{\sigma}}}{3 - tr \, \boldsymbol{D}} (\mathbf{1} - \boldsymbol{D}) + \frac{1}{3} \left[(1 - tr \, \boldsymbol{D}) \langle tr \, \tilde{\boldsymbol{\sigma}} \rangle - \langle -tr \, \tilde{\boldsymbol{\sigma}} \rangle \right] \mathbf{1} \tag{46}$$

There is a total of only 5 material parameters for the final constitutive model with induced damage anisotropy (plus a characteristic length l_c for the nonlocal model): E, ν for elasticity, κ_0 as damage threshold, A and a for damage evolution. A single set of material parameters is valid for both tension and compression. A critical damage D_c may also be considered: when the maximum principal damage reaches D_c there is rupture [23, 44]. There is often no need to introduce a critical damage for nonlocal computations as they need the stress-strain response of the material up to vanishing stresses and as the mesoscopic cracks are represented by localized zones or shear bands.

Fig. 2. Stress-strain curves for concrete (tensile strength $f_t = 4$ MPa, compressive strength $f_c = 38$ MPa)

Figure 2 shows for both damage laws 1 and 2 the monotonic stress-strain curves for concrete obtained either in tension or in compression. Note that for the proposed anisotropic damage models the stress goes to zero in all cases for highly damaged materials. The material parameters describing well concrete behavior (using law 2) are: $E = 42$ GPa, $\nu = 0.2$, $\kappa_0 = 5 \; 10^{-5}$, $A = 5 \; 10^3$, $a = 2.93 \; 10^{-4}$. The same value for the damage parameter A is used for law 1 illustrating the fact that there is an influence of parameter a on tension. An adequate choice for A will allow to make both tensile responses strictly match but not the compressive responses. Important point, one can see on such curves that damage anisotropy is truly responsible for the dissymmetry tension/compression, even if not in an adequate manner for the damage law 1. The snapback exhibited for law 1 is not physical and corresponds to a damage rate too high in compression, rate corrected through the consideration of parameter a. This kind of unstable compressive response was also obtained for Bažant and Gambarova initial microplane model [45, 46] in which the local shear behavior was neglected (only local shear representation leading to correct concrete response in compression [47]). The property of shear/compression related behaviors is illustrated here for the anisotropic damage model by plotting the shear stress τ vs engineering shear strain γ monotonic model response (Fig. 3). To introduce the parameter $a = 2.93 \; 10^{-4}$ in law 2 and then to model properly the concrete compressive response induce a less brittle response in shear. Note that this effect due to a lower damage growth will be even more pronounced if the quasi-unilateral conditions of microcracks closure are written on the deviatoric stresses also [22, 48] when in the present case they are written on the hydrostatic stress only.

Fig. 3. Model response in shear for damage laws 1 and 2

Last, the fact that the thermodynamics potential can be continuously differentiated leads to the natural continuity of each component of the stress and strain tensors, even for non proportional loading paths. This property is essential for the ability of anisotropic damage models to deal with complex loading as encountered at Finite Element Gauss points.

6 Numerical implementation

The anisotropic damage model is in fact quite simple to implement in a Finite Element code: local iterations may be avoided even if Euler backward scheme is used, as proposed in present implementation procedure.

A global resolution of the equilibrium equations gives the displacements at time t_{n+1} with the internal damage variable $\boldsymbol{D} = \boldsymbol{D}_n$ kept unchanged from the last computed increment t_n. The strains $\boldsymbol{\epsilon}_{n+1} = \boldsymbol{\epsilon}(t_{n+1})$ at each Gauss point are calculated from the elements interpolation functions. To integrate the constitutive equations means to determine the stress $\boldsymbol{\sigma}_{n+1}$ and the damage $\boldsymbol{D}_{n+1}$ at time t_{n+1}. An iterative process, not described here, made of global equilibrium resolutions followed by local time integration of the constitutive equations often takes place [49, 50, 51, 52, 53, 54]. One focus here on the numerical scheme for the local integration of the damage law.

6.1 Exact resolution of Euler backward discretization

The inputs of the time integration subroutine are the strain tensor $\boldsymbol{\epsilon}_{n+1}$ and the damage tensor $\boldsymbol{D}_n$ at a FE Gauss point; the ouputs are the damage and stress tensors $\boldsymbol{D}_{n+1}$, $\boldsymbol{\sigma}_{n+1}$ at the same point. Instead of using the elastic stiffness, to calculate the secant operator $\underline{\underline{\boldsymbol{E}}}^S$ helps the global resolutions, it helps a lot for the highly damaged parts subject to elastic unloadings. Computations will generally be more efficient if one uses the consistent tangent operator. But note that as one has to take the positive part of the strain tensor in previous constitutive equations, the stress-strain law cannot be continuously differentiated (the thermodynamics potential cannot be derivated twice) so that the tangent operator does not exist at some points, for instance for non proportional loading conditions with changes of sign of the strains. One has also in mind nonlocal integral computations for which it even more difficult to define a tangent operator [55] and this key point for convergency acceleration with risk of loss of robustness is left to further developpments.

Euler backward scheme is used, i.e. the variables are replaced by their value at time t_{n+1} in the constitutive equations when the damage rate $\dot{\boldsymbol{D}}$ and the damage multiplier $\dot{\lambda}$ are replaced by $\Delta \boldsymbol{D} = \boldsymbol{D}_{n+1} - \boldsymbol{D}_n$ and $\Delta\lambda = \lambda_{n+1} - \lambda_n$ in the damage law. In order to integrate the damage model proceed as follows:

(1) Compute the equivalent strain,

$$\hat{\epsilon}_{n+1} = \sqrt{\langle \boldsymbol{\epsilon}_{n+1} \rangle_+ : \langle \boldsymbol{\epsilon}_{n+1} \rangle_+} \tag{47}$$

(2) Make a test on the criterion function $f = \hat{\epsilon}_{n+1} - \kappa(tr\ \boldsymbol{D}_n)$
If $f \leq 0$, $\boldsymbol{D}_{n+1} = \boldsymbol{D}_n$, the material behaves elastically.
If $f > 0$, the damage must be corrected by using the damage evolution law discretized as ($\alpha = 2$ is set),

$$\Delta \boldsymbol{D} = \boldsymbol{D}_{n+1} - \boldsymbol{D}_n = \Delta\lambda\ \langle \boldsymbol{\epsilon}_{n+1} \rangle_+^2 \tag{48}$$

The damage multiplier is determined from the consistency condition numerically written $f_{n+1} = \hat{\epsilon}_{n+1} - \kappa(tr\ \boldsymbol{D}_{n+1}) = 0$ so that:

$$tr\ \boldsymbol{D}_{n+1} = \kappa^{-1}\left(\hat{\epsilon}_{n+1}\right) \tag{49}$$

which leads to the explicit expression of $\Delta\lambda$ even for Euler backward scheme,

$$\Delta\lambda = \frac{tr\ \boldsymbol{D}_{n+1} - tr\ \boldsymbol{D}_n}{\hat{\epsilon}_{n+1}^2} \tag{50}$$

and to the exact actualization of $\boldsymbol{D}$,

$$\boldsymbol{D}_{n+1} = \boldsymbol{D}_n + \Delta\lambda\ \langle \boldsymbol{\epsilon}_{n+1} \rangle_+^2 \tag{51}$$

(3) Compute then the stresses using first the elasticity law written

$$\tilde{\boldsymbol{\sigma}}_{n+1} = \underline{\boldsymbol{E}} : \boldsymbol{\epsilon}_{n+1} \tag{52}$$

using then eq. (46),

$$\begin{aligned} \boldsymbol{\sigma}_{n+1} = (\mathbf{1} - \boldsymbol{D}_{n+1})^{1/2}\ \tilde{\boldsymbol{\sigma}}_{n+1}\ (\mathbf{1} - \boldsymbol{D}_{n+1})^{1/2} - \frac{(\mathbf{1} - \boldsymbol{D}_{n+1}) : \tilde{\boldsymbol{\sigma}}_{n+1}}{3 - tr\ \boldsymbol{D}_{n+1}}(\mathbf{1} - \boldsymbol{D}_{n+1}) \\ + \frac{1}{3}\left[(1 - tr\ \boldsymbol{D}_{n+1})\langle tr\ \tilde{\boldsymbol{\sigma}}_{n+1}\rangle - \langle -tr\ \tilde{\boldsymbol{\sigma}}_{n+1}\rangle\right]\mathbf{1} \end{aligned} \tag{53}$$

The numerical scheme is Euler backward scheme, therefore robust at the Gauss point level, and it has here the main advantage of the explicit schemes: there is no need of a local iterative process as the exact solution of the discretized constitutive equations can be gained.

Important feature, due to the use of a nonlocal damage criterion written in terms of strains, the nonlocal damage model is easily implemented in FE codes already working with nonlocal variables. For instance the existing subroutines computing nonlocal Mazars strain $\hat{\epsilon}^{nl}$ can be used with no change. Once the strain $\boldsymbol{\epsilon}_{n+1}$ is known from the global equilibrium, compute first $\hat{\epsilon}^{nl}_{n+1}$ using for example the integral form (12). The only change in the integration of the constitutive law is then in equation (49) which has to be replaced by

$$tr\, \boldsymbol{D}_{n+1} = \kappa^{-1}\left(\hat{\epsilon}^{nl}_{n+1}\right) \tag{54}$$

The above procedure still applies.

6.2 *Secant operator*

The secant operator is in fact the effective (damaged) fourth order elasticity tensor $\tilde{\underline{\boldsymbol{E}}}$ as when no permanent stresses are modelled,

$$\boldsymbol{\sigma} = \underline{\boldsymbol{E}}^S : \boldsymbol{\epsilon} = \tilde{\underline{\boldsymbol{E}}} : \boldsymbol{\epsilon}, \qquad \tilde{\underline{\boldsymbol{E}}} = \tilde{\underline{\boldsymbol{E}}}(\boldsymbol{D}) \tag{55}$$

In our case it has the same expression for both local and nonlocal anisotropic damage models.

Using eq. (46) allows to derive a closed form for $\tilde{\underline{\boldsymbol{E}}}$,

- if $tr\, \boldsymbol{\epsilon} > 0$,

$$\tilde{\underline{\boldsymbol{E}}} = 2G\left[(\boldsymbol{1}-\boldsymbol{D})^{1/2}\underline{\otimes}(\boldsymbol{1}-\boldsymbol{D})^{1/2} - \frac{(\boldsymbol{1}-\boldsymbol{D})\otimes(\boldsymbol{1}-\boldsymbol{D})}{3-tr\,\boldsymbol{D}}\right] + K(1-tr\,\boldsymbol{D})\,\boldsymbol{1}\otimes\boldsymbol{1} \tag{56}$$

- if $tr\, \boldsymbol{\epsilon} < 0$,

$$\tilde{\underline{\boldsymbol{E}}} = 2G\left[(\boldsymbol{1}-\boldsymbol{D})^{1/2}\underline{\otimes}(\boldsymbol{1}-\boldsymbol{D})^{1/2} - \frac{(\boldsymbol{1}-\boldsymbol{D})\otimes(\boldsymbol{1}-\boldsymbol{D})}{3-tr\,\boldsymbol{D}}\right] + K\,\boldsymbol{1}\otimes\boldsymbol{1} \tag{57}$$

where $G = E/2(1+\nu)$ and $K = E/3(1-2\nu)$ are respectively the shear and the bulk moduli and where the special tensorial product $\underline{\otimes}$ is used: for two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, $(\boldsymbol{A}\underline{\otimes}\boldsymbol{B})_{ijkl} = A_{ik}B_{jl}$.

To know the expression for the secant tensor allows to built the global Finite Element secant matrix and to solve the equilibrium equations by use of quasi-Newton method. The secant method applies in the same manner to both local and nonlocal models. Note that with the gradient form (13) for nonlocal Mazars strain, a tangent operator can still be derived at Gauss points level [41]. This is not the case anymore for the integral form (12) and refer then to [55] for the definition and use of a global tangent matrix.

7 Numerical control of rupture with anisotropic damage

Induced damage anisotropy often leads to numerical difficulties, more or less important, depending on the model, which can sometimes stop the computations very early (an example is given in next section for the present model, see figures 7 and 9).

As numerical difficulties one often thinks of a poorly estimated consistent tangent operator but for most models with no permanent strains, the use of the global secant stiffness proves sufficient [56]. The main difficulties are in fact due to a bad numerical control of local rupture once a high level of damage, anisotropic, is reached. In order to continue the computation up to complete structural failure, one must ensure the local damaged Hooke operator to keep positive stiffnesses (i.e. positive Kelvin moduli) or in other words to remain positive definite [6].

7.1 Broken bulk modulus

In the models proposed in previous sections, the trace of the damage tensor acts on the hydrostatic stress as:

$$tr\,\boldsymbol{\epsilon} = \frac{\langle tr\,\boldsymbol{\sigma}\rangle}{3K(1 - tr\,\boldsymbol{D})} - \frac{\langle -tr\,\boldsymbol{\sigma}\rangle}{3K} \tag{58}$$

with K the bulk modulus of the undamaged (isotropic) material. There are situations such as compression leading to $tr\,\boldsymbol{D} > 1$. These are admissible as long as they are mainly compressive, more precisely as long as $tr\,\boldsymbol{\sigma} < 0$, as the hydrostatic stiffness remains equal to $3K$. But they become critical when a change in the loading sign occurs if the broken material behavior is not properly defined as then $3K(1 - tr\,\boldsymbol{D})$ is negative and cannot therefore act as a material stiffness. The physical meaning of a large damage trace due to compression is that the material, highly damaged, still resists for compressive states of stresses but will be broken in pieces as soon tensile stresses are applied on it. It is nevertheless possible to continue the computation by defining a fictitious hydrostatic broken behavior for the material such as

- a linear broken behavior,

$$tr\,\boldsymbol{\sigma} = 3K_{broken}\langle tr\,\boldsymbol{\epsilon}\rangle - 3K\langle -tr\,\boldsymbol{\epsilon}\rangle \tag{59}$$

  where the "broken" bulk modulus K_{broken} is very small and is related to the trace of the damage reached when the broken behavior is activated. If $tr\,\boldsymbol{D} \geq D_c$ at this time,

$$K_{broken} = K(1 - D_c) \tag{60}$$

- a constant broken behavior,

$$\begin{aligned}
&tr\,\boldsymbol{\sigma} = \sigma_{broken} && \text{if} \quad tr\,\boldsymbol{\epsilon} > \frac{\sigma_{broken}}{3K_{broken}} \\
&tr\,\boldsymbol{\sigma} = 3K_{broken} tr\,\boldsymbol{\epsilon} && \text{if} \quad 0 < tr\,\boldsymbol{\epsilon} \leq \frac{\sigma_{broken}}{3K_{broken}} \\
&tr\,\boldsymbol{\sigma} = 3K_{broken} tr\,\boldsymbol{\epsilon} && \text{if} \quad tr\,\boldsymbol{\epsilon} \leq 0
\end{aligned} \tag{61}$$

  with the residual stress σ_{broken} as material parameter.
- or even a softening broken behavior ensuring $tr\,\boldsymbol{\sigma} \rightarrow 0$ for "large" positive hydrostatic strains.

Concerning the other (shear) moduli, the numerical difficulties can be avoided by ensuring the principal damages to remain bounded by 1 or, better, by the critical damage D_c. Once the largest principal damage D_I reaches D_c, a solution is to keep the damage evolution law unchanged, here $\dot{\boldsymbol{D}} = \dot{\lambda}\langle\boldsymbol{\epsilon}\rangle_+^2$, but projected in such a manner that D_I remains constant equal to D_c. Such a projected damage law reads:

$$\dot{\boldsymbol{D}} = \dot{\lambda}\, \Pi_{\perp n^I} \langle\boldsymbol{\epsilon}\rangle_+^2 \tag{62}$$

with $\Pi_{\perp n^I}$ the projection operator. It has the main advantage to consider the same damage law (with no need of additional parameters) as for the 3D case but in fact restricted to 2D as the material cannot damage anymore in the direction n^I. The eigenvector n^I corresponds to the eigenvalue D_I, it is the normal of the main mesocrack initiated and is determined by solving

$$\boldsymbol{D}\, n^I = D_I\, n^I = D_c\, n^I, \qquad \left\| n^I \right\| = 1 \tag{63}$$

The two other eigenvectors, also normalized, are denoted n^{II} and n^{III} and are associated with the principal damages D_{II}, D_{III} smaller than D_c (consider here $D_I \geq D_{II} \geq D_{III}$) so that a less formal expression for the projected damage law is derived as:

$$\dot{\boldsymbol{D}} = \dot{\lambda} \left[\langle\boldsymbol{\epsilon}\rangle_+^2 - (n^I \cdot \langle\boldsymbol{\epsilon}\rangle_+^2 n^I)\, n^I \otimes n^I - (n^{II} \cdot \langle\boldsymbol{\epsilon}\rangle_+^2 n^I) \left(n^I \otimes n^{II}\right)_{sym} - (n^{III} \cdot \langle\boldsymbol{\epsilon}\rangle_+^2 n^I) \left(n^I \otimes n^{III}\right)_{sym} \right] \tag{64}$$

and is then valid for fixed n^I but for updated D_{II}, D_{III}, n^{II}, n^{III} as long as D_{II} has not reached D_c. Last, only D_{III} will evolve as $\dot{D}_{III} = \dot{\lambda}\, n^{III} \cdot \langle\boldsymbol{\epsilon}\rangle_+^2 n^{III}$ at fixed n^I and n^{II} up to $D_{III} = D_c$ so that one ends up to a diagonal (isotropic) damage $\boldsymbol{D} = D_c\, \mathbf{1}$. A 3D broken behavior is finally obtained, isotropic, dissymetric (linear case, D_c constant close to 1),

$$\boldsymbol{\sigma} = 2G(1 - D_c)\boldsymbol{\epsilon}^D + K\left[(1 - D_c)\langle tr\ \boldsymbol{\epsilon}\rangle - \langle -tr\ \boldsymbol{\epsilon}\rangle\right]\ \mathbf{1} \tag{65}$$

with $\boldsymbol{\epsilon}^D = \boldsymbol{\epsilon} - \frac{1}{3} tr\ \boldsymbol{\epsilon}\ \mathbf{1}$ the deviatoric strain.

The numerical implementation is similar to the one described in section 6.1 as the projected damage evolution law (64) is discretized in

$$\Delta\boldsymbol{D} = \boldsymbol{D}_{n+1} - \boldsymbol{D}_n = \Delta\lambda\, \Pi_{\perp n^I} \langle\boldsymbol{\epsilon}_{n+1}\rangle_+^2 \tag{66}$$

instead of Eq. (48) with a damage multiplier given by

$$\Delta\lambda = \frac{tr\ \boldsymbol{D}_{n+1} - tr\ \boldsymbol{D}_n}{tr\ (\Pi_{\perp n^I} \langle\boldsymbol{\epsilon}_{n+1}\rangle_+^2)} \tag{67}$$

instead of Eq. (50), so that the damage is updated as $\boldsymbol{D}_{n+1} = \boldsymbol{D}_n + \Delta\lambda\, \Pi_{\perp n^I} \langle\boldsymbol{\epsilon}_{n+1}\rangle_+^2$.

The risk of obtaining negative (non physical!) stiffnesses is in fact strong for structures computations with any anisotropic damage model introducing $\mathbf{1} - \boldsymbol{D}$ (2nd order) terms or $\underline{\boldsymbol{I}} - \underline{\boldsymbol{D}}$ (4th order) terms as one principal damage may reach 1 at some Gauss points for locally severe loading. This is usual for instance for reinforced concrete structures with perfect interfaces between steel and concrete so that severe shear easily occurs. The $\mathbf{1} - \boldsymbol{D}$ and $\underline{\boldsymbol{I}} - \underline{\boldsymbol{D}}$ terms must remain positive and the projection procedure just proposed can be applied to ensure this feature. Write the considered anisotropic damage evolution law in the general form $\dot{\boldsymbol{D}} = \dot{\lambda}\boldsymbol{Q}$ (initial case of principal damages smaller than unity), with $\boldsymbol{Q}$ often a normal to a damage surface, i.e. a tensorial function of the stresses, strains and other variables. To replace the term $\langle \boldsymbol{\epsilon} \rangle_+^2$ by $\boldsymbol{Q}$ in equation (62) define in a general manner the projected damage law as $\dot{\boldsymbol{D}} = \dot{\lambda}\ \Pi_{\perp n^I}\boldsymbol{Q}$. Note that for models based on compliance increase due to damage the problem just mentioned shall not occur, except once more if $(\mathbf{1} - \boldsymbol{D})^{-1}$ or $(\underline{\boldsymbol{I}} - \underline{\boldsymbol{D}})^{-1}$ terms are introduced. For micromechanics based models such as the microplane damage model [45, 46, 47] this difficulty does not occur if the local scalar damages over each microcrack or over each microplane direction does not reach unity. This can be gained by using local uniaxial stress-strain relations having a softening part asymptotically equivalent to a monotonically decreasing function approaching 0 at large strains or by using local scalar damage laws monotonically increasing functions approaching 1 at large strains [57, 58]. This feature is more complex to gain for a tensorial writing of the damage evolution law.

8 Finite Element computations of structures

Once a constitutive model is developed it has to prove its efficiency in Finite Element computations of engineering structures, or at least on structural components. The difficulties encountered are numerous. The strain-damage localization due to stress softening is of course one of them, phenomenon usually regularized with use of nonlocal models, but at expensive computation cost. There are also important difficulties related to interfaces and steels yielding, difficulties preponderant in reinforced and pre-stressed concrete structures. Once illustrates here the ability of the anisotropic damage model to deal with such structures, starting first with Nooru-Mohamed test exhibiting rotations of the loading principal directions.

8.1 Plain concrete mixed-mode fracture

The double edge notched specimen tested by Nooru-Mohamed [59] is analysed using the implementation of the model developed in the previous section. The specimen geometry and the experimental testing set up are shown in Figure 4. It is a symmetric 200 mm$\times$ 200 mm mortar square with two notches, 30 mm long and 5 mm thick.

The rotation of the external boundary of the plate is restricted around the Oz axis (thick borders). The concrete specimen is first loaded by an increasing shear force $F(t)$ applied on the lateral surface. During the application of the shear force, the vertical displacement of the upper surface is totally free. In a second time, a vertical displacement

$U(t)$ is applied up to failure at constant $F = F_{Max}$, the higher F_{Max} the more curved the crack path [59].

Fig. 4. Nooru-Mohamed test

The case study is here carried out for $F_{Max} = 22.5$ kN. The FE discretization of the specimen is made by the use of four node tetrahedron elements with one integration point. In order to perform the computations in 3D at reasonable cost, a FE mesh with a 5 mm width is used when the real width of the specimen is 50 mm. The model parameters used for the simulation are those of section 5 for concrete: $E = 42000$ MPa, $\nu = 0.2$, $\kappa_0 = 5\ 10^{-5}$, $A = 5\ 10^3$, $a = 2.93\ 10^{-4}$. The nonlocal length used in the integral weight function (Gaussian type, Eq. 43) is $l_c = 8$ mm, small value indeed justified by the fact that the material is a mortar with very small constituents [60, 61]. Three meshes are used (Figure 5): a coarse mesh with a total of 862 elements, a medium mesh with 2428 elements, and a fine mesh with 4558 elements. The characteristic length corresponds then close to the notch to 2 elements of the coarse mesh, to 5 elements of the medium mesh, to 10 elements of the fine mesh. The above procedure for the control of rupture with anisotropic damage is considered with a linear broken behavior ($D_c = 0.99$).

Fig. 5. Different meshes of Nooru-Mohamed structure

Figure 6 shows the anisotropic damage patterns computed. The left column computations correspond to local results, the right one to nonlocal results. The application of the shear load up to F_{Max} yields localized damage at the notch tip. The structural failure is then due to the application of the vertical displacement $U(t)$ with mainly mode I cracks represented here in the Continuum Damage Mechanics framework by large D_{22} values. The damage patterns computed corresponds well to the crack patterns experimentally observed [59]. They are symmetric with respect to the center of the specimen (two main cracks obtained) so that the damage D_{22} is drawn here in the upper part of the structure, D_{11} in the lower part (most D_{11}-damage occurs due to the control of rupture procedure). The right column computations exhibit the now classical convergence and mesh independence of the results obtained with a nonlocal model. Due to instabilities related to the too brittle response in local computations (induced by the mesh refinement), no convergence has been obtained for the local model with the fine mesh. Note that no specific procedure such as arclength methods has been used to overpass the numerical difficulties as the conceptually satisfactory introduction of a characteristic length proves to be efficient (convergence is obtained in nonlocal for the fine mesh). The importance of the procedure for the control of rupture must be emphasized: without its use the computations stop before the shear load F_{Max} is reached.

Fig. 6. Damage maps for Nooru-Mohamed test at $U = 3.5\ 10^{-3}$ mm– (a) left column: results with local anisotropic damage model, (b) right column: results with nonlocal anisotropic damage model

Fig. 7. Structure response for nonlocal computations

Figure 7 shows the computed global responses of the loading plate as the tensile load T versus the normal displacement U curves once $F = F_{Max}$ is imposed. The post-peak response is of course mesh-dependent in case of local computations (not plotted), the finer the mesh the more brittle the structure behavior. This is classically not the case anymore for the nonlocal computations as convergence is reached for the nonlocal anisotropic damage model by mesh refinement.

To conclude, the nonlocal anisotropic damage model with an adequate control of rupture correctly represents the crack propagation for mixed mode fracture.

8.2 Reinforced concrete structure

The objectives of this section are to evaluate the ability of the anisotropic damage model to deal with a reinforced concrete element subject to flexion. The structure is a reinforced square cross section beam, subject to three point bend loading. Figure 8 shows geometric features for concrete and steel. During loading, multiple loading paths are encountered in different parts of the beam: tension on the lower part, compression on the upper part, shear near the edge and along the reinforcing bars. The corresponding different features of the constitutive equations are activated at the same time and the occurrence during loading of several competitive cracks usually makes difficult the global convergence scheme. For these reasons, this case-study was part of the international MECA benchmark, launched by E.D.F. to compare and discriminate different 3D constitutive models for concrete [62].

Fig. 8. Reinforced concrete beam

For concrete, the material parameters used in the following computations are those of section 5 and used for previous computations of Nooru-Mohamed test. For steel, elasto-plasticity with linear hardening is considered and the material parameters are imposed by the benchmark: Young's modulus $E = 200000$ MPa, Poisson ratio $\nu = 0.3$, yield stress of 480 MPa, plastic modulus of 20000 MPa. For the computation, a 3D specimen has been meshed with 2 elements in the thickness for a total of 600 eight node parallelepipedic elements. Accounting for the different symmetries of the problem, only one reinforcing steel bar is modelled as shown in Figure 9. The mean dimension of the finite element size is 50 mm.

The monotonic loading is applied up to failure. Two computations with two different characteristic lengths are performed in order to appreciate the effect of the nonlocal length (Gaussian weight function, Eq. (43), $l_c = 150$ mm and $l_c = 250$ mm). The choice $l_c = 150$ mm is the more physical as it corresponds here to a characteristic length equal to 3 or 4 times the maximum aggregate size [60].

Figure 9 shows comparisons between experiment and modeling in terms of global response, i.e. the load applied at the center of the beam versus the deflection. As often for numerical computations using the displacement based FE method, the initial beam stiffness is larger than the experimental one, phenomenon here enhanced by the high value of the concrete Young's modulus. As one also can see in Figure 9, due to the reinforcing steel bar implying a flexural rupture the effect of the characteristic length is small in terms of global response, but not of cracking pattern as it can be seen from damage maps. The choice $l_c = 150$ mm is the more appropriate here, as expected.

Fig. 9. Comparisons experiment/computations

Fig. 10. Mesh and D_{11} damage field obtained at the begining of steels yielding (left : $l_c = 150$ mm, right: $l_c = 250$ mm - left and right correspond to two different computations)

The maps of the damage variable D_{11} are given in Figure 10. The computation represents quite well the multiple cracks propagation for the characteristic length $l_c = 150$ mm. The three main stages of a reinforced concrete structure subject to flexion are then recovered: elasticity, cracking in tension, and yielding of the reinforcement.

Due to the high number of competitive cracks, the convergence of the local model stop very early during the analysis. Only the nonlocal approach allows for the calculations achievement without local instabilities. Last and again, the computations performed without the procedure for the rupture control stop too early when the computations with numerical rupture control (linear broken behavior) run up to complete failure of the beam.

8.3 Pre-stressed powerplant ring

Consider last the response under pressure of the concrete structure of Figure 11 representative of a ring of a powerplant containment vessel reinforced by half circular steel bars and pre-stressed by metallic cables. The diameter of the ring is 46.8 m, its thickness 0.9 m and only a height of 0.4 m is meshed. A vertical compressive stress of 8.5 MPa is applied to model the weight of the upper part of the structure (not meshed). There are two symmetric anchoring parts and the steel reinforcement is made of 88 rectangular stirrups (not considered in the computations), of 4 circular reinforcement bars and of 2 pre-stressed half-circular cables. A more detailled description of the studied structure can be found in [63] and in [22].

The FE analysis of the ring is made in 3D with the bars and the cables modelled with 812 two node bar elements and the concrete part with 4420 eight node bricks. The prestressing tension is applied through an equivalent loading allowing for the introduction of a constant tensile stress all along the cables. The equivalent loading of the steel cable on the concrete part of the structure corresponds to an initial tension of 4.4 MN for each cable.

Both the passive steel bars and the active metallic cables are assumed elastic-perfectly plastic. Concrete is modelled with the anisotropic damage model and the material parameters of previous sections. The material parameters for the steel and the cables are: $E =$ 190000 MPa, ν= 0.3, yield stress of 500 MPa for the steel bars of diameter 25 mm ; $E =$ 190000 MPa, ν= 0.3, yield stress of 1814 MPa for the active cables of cross section 5143 mm^2. The deformed shape of the ring after the prestressing stage is shown in Figure 11a.

Fig. 11. Deformed shape of the ring (a) after prestressing and (b) after internal pressure - Points A and B of the anchoring part

An internal pressure $P(t)$ is then applied. It corresponds to the every ten years experimental loading applied on powerplant containement vessels in order to measure the air leakage of the structure. The damage fields are computed up to $P_{Max} = 2$ MPa for which the average damage intensity corresponds to a structure already collapsed (of course real tests are performed with a much lower pressure level). Due to steel reinforcement a non uniformly deformed mesh is obtained (Figure 11b) in which the displacements have an amplification factor of 633).

Fig. 12. Anisotropic damage fields at $P = 2$ MPa

The anisotropic damage maps show an oriented microcracking pattern, mainly radial (truly represented by $D_{\theta\theta}$ field, D_{22} is the sectors $\theta \approx 0°$ and 180°, D_{11} in the sectors $\theta \approx$ 90° or -90°), which will be preferable paths for air leakage under pressure. The evolution of the principal damages at points A (external point of the anchoring part) and B (internal point of the anchoring) are plotted in Figure 13 under of the pre-stressing loading. Point A is the most loaded point after pre-stressing. During the pressure increase, the microcracks spread around the whole ring as the damage field tends to the uniform level $D_{\theta\theta} \approx 1$ for $P = P_{Max} = 2$ MPa.

Fig. 13. Principal damages at points A and B during prestressing

The procedure for the rupture control of section 7 proves again efficient as without it one cannot numerically apply pre-stresses larger than 2 MN in the cables. The steel reinforcements prevent the localization modes, at least with the large elements used. In order to obtain large mesocracks initiation by strain-damage localization, nonlocal computations using smaller elements shall be performed but at a much expensive cost!

Conclusion

To conclude, an anisotropic damage model is proposed for quasi-brittle materials. The damage state corresponds to a given microcracks pattern and, accordingly to the definition of thermodynamics state variables, it is represented by a single 2nd order tensorial damage variable $\boldsymbol{D}$ whatever the sign of the loading. The strong dissymetry of concrete behavior is represented and is shown to be mainly due to induced damage anisotropy. A single set of damage parameters A and a is introduced and is valid for both tensile or compressive loading. The model is 3D and allows for computations of structures under either local and nonlocal assumptions. A proof of the positivity of the intrinsic dissipation is given.

From numerical examples up to structural failure, the anisotropic damage model shows its ability to be used in large scale engineering computations at reasonable computational cost. Reinforced and pre-stressed concrete structures can be studied up to a very high damage level and yielding of the reinforcement steels. Note that the numerical calculations are efficient due to three main features:

- a thermodynamics formulation (a state potential continuously differentiable, positive dissipation for any complex loading),
- Euler backward scheme solved in an exact manner for the time integration of the constitutive equations (no local iterations),
- the use of a procedure for the numerical control of rupture probably always necessary in case of anisotropic damage.

Last, even if anisotropic damage has not proven yet its superiority over the isotropic damage consideration for structural (mechanical) applications, it is nevertheless a more thermodynamically consistent framework. But probably the most promising point concerns the mechanical coupling with diffusion problems such as thermo-hydro-mechanical analyses. For instance, when dealing with the evaluation of the gas or liquid leakage of a cracked structure, one shall take advantage of the anisotropic description of the damage in the coupling with material permeability [64].

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