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PERTURBATION OF ℓ_1-COPIES IN PREDUALS OF JBW*-TRIPLES

ANTONIO M. PERALTA AND HERMANN PFITZNER

Abstract. Two normal functionals on a JBW*-triple are known to be orthogonal if and only if they are L-orthogonal (meaning that they span an isometric copy of $\ell_1(2)$). This is shown to be stable under small norm perturbations in the following sense: if the linear span of the two functionals is isometric up to $\delta > 0$ to $\ell_1(2)$, then the functionals are less far (in norm) than $\varepsilon > 0$ from two orthogonal functionals, where $\varepsilon \to 0$ as $\delta \to 0$. Analogous statements for finitely and even infinitely many functionals hold as well. And so does a corresponding statement for non-normal functionals. Our results have been known for C*-algebras.

1. Introduction

The starting point of this note consists in two well-known facts. First, two elements in the predual of a JBW*-triple are orthogonal if and only if they span the two-dimensional $\ell_1(2)$ isometrically and second, in preduals of von Neumann algebras this still makes sense after small norm perturbations, moreover not only for two but for finitely and, up to subsequences, even infinitely many elements.

For example, if a sequence (φ_n) in $L^1([0,1])$ is such that

$$\sum |\alpha_n| \geq \| \sum \alpha_n \varphi_n \| \geq r \sum |\alpha_n|,$$

then there are pairwise orthogonal $\widetilde{\varphi}_n$ such that $\|\varphi_n - \widetilde{\varphi}_n\| < \varepsilon$, with $\varepsilon \to 0$ as $r \to 1$ (and, of course, with $\| \sum \alpha_n \widetilde{\varphi}_n \| = \sum |\alpha_n|$), see [16]. Briefly, in L^1 a sequence near to an isometric copy of ℓ_1 is near to an orthogonal sequence. Up to subsequences the same follows from [36, Th. 1.2] for arbitrary von Neumann preduals. Analogous non-normal versions hold, too: it can be deduced from [36, Prop. 1.3] that if a sequence (φ_n) in the dual of a C*-algebra A is as above then for any $\varepsilon > 0$ there are pairwise orthogonal elements $c_k \in A$ such that $\varphi_{n_k}(c_k) > (1-\varepsilon)r$ for some subsequence (φ_{n_k}); a similar formulation (reminiscent of Pełczyński's property (V) or Grothendieck's criterion of weak compactness in the dual of a $C(K)$-space) is that if one accepts to replace r by a worse constant (e.g. $r^2/2$) in the last

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inequality, then the c_k's can be considered to be selfadjoint elements of a commutative subalgebra of A [36, §6, Lem. 6.3].

In view of the numerous generalizations of geometric (=Banach space theoretic) properties from C*-algebras to JB*-triples it is natural to conjecture similar results for JB*-triples. The aim of this article is to state and prove them.

Let us describe these results. They divide into two parts, contained in Sections 3 and 4, respectively, depending on whether the ℓ_1-copies are of finite or infinite dimension. Basic to all this, as already alluded to in the first paragraph, is the result of Y. Friedman, B. Russo [26, Lem. 2.3], according to which two functionals are algebraically orthogonal if and only if they are L-orthogonal where the latter means that the two functionals span an isometric copy of $\ell_1(2)$ (see Section 2 for definitions).

The main result of Section 3, Theorem 3.6, yields a quantification of algebraic orthogonality for finitely many arbitrary elements in the dual of a JB*-triple E: if functionals $\varphi_1, \ldots, \varphi_n$ in E^* span $\ell_1(n)$ $(1-\delta)$-isomorphically then they are near to pairwise orthogonal functionals as δ is near to 0 and moreover they attain their norm up to a given $\varepsilon > 0$ on some pairwise orthogonal elements in E. A quantification of orthogonality in E is not possible in general but it is for tripotents, see Proposition 3.10.

Section 4 contains what has been described in the second paragraph. More specifically, if a bounded sequence (φ_n) in a JBW*-predual W_* spans ℓ_1 almost isometrically, then according to Theorem 4.1 there are pairwise orthogonal $\widetilde{\varphi}_k$ such that $\|\varphi_{n_k} - \widetilde{\varphi}_k\| \to 0$ for some subsequence φ_{n_k}. The non-normal case, treated in Theorem 4.2, can be resumed by saying that if the φ_n's span ℓ_1 r-isomorphically in the dual of a JB*-triple E, then E contains an abelian subtriple such that the restrictions of an appropriate subsequence of the φ_n to this subtriple still span ℓ_1 $(1-\varepsilon)$-isomorphically for any given $\varepsilon > 0$. A quantitative version of Theorem 4.2 is already contained in [22, Th. 2.3] and, what is more, the arguments in [22] and [13] seem to lend themselves to a quantification that gives our Theorem 4.2. We refrained from pursuing this approach for it seems more natural to deduce the infinite dimensional case from the finite dimensional one, all the more because the latter has an interest in its own.

2. Preliminaries

We shall follow the standard notation employed, for example in [21], [22] or [10]. For Banach space theory we refer, e.g., to [14, 20, 29].

We recall that a JB*-triple [30] is a complex Banach space E equipped with a continuous ternary product $\{.,.,.\}$ symmetric and bilinear in the outer variables and conjugate linear in the middle one satisfying

$$L(x,y)\{a,b,c\} = \{L(x,y)a,b,c\} - \{a,L(y,x)b,c\} + \{a,b,L(x,y)c\}\,, \tag{2.1}$$

such that $\|L(a,a)\| = \|a\|^2$ and $L(a,a)$ is an hermitian operator on E with non-negative spectrum, where $L(a,b)$ is given by $L(a,b)y = \{a,b,y\}$.

Every C*-algebra is a JB*-triple with respect to the triple product given by $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$. The same triple product equipes the space $B(H,K)$, of all bounded linear operators between complex Hilbert spaces H and K, with a structure of JB*-triples. Among the examples involving Jordan algebras, we can say that every JB*-algebra is a JB*-triple under the triple product $\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$.

An element u in a JB*-triple E is said to be a *tripotent* when it is a fixed point of the triple product, that is, when $u = \{u,u,u\}$. Given a tripotent $u \in E$, the mappings $P_i(u) : E \to E_i(u)$, $(i = 0,1,2)$, defined by

$$P_2(u) = L(u,u)(2L(u,u) - id_E),\ P_1(u) = 4L(u,u)(id_E - L(u,u)),$$
$$\text{and } P_0(u) = (id_E - L(u,u))(id_E - 2L(u,u)),$$

are contractive linear projections, called the *Peirce projections* associated with u. The range of $P_i(u)$ is the eigenspace $E_i(u)$ of $L(u,u)$ corresponding to the eigenvalue $\frac{i}{2}$, and

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u)$$

is the *Peirce decomposition* of E relative to u. Furthermore, the following Peirce rules are satisfied,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = \{0\}, \tag{2.2}$$

$$\{E_i(u), E_j(u), E_k(u)\} \subseteq E_{i-j+k}(u), \tag{2.3}$$

where $E_{i-j+k}(u) = \{0\}$ whenever $i - j + k \notin \{0,1,2\}$ ([24] or [12, Th. 1.2.44]). For x,y,z in a JB*-triple E we have [25, Cor. 3]

$$\|\{x,y,z\}\| \leq \|x\|\|y\|\|z\|. \tag{2.4}$$

A tripotent u is called *complete* if $E_0(u)$ reduces to $\{0\}$.

The Peirce-2 subspace $E_2(u)$ is a unital JB*-algebra with unit u, product $a \circ_u b = \{a,u,b\}$ and involution $a^{\sharp_u} = \{u,a,u\}$ (c.f. [7, Theorem 2.2] and [31, Theorem 3.7]; [12, p. 185]).

A JBW*-triple is a JB*-triple which is also a dual Banach space. Every JBW*-triple admits a unique isometric predual and its triple product is separately weak*-continuous ([5], [28], [12, Th. 3.3.9]). Consequently, the Peirce projections associated with a tripotent in a JBW*-triple are weak*-continuous. The second dual of a JB*-triple is a JBW*-triple such that its triple product reduces to the original one (cf. [15], [12, Cor. 3.3.5]). The class of JBW*-triples includes all von Neumann algebras. Functionals on a JBW*-triple W are called normal if they belong to the predual W_*.

JBW*-triples play, in the category of JB*-triples, a similar role to that played by von Neumann algebras in the setting of C*-algebras. A JB*-triple need not have any non-zero tripotent. However, since the complete tripotents of a JB*-triple E coincide with the complex and the real extreme

points of its closed unit ball (cf. [7, Lem. 4.1], [31, Prop. 3.5], [12, Th. 3.2.3]), the Krein-Milman theorem implies that every JBW*-triple contains an abundant set of tripotents.

Given elements a, b in a JB*-triple E, the symbol $Q(a,b)$ will denote the conjugate linear operator on E defined by $Q(a,b)(x) := \{a,x,b\}$. We write $Q(a)$ instead of $Q(a,a)$. The *Bergmann operator* $B(a,b) : E \to E$ is the mapping given by $B(a,b)(z) = z - 2L(a,b)(z) + Q(a)Q(b)(z)$, for all z in E (compare [32] or [39, page 305]). In the particular case of u being a tripotent, we have $P_0(u) = B(u,u)$.

Throughout the paper, given a Banach space X, we consider X as a closed subspace of X^{**}, via its natural isometric embedding, and for each closed subspace Y of X we shall identify $\overline{Y}^{\sigma(X^{**},X^*)}$, the weak*-closure of Y in X^{**}, with Y^{**}.

A normalized sequence (x_n) in a Banach space X is said to span ℓ_1 r-isomorphically if $\| \sum_n \alpha_n x_n \| \geq r \sum_n |\alpha_n|$ for all scalars α_n. If there is a sequence (δ_m) such that $0 \leq \delta_m \to 0$ and $(x_n)_{n \geq m}$ spans ℓ_1 δ_m-isomorphically for all m then (x_n) is said to span ℓ_1 almost isometrically.

The strong-*-topology. Given a norm-one element φ in the predual W_* of a JBW*-triple W, and a norm-one element z in W with $\varphi(z) = 1$, it follows from [4, Proposition 1.2] that the assignment

$$(x,y) \mapsto \varphi\{x,y,z\}$$

defines a positive sesquilinear form on W. Moreover, for every norm-one element w in W satisfying $\varphi(w) = 1$, we have $\varphi\{x,y,z\} = \varphi\{x,y,w\}$, for all $x, y \in W$. The mapping $x \mapsto \|x\|_\varphi := (\varphi\{x,x,z\})^{\frac{1}{2}}$, defines a prehilbertian seminorm on W. The *strong*-topology* of W, introduced by T.J. Barton and Y. Friedman in [4], is the topology on W generated by the family $\{\|\cdot\|_\varphi : \varphi \in W_*, \|\varphi\| = 1\}$, and will be denoted by $s^*(W, W_*)$. From [4, page 258] we get $|\varphi(x)| \leq \|x\|_\varphi$ for any $x \in W$, and it is clear from this that $s^*(W, W_*)$ is stronger than the weak*-topology of W.

It is known that the triple product of a JBW*-triple is jointly strong*-continuous on bounded sets ([34, Th. 9], [37, Th., page 103]). Another interesting property tells us that the strong*-topology of a JBW*-triple W is compatible with the duality (W, W_*) (i.e. a linear functional on W is weak*-continuous if, and only if, it is strong*-continuous, see [34, Th. 9]). The bipolar theorem implies that for convex sets of W, weak*-closure and strong*-closure coincide. It follows that the closed unit ball of a weak*-dense JB*-subtriple E of a JBW*-triple W is strong*-dense in the closed unit ball of W. This result, known as *Kaplansky Density theorem for JBW*-triples*, was established by J.T. Barton and Y. Friedman in [4, Cor. 3.3].

In 2001, L.J. Bunce culminated the description of the fundamental properties of the strong*-topology showing that for every JBW*-subtriple F of a JBW*-triple W, the strong*-topology of F coincides with the restriction

to F of the strong*-topology of W, that is, $s^*(F, F_*) = s^*(W, W_*)|_F$ [8]. It is also known that a linear map between JBW*-triples is strong*-continuous if, and only if, it is weak*-continuous (compare [34, page 621]).

Functional calculus, open and closed tripotents. Let x be an element in a JB*-triple E. Throughout the paper, the symbol E_x will stand for the norm-closed subtriple of E generated by x. It is known that E_x is JB*-triple isomorphic to the abelian C*-algebra $C_0(L)$ of all complex-valued continuous functions on L vanishing at 0, where L is a locally compact subset of $(0, \|x\|]$ satisfying that $L \cup \{0\}$ is compact. Further, there exists a JB*-triple isomorphism $\Psi : E_x \to C_0(L)$ satisfying $\Psi(x)(t) = t$, for all t in L (compare [30, 1.15]). Given a continuous complex-valued function $f : L \cup \{0\} \to \mathbb{C}$ vanishing at 0, the *continuous triple functional calculus* $f(x)$ will have its usual meaning $f(x) = \Psi^{-1}(f)$.

We define $x^{[1]} := x$ and $x^{[2n+1]} = \{x, x, x^{[2n-1]}\}$ for every $n \in \mathbb{N}$. JB*-triples are power associative, that is,

$$\left\{x^{[2k-1]}, x^{[2l-1]}, x^{[2m-1]}\right\} = x^{[2(k+l+m)-3]},$$

for every $k, l, m \in \mathbb{N}$ (cf. [32, §3.3] or [12, Lem. 1.2.10] or simply apply the Jordan identity).

Suppose now that $\|x\| = 1$ and that E is a subtriple of a JBW*-triple W, for example $W = E^{**}$. It is known that $(x^{[2n+1]})$ converges in the strong*-topology to the tripotent $u(x) = \chi_{\{1\}}(x) \in \overline{E_x}^{w^*} \subset W$, which is called the *support tripotent* of x ([18, Lem. 3.3]). (By χ_A we denote the characteristic function of a set A.) By functional calculus there exist, for each $n \in \mathbb{N}$, unique elements $x^{[\frac{1}{2n-1}]}$ in $E_x \cong C_0(L)$ satisfying $\left(x^{[\frac{1}{2n-1}]}\right)^{[2n-1]} = x$. The latter are strong*-convergent to the tripotent $r(x) = \chi_{(0,1]}(x) \in \overline{E_x}^{w^*} \subset W$, which is called the *range tripotent of* x. The tripotent $r(x)$ is the smallest tripotent $e \in W$ satisfying that x is positive in the JBW*-algebra $W_2(e)$ (see, for example, [17, comments before Lemma 3.1] or [9, §2]). The inequalities

$$u(x) \leq x^{[2n+1]} \leq x \leq r(x)$$

hold in $W_2(r(x))$ for every norm-one element $x \in E$.

A tripotent u, in a JB*-triple E, is said to be *bounded* if there exists a norm-one element $x \in E$ such that $L(u, u)x = u$. The element x is called a bound of u and we write $u \leq x$. We shall write $y \leq u$ whenever y is a positive element in the JB*-algebra $E_2(u)$ (cf. [21, pages 79-80]). A JB*-triple E need not have a cone of positive elements and the lacking of order implies that the symbol $x \leq y$ makes no sense for general elements $x, y \in E$. However, it should be remarked that, given $x, y \in E$ and tripotents $u, v \in E$ with $x \leq u$ and $u \leq y \leq v$, we have $x \leq u \leq y \leq v$ with respect to the natural order of the JB*-algebra $E_2(v)$. Note further, for Lemma 3.3 below,

that we have $u \leq x$ in the just mentioned sense if we have $u \leq x$ in the JBW*-algebra $E_2^{**}(r(x))$.

Inspired by the notion of open projection in the bidual of a C*-algebra introduced and studied by C. Akemann, L. Brown, and G.K. Pedersen (cf. [1] or [2, 3] or [33, Proposition 3.11.9]), C.M. Edwards and G.T. Rüttimann develop the notion of open tripotent in the bidual of a JB*-triple E: we say that a tripotent e in E^{**} is *open* if $E_2^{**}(e) \cap E$ is weak*-dense in $E_2^{**}(e)$ (see [18, page 167]). It is known that the range tripotent of a norm-one element of E is open (cf. [9, Proposition 2.1]). A tripotent e in E^{**} is said to be *compact-G_δ* (relative to E) if there exists a norm-one element x in E such that e coincides with $u(x)$, the support tripotent of x. A tripotent e in E^{**} is said to be *compact* (relative to E) if there exists a decreasing net (e_λ) of tripotents in E^{**} which are compact-G_δ with infimum e, or if e is zero (cf. [18, pages 163-164]). In the terminology of [21], we say that a tripotent u in E^{**} is closed if $E \cap E_0^{**}(u)$ is weak*-dense in $E_0^{**}(u)$. The equivalence established in [21, Th. 2.6] shows that a tripotent $e \in E^{**}$ is compact if and only if e is closed and bounded by an element of E.

Small perturbation of a normal functional. Let $\varphi \in W_*$ be a functional in the predual of a JBW*-triple W and let e be a tripotent in W. In [24, Proposition 1], Y. Friedman and B. Russo prove that $\|\varphi P_2(e)\| = \|\varphi\|$ if and only if $\varphi = \varphi P_2(e)$. Using the techniques of ultraproducts of Banach spaces, J. Becerra-Guerrero and A. Rodríguez Palacios obtained the following quantitative version of the above property which will be used throughout this article.

Lemma 2.1. [6, Lem. 2.2] *Given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that, for every JB*-triple E, every non-zero tripotent e in E, and every φ in E^* with $\|\varphi\| \leq 1$ and $\|\varphi P_2(e)\| \geq 1 - \eta$, we have $\|\varphi - \varphi P_2(e)\| < \varepsilon$.* □

Orthogonality and geometric M- and L-orthogonality. We recall that elements a, b in a JB*-triple E are said to be *algebraically orthogonal* or simply *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. If we consider a C*-algebra as a JB*-triple then two elements are orthogonal in the C*-sense if and only if they are orthogonal in the triple sense. It is known (compare [11, Lem. 1]) that $a \perp b$ if and only if one of the following statements holds:

$$\{a, a, b\} = 0; \qquad a \perp r(b); \qquad r(a) \perp r(b);$$

$$E_2^{**}(r(a)) \perp E_2^{**}(r(b)); \quad r(a) \in E_0^{**}(r(b)); \quad a \in E_0^{**}(r(b));$$

$$b \in E_0^{**}(r(a)); \qquad E_a \perp E_b.$$

It follows from Peirce rules (2.2) that, for each tripotent u in a JB*-triple E, $E_0(u) \perp E_2(u)$.

For each norm one functional φ in the predual of a JBW*-triple W, the square of the prehilbertian seminorm $\|\cdot\|_\varphi$ is additive on orthogonal elements:

$$\|a+b\|_\varphi^2 = \|a\|_\varphi^2 + \|b\|_\varphi^2, \quad \forall a \perp b.$$

We recall that a functional ϕ in the predual of a JBW*-algebra M is said to be *faithful* if for each $a \geq 0$ in M, $\phi(a) = 0$ implies $a = 0$.

Let φ be a norm-one functional in the predual of a JBW*-triple W. By [24, Prop. 2], there exists a unique tripotent $e = e(\varphi) \in W$ satisfying $\varphi = \varphi P_2(e)$ and $\varphi|_{W_2(e)}$ is a faithful normal state of the JBW*-algebra $W_2(e)$. This unique tripotent e is called the *support tripotent* of φ, and will be denoted by $e(\varphi)$. (Note that at the time of the writing of [24] condition [24, (1.13)] was not yet known to hold for all JBW*-triples.)

Now, according to [26] and [19], we define two functionals φ and ψ in the predual of a JBW*-triple W to be *algebraically orthogonal* or simply *orthogonal*, denoted by $\varphi \perp \psi$, if their support tripotents are orthogonal in W, that is $e(\varphi) \perp e(\psi)$.

Elements x, y in a normed space X are said to be *L-orthogonal* (and we write $x \perp_L y$) if $\|x \pm y\| = \|x\| + \|y\|$, and are said to be *$M$-orthogonal* (denoted by $x \perp_M y$) if $\|x \pm y\| = \max\{\|x\|, \|y\|\}$.

Given a, b in E, it follows from [24, Lem. 1.3(a)] that $a \perp_M b$ whenever $a \perp b$. In general the reverse implication does not hold, for example $(1/2, 1, 0)$ and $(1/2, 0, 1)$ in the C*-algebra $l^\infty(3)$ are M-orthogonal but not orthogonal. The following result is borrowed from [26] and [19].

Lemma 2.2. [26, Lem. 2.3] *and* [19, Theorem 5.4 and Lemma 5.5] *Let φ and ψ be two functionals in the predual of a JBW*-triple W. Then $\varphi \perp \psi$ if, and only if, $\varphi \perp_L \psi$. Furthermore, given two tripotents e and u in W, then $e \perp u$ if, and only if, $e \perp_M u$.* □

Note that $\varphi \perp_L \psi$ in the lemma (with $\|\psi\|, \|\varphi\| \neq 0$) is equivalent to

$$\|\alpha \frac{\varphi}{\|\varphi\|} + \beta \frac{\psi}{\|\psi\|}\| = |\alpha| + |\beta|,$$

for any $\alpha, \beta \in \mathbb{C}$.

3. Quantitative versions of M- and L-orthogonality in JBW*triples and their predual spaces

The main goal of this section is to establish quantitative versions of Lemma 2.2 (see Propositions 3.5, 3.10 and Theorem 3.6 below). The proof will follow from a series of technical results. The next two lemmas, which are included here for the sake of completeness, are borrowed from [22].

Lemma 3.1. [22, Lem. 1.2] *Let E be a JB*-triple, e a tripotent in E, and x a norm-one element in E with $e \leq x$. Then $B(x, x)$ is a contractive operator and $B(x, x)(y)$ belongs to $E_0(e)$, for every y in E.* □

Lemma 3.2. [22, Lem. 2.1] *Let E be a JB*-triple, v be a tripotent in E, and φ an element in the closed unit ball of E^*. Then for each $y \in E_2(v)$ with $\|y\| \leq 1$ we have*

$$|\varphi(x - B(y,y)(x))| \leq 21\|x\|\|v\|_{\varphi}, \tag{3.1}$$

for every $x \in E$. □

We shall also need an appropriate version of [22, Lem. 2.2], the argument is taken from the just quoted paper.

Lemma 3.3. *Let E be a JB*-triple, $\theta > 0$, $1 > \delta > 0$, and let φ_1, φ_2 be two norm-one functionals in E^*. Suppose x is an element in the closed unit ball of E, satisfying $|\varphi_1(x)| \geq 1 - \delta$ and $\|x\|_{\varphi_2} \leq \theta$. Then, for every $\varepsilon > 0$ with $1 - \delta \geq 2\varepsilon$ there exist two elements $\widetilde{a}, y$ in the unit ball of E_x, and two tripotents u, v in $(E_x)^{**}$ such that $\widetilde{a} \leq u \leq y \leq v = r(y)$, $1 \geq |\varphi_1(\widetilde{a})| > 1 - \delta - \varepsilon$, and $\|v\|_{\varphi_2} < \frac{3\theta}{\varepsilon}$. We can further find $a \in E_2^{**}(u)$ such that $1 \geq \varphi_1(a) > 1 - \delta - \varepsilon$.*

Proof. Let $\alpha > 0$ and define $f_\alpha, g_\alpha \in C_0(L)$ by

$$f_\alpha = \begin{cases} 0, & \text{if } 0 \leq t \leq \alpha \\ \text{affine}, & \text{if } \alpha \leq t \leq 2\alpha \\ t, & \text{if } 2\alpha \leq t \leq \|x\|, \end{cases} \qquad g_\alpha = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\alpha}{2} \\ \text{affine}, & \text{if } \frac{\alpha}{2} \leq t \leq \alpha \\ 1, & \text{if } \alpha \leq t \leq \|x\|. \end{cases}$$

Let $2\varepsilon/3 < \varepsilon' < \varepsilon$ and define $\widetilde{a} = f_{\varepsilon'}(x)$ and $y = g_{\varepsilon'}(x)$ by the functional calculus recalled in Section 2. Then $\|\widetilde{a}\| \leq 1$ because $2\varepsilon' \leq 1 - \delta \leq \|x\|$. Since $\|x - \widetilde{a}\| \leq \varepsilon'$ and $|\varphi_1(x)| \geq 1 - \delta$ it follows that $|\varphi_1(\widetilde{a})| > 1 - \delta - \varepsilon$.

We set $u = \chi_{[\varepsilon', \|x\|]}$, and $v = r(y) = \chi_{(\frac{\varepsilon'}{2}, \|x\|]}$ (also in $(E_x)^{**}$) and get $\widetilde{a} \leq u \leq y \leq v$. Let us take $\alpha \in \mathbb{R}$ such that $\varphi_1(e^{i\alpha}\widetilde{a}) > 0$ and define $a = e^{i\alpha}\widetilde{a}$. Then $a \in E_2^{**}(u)$ because $\widetilde{a} \leq u$ and we have

$$1 \geq \varphi_1(a) > 1 - \delta - \varepsilon.$$

Since $\|\cdot\|_{\varphi}$ is an order-preserving map on the set of positive elements in $(E_x)^{**}$ (cf. [21, Lem. 3.3]), we deduce that

$$\|v\|_{\varphi_2} \leq \left\| \frac{2}{\varepsilon'} x \right\|_{\varphi_2} \leq \frac{2\theta}{\varepsilon'} < \frac{3\theta}{\varepsilon}.$$

□

Proposition 3.4. *Let E be a JB*-triple, and let φ_1 and φ_2 be two orthogonal norm-one functionals in E^*. Then for every $\varepsilon > 0$ there exist norm-one elements a, b in E satisfying $a \perp b$, $\varphi_1(a) > 1 - \varepsilon$ and $\varphi_2(b) > 1 - \varepsilon$.*

Proof. Let us fix an arbitrary $\varepsilon > 0$. Take $\eta > 0$ satisfying $\eta < \min\{\frac{1}{3}, \frac{\varepsilon}{2}\}$. We can also find $0 < \delta < \frac{\varepsilon\eta}{66}$. We note that η and δ satisfy $2\eta < 1 - \eta$, $1 - 2\eta > 1 - \varepsilon$, and $22\frac{3}{\eta}\delta < \varepsilon$.

Let e_j in E^{**} be the support tripotent of φ_j, $j = 1, 2$. Since $e_1 \perp e_2$, $e_1 + e_2$ and $e_1 - e_2$ are the support tripotents of $\phi = \varphi_1 + \varphi_2$ and $\psi = \varphi_1 - \varphi_2$, respectively (see [19, Th. 5.4]). In particular, $\varphi_1(e_2) = 0 = \varphi_2(e_1)$.

By the Kaplansky Density theorem for JBW*-triples [4, Cor. 3.3] (i.e. by the strong*-density of the closed unit ball of E in the one of E^{**}), there are two nets (z_λ) and $(\widetilde{z}_\mu)$ in the closed unit ball of E converging in the strong*-topology of E^{**} to e_1 and e_2, respectively. Since $s^*(E^{**}, E^*)$ is stronger than the weak*-topology of E^{**}, we deduce that $(z_\lambda) \to e_1$ and $(\widetilde{z}_\mu) \to e_2$ in the weak*-topology of E^{**}. In particular,

$$\varphi_1(z_\lambda) \to \varphi_1(e_1) = 1,\ \varphi_1(\widetilde{z}_\mu) \to \varphi_1(e_2) = 0,$$

$$\varphi_2(z_\lambda) \to \varphi_2(e_1) = 0,\ \varphi_2(\widetilde{z}_\mu) \to \varphi_2(e_2) = 1,$$

$$\|z_\lambda\|_{\varphi_1} \to \|e_1\|_{\varphi_1} = 1,\ \|\widetilde{z}_\mu\|_{\varphi_1} \to \|e_2\|_{\varphi_1} = 0,$$

$$\|z_\lambda\|_{\varphi_2} \to \|e_1\|_{\varphi_2} = 0, \text{ and } \|\widetilde{z}_\mu\|_{\varphi_2} \to \|e_2\|_{\varphi_2} = 1.$$

Find indices λ_0 and μ_0 such that

$$|\varphi_1(z_{\lambda_0})| > 1 - \eta, \quad \|z_{\lambda_0}\|_{\varphi_2} < \delta, \quad |\varphi_2(\widetilde{z}_{\mu_0}) - 1| < \frac{3}{\eta}\delta. \tag{3.2}$$

Applying Lemma 3.3 (with $\delta, \eta, z_{\lambda_0}$ for $\theta, \delta = \varepsilon, x$) we can find $a_0, \widetilde{a}, y$ in the closed unit ball of $E_{z_{\lambda_0}}$ and two tripotents u, v in $E^{**}_{z_{\lambda_0}}$ satisfying

$$\widetilde{a} \le u \le y \le v, \qquad 1 \ge \varphi_1(a_0) > 1 - 2\eta > 1 - \varepsilon,$$

$$\|v\|_{\varphi_2} < \frac{3}{\eta}\delta, \text{ and } a_0 \in E_2^{**}(u).$$

Define $a = a_0/\|a_0\| \in E_0^{**}(u)$. Then $\varphi_1(a) > 1-\varepsilon$. By Lemma 3.2 we obtain

$$\Big|\varphi_2\Big(\widetilde{z}_{\mu_0} - B(y,y)(\widetilde{z}_{\mu_0})\Big)\Big| < 21\|\widetilde{z}_{\mu_0}\|\|v\|_{\varphi_2} < 21\frac{3}{\eta}\delta,$$

and by the third inequality in (3.2) we deduce that

$$\Big|\varphi_2\Big(B(y,y)(\widetilde{z}_{\mu_0})\Big) - 1\Big| < 22\frac{3}{\eta}\delta < \varepsilon.$$

Setting $\widetilde{b} = e^{i\beta}B(y,y)(\widetilde{z}_{\mu_0})$ for a suitable $\beta \in \mathbb{R}$ we have $\varphi_2(\widetilde{b}) > 1 - \varepsilon$ and setting $b = \widetilde{b}/\|\widetilde{b}\|$ we still have $\varphi_2(b) > 1 - \varepsilon$.

By Lemma 3.1, $b \in B(y,y)(E) \subseteq E_0^{**}(u)$. Since, by construction, a lies in $E_2^{**}(u)$, it follows that $a \perp b$. □

Remark. Proposition 3.4 remains valid (with practically the same proof) if the first sentence is replaced by "Let E be a weak*-dense subtriple of a JBW*-triple W and let φ_1, φ_2 be two orthogonal norm-one functionals in W_*."

We shall require some results in the theory of ultraproducts of Banach spaces [27]. To this end, we recall some basic facts and definitions. Let $\mathcal{U}$ be an ultrafilter on a non-empty set I, and let $\{X_i\}_{i\in I}$ be a family of Banach

spaces. Let $\ell_\infty(I, X_i) = \ell_\infty(X_i)$ denote the Banach space obtained as the ℓ_∞-sum of the family $\{X_i\}_{i\in I}$, and let

$$c_0\left(X_i\right) := \left\{ (x_i) \in \ell_\infty(X_i) : \lim_{\mathcal{U}} \|x_i\| = 0 \right\}.$$

The ultraproduct of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$, denoted by $(X_i)_{\mathcal{U}}$, is the quotient Banach space $\ell_\infty(X_i)/c_0\left(X_i\right)$ equipped with the quotient norm. Let $[x_i]_{\mathcal{U}}$ be an equivalence class in $(X_i)_{\mathcal{U}}$ represented by a family $(x_i)_i \in \ell_\infty(X_i)$. It is known that

$$\|[x_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|,$$

independently of the representative of $[x_i]_{\mathcal{U}}$. In general, the ultraproduct of a family of dual Banach spaces is not a dual Banach space (not even in the case of von Neumann algebras). The ultraproduct $(X_i^*)_{\mathcal{U}}$ of the duals can be identified isometrically with a closed subspace of the dual $((X_i)_{\mathcal{U}})^*$ via the canonical mapping

$$\mathcal{J} : (X_i^*)_{\mathcal{U}} \to ((X_i)_{\mathcal{U}})^*$$
$$\mathcal{J}[\varphi_i]_{\mathcal{U}}([x_i]_{\mathcal{U}}) = \lim_{\mathcal{U}} \varphi_i(x_i).$$

In [15, Cor. 10] S. Dineen establishes that the class of JB*-triples (analogously to the class of C*-algebras [27, Prop. 3.1]), is stable under ultraproducts via the canonical triple product $\{[u_i]_{\mathcal{U}}, [v_i]_{\mathcal{U}}, [w_i]_{\mathcal{U}}\} = [\{u_i, v_i, w_i\}]_{\mathcal{U}}$.

Here is a simple argument to prove Dineen's theorem (cf. also [12, proof of Cor. 3.3.5]). Let $\{E_i\}_{i\in I}$ be a family of JB*-triples. Then the Banach space $\ell_\infty(E_i)$ is a JB*-triple with pointwise operations ([30, page 523] or [12, Ex. 3.1.4]). Let E be a JB*-triple. A subtriple $\mathcal{I}$ of a JB*-triple a JB*-triple E is said to be an *ideal* or a *triple ideal* of E if $\{E, E, \mathcal{I}\}+\{E, \mathcal{I}, E\} \subseteq \mathcal{I}$. It is easy to see, under the above conditions, that $\{\ell_\infty(E_i), \ell_\infty(E_i), c_0\left(E_i\right)\} \subseteq c_0\left(E_i\right)$ and $\{\ell_\infty(E_i), c_0\left(E_i\right), \ell_\infty(E_i)\} \subseteq c_0\left(E_i\right)$, and hence $c_0\left(E_i\right)$ is a closed triple ideal of $\ell_\infty(E_i)$. Since the quotient of a JB*-triple by a closed triple ideal is a JB*-triple ([30] or [12, Cor. 3.1.18]), we deduce that $(E_i)_{\mathcal{U}} = \ell_\infty(E_i)/c_0\left(E_i\right)$ is a JB*-triple.

Proposition 3.5. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every JB*-triple E and every pair of functionals φ_1 and φ_2 in the closed unit ball of E^* with $2 \geq \|\varphi_1 \pm \varphi_2\| \geq 2(1-\delta)$ there exist orthogonal norm-one elements a, b in E satisfying $\varphi_1(a) > 1-\varepsilon$ and $\varphi_2(b) > 1-\varepsilon$.*

Proof. Suppose, to the contrary, that there exists $\varepsilon_0 > 0$ such that for each natural n, we can find a JB*-triple E_n and functionals $\varphi_{1,n}$ and $\varphi_{2,n}$ in the closed unit ball of E_n^* with $2 \geq \|\varphi_{1,n} \pm \varphi_{2,n}\| \geq 2(1-\frac{1}{n})$ satisfying $|\varphi_{1,n}(a)| \leq 1-\varepsilon_0$ and $|\varphi_{2,n}(b)| \leq 1-\varepsilon_0$, whenever a, b are elements of norm one in E_n with $a \perp b$.

Take a non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$, let $0 < \varepsilon_1 < \varepsilon$ and let $\mathcal{J} : (E_n^*)_{\mathcal{U}} \to ((E_n)_{\mathcal{U}})^*$ be the canonical isometric embedding defined by $\mathcal{J}[\varphi_i]_{\mathcal{U}}([x_n]_{\mathcal{U}}) =$

$\lim_{\mathcal{U}} \varphi_n(x_n)$. Then $\mathcal{J}[\varphi_{1,n}]_{\mathcal{U}}$ and $\mathcal{J}[\varphi_{2,n}]_{\mathcal{U}}$ have norm one and are L-orthogonal in $((E_n)_{\mathcal{U}})^*$ because so are $[\varphi_{1,n}]_{\mathcal{U}}$ and $[\varphi_{2,n}]_{\mathcal{U}}$ in $(E_n^*)_{\mathcal{U}}$. As explained above, $((E_n)_{\mathcal{U}})^*$ is a JB*-triple and Proposition 3.4 applies: there exist norm-one elements $[a_n]_{\mathcal{U}}, [b_n]_{\mathcal{U}}$ in $(E_n)_{\mathcal{U}}$ satisfying $[a_n]_{\mathcal{U}} \perp [b_n]_{\mathcal{U}}$, $\mathcal{J}[\varphi_{1,n}]_{\mathcal{U}}([a_n]_{\mathcal{U}}) > 1-\varepsilon_1$ and $\mathcal{J}[\varphi_{2,n}]_{\mathcal{U}}([b_n]_{\mathcal{U}}) > 1-\varepsilon_1$.

We note that the elements $[a_n]_{\mathcal{U}}, [b_n]_{\mathcal{U}}$ are orthogonal in the quotient $(E_n)_{\mathcal{U}} = \ell_\infty(E_n)/c_0(E_n)$. Since the quotient mapping $\pi : \ell_\infty(E_n) \to \ell_\infty(E_n)/c_0(E_n)$ is a triple homomorphism between JB*-triples and $\pi((a_n)_n) = [a_n]_{\mathcal{U}} \perp \pi((b_n)_n) = [b_n]_{\mathcal{U}}$, by [10, Proposition 4.7] there exist orthogonal elements $(\widetilde{a}_n)_n$ and $(\widetilde{b}_n)_n$ in $\ell_\infty(E_n)$ satisfying $\pi((\widetilde{a}_n)_n) = [a_n]_{\mathcal{U}}$ and $\pi((\widetilde{b}_n)_n) = [b_n]_{\mathcal{U}}$. We have $\lim_{\mathcal{U}} \|\widetilde{a}_n\| = \lim_{\mathcal{U}} \|a_n\| = 1$ and likewise for $(b_n)_n$, $(\widetilde{b}_n)_n$.

Now, $1-\varepsilon_1 < \mathcal{J}[\varphi_{1,n}]_{\mathcal{U}}([\widetilde{a}_n]_{\mathcal{U}}) = \lim_{\mathcal{U}} \varphi_{1,n}(\widetilde{a}_n)$, $1-\varepsilon_1 < \mathcal{J}[\varphi_{2,n}]_{\mathcal{U}}([\widetilde{b}_n]_{\mathcal{U}}) = \lim_{\mathcal{U}} \varphi_{2,n}(\widetilde{b}_n)$, and, for every n, $\widetilde{a}_n \perp \widetilde{b}_n$. Hence $\varphi_{1,n}(\widetilde{a}_n/\|\widetilde{a}_n\|) > 1-\varepsilon$ or $\varphi_{2,n}(\widetilde{b}_n/\|\widetilde{b}_n\|) > 1-\varepsilon$ can be achieved for infinitely many n which contradicts the assumption made in the beginning of the proof. □

We shall establish now an analogous version of Proposition 3.5 for finite sets of functionals in the dual of a JB*-triple.

Theorem 3.6. *For each $\varepsilon > 0$ and each natural n, there exists $\delta = \delta(n,\varepsilon) > 0$ with the following property. Let E be a JB*-triple and let $\varphi_1, \dots, \varphi_n$ be functionals in E^* such that*

$$\sum_{j=1}^n |\alpha_j| \geq \left\| \sum_{j=1}^n \alpha_j \varphi_j \right\| \geq (1-\delta(n,\varepsilon)) \sum_{j=1}^n |\alpha_j| \quad \forall \alpha_j \in \mathbb{C}. \tag{3.3}$$

Then there exist mutually orthogonal elements $a_1, \dots, a_n$ of norm one in E and mutually orthogonal functionals $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n$ of norm one in E^* satisfying

$$\varphi_j(a_j) > 1-\varepsilon \quad \text{and} \quad \|\varphi_j - \widetilde{\varphi}_j\| < \varepsilon \quad \forall j = 1, \dots, n \tag{3.4}$$
$$\text{where} \quad \widetilde{\varphi}_j = \frac{\varphi_j P_2(r(a_j))}{\|\varphi_j P_2(r(a_j))\|}.$$

Proof. We shall proceed by induction over $n \geq 1$. For $n = 1$ there is nothing to prove. Let us fix $\varepsilon > 0$ and $n \in \mathbb{N}$.

We CLAIM that there is $\varepsilon' \in (0,\varepsilon)$ such that if an element b in a JB*-triple E and $\xi \in E^*$ satisfy

$$\|b\| \leq 1, \quad \|\xi\| \leq 1 \quad \text{and} \quad |\xi(b)| > 1-\varepsilon' \tag{3.5}$$

then $\left\| \xi - \frac{\xi P_2(r(b))}{\|\xi P_2(r(b))\|} \right\| < \varepsilon$ In fact, define $\varepsilon' \in (0, \varepsilon/2)$ by Lemma 2.1 such that (3.5) entails $\|\xi - \psi\| < \varepsilon/2$ where $\psi = \xi P_2(r(b))$. Hence, if (3.5) holds

then $|\psi(b)| = |\xi((b)| > 1 - \varepsilon/2$ and

$$\begin{aligned} \left\|\xi - \frac{\psi}{\|\psi\|}\right\| &\leq \|\xi - \psi\| + \left\|\psi - \frac{\psi}{\|\psi\|}\right\| < \frac{\varepsilon}{2} + \left(\frac{1}{\|\psi\|} - 1\right)\|\psi\| \\ &= \frac{\varepsilon}{2} + 1 - \|\psi\| < \varepsilon \end{aligned}$$

which proves the claim.

Choose $\delta(n, \varepsilon') > 0$ according to the induction hypothesis, choose $\eta_0 = \eta(\frac{\delta(n,\varepsilon')}{2}) > 0$ according to Lemma 2.1 and choose

$$\delta_0 = \delta\left(\min\left\{\varepsilon', \frac{1}{n}\eta_0\right\}\right) > 0$$

according to Theorem 3.5. Furthermore, let $\delta(n+1, \varepsilon')$ be such that

$$0 < \delta(n+1, \varepsilon') < \min\left\{\frac{\delta(n, \varepsilon')}{2}, \delta_0\right\}.$$

Let E be a JB*-triple and let $\varphi_1, \ldots, \varphi_{n+1} \in E^*$ satisfy

$$\sum_{j=1}^{n+1} |\alpha_j| \geq \left\|\sum_{j=1}^{n+1} \alpha_j \varphi_j\right\| \geq \Big(1 - \delta(n+1, \varepsilon')\Big) \sum_{j=1}^{n+1} |\alpha_j| \quad \forall \alpha_j \in \mathbb{C}.$$

Let us define $\phi = \displaystyle\sum_{j=1}^{n} \frac{1}{n}\varphi_j$. Clearly,

$$2 \geq \|\phi \pm \varphi_{n+1}\| \geq \Big(1 - \delta(n+1, \varepsilon')\Big)(n\frac{1}{n} + 1) > 2(1 - \delta_0).$$

Thus, by the choice of δ_0 (see Theorem 3.5) there exist $a \perp a_{n+1}$ of norm one in E satisfying

$$\begin{aligned} \varphi_{n+1}(a_{n+1}) &> 1 - \min\left\{\varepsilon', \frac{1}{n}\eta_0\right\} > 1 - \varepsilon', \\ \phi(a) &> 1 - \min\left\{\varepsilon', \frac{1}{n}\eta_0\right\} > 1 - \frac{1}{n}\eta_0. \end{aligned} \tag{3.6}$$

Since $\sum_{j=1}^{n} \varphi_j(a) = n\phi(a) > n - \eta_0$ and $\|\varphi_j\| \leq 1$ we have

$$|\varphi_j(a)| > n - \eta_0 - (n-1) = 1 - \eta_0 \quad \forall j = 1, \ldots, n.$$

Thus, $|\varphi_j P_2(r(a))(a)| = |\varphi_j(a)| > 1 - \eta_0$ and by the choice of η_0 we deduce

$$\|\varphi_j - \varphi_j P_2(r(a))\| < \frac{\delta(n, \varepsilon')}{2} \quad \forall j = 1, \ldots, n.$$

Therefore we have

$$\sum_{j=1}^{n} |\alpha_j| \geq \left\|\sum_{j=1}^{n} \alpha_j \varphi_j P_2(r(a))\right\|$$

$$\geq \left\|\sum_{j=1}^n \alpha_j\varphi_j\right\| - \left\|\sum_{j=1}^n \alpha_j\Big(\varphi_j - \varphi_j P_2(r(a))\Big)\right\|$$

$$\geq \Big(1-\delta(n+1,\varepsilon')\Big)\sum_{j=1}^n|\alpha_j| - \frac{\delta(n,\varepsilon')}{2}\sum_{j=1}^n|\alpha_j| \geq \Big(1-\delta(n,\varepsilon')\Big)\sum_{j=1}^n|\alpha_j|$$

for all scalars α_j. Recall that $r(a)$ is an open tripotent which means that the subtriple $F := E\cap E_2^{**}(r(a))$ is weak*-dense in $E_2^{**}(r(a))$. Set $\psi_j = \varphi_j P_2(r(a))_{|F}$ for $j\leq n$. Then

$$\sum_{j=1}^n|\alpha_j| \geq \left\|\sum_{j=1}^n\alpha_j\psi_j\right\| = \left\|\sum_{j=1}^n\alpha_j\varphi_j P_2(r(a))\right\| \geq \Big(1-\delta(n,\varepsilon')\Big)\sum_{j=1}^n|\alpha_j|,$$

for all $\alpha_j\in\mathbb{C}$, and by the induction hypothesis, applied to F, there exist mutually orthogonal norm-one elements $a_1,\dots,a_n\in F$ satisfying $\psi_j(a_j) = \varphi_j(a_j) > 1-\varepsilon'$, for every $j=1,\dots,n$. They are orthogonal to a_{n+1} because a is. Together with (3.6) this shows the first half of (3.4) (for $n+1$) because $1-\varepsilon' > 1-\varepsilon$. The second half follows from the claim. This ends the induction and the proof. □

In passing we note an obvious reformulation of the conclusion of Theorem 3.6: There exists an abelian subtriple $\mathcal{C}$ of E such that if we set $\psi_j = \varphi_{j|\mathcal{C}}$ then $(\psi_j)_{j=1}^n$ spans $\ell_1(n)$ $(1-\varepsilon)$-isomorphically in $\mathcal{C}^*$.

For the proofs of Theorems 4.1 and 4.2 we need the following technical strengthening of Theorem 3.6.

Lemma 3.7. *In Theorem 3.6 the a_j can be constructed such that additionally there are mutually orthogonal compact tripotents $u_1,\dots,u_n$ in E^{**} such that $a_j\in E_2^{**}(u_j)$ for $j=1,\dots,n$.*

Proof. We CLAIM that if a JB*-triple E, $a\in E$, $\varphi\in E^*$, and $\varepsilon'>0$ are given such that

$$\|\varphi\|\leq 1,\quad \|a\|\leq 1,\quad \varphi(a) > 1-\varepsilon' \tag{3.7}$$

then there exist a compact tripotent $u\in E_a^{**}\subset E_2^{**}(e)$ and $b\in E_a\cap E_2^{**}(u)$ such that $\|b\|=1$ and $\varphi(b)>1-\varepsilon'$.

In order to show the claim suppose (3.7) holds. Define $z_m = f_\alpha(a)\in E_a$ for $\alpha = \|b_j\|/m$ where f_α is as in the proof of Lemma 3.3. Since $\|z_m - a\|\to 0$ there is m_0 such that $\varphi(e^{i\theta}z_{m_0}) > 1-\varepsilon'$ for an appropriate $\theta\in\mathbb{R}$. Also $\|z_{m_0}\|\leq 1$. It remains to set $b = e^{i\theta}z_{m_0}/\|z_{m_0}\|$ and $u = \chi_{[\|b_j\|/2m_0,\|b_j\|]\cap L}$ and the claim is proved.

Now we apply the claim n times to pairwise orthogonal a_j and note that $u_j\in E_{a_j}^{**}\perp E_{a_k}^{**}\ni u_k$ if $j\neq k$. The claim in the proof of Theorem 3.6 shows that it is enough to replace a_j by b_j in (3.4) in order to finish the proof. □

Recalling that Peirce projections associated with a tripotent in a JBW*-triple are weak*-continuous, and the fact that the range tripotent of an element in a JBW*-triple always lies in the JBW*-triple, the arguments given above show:

Corollary 3.8. *For each $\varepsilon > 0$ and each natural n, there exists a positive $\delta = \delta(n, \varepsilon)$ such that for every JBW*-triple W, and every finite set of functionals $\varphi_1, \ldots, \varphi_n$ in W_* satisfying (3.3) there exist orthogonal norm one elements $a_1, \ldots, a_n \in W$ and orthogonal functionals $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n \in W_*$ of norm one such that (3.4) holds.*

The following corollary will not be needed in the sequel but could perhaps be useful elsewhere. The argument leading to part (a) has already been used in the proof of Theorem 3.6.

Corollary 3.9. *For each $\varepsilon > 0$ and each natural n, there exists a positive $\delta = \delta(n, \varepsilon)$ with the following properties.*

(a) Let E be a JB-triple, $e \in E^{**}$ an open tripotent and let $\varphi_1, \ldots, \varphi_n$ be functionals in the closed unit ball of E^*. If*

$$\sum_{j=1}^{n} |\alpha_j| \geq \left\| \sum_{j=1}^{n} \alpha_j \varphi_j P_2(e) \right\| \geq (1 - \delta(n, \varepsilon)) \sum_{j=1}^{n} |\alpha_j|, \quad \forall \alpha_j \in \mathbb{C} \tag{3.8}$$

then there exist orthogonal norm-one elements $a_1, \ldots, a_n$ in $E \cap E_2^{**}(e)$ and mutually orthogonal norm-one functionals $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n$ in E^* satisfying (3.4).

(b) Let W be a JBW-triple, $e \in E^{**}$ an arbitrary tripotent and let $\varphi_1, \ldots, \varphi_n$ be functionals in the closed unit ball of W_* satisfying (3.8). Then there exist mutually orthogonal elements $a_1, \ldots, a_n$ of norm one in $W_2(e)$ and mutually orthogonal norm-one functionals $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n$ in W_* satisfying (3.4).*

Proof. (a) Set $F = E \cap E_2^{**}(e)$ and $\psi_j = \varphi_j P_2(e)_{|F}$. Then $\overline{F}^{w^*} = E_2^{**}(e)$ because e is open. The ψ_j satisfy (3.3) and it is enough to apply Theorem 3.6 to F. Similarly, for part (b) identify $\varphi_j P_2(e)$ with $\varphi_{|W_2(e)} \in (W_2(e))_*$ and apply Corollary 3.8 to $W_2(e)$. □

We can now establish the promised quantitative version of the last statement in Lemma 2.2.

Proposition 3.10. *Given $\varepsilon > 0$ there exits $\delta = \delta(\varepsilon)$ satisfying that for every JB*-triple E and every couple of tripotents u, v in E with*

$$1 - \delta < \|u \pm v\| < 1 + \delta,$$

we have $\|u - P_0(v)(u)\| < \varepsilon$ and $\|v - P_0(u)(v)\| < \varepsilon$.

Proof. Let us note that the statement is true whenever the set of tripotents in a JB*-triple E reduces to the zero element. Suppose, contrary to our claim, that there exists $\varepsilon_0 > 0$ such that for each natural n, we can find a

JB*-triple E_n and tripotents u_n and v_n with $1-\frac{1}{n} < \|u_n \pm v_n\| < 1+\frac{1}{n}$ satisfying $\|u_n - P_0(v_n)(u_n)\| \geq \varepsilon_0$ or $\|v_n - P_0(u_n)(v_n)\| \geq \varepsilon_0$.

Take a non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$. The elements $[u_n]_{\mathcal{U}}$ and $[v_n]_{\mathcal{U}}$ are non-zero tripotents in $(E_n)_{\mathcal{U}}$ with $\|[u_n]_{\mathcal{U}} \pm [v_n]_{\mathcal{U}}\| = 1$, that is, $[u_n]_{\mathcal{U}} \perp_M [v_n]_{\mathcal{U}}$ in $(E_n)_{\mathcal{U}}$. The final statement in Lemma 2.2 implies that $[u_n]_{\mathcal{U}} \perp [v_n]_{\mathcal{U}}$ in $(E_n)_{\mathcal{U}}$. In particular

$$[P_0(u_n)(v_n)]_{\mathcal{U}} = P_0([u_n]_{\mathcal{U}})([v_n]_{\mathcal{U}}) = [v_n]_{\mathcal{U}}$$

(because $P_0([u_n]_{\mathcal{U}}) = [P_0(u_n)]_{\mathcal{U}}$) and

$$[P_0(v_n)(u_n)]_{\mathcal{U}} = P_0([v_n]_{\mathcal{U}})([u_n]_{\mathcal{U}}) = [u_n]_{\mathcal{U}},$$

which implies that $\lim\limits_{\mathcal{U}} \|P_0(u_n)(v_n) - v_n\| = 0$ and $\lim\limits_{\mathcal{U}} \|P_0(v_n)(u_n) - u_n\| = 0$, contradicting our assumptions. □

4. Infinite dimensional copies of ℓ_1 in preduals of JBW*-triples

Theorem 4.1. *Let W be a JBW*-triple and let (φ_m) be a bounded sequence in its predual W_*. If (φ_m) spans ℓ_1 almost isometrically then there are a subsequence (φ_{m_n}) of (φ_m) and a sequence $(\widetilde{\varphi}_n)$ of pairwise orthogonal functionals in W_* such that $\|\varphi_{m_n} - \widetilde{\varphi}_n\| \to 0$ when $l \to \infty$.*

Proof. We can assume that $\|\varphi_m\| = 1$, for every m. Let (ν_n) be a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} \nu_n < \infty$. We shall prove, by induction over n, the existence of $m_n \in \mathbb{N}$, and $\phi_{m_1}^{(n)}, \ldots, \phi_{m_n}^{(n)}$ in W_* satisfying $m_{n-1} < m_n$, and for each natural n

$$\phi_{m_k}^{(n)} \perp \phi_{m_l}^{(n)} \quad \forall k \neq l \in \{1, \ldots, n\},$$

$$\|\phi_{m_k}^{(n)}\| = 1, \quad \forall k \leq n,$$

$$\|\phi_{m_k}^{(n)} - \phi_{m_k}^{(n-1)}\| < \nu_n, \quad \forall k = 1, \ldots, n-1 \quad \text{if } n \geq 2,$$

and $\|\phi_{m_n}^{(n)} - \varphi_{m_n}\| < \nu_n$.

When $n = 1$ we set $m_1 = 1$, $\phi_1^{(1)} = \varphi_1$ and the statement is clear. Suppose that $m_1 < m_2 < \ldots < m_n$, $\{\phi_{m_1}^{(1)}\}$, $\{\phi_{m_1}^{(2)}, \phi_{m_2}^{(2)}\}$, ..., $\{\phi_{m_1}^{(n)}, \ldots, \phi_{m_n}^{(n)}\}$ have been defined satisfying the above properties.

By Corollary 3.8, there exists $\delta_1 = \min\{\delta(n, \nu_{n+1}/2), \nu_{n+1}/2\} > 0$. Choose a natural j satisfying $\frac{21}{\sqrt{j}} < \delta_1$. We use Corollary 3.8 again in order to choose $\delta_0 = \delta(nj, \nu_{n+1}) > 0$. Since (φ_m) spans ℓ_1 almost isometrically there exists $m_0 > m_n$ satisfying

$$(1-\delta_0) \sum_{m=m_0}^{\infty} |\alpha_m| \leq \left\| \sum_{m=m_0}^{\infty} \alpha_m \varphi_m \right\| \quad \forall \alpha_m \in \mathbb{C}.$$

Set $N = \{m_0 + 1, \ldots, m_0 + nj\} \subseteq \mathbb{N}$. Since

$$(1-\delta(nj, \nu_{n+1})) \sum_{m=m_0+1}^{m_0+nj} |\alpha_m| \leq \left\| \sum_{m=m_0+1}^{m_0+nj} \alpha_m \varphi_m \right\| \quad \forall \alpha_m \in \mathbb{C},$$

Corollary 3.8 implies the existence of mutually orthogonal elements $a_1, \ldots, a_{nj}$ in the closed unit ball of W such that

$$\left\| \varphi_m - \frac{\varphi_m P_2(r(a_m))}{\|\varphi_m P_2(r(a_m))\|} \right\| < \nu_{n+1} \quad \forall m \in N. \tag{4.1}$$

On the other hand, it is clear, by orthogonality, that

$$0 \leq \sum_{m \in N} \sum_{k=1}^{n} \|r(a_m)\|^2_{\phi^{(n)}_{m_k}} = \sum_{k=1}^{n} \sum_{m \in N} \|r(a_m)\|^2_{\phi^{(n)}_{m_k}} = \sum_{k=1}^{n} \left\| \sum_{m \in N} r(a_m) \right\|^2_{\phi^{(n)}_{m_k}} \leq n.$$

Thus, there exists $m_{n+1} \in N$ satisfying

$$\left\| r(a_{m_{n+1}}) \right\|^2_{\phi^{(n)}_{m_k}} \leq \frac{1}{j} \quad \forall k = 1, \ldots, n$$

hence, by Lemma 3.2,

$$\|\phi^{(n)}_{m_k} - \phi^{(n)}_{m_k} P_0(r(a_{m_{n+1}}))\| \leq 21 \frac{1}{\sqrt{j}} \quad \forall k = 1, \ldots, n. \tag{4.2}$$

We define $\widetilde{\phi}^{(n+1)}_{m_k} = \phi^{(n)}_{m_k} P_0(r(a_{m_{n+1}}))$, for $k = 1, \ldots, n$ and

$$\phi^{(n+1)}_{m_{n+1}} = \frac{\varphi_{m_{n+1}} P_2(r(a_{m_{n+1}}))}{\|\varphi_{m_{n+1}} P_2(r(a_{m_{n+1}}))\|}.$$

By (4.1), $\left\| \varphi_{m_{n+1}} - \phi^{(n+1)}_{m_{n+1}} \right\| < \nu_{n+1}$, and by (4.2)

$$\|\phi^{(n)}_{m_k} - \widetilde{\phi}^{(n+1)}_{m_k}\| \leq 21 \frac{1}{\sqrt{j}} < \delta_1 \leq \frac{\nu_{n+1}}{2} \quad \forall k = 1, \ldots, n. \tag{4.3}$$

Therefore we have

$$\begin{aligned} \sum_{k=1}^{n} |\alpha_k| &\geq \left\| \sum_{k=1}^{n} \alpha_k \widetilde{\phi}^{(n+1)}_{m_k} \right\| \geq \left\| \sum_{k=1}^{n} \alpha_k \phi^{(n)}_{m_k} \right\| - \left\| \sum_{k=1}^{n} \alpha_k (\widetilde{\phi}^{(n+1)}_{m_k} - \phi^{(n)}_{m_k}) \right\| \\ &\geq \left\| \sum_{k=1}^{n} \alpha_k \phi^{(n)}_{m_k} \right\| - \delta_1 \sum_{k=1}^{n} |\alpha_k| = \sum_{k=1}^{n} |\alpha_k| - \delta_1 \sum_{k=1}^{n} |\alpha_k| \\ &\geq (1 - \delta(n, \nu_{n+1}/2)) \sum_{k=1}^{n} |\alpha_k| \qquad \forall \alpha_m \in \mathbb{C}. \end{aligned}$$

By Corollary 3.8, applied to the JBW*-triple $W_0(r(a_{m_{n+1}}))$ and the functionals $\{\widetilde{\phi}^{(n+1)}_{m_1}, \ldots, \widetilde{\phi}^{(n+1)}_{m_n}\} \subset \left(W_0(r(a_{m_{n+1}}))\right)_*$, we can find mutually orthogonal elements $b_1, \ldots, b_n$ in the closed unit ball of $W_0(r(a_{m_{n+1}}))$ such that

$$\left\| \widetilde{\phi}^{(n+1)}_{m_k} - \frac{\widetilde{\phi}^{(n+1)}_{m_k} P_2(r(b_j))}{\|\widetilde{\phi}^{(n+1)}_{m_k} P_2(r(b_j))\|} \right\| < \frac{\nu_{n+1}}{2} \quad \forall k = 1, \ldots, n. \tag{4.4}$$

We define $\phi_{m_k}^{(n+1)} := \frac{\widetilde{\phi}_{m_k}^{(n+1)} P_2(r(b_j))}{\|\widetilde{\phi}_{m_k}^{(n+1)} P_2(r(b_j))\|}$ for $k = 1, \ldots, n$. The inequalities (4.3) and (4.4) show $\|\phi_{m_k}^{(n)} - \phi_{m_k}^{(n+1)}\| < \nu_n$ $(k = 1, \ldots, n)$, which finishes the induction argument.

Fix a natural k and consider the sequence $(\phi_{m_k}^{(n)})_{n \geq k}$. The inequalities $\|\phi_{m_k}^{(n)} - \phi_{m_k}^{(n-1)}\| < \nu_n$ and

$$\|\phi_{m_k}^{(n)} - \phi_{m_k}^{(i)}\| < \sum_{j=i+1}^{n} \nu_j \to 0 \text{ if } n > i \to \infty$$

show that $(\phi_{m_k}^{(n)})_{n \geq k}$ is a Cauchy sequence which converges to some $\widetilde{\varphi}_k \in W_*$. By construction $\phi_{m_k}^{(n)} \perp \phi_{m_j}^{(n)}$ for every $k \neq j$, and every $n \geq \max\{j, k\}$, therefore

$$\|\widetilde{\varphi}_k \pm \widetilde{\varphi}_j\| = \lim_{n \to \infty} \|\phi_{m_k}^{(n)} \pm \phi_{m_j}^{(n)}\| = 2$$

for every $k \neq j$ in $\mathbb{N}$. This implies $\widetilde{\varphi}_k \perp \widetilde{\varphi}_j$ for every $j \neq k$ (cf. Lemma 2.2). Finally, the inequality

$$\|\varphi_{m_n} - \widetilde{\varphi}_n\| \leq \|\phi_{m_n}^{(n)} - \varphi_{m_n}\| + \|\phi_{m_n}^{(n)} - \widetilde{\varphi}_n\|$$

$$= \|\phi_{m_n}^{(n)} - \varphi_{m_n}\| + \|\phi_{m_n}^{(n)} - \lim_{k \to \infty} \phi_{m_n}^{(k)}\| < \nu_n + \sum_{k=n+1}^{\infty} \nu_k,$$

gives the desired statement $\lim_{n \to \infty} \|\varphi_{m_n} - \widetilde{\varphi}_n\| = 0$. □

The study of isomorphic copies in the dual space of a JB*-triple requires an extra effort. It should be remarked here that the next proposition can be considered as a quantitative version of [35, Theorem 1] and [22, Theorem 2.3].

Theorem 4.2. *Let E be a JB*-triple and let (φ_m) be a normalized sequence in E^* spanning ℓ_1 r-isomorphically (with $0 < r \leq 1$). Then for each $\varepsilon > 0$ there exist a subsequence (φ_{m_n}) of (φ_m) and a sequence (c_n) of mutually orthogonal elements of norm one in E such that*

$$\varphi_{m_n}(c_n) > r(1 - \varepsilon), \qquad \forall n \in \mathbb{N}, \tag{4.5}$$

and such that the restrictions $\varphi_{m_n|\mathcal{C}}$ span ℓ_1 $(r(1-\varepsilon))$-isomorphically where $\mathcal{C}$ is the abelian subtriple of E generated by the c_n's and isometric to a commutative C-algebra.*

Proof. We may assume that $1 \geq \varepsilon > 0$, we consider a series $\sum_{n \geq 1} \varepsilon_n$ with $\varepsilon_n > 0$ and $\sum_{n=1}^{\infty} \varepsilon_n = \frac{\varepsilon}{2}$. By induction on n we shall define a strictly monotone sequence (m_n) in $\mathbb{N}$, a strictly decreasing sequence (N_n) of infinite subsets

of $\mathbb{N}$ (i.e. $N_n \supsetneq N_{n+1}$), a sequence (a_n) of mutually orthogonal norm-one elements in E, and a sequence (u_n) of mutually orthogonal compact tripotents in E^{**} satisfying, for all $n \in \mathbb{N}$,

$$m_n < \min N_n, \tag{4.6}$$

$$a_n \in E_2^{**}(u_n), \tag{4.7}$$

$$\|u_n\|_{\varphi_m} < r\frac{\varepsilon_n}{63} \quad \forall m \in N_n, \tag{4.8}$$

$$|\varphi_{m_n}(a_n)| > \left(1 - r\sum_{i=1}^{n-1}\frac{\varepsilon_i}{3}\right) r \left(1 - \sum_{i=1}^{n}\varepsilon_i\right) \tag{4.9}$$

(where $\sum_{i=1}^{0} = 0$) and, for $v_n = u_1 + \ldots + u_n$,

$$\left\| \sum_{m\in N_n} \alpha_m \frac{\varphi_m P_0(v_n)}{\|\varphi_m P_0(v_n)\|} \right\| \geq r\left(1 - \sum_{i=1}^{n}\varepsilon_i\right) \sum_{m\in N_n} |\alpha_m| \quad \forall \alpha_m \in \mathbb{C}. \tag{4.10}$$

Let us note that since

$$\left(1 - r\sum_{i=1}^{n-1}\frac{\varepsilon_i}{3}\right) r \left(1 - \sum_{i=1}^{n}\varepsilon_i\right) > r\left(1 - 2\sum_{i=1}^{n}\varepsilon_i\right)$$

$$> r\left(1 - 2\sum_{i=1}^{\infty}\varepsilon_i\right) = r\,(1-\varepsilon)\,,$$

the inequality in (4.9) proves $|\varphi_{m_n}(a_n)| > r\,(1-\varepsilon)$, for every $n \in \mathbb{N}$. The statement of the proposition will follow for $c_n = e^{i\theta_n} a_n$ for a suitable choice of $\theta_n \in \mathbb{R}$ such that $\varphi_{m_n}(c_n) = |\varphi_{m_n}(a_n)|$ for every natural n.

We deal first with the case $n = 1$. Set $N_0 = \mathbb{N}$. Let us take a natural number j_1 such that $3\frac{21}{\sqrt{j_1}} < r\varepsilon_1$. Let $\delta_1 = \widetilde{\delta}(j_1, \varepsilon_1/2) > 0$ be given by Lemma 3.7. By James' distortion theorem there exist mutually disjoint finite subsets $G_k^{(1)} \subset N_0$, finite sequences $(\lambda_m^{(1)})_{m\in F_k^{(1)}} \subset \mathbb{C}$ such that

$$\sum_{m\in G_k^{(1)}} |\lambda_m^{(1)}| \leq \frac{1}{r}, \text{ for every } k \in \mathbb{N}, \tag{4.11}$$

and the functionals $\phi_k^{(1)} = \sum_{m\in G_k^{(1)}} \lambda_m^{(1)} \varphi_m$ satisfy

$$\sum_{k\in N_0} |\alpha_k| \geq \left\| \sum_{k\in N_0} \alpha_k \phi_k^{(1)} \right\| \geq (1-\delta_1) \sum_{k\in N_0} |\alpha_k| \quad \forall \alpha_k \in \mathbb{C}. \tag{4.12}$$

By Lemma 3.7 and the choice of δ_1, we find mutually orthogonal elements $a_1^{(1)},\dots,a_{j_1}^{(1)}$ of norm one in E and mutually orthogonal compact tripotents $u_1^{(1)},\dots,u_{j_1}^{(1)}$ in E^{**} satisfying $a_k^{(1)}\in E_2^{**}(u_k^{(1)})$,

$$\left\|\phi_k^{(1)}-\frac{\phi_k^{(1)}P_2(r(a_k^{(1)}))}{\|\phi_k^{(1)}P_2(r(a_k^{(1)}))\|}\right\|<\frac{\varepsilon_1}{2},\ \text{and}\ \phi_k^{(1)}(a_k^{(1)})>1-\varepsilon_1/2,$$

for every $k=1,\dots,j_1$. Keeping in mind that $u_1^{(1)},\dots,u_{j_1}^{(1)}$ are mutually orthogonal we deduce that

$$0\leq\sum_{k=1}^{j_1}\|u_k^{(1)}\|_{\varphi_m}^2=\left\|\sum_{k=1}^{j_1}u_k^{(1)}\right\|_{\varphi_m}^2\leq 1\quad\forall m\in N_0.$$

It follows that there exist $k_1\leq j_1$ in N_0 and an infinite subset $N_1\subset N_0$ such that $k_1\notin N_1$, $\max G_k^{(1)}<\min N_1$, for every $k=1,\dots,j_1$, and

$$\|u_{k_1}^{(1)}\|_{\varphi_m}^2\leq\frac{1}{j_1}\quad\forall m\in N_1. \tag{4.13}$$

Lemma 3.2 implies that

$$\left\|\varphi_m-\varphi_m P_0(u_{k_1}^{(1)})\right\|\leq\frac{21}{\sqrt{j_1}}\quad\forall m\in N_1. \tag{4.14}$$

Define $a_1=a_{k_1}^{(1)}$ and $u_1=u_{k_1}^{(1)}$. By (4.13) and the choice of j_1 we obtain (4.8) for $n=1$.

We claim that there exists $m_1\in G_{k_1}^{(1)}$ satisfying $|\varphi_{m_1}(a_1)|>r(1-\varepsilon_1)$. Otherwise, $|\varphi_m(a_1)|\leq r(1-\varepsilon_1)$, for every $m\in G_{k_1}^{(1)}$. Recalling that $\phi_{k_1}^{(1)}(a_1)>1-\frac{\varepsilon_1}{2}$, we have

$$1-\frac{\varepsilon_1}{2}<\phi_{k_1}^{(1)}(a_1)=\left|\sum_{m\in G_{k_1}^{(1)}}\lambda_m^{(1)}\varphi_m(a_1)\right|\leq r(1-\varepsilon_1)\sum_{m\in G_{k_1}^{(1)}}|\lambda_m^{(1)}|\leq 1-\varepsilon_1,$$

which is impossible. This defines $m_1\notin N_1$ as in (4.6) and (4.9) for $n=1$.

Now, we set $\widetilde{\varphi}_m^{(1)}:=\varphi_m P_0(u_1)$. By (4.14) we have $\left\|\varphi_m-\widetilde{\varphi}_m^{(1)}\right\|\leq\frac{21}{\sqrt{j_1}}<\varepsilon_1/3<1/3$, for every $m\in N_1$. The inequalities

$$1=\|\varphi_m\|\geq\|\widetilde{\varphi}_m^{(1)}\|\ \text{and}\ 0\leq 1-\|\widetilde{\varphi}_m^{(1)}\|\leq\|\varphi_m-\widetilde{\varphi}_m^{(1)}\|<\frac{21}{\sqrt{j_1}}$$

imply that $\|\widetilde{\varphi}_m^{(1)}\|\geq 1-\frac{21}{\sqrt{j_1}}>1-\frac{1}{3}>\frac{1}{2}$. Therefore,

$$\left\|\widetilde{\varphi}_m^{(1)}-\frac{\widetilde{\varphi}_m^{(1)}}{\|\widetilde{\varphi}_m^{(1)}\|}\right\|=\|\widetilde{\varphi}_m^{(1)}\|\left|1-\frac{1}{\|\widetilde{\varphi}_m^{(1)}\|}\right|\leq\frac{1}{\|\widetilde{\varphi}_m^{(1)}\|}-1$$

$$= \frac{1}{\|\widetilde{\varphi}_m^{(1)}\|}(1 - \|\widetilde{\varphi}_m^{(1)}\|) < 2\frac{21}{\sqrt{j_1}},$$

which shows that

$$\left\|\varphi_m - \frac{\varphi_m P_0(u_1)}{\|\varphi_m P_0(u_1)\|}\right\| = \left\|\varphi_m - \frac{\widetilde{\varphi}_m^{(1)}}{\|\widetilde{\varphi}_m^{(1)}\|}\right\|$$

$$\leq \left\|\varphi_m - \widetilde{\varphi}_m^{(1)}\right\| + \left\|\widetilde{\varphi}_m^{(1)} - \frac{\widetilde{\varphi}_m^{(1)}}{\|\widetilde{\varphi}_m^{(1)}\|}\right\| < 3\frac{21}{\sqrt{j_1}} < r\varepsilon_1.$$

By hypothesis, (φ_m) is a normalized sequence in E^* spanning ℓ_1 r-isomorphically, hence

$$\left\|\sum_{m\in N_1} \alpha_m \frac{\varphi_m P_0(u_1)}{\|\varphi_m P_0(u_1)\|}\right\| \geq \left\|\sum_{m\in N_1} \alpha_m \varphi_m\right\| - \left\|\sum_{m\in N_1} \alpha_m \left(\varphi_m - \frac{\widetilde{\varphi}_m^{(1)}}{\|\widetilde{\varphi}_m^{(1)}\|}\right)\right\|$$

$$\geq r \sum_{m\in N_1} |\alpha_m| - r\varepsilon_1 \sum_{m\in N_1} |\alpha_m| = r\,(1-\varepsilon_1) \sum_{m\in N_1} |\alpha_m| \quad \forall \alpha_m \in \mathbb{C}.$$

This proves (4.10) for $n = 1$, which concludes the first induction step.

Suppose now, by the induction hypothesis, that m_k, N_k, a_k, and u_k have been defined for $k \leq n$ according to (4.6) – (4.10). By [23, Theorem 3.9] the element $v_n = \sum_{k=1}^{n} u_k$ is a compact tripotent in E^{**}, therefore $F_n := E \cap E_0^{**}(v_n)$ is a weak*-dense subtriple of $E_0^{**}(v_n)$ whose second dual, F_n^{**}, identifies with $E_0^{**}(v_n)$.

To simplify notation, we write $\psi_m^{(n)} = \dfrac{\varphi_m P_0(v_n)}{\|\varphi_m P_0(v_n)\|}$ $(m \in N_n)$, and we regard $(\psi_m^{(n)})$ as a normalized sequence in F_n^*. By (4.10)

$$\sum_{m\in N_n} |\alpha_m| \geq \left\|\sum_{m\in N_n} \alpha_m \psi_m^{(n)}\right\| \geq r\left(1 - \sum_{i=1}^{n} \varepsilon_i\right) \sum_{m\in N_n} |\alpha_m| \quad \forall \alpha_m \in \mathbb{C}$$

that is, $(\psi_m^{(n)})$ is a normalized basis spanning ℓ_1 $(r\,(1 - \sum_{i=1}^{n} \varepsilon_i))$-isomorphically.

Let us take a natural number j_{n+1} such that $3\frac{21}{\sqrt{j_{n+1}}} < r\varepsilon_{n+1}$. Let $\delta_{n+1} = \widetilde{\delta}(j_{n+1}, \varepsilon_{n+1}/2) > 0$ given by Lemma 3.7. By James' distortion theorem there exist mutually disjoint finite subsets $G_k^{(n)} \subset N_n$ $(k \in \mathbb{N})$, finite sequences $(\lambda_m^{(n)})_{m\in G_k^{(n)}} \subset \mathbb{C}$ such that

$$\sum_{m\in G_k^{(n)}} |\lambda_m^{(n)}| \leq \frac{1}{r\left(1 - \sum_{i=1}^{n} \varepsilon_i\right)} \text{ for all } k \in \mathbb{N}, \tag{4.15}$$

and the functionals $\phi_k^{(n)} = \sum_{m\in G_k^{(n)}} \lambda_m^{(n)}\psi_m^{(n)}$ satisfy

$$\sum_{k\in N_n} |\alpha_k| \geq \left\| \sum_{k\in N_n} \alpha_k \phi_k^{(n)} \right\| \geq (1-\delta_{n+1}) \sum_{k\in N_n} |\alpha_k| \quad \forall \alpha_m \in \mathbb{C}. \tag{4.16}$$

By Lemma 3.7 and the choice of δ_{n+1}, we find mutually orthogonal elements $a_1^{(n)}, \ldots, a_{j_{n+1}}^{(n)}$ of norm one in $F_n = E \cap E_0^{**}(v_n)$ and mutually orthogonal compact tripotents $u_1^{(n)}, \ldots, u_{j_{n+1}}^{(n)}$ in $F_n^{**} \cong E_0^{**}(v_n)$ satisfying $a_k^{(n)} \in E_2^{**}(u_k^{(n)})$, and

$$\left\| \phi_k^{(n)} - \frac{\phi_k^{(n)} P_2(r(a_k^{(n)}))}{\|\phi_k^{(n)} P_2(r(a_k^{(n)}))\|} \right\| < \frac{\varepsilon_{n+1}}{2}, \text{ and } \phi_k^{(n)}(a_k^{(n)}) > 1 - \frac{\varepsilon_{n+1}}{2},$$

for every $k = 1, \ldots, j_{n+1}$. We remark that $v_n \perp u_k^{(n)}$, for every $k = 1, \ldots, j_{n+1}$ because $u_k^{(n)} \in E_0^{**}(v_n)$.

Keeping in mind that $u_1^{(n)}, \ldots, u_{j_{n+1}}^{(n)}$ are mutually orthogonal we deduce that

$$0 \leq \sum_{k=1}^{j_{n+1}} \|u_k^{(n)}\|_{\varphi_m}^2 = \left\| \sum_{k=1}^{k_{n+1}} u_k^{(n)} \right\|_{\varphi_m}^2 \leq 1 \quad \forall m \in N_n.$$

It follows that there exist $k_{n+1} \leq j_{n+1}$ in N_n, and an infinite subset $N_{n+1} \subset N_n$ such that $k_{n+1} \notin N_{n+1}$, $\max G_k^{(n)} < \min N_{n+1}$, for $k = 1, \ldots, j_{n+1}$, and

$$\|u_{k_{n+1}}^{(n)}\|_{\varphi_m}^2 \leq \frac{1}{j_{n+1}}, \text{ for all } m \in N_{n+1}. \tag{4.17}$$

Lemma 3.2 implies that

$$\left\| \varphi_m - \varphi_m P_0(u_{k_{n+1}}^{(n)}) \right\| \leq \frac{21}{\sqrt{j_{n+1}}} \quad \forall m \in N_{n+1}. \tag{4.18}$$

Define $a_{n+1} = a_{k_{n+1}}^{(n)}$ and $u_{n+1} = u_{k_{n+1}}^{(n)}$. By (4.17) and the choice of j_{n+1} we obtain (4.8) for $n+1$. Since $a_{n+1} \in F_n = E \cap E_0^{**}(v_n)$, and $a_1, \ldots, a_n \in E_2^{**}(v_n)$, it follows that $a_{n+1} \perp a_k$, for $k = 1, \ldots, n$. Therefore, $a_1, \ldots, a_{n+1}$ are mutually orthogonal elements in the closed unit ball of E.

We claim that there exists $m_{n+1} \in G_{k_{n+1}}^{(n)}$ satisfying

$$\frac{1}{\|\varphi_{m_{n+1}} P_0(v_n)\|} |\varphi_{m_{n+1}}(a_{n+1})| > r\left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right). \tag{4.19}$$

Otherwise, $\frac{1}{\|\varphi_m P_0(v_n)\|} |\varphi_m(a_{n+1})| \leq r\left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right)$, for all $m \in G_{k_{n+1}}^{(n)}$. Recalling that $\phi_{k_{n+1}}^{(n)}(a_{n+1}) > 1 - \frac{\varepsilon_{n+1}}{2}$, and $a_{n+1} \in F_n = E \cap E_0^{**}(v_n)$, we

have

$$\psi_m^{(n)}(a_{n+1}) = \frac{\varphi_m P_0(v_n)}{\|\varphi_m P_0(v_n)\|}(a_{n+1}) = \frac{1}{\|\varphi_m P_0(v_n)\|}\varphi_m(a_{n+1}),$$

which gives

$$1 - \frac{\varepsilon_{n+1}}{2} < \phi_{k_{n+1}}^{(n)}(a_{n+1}) = \left| \sum_{m \in G_{k_{n+1}}^{(n)}} \lambda_m^{(n)} \psi_m^{(n)}(a_{n+1}) \right|$$

$$\leq r\left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right) \sum_{m \in G_{k_{n+1}}^{(n)}} |\lambda_m^{(n)}| \leq r\left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right) \frac{1}{r\left(1 - \sum_{i=1}^{n} \varepsilon_i\right)}$$

$$= 1 - \frac{\varepsilon_{n+1}}{\left(1 - \sum_{i=1}^{n} \varepsilon_i\right)} < 1 - \frac{\varepsilon_{n+1}}{2}$$

which is impossible. This proves the claim in (4.19).

By (4.8) for $i \leq n$, we have

$$\|v_n\|_{\varphi_m} = \left\| \sum_{i=1}^{n} u_i \right\|_{\varphi_m} \leq \sum_{i=1}^{n} \|u_i\|_{\varphi_m} < r \sum_{i=1}^{n} \frac{\varepsilon_i}{63} \quad \forall m \in N_{n+1}.$$

Now, since $1 = \|\varphi_{m_{n+1}}\| \geq \|\varphi_{m_{n+1}} P_0(v_n)\|$ we deduce via Lemma 3.2 that

$$0 \leq 1 - \|\varphi_{m_{n+1}} P_0(v_n)\| \leq \|\varphi_{m_{n+1}} - \varphi_{m_{n+1}} P_0(v_n)\| < 21r \sum_{i=1}^{n} \frac{\varepsilon_i}{63}$$

which implies that

$$1 - r \sum_{i=1}^{n} \frac{\varepsilon_i}{3} < \|\varphi_{m_{n+1}} P_0(v_n)\|$$

hence

$$|\varphi_{m_{n+1}}(a_{n+1})| > \left(1 - r \sum_{i=1}^{n} \frac{\varepsilon_i}{3}\right) r \left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right)$$

by (4.19). We have thus defined $m_{n+1} \notin N_{n+1}$ such that $m_n < m_{n+1}$ and such that (4.9) holds for $n+1$.

Finally we show (4.10) for $n+1$. Since (4.8) holds for $i \leq n+1$ we get

$$\|v_{n+1}\|_{\varphi_m} = \left\| \sum_{i=1}^{n+1} u_i \right\|_{\varphi_m} \leq \sum_{i=1}^{n+1} \|u_i\|_{\varphi_m} < r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{63} \quad \forall m \in N_{n+1}. \tag{4.20}$$

To simplify notation, let us denote $\widetilde{\varphi}_m^{(n+1)} := \varphi_m P_0(v_{n+1})$, $m \in N_{n+1}$. Lemma 3.2 and (4.20) imply that

$$\left\|\varphi_m - \widetilde{\varphi}_m^{(n+1)}\right\| < 21r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{63} = r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{3} < r\frac{\varepsilon}{6} < \frac{1}{6} \quad \forall m \in N_{n+1}.$$

The inequalities

$$1 = \|\varphi_m\| \geq \|\widetilde{\varphi}_m^{(n+1)}\| \text{ and } 0 \leq 1 - \|\widetilde{\varphi}_m^{(n+1)}\| \leq \|\varphi_m - \widetilde{\varphi}_m^{(n+1)}\| < r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{3},$$

imply that $\|\widetilde{\varphi}_m^{(n+1)}\| \geq 1 - r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{3} > 1 - \frac{1}{6} > \frac{1}{2}$. Therefore,

$$\left\|\widetilde{\varphi}_m^{(n+1)} - \frac{\widetilde{\varphi}_m^{(n+1)}}{\|\widetilde{\varphi}_m^{(n+1)}\|}\right\| = \|\widetilde{\varphi}_m^{(n+1)}\| \left|1 - \frac{1}{\|\widetilde{\varphi}_m^{(n+1)}\|}\right| \leq \frac{1}{\|\widetilde{\varphi}_m^{(n+1)}\|} - 1$$

$$= \frac{1}{\|\widetilde{\varphi}_m^{(n+1)}\|}(1 - \|\widetilde{\varphi}_m^{(n+1)}\|) < 2r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{3}$$

which shows that

$$\left\|\varphi_m - \frac{\varphi_m P_0(v_{n+1})}{\|\varphi_m P_0(v_{n+1})\|}\right\| = \left\|\varphi_m - \frac{\widetilde{\varphi}_m^{(n+1)}}{\|\widetilde{\varphi}_m^{(n+1)}\|}\right\|$$

$$\leq \left\|\varphi_m - \widetilde{\varphi}_m^{(n+1)}\right\| + \left\|\widetilde{\varphi}_m^{(n+1)} - \frac{\widetilde{\varphi}_m^{(n+1)}}{\|\widetilde{\varphi}_m^{(n+1)}\|}\right\| < 3r \sum_{i=1}^{n+1} \frac{\varepsilon_i}{3} = r \sum_{i=1}^{n+1} \varepsilon_i,$$

for all $m \in N_{n+1}$. By hypothesis, (φ_m) is a normalized sequence in E^* spanning ℓ_1 r-isomorphically, hence

$$\left\|\sum_{m \in N_{n+1}} \alpha_m \frac{\varphi_m P_0(v_{n+1})}{\|\varphi_m P_0(v_{n+1})\|}\right\| \geq$$

$$\left\|\sum_{m \in N_{n+1}} \alpha_m \varphi_m\right\| - \left\|\sum_{m \in N_{n+1}} \alpha_m \left(\varphi_m - \frac{\varphi_m P_0(v_{n+1})}{\|\varphi_m P_0(v_{n+1})\|}\right)\right\|$$

$$\geq r \sum_{m \in N_{n+1}} |\alpha_m| - r\left(\sum_{i=1}^{n+1} \varepsilon_i\right) \sum_{m \in N_{n+1}} |\alpha_m| = r\left(1 - \sum_{i=1}^{n+1} \varepsilon_i\right) \sum_{m \in N_{n+1}} |\alpha_m|$$

for all $\alpha_m \in \mathbb{C}$. This proves (4.10) for $n+1$ and shows (4.5).

By an extraction lemma of Simons [38] we may (after passing to appropriate subsequences of (φ_{m_n}) and (c_n) which we still denote by (φ_{m_n}) and (c_n)) suppose that $\sum_{k \neq n} |\varphi_{m_n}(c_k)| < \varepsilon'$ for all n where $\varepsilon' > 0$ is such that $r(1-\varepsilon) - \varepsilon' > r(1-2\varepsilon)$. That the subtriple $\mathcal{C}$ generated by the c_n is isometric to a commutative C*-algebra can be seen as in [22, Th. 2.3b) $\Rightarrow$ b')].

Fix $(\alpha_n) \in \ell_1$, choose $\theta_n \in \mathbb{C}$ such that $\theta_n \alpha_n = |\alpha_n|$ and set $c = \sum_{k\geq 1} \theta_k c_k$. Then $\|c\| \leq 1$ and

$$\begin{aligned}
\left\| \sum_{n\geq 1} \alpha_n \varphi_{m_n|\mathcal{C}} \right\| &\geq \left| \sum_{n\geq 1} \alpha_n \varphi_{m_n}(c) \right| \\
&= \left| \sum_n |\alpha_n| \varphi_{m_n}(c_n) + \sum_n \alpha_n \left(\sum_{k\neq n} \varphi_{m_n}(\theta_k c_k) \right) \right| \\
&\geq r(1-\varepsilon) \sum_n |\alpha_n| - \sum_n |\alpha_n| \left(\sum_{k\neq n} |\varphi_{m_n}(c_k)| \right) \\
&\geq (r(1-\varepsilon) - \varepsilon') \sum_n |\alpha_n| \geq r(1-2\varepsilon) \sum_n |\alpha_n|.
\end{aligned}$$

Up to an adjustment of ε this ends the proof. □

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