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Abstract. We present general transience bounds in tropical linear algebra based on Nachtigall's matrix decomposition. Our approach is also applicable to reducible matrices. The core technical novelty are general bounds on the transient of the maximum of two eventually periodic sequences. Our proof is algebraic in nature, in contrast to the existing purely graph-theoretic approaches.

1. Introduction

Tropical linear systems describe the behavior of transportation systems, manufacturing plants, network synchronizers, as well as certain distributed algorithms for resource allocation and routing. It is known that the sequence of tropical matrix powers, and hence every linear system, becomes periodic after an initial *transient*. In applications it is of interest to have upper bounds on the transient, to which we contribute with this work.

We use the Nachtigall decomposition [**5**] of square matrices in tropical algebra to show new transience bounds for sequences of matrix powers. The Nachtigall decomposition is a representation of the sequence of matrix powers of a square matrix as a maximum of a bounded number of sequences of bounded transients and bounded periods. Transience bounds for the sequence of matrix powers have been given, amongst others, by Hartmann and Arguelles [**4**] and Charron-Bost et al. [**1**]. Their proofs are purely graph-theoretic. They consider the edge-weighted graph described by the matrix as an adjacency matrix and argue about existence of walks of certain weights. We, too, use this graph interpretation of a matrix in two supplementary results (Lemma 2 and Lemma 6). However, the rest of our proof is algebraic. Because our proof is based on the Nachtigall decomposition, which is also applicable to reducible matrices, so is our proof. To the best of our knowledge, we are the first to give transience bounds for reducible matrices. An example by Even and Rajsbaum [**3**, Fig. 2] shows that our new bounds are asymptotically optimal.

2. Preliminaries

We consider the max-plus semiring on the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. That is, we consider the addition $(x, y) \mapsto \max(x, y)$ and the multiplication $(x, y) \mapsto x + y$. The semiring's zero element is $-\infty$ and its unit is 0. The matrix multiplication of two matrices A and B of compatible size satisfies $(AB)_{i,j} = \max_k(A_{i,k} + B_{k,j})$.

We call a sequence $f : \mathbb{N} \to \overline{\mathbb{R}}$ *eventually periodic* if there exist numbers p, T, and ϱ such that:

$$\forall n \geqslant T \ : \ f(n+p) = f(n) + p \cdot \varrho \tag{1}$$

In this case we call p a *period*, T a *transient*, and ϱ a *ratio* of the sequence f. It is easy to see that the ratio is unique and finite if the sequence is not eventually constantly infinite. For every period p, there exists a unique minimal transient T that satisfies (1). The next fundamental lemma shows that these minimal transients do, in fact, not depend on p:

LEMMA 1. *Let $f : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic. Let p and $\hat{p}$ be periods of f with respective minimal transients T_p and $\widehat{T_p}$. Then $T_p = \widehat{T_p}$.*

We will henceforth call this unique minimal transient *the* transient of f. Also, we will call the minimal period *the* period of f.

Cohen et al. [**2**] established that the entrywise sequences of matrix powers A^n of a square matrix are eventually periodic. More generally, we say that a sequence of matrices is eventually periodic if every entrywise sequence is eventually periodic. Period and transience of a matrix sequence is a period and transience for all entrywise sequences.

The tropical *convolution* $f \otimes g$ of two sequences f and g is defined as

$$(f \otimes g)(n) = \max_{n_1+n_2=n} \big(f(n_1) + g(n_2) \big) \ . \tag{2}$$

To a square matrix A naturally corresponds an edge-weighted digraph $G(A)$. The weight $p(W)$ of a walk W in $G(A)$ is the sum of the weights of its edges. The matrix powers of A satisfy the correspondence

$$\big(A^n\big)_{i,j} = \max \big\{ p(W) \mid W \text{ has length } n \text{ and is from } i \text{ to } j \big\} \ . \tag{3}$$

3. Nachtigall Decomposition

Nachtigall [**5**] introduced a representation of the sequence of matrix powers of an $N \times N$ square matrix as the maximum of at most N matrix sequences whose transients are at most $3N^2$. He showed that this representation can be computed efficiently. However, no results on the transient of the original matrix were obtained. The core of the representation is a decomposition of the original matrix into components corresponding to cycles in the matrix' digraph, The matrix sequences are defined as convolutions corresponding to this decomposition. The following lemma shows the utility of *maximum* mean cycles for the decomposition.

Lemma 2 ([**5**, Lemma 3.2]). *If A is an $N \times N$ matrix and k is a node of a maximum mean cycle C in $G(A)$, then both sequences $\left(A^n\right)_{i,k}$ and $\left(A^n\right)_{k,i}$ are eventually periodic with period $\ell(C)$, ratio $p(C)/\ell(C)$, and transient at most $\ell(C) \cdot (N-1)$.*

Given an $N \times N$ matrix A and a set $I \subseteq \{1, \ldots, N\}$ of indices, we define the *deletion* of I in A as the matrix B whose entries satisfy $B_{i,j} = -\infty$ if $i \in I$ or $j \in I$, and $B_{i,j} = A_{i,j}$ otherwise.

The following lemmas are used to prove the upper bound on the transient of each matrix sequence in the Nachtigall decomposition.

Lemma 3. *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with common period p, common ratio ϱ, and respective transients T_f and T_g. Then the sequence $\max(f, g)$ is eventually periodic with period p, ratio ϱ, and transient at most $\max(T_f, T_g)$.*

Lemma 4 ([**5**, Lemma 6.1]). *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with common period p, common ratio ϱ, and respective transients T_f and T_g. Then the convolution $f \otimes g$ is eventually periodic with period p, ratio ϱ, and transient at most $T_f + T_g + p - 1$.*

We now state an improvement of Nachtigall's theorem using essentially the same arguments as the original version. The improvement lies in an upper bound of $2N^2 - N$ on the sequences' transients instead of $3N^2$.

Theorem 1 (Nachtigall decomposition [**5**, Theorem 3.3]). *Let A be an $N \times N$ matrix. Then there exist eventually periodic matrix sequences $A_1(n), A_2(n), \ldots, A_N(n)$ with periods at most N and transients at most $2N^2 - N$ such that for all n:*

$$A^n = \max\left(A_1(n), A_2(n), \ldots, A_N(n)\right) \tag{4}$$

The proof proceeds by induction on N. The case $N = 1$ is trivial.

If $G(A)$ does not contain a cycle, then $(A^n)_{i,j} = -\infty$ for all $n \geqslant N$, so the transient of A is at most N and the theorem's statement is trivially fulfilled when choosing all matrix sequences $A_m(n)$ equal to A^n.

We hence suppose that $G(A)$ contains a cycle. By the definition of the matrix multiplication, whenever $n = n_1 + n_2$, we have:

$$(A^n)_{i,j} = \max_k \big((A^{n_1})_{i,k} + (A^{n_2})_{k,j}\big) \tag{5}$$

Let C be a maximum mean cycle in $G(A)$. Denote by B the deletion of the set of C's nodes in A. It follows from the definition of deletion and from the graph interpretation (3) that

$$(A^n)_{i,j} = \max\Big(\max_{k \text{ in } C} \big((A^{n_1})_{i,k} + (A^{n_2})_{k,j}\big)\ ,\ (B^n)_{i,j}\Big)\ . \tag{6}$$

In particular, (6) continues to hold when forming the maximum over all n_1 and n_2 such that $n = n_1 + n_2$. By writing $A_{i,j}(n) = (A^n)_{i,j}$ and recalling the definition (2) of convolution, we can hence write

$$(A^n)_{i,j} = \max\Big(\max_{k \text{ in } C} \big(A_{i,k} \otimes A_{k,j}\big)(n)\ ,\ (B^n)_{i,j}\Big)\ . \tag{7}$$

Lemmas 2, 3, and 4 imply that the transient of the inner maximum in (7) is at most

$$2 \cdot \ell(C) \cdot (N-1) + \ell(C) - 1 \leqslant 2N^2 - N - 1 \tag{8}$$

and its period is at most $\ell(C) \leqslant N$. Choose the matrix sequence $A_1(n)$ equal to this inner maximum, i.e., $A_1(n) = \max_{k \text{ in } C} \big(A_{i,k} \otimes A_{k,j}\big)(n)$.

By induction hypothesis, there exist matrix sequences $A_2(n),\dots,A_N(n)$ with periods at most N and transients at most $2N^2 - N$ such that $B^n = \max\big(A_2(n), \dots, A_N(n)\big)$. This then concludes the proof's inductive step.

4. Transience Bounds

Note that Theorem 1 does *not* imply that the transient of any sequence of matrix powers is at most $2N^2 - N$. The reason for this is that Lemma 3 is not applicable to the maximum in the Nachtigall decomposition because the involved sequences can have different ratios.

The following lemma is our main technical novelty and provides a tool for bounding the transient of a maximum of two eventually periodic sequences if their ratios are not equal.

LEMMA 5. *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with the same period p, respective ratios ϱ_f and ϱ_g, and respective transients T_f and T_g.*

Assume that $\varrho_f \geqslant \varrho_g$ *and that for all* $n \geqslant S$ *we have* $g(n) = -\infty$ *whenever* $f(n) = -\infty$*. Then the sequence* $\max(f, g)$ *is eventually periodic with period* p*, ratio* $\max(\varrho_f, \varrho_g)$*, and transient at most*

$$\max(T_f, T_g, S) + \frac{\Gamma}{|\varrho_f - \varrho_g|}$$

where

$$\Gamma = \max\big\{ f(m) - g(m) \mid S \leqslant m < S + p\,,\, g(m) \neq -\infty \big\} \ .$$

In the rest of this section, we provide general transience bounds for sequences of matrix powers. We do this by applying Lemma 5 entrywise to the maximum in (4). Evidently, given a Nachtigall decomposition (which is not unique), one can use Lemma 5 to bound the transient of the matrix powers. In this section, we proceed to use the existence of a Nachtigall decomposition to prove general transience bounds independent of an explicit decomposition.

We first assume that the graph $G(A)$ is strongly connected. In this case, all sequences $(A^n)_{i,j}$ are eventually periodic with ratio equal to the maximum cycle mean in $G(A)$. For every pair (i, j) of indices, the ratio of the sequence $\big(A_m(n)\big)_{i,j}$ is greater or equal to that of $\big(A_{m+1}(n)\big)_{i,j}$. We want to apply Lemma 5 to every pair of (i) a sequence of maximum ratio and (ii) a sequence of another ratio. Afterwards, we apply Lemma 3 to the resulting maxima (which all have the same ratio). But to effectively apply Lemma 5, we have to bound its parameter S for every pair.

We show that the parameter S is at most $N(N+1)$ with a graph-theoretic argument: The *exploration penalty* of graph G is the least integer *ep* such that there exists a closed walk of length n at every node in G for all multiples n of G's cyclicity[1] that satisfy $n \geqslant ep$.

LEMMA 6 ([**1**, Theorem 3]). *The exploration penalty of a graph with* N *nodes is at most* $N(N-1)$*.*

It is well-known that the difference of lengths of two walks from i to j in a strongly connected graph is always a multiple of the graph's cyclicity. In each step of the Nachtigall recursion, there exists a path from i to j via a node k of C of length at most $2N$. Hence Lemma 6 implies that if there exists *any* path from i to j of length $n \geqslant 2N + N(N-1) = N(N+1)$, then there exists also a path from i to j via k of length n. This shows that S can be chosen to be at most $N(N+1)$.

[1]The *cyclicity* of a strongly connected graph is the greatest common divisor of its cycle lengths.

A common period p of a pair of sequences is the least common multiple of the two periods. Because in the Nachtigall decomposition the periods are at most N, there exists a common period less or equal to $N(N-1)$. Hence $S+p$ can be bounded by $2N^2$.

We thus arrive at the following theorem bounding the transient of the sequence of matrix powers. Denote by $\|A\|$ the difference of the maximum and the minimum finite entry in matrix A.

THEOREM 2. *Let A be an $N \times N$ matrix such that $G(A)$ is strongly connected. Denote by λ the maximum cycle mean in $G(A)$ and λ' the second largest cycle mean weight. Then the sequence of matrix powers A^n has transient at most*

$$2N^2 + \frac{\|A\| 2N^2}{\lambda - \lambda'} \ .$$

Hartmann and Arguelles [**4**, Theorem 10] arrived at a bound of $\|A\|2N^2/(\lambda - \lambda^0)$ on the transient, where λ^0 is a parameter of the max-balanced graph of $G(A)$. Their bound is always smaller than ours in Theorem 2. However, in the worst case, both are asympotically in the same order of growth. Also, our technique is different than that of Harmann and Arguelles, and can also be used to provide sharper bounds if information on the Nachtigall decomposition is available.

Our technique is also applicable to the general case where $G(A)$ is not necessarily strongly connected. In the general case, the ratio $\lambda_{i,j}$ of the sequence of entries $(A^n)_{i,j}$ is equal to the maximum mean of cycles reachable from i and from which j is reachable. Denote by $\lambda'_{i,j}$ the second largest mean of cycles reachable from i and from which j is reachable. By analogous arguments, Theorem 2 continues to hold in the general case if we replace the denominator $\lambda - \lambda'$ by $\min_{i,j}(\lambda_{i,j} - \lambda'_{i,j})$.

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