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# SUBEXPONENTIALLY INCREASING SUM OF PARTIAL QUOTIENTS IN CONTINUED FRACTION EXPANSIONS

LINGMIN LIAO AND MICHAŁ RAMS

ABSTRACT. We investigate from multifractal analysis point of view the increasing rate of the sum of partial quotients  $S_n(x) = \sum_{j=1}^n a_j(x)$ , where  $x = [a_1(x), a_2(x), \dots]$  is the continued fraction expansion of an irrational  $x \in (0, 1)$ . Precisely, for an increasing function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ , one is interested in the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0, 1) : \lim_{n \rightarrow \infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

Several cases are solved by Iommi and Jordan, Wu and Xu, and Xu. We attack the remaining subexponential case  $\exp(n^\gamma)$ ,  $\gamma \in [1/2, 1)$ . We show that when  $\gamma \in [1/2, 1)$ ,  $E_\varphi$  has Hausdorff dimension  $1/2$ . Thus surprisingly the dimension has a jump from 1 to  $1/2$  at the increasing rate  $\exp(n^{1/2})$ . In a similar way, the distribution of the largest partial quotients is also studied.

## 1. INTRODUCTION

Each irrational number  $x \in [0, 1)$  admits a unique infinite continued fraction expansion of the form

$$(1.1) \quad x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \dots}}},$$

where the integers  $a_n(x)$  are called the partial quotients of  $x$ . Usually, (1.1) is written as  $x = [a_1, a_2, \dots]$  for simplicity. The  $n$ -th finite truncation of (1.1):  $p_n(x)/q_n(x) = [a_1, \dots, a_n]$  is called the  $n$ -th convergent of  $x$ . The continued fraction expansions can be induced by the Gauss transformation  $T : [0, 1) \rightarrow [0, 1)$  defined by

$$T(0) := 0, \quad T(x) = \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1).$$

It is well known that  $a_1(x) = \lfloor x^{-1} \rfloor$  ( $\lfloor \cdot \rfloor$  stands for the integer part) and  $a_n(x) = a_1(T^{n-1}(x))$  for  $n \geq 2$ .

For any  $n \geq 1$ , write  $S_n(x) = \sum_{j=1}^n a_j(x)$  the sum of the  $n$  first partial quotients. It is proved by Khintchine [5] in 1935 that  $S_n(x)/(n \log n)$  converges in measure (Lebesgue measure) to the constant  $1/\log 2$ . In 1988, Philipp [7] showed that there is no reasonable normalizing sequence  $\varphi(n)$

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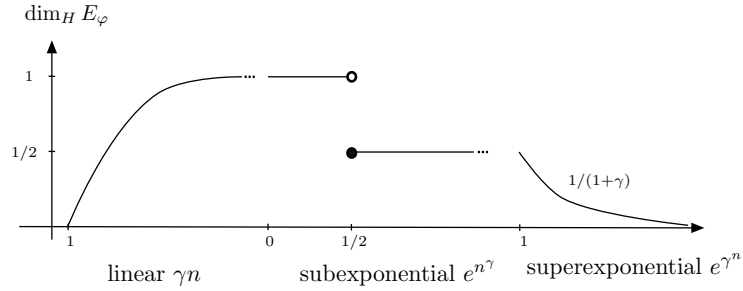


FIGURE 1.  $\dim_H E_\varphi$  for  $\varphi$  with different increasing rate.

such that a strong law of large numbers is satisfied, i.e.,  $S_n(x)/\varphi(n)$  will never converge to a positive constant almost surely.

From the point of view of multifractal analysis, one considers the Hausdorff dimension of the sets

$$E_\varphi = \left\{ x \in (0, 1) : \lim_{n \rightarrow \infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

where  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  is an increasing function.

The case  $\varphi(n) = \gamma n$  with  $\gamma \in [1, \infty)$  was studied by Iommi and Jordan [3]. It is proved that with respect to  $\gamma$ , the Hausdorff dimension of  $E_\varphi$  is analytic, increasing from 0 to 1, and tends to 1 when  $\gamma$  goes to infinity. In [9], Wu and Xu proved that if  $\varphi(n) = n^\gamma$  with  $\gamma \in (1, \infty)$  or  $\varphi(n) = \exp\{n^\gamma\}$  with  $\gamma \in (0, 1/2)$ , the Hausdorff dimension of  $E_\varphi$  is always 1. It was shown by Xu [10], that if  $\varphi(n) = \exp\{n\}$  then the Hausdorff dimension of  $E_\varphi$  is  $1/2$  and if  $\varphi(n) = \exp\{\gamma^n\}$  with  $\gamma > 1$  then the Hausdorff dimension is  $1/(\gamma+1)$ . The same proofs of [10] also imply that for  $\varphi(n) = \exp\{n^\gamma\}$  with  $\gamma \in (1, \infty)$  the Hausdorff dimension of  $E_\varphi$  stays at  $1/2$ . So, only the subexponentially increasing case:  $\varphi(n) = \exp\{n^\gamma\}$ ,  $\gamma \in [1/2, 1)$  was left unknown. In this paper, we fill this gap.

**Theorem 1.1.** *Let  $\varphi(n) = \exp\{n^\gamma\}$  with  $\gamma \in [1/2, 1)$ . Then the Hausdorff dimension of  $E_\varphi$  is one-half.*

We also show that for increasing rates slightly slower than  $e^{\sqrt{n}}$ , for example  $\varphi(n) = e^{\sqrt{n}(\log n)^{-1}}$ , the Hausdorff dimension will jump.

**Theorem 1.2.** *Let  $\varphi(n) = e^{\sqrt{n} \cdot \psi(n)}$  be an increasing function with  $\psi$  being a  $\mathcal{C}^1$  positive function on  $\mathbb{R}_+$  satisfying*

$$\lim_{n \rightarrow \infty} \psi(n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{n\psi'(n)}{\psi(n)} = 0.$$

*Then the Hausdorff dimension of  $E_\varphi$  is equal to one.*

Theorems 1.1 and 1.2 show that, surprisingly, there is a jump of the Hausdorff dimensions from 1 to  $1/2$ .

By the same method, we also prove some similar results on the distribution of the largest partial quotients in continued fraction expansions. For  $x \in [0, 1) \setminus \mathbb{Q}$ , define

$$T_n(x) := \max\{a_k(x) : 1 \leq k \leq n\}.$$

One is interested in the following lower limit:

$$T(x) := \liminf_{n \rightarrow \infty} \frac{T_n(x) \log \log n}{n}.$$

It was conjectured by Erdős that almost surely  $T(x) = 1$ . However, it is proved by Philipp [6] that for almost all  $x$ , one has  $T(x) = 1/\log 2$ . Recently, Wu and Xu [8] showed that for all  $\alpha \geq 0$  the level set

$$\left\{ x \in [0, 1) \setminus \mathbb{Q} : \lim_{n \rightarrow \infty} \frac{T_n(x) \log \log n}{n} = \alpha \right\}$$

has Hausdorff dimension 1. They also proved that if the denominator  $n$  is replaced by a polynomial the same result holds. In this paper, we show the following theorem.

**Theorem 1.3.** *For all  $\alpha > 0$ ,*

$$\left\{ x \in [0, 1) \setminus \mathbb{Q} : \lim_{n \rightarrow \infty} T_n(x)/e^{n^\gamma} = \alpha \right\}$$

*is of Hausdorff dimension 1 if  $\gamma \in (0, 1/2)$ , and is of Hausdorff dimension  $1/2$  if  $\gamma \in (1/2, \infty)$ .*

We do not know what happens in the case  $a = 1/2$ .

## 2. PRELIMINARY

For any  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , call

$$I_n(a_1, \dots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}$$

a *rank- $n$  basic interval*. Denote by  $I_n(x)$  the rank- $n$  basic interval containing  $x$ . Write  $|I|$  the length of an interval  $I$ . The length of the basic interval  $I_n(a_1, a_2, \dots, a_n)$  satisfies

$$(2.1) \quad \prod_{k=1}^n (a_k + 1)^{-2} \leq |I_n(a_1, \dots, a_n)| \leq \prod_{k=1}^n a_k^{-2}.$$

Let  $A(m, n) := \{(i_1, \dots, i_n) \in \{1, \dots, m\}^n : \sum_{k=1}^n i_k = m\}$ . Let  $\zeta(\cdot)$  be the Riemann zeta function.

**Lemma 2.1.** *For any  $s > 1/2$ , for all  $n \geq 1$  and for all  $m \geq n$ , we have*

$$\sum_{(i_1, \dots, i_n) \in A(m, n)} \prod_{k=1}^n i_k^{-2s} \leq \left( \frac{9}{2} (1 + \zeta(2s)) \right)^n m^{-2s}.$$

*Proof.* The proof goes by induction. First consider the case  $n = 2$ . We will estimate the sum  $\sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s}$ . We have

$$\begin{aligned} & \sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} \\ &= 2 \sum_{i=1}^{u-1} i^{-2s}(m-i)^{-2s} + \sum_{i=u}^{m-u} i^{-2s}(m-i)^{-2s} \\ &\leq 2 \left( \sum_{i=1}^{u-1} i^{-2s} \right) (m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s} \\ &\leq 2\zeta(2s)(m-u)^{-2s} + (m-2u+1)u^{-2s}(m-u)^{-2s}. \end{aligned}$$

Take  $u = \lfloor m/3 \rfloor$ . Then for  $m$  large enough, one has

$$(m-2u+1)u^{-2s} \leq \left( m - 2\left(\frac{m}{3} - 1\right) + 1 \right) \left(\frac{m}{3}\right)^{-2s} \leq 2.$$

Hence, the above sum is bounded from above by

$$(2 + 2\zeta(2s)) \cdot \left(\frac{2m}{3}\right)^{-2s} \leq \frac{9}{2}(1 + \zeta(2s)) \cdot m^{-2s}.$$

Suppose that we have the estimation for  $n$ . Then for  $n+1$ , we have

$$\begin{aligned} & \sum_{(i_1, \dots, i_{n+1}) \in \{1, \dots, m\}^{n+1}, \sum i_k = m} \prod_{k=1}^{n+1} i_k^{-2s} \\ &= \sum_{i=1}^{m-1} i^{-2s} \sum_{(i_1, \dots, i_n) \in \{1, \dots, m\}^n, \sum i_k = m-i} \prod_{k=1}^n i_k^{-2s} \\ &\leq \sum_{i=1}^{m-1} i^{-2s} \left( \frac{9}{2}(1 + \zeta(2s)) \right)^n (m-i)^{-2s} \\ &= \left( \frac{9}{2}(1 + \zeta(2s)) \right)^n \cdot \sum_{i=1}^{m-1} i^{-2s}(m-i)^{-2s} \\ &\leq \left( \frac{9}{2}(1 + \zeta(2s)) \right)^n \cdot \left( \frac{9}{2}(1 + \zeta(2s)) \right) m^{-2s} \\ &= \left( \frac{9}{2}(1 + \zeta(2s)) \right)^{n+1} m^{-2s}. \end{aligned}$$

□

Let

$$A(\gamma, c_1, c_2, N) := \left\{ x \in (0, 1) : c_1 < \frac{a_n(x)}{e^{n\gamma}} < c_2, \forall n \geq N \right\}.$$

Denote by  $N_0$  the smallest integer  $n$  such that  $(c_2 - c_1) \cdot e^{n\gamma} > 1$ . Then the set  $A(\gamma, c_1, c_2, N)$  is non-empty when  $N \geq N_0$ .

**Lemma 2.2.** *For any  $\gamma > 0$ , any  $N \geq N_0$  and any  $0 < c_1 < c_2$ ,*

$$\dim_H A(\gamma, c_1, c_2, N) = \frac{1}{2}.$$

*Proof.* This lemma is only a simplest special case of [2, Lemma 3.2], but we will sketch the proof (based on [4]), needed for the next lemma. Without loss of generality, we suppose  $N_0 = 1$  and let  $N = 1$  (the proof for other  $N$  is almost identical).

Let  $a_1, a_2, \dots, a_n$  satisfy  $c_1 < a_j e^{-j^\gamma} < c_2$  for all  $j$ . Those are exactly the possible sequences for which the basic interval  $I_n(a_1, \dots, a_n)$  has nonempty intersection with  $A(\gamma, c_1, c_2, 1)$ .

There are approximately

$$(2.2) \quad \prod_{j=1}^n (c_2 - c_1) e^{j^\gamma} \approx e^{\sum_1^n j^\gamma}$$

of such basic intervals, each of diameter

$$(2.3) \quad |I_n(a_1, \dots, a_n)| \approx e^{-2 \sum_1^n j^\gamma},$$

(both estimations are up to a factor exponential in  $n$ ). Hence, using the intervals  $\{I_n(a_1, \dots, a_n)\}$  as a cover, we obtain

$$\dim_H A(\gamma, c_1, c_2, 1) \leq \frac{1}{2}.$$

To get the lower bound, we consider a probabilistic measure  $\mu$  uniformly distributed on  $A(\gamma, c_1, c_2, 1)$ , in the following sense: given  $a_1, \dots, a_{n-1}$ , probability of  $a_n$  taking any particular value between  $c_1 e^{n^\gamma}$  and  $c_2 e^{n^\gamma}$  is the same.

The basic intervals  $I_n(a_1, \dots, a_n)$  have length  $e^{-2 \sum_1^n j^\gamma}$  and measure  $e^{-\sum_1^n j^\gamma}$ , each (up to a factor  $c^n$ ). They are distributed in clusters: all  $I_n(a_1, \dots, a_n)$  contained in single  $I_n(a_1, \dots, a_{n-1})$  form an interval of length  $e^{-2 \sum_1^{n-1} j^\gamma - n^\gamma}$  (up to a factor  $c^n$ , with  $c$  being a constant), then there is a gap, then there is another cluster. Hence, for any  $r \in (e^{-2 \sum_1^n j^\gamma}, e^{-2 \sum_1^{n-1} j^\gamma})$  and any  $x \in A(\gamma, c_1, c_2, 1)$  it is easy to estimate the measure of  $B(x, r)$ :

$$\mu(B(x, r)) \approx \begin{cases} r \cdot e^{-\sum_1^n j^\gamma} & \text{if } r < e^{-2 \sum_1^{n-1} j^\gamma - n^\gamma} \\ e^{-\sum_1^{n-1} j^\gamma} & \text{if } r > e^{-2 \sum_1^{n-1} j^\gamma - n^\gamma} \end{cases}$$

(up to a factor  $c^n$ ). The minimum of  $\log \mu(B(x, r)) / \log r$  is thus achieved for  $r = e^{-2 \sum_1^{n-1} j^\gamma - n^\gamma}$ , and this minimum equals

$$\frac{-\sum_1^{n-1} j^\gamma}{-2 \sum_1^{n-1} j^\gamma - n^\gamma} \approx \frac{-n^{\gamma+1}/(\gamma+1)}{-2n^{\gamma+1}/(\gamma+1) - n^\gamma} = \frac{1}{2} - O(1/n).$$

Hence, the lower local dimension of  $\mu$  equals  $1/2$  at each point of  $A(\gamma, c_1, c_2, 1)$ , which implies

$$\dim_H A(\gamma, c_1, c_2, 1) \geq \frac{1}{2}$$

by the Frostman Lemma (see [1]). □

Let now  $c_1$  and  $c_2$  not be constant but depend on  $n$ :

$$B(\gamma, c_1, c_2, N) = \left\{ x \in (0, 1) : c_1(n) < \frac{a_n(x)}{e^{n^\gamma}} < c_2(n) \ \forall n \geq N \right\}.$$

A slight modification of the proof of Lemma 2.2 gives the following.

**Lemma 2.3.** Fix  $\gamma > 0$ . Assume  $0 < c_1(n) < c_2(n)$  for all  $n$ . Assume also that

$$\lim_{n \rightarrow \infty} \frac{\log(c_2(n) - c_1(n))}{n^\gamma} = 0$$

and

$$\liminf_{n \rightarrow \infty} \frac{\log c_1(n)}{\log n} > -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log c_2(n)}{\log n} < +\infty.$$

Then there exists an integer  $N_1$  such that  $(c_2(n) - c_1(n)) \cdot e^{n^\gamma} > 1$  for all  $n \geq N_1$ , and for all  $N \geq N_1$ ,

$$\dim_H B(\gamma, c_1, c_2, N) = 1/2.$$

*Proof.* We need only to replace the constants  $c_1$  and  $c_2$  by  $c_1(n)$  and  $c_2(n)$  in the proof of Lemma 2.2. Notice that by the assumptions of Lemma 2.3, the formulas (2.2) and (2.3) still hold, up to a factor  $e^{cn \log n}$  for some bounded  $c$ , much smaller than the main term  $e^{\sum_1^n j^\gamma}$  which is of order  $e^{n^{1+\gamma}}$ . The rest of the proofs are the same.  $\square$

### 3. PROOFS

*Proof of Theorem 1.1.* Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  be defined by  $\varphi(n) = \exp\{n^\gamma\}$  with  $\gamma > 0$ . For this case, we will denote  $E_\varphi$  by  $E_\gamma$ .

Let us start from some easy observations, giving (among other things) a simple proof of  $\dim_H E_\gamma = 1/2$  for  $\gamma \geq 1$ .

Consider first  $\gamma \geq 1/2$ . If  $x \in E_\gamma$  then for any  $\varepsilon > 0$  for  $n$  large enough,

$$(1 - \varepsilon)e^{n^\gamma} \leq S_n(x) \leq (1 + \varepsilon)e^{n^\gamma}$$

and

$$(1 - \varepsilon)e^{(n+1)^\gamma} \leq S_{n+1}(x) \leq (1 + \varepsilon)e^{(n+1)^\gamma}$$

Hence,

$$(1 - \varepsilon)e^{(n+1)^\gamma} - (1 + \varepsilon)e^{n^\gamma} \leq a_{n+1}(x) \leq (1 + \varepsilon)e^{(n+1)^\gamma} - (1 - \varepsilon)e^{n^\gamma}.$$

For  $\gamma \geq 1$  this implies

$$E_\gamma \subset \bigcup_N A(\gamma, c_1, c_2, N)$$

for some constants  $c_1, c_2$ . By Lemma 2.2,

$$\dim_H E_\gamma \leq \frac{1}{2}, \quad \forall \gamma \geq 1.$$

Consider now any  $\gamma > 0$ . Set

$$c_1(n) = (e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \quad \text{and} \quad c_2(n) = \frac{n+1}{n}c_1(n).$$

As

$$(e^{n^\gamma} - e^{(n-1)^\gamma})e^{-n^\gamma} \approx \gamma n^{\gamma-1},$$

the assumptions of Lemma 2.3 are satisfied. As  $B(\gamma, c_1, c_2, N) \subset E_\gamma$ , by Lemma 2.3,

$$\dim_H E_\gamma \geq \frac{1}{2}, \quad \forall \gamma > 0.$$

Thus we have obtained  $\dim_H E_\gamma = 1/2$  for  $E_\gamma \geq 1$  and  $\dim_H E_\gamma \geq 1/2$  for  $\gamma > 0$ . What is left to prove is that for  $\gamma \in [1/2, 1)$  we have  $\dim_H E_\gamma \leq 1/2$ .

Let us first assume that  $\gamma > 1/2$ . We will once again use the fact that for any  $\varepsilon > 0$ , if  $x \in E_\gamma$ , then for  $n$  large enough,

$$(1 - \varepsilon)e^{n^\gamma} \leq S_n(x) \leq (1 + \varepsilon)e^{n^\gamma}.$$

Take a subsequence  $n_0 = 1$ , and  $n_k = k^{1/\gamma}$  ( $k \geq 1$ ). Then there exists an integer  $N \geq 1$  such that for all  $k \geq N$ ,

$$(1 - \varepsilon)e^k \leq S_{n_k}(x) \leq (1 + \varepsilon)e^k,$$

and

$$(1 - \varepsilon)e^k - (1 + \varepsilon)e^{k-1} \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq (1 + \varepsilon)e^k - (1 - \varepsilon)e^{k-1}.$$

Thus

$$E_\gamma \subset \bigcup_N A(\gamma, N),$$

with  $A(\gamma, N)$  being the union of the intervals  $\{I_{n_k}(a_1, a_2, \dots, a_{n_k})\}_{k \geq N}$  such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \leq \ell \leq k,$$

where  $D_\ell := [(1 - \varepsilon)e^\ell - (1 + \varepsilon)e^{\ell-1}, (1 + \varepsilon)e^\ell - (1 - \varepsilon)e^{\ell-1}]$ .

Now, we are going to estimate the upper bound of the Hausdorff dimension of  $A(\gamma, 1)$ . For  $A(\gamma, N)$  with  $N \geq 2$ , we will have the same bound and the proofs are almost the same.

For any  $s > 1/2$  we can apply Lemma 2.1 together with the formula

$$|I_{n_k}|^s \leq \prod_{\ell=1}^k (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s}$$

to obtain

$$\begin{aligned} \sum_{I_{n_k} \in \mathcal{A}} |I_{n_k}|^s &\leq \prod_{\ell=1}^k \sum_{m \in D_\ell} \left( \frac{9}{2} (1 + \zeta(2s)) \right)^{n_\ell - n_{\ell-1}} m^{-2s} \\ &\leq \prod_{\ell=1}^k 2\varepsilon \left(1 + \frac{1}{e}\right) e^\ell \cdot \left( \frac{9}{2} (1 + \zeta(2s)) \right)^{\ell^{1/\gamma} - (\ell-1)^{1/\gamma}} \cdot \left( \frac{e}{e-1-\varepsilon e - \varepsilon} \right)^{2s} e^{-2s\ell}. \end{aligned}$$

We have  $\ell^{1/\gamma} - (\ell-1)^{1/\gamma} \approx \ell^{1/\gamma-1}$ . As  $\gamma > 1/2$ , we have  $1/\gamma - 1 < 1$ , and the main term in the above estimate is  $e^{(1-2s)\ell}$ . Thus for any  $s > 1/2$ , the product is uniformly bounded and the Hausdorff dimension of  $A(\gamma, 1)$  is not greater than  $1/2$ .

If  $\gamma = 1/2$ , we take  $n_k = k^2/L$  with  $L$  being a constant and we repeat the same argument. Then the same estimation will lead to

$$\sum_{I_{n_k} \in \mathcal{A}} |I_{n_k}|^s \leq \prod_{\ell=1}^k C \cdot e^{\ell/\sqrt{L}} \left( \frac{9}{2} (1 + \zeta(2s)) \right)^{(\ell^2 - (\ell-1)^2)/L} e^{-2s\ell/\sqrt{L}}.$$



The main term of the right side of the above inequality should be

$$\left(\frac{9}{2}(1 + \zeta(2s))\right)^{2\ell/L} \cdot e^{(1-2s)\ell/\sqrt{L}}.$$

We solve the equation

$$\left(\frac{9}{2}(1 + \zeta(2s))\right)^{2/L} \cdot e^{(1-2s)/\sqrt{L}} = 1,$$

which is equivalent to

$$(3.1) \quad \left(\frac{9}{2}(1 + \zeta(2s))\right) = e^{\frac{2s-1}{2}\sqrt{L}}.$$

Observe that the two curves (of the variable  $s$ ) of the two sides of (3.1) always have a unique intersection for some  $s_L \in [1/2, 1]$ , when  $L$  is large enough. These  $s_L$  are all upper bounds for the Hausdorff dimension of  $A(\gamma, 1)$ . Notice that the intersecting point  $s_L \rightarrow 1/2$  as  $L \rightarrow \infty$  since the zeta function  $\zeta$  has a pole at 1. Thus the dimension of  $A(\gamma, 1)$  is not greater than  $1/2$ .

So, in both cases, we have obtained  $\dim_H E_\gamma \leq 1/2$ .  $\square$

*Sketch proof of Theorem 1.2.* The proof goes like Section 4 of [9] with the following changes. We choose  $\varepsilon_k = \psi(k)$ . Then by the hypothesis on the function  $\psi$ , we have

$$\sum_{k=1}^{r(n)} \varepsilon_k \approx r(n)\psi(r(n)),$$

and we obtain the key formula (10) in [9] in the form

$$r(n)\psi(r(n)) \ll \sqrt{n}\psi(n).$$

The other key point, the formula (15) in [9] follows by the estimation

$$\log(a_{n_1} a_{n_2} \cdots a_{n_{r(n)}}) \ll r(n)\sqrt{n}\psi(n) + r(n) \ll \frac{n\psi^2(n)}{\psi(r(n))} + r(n) \ll n.$$

$\square$

*Proof of Theorem 1.3.* For the case  $\gamma < 1/2$ , the set constructed in Section 4 of [9] (as a subset of the set of points for which  $S_n(x) \approx e^{n^\gamma}$ ) satisfies also  $T_n(x) \approx e^{n^\gamma}$  and has Hausdorff dimension one. We proceed to the case  $\gamma > 1/2$ .

The lower bound is a corollary of Lemma 2.3. Take  $c_1(n) = \alpha(1 - \frac{1}{n})$  and  $c_2(n) = \alpha$ . Let  $N_1$  be the smallest integer  $n$  such that  $\frac{\alpha}{n}e^{n^\gamma} > 1$ . Then the conditions of Lemma 2.3 are satisfied, and for all points  $x$  such that  $c_1(n)e^{n^\gamma} < a_n(x) < c_2(n)e^{n^\gamma}$ , we have

$$T_n(x)/e^{n^\gamma} \geq c_1(n) = \alpha \left(1 - \frac{1}{n}\right),$$

and

$$T_n(x)/e^{n^\gamma} = a_k/e^{n^\gamma} \leq \alpha e^{k^\gamma}/e^{n^\gamma} \leq \alpha$$

(where  $k \leq n$  is the position at which the sequence  $a_1, \dots, a_n$  achieves a maximum). Thus we have

$$\lim_{n \rightarrow \infty} T_n(x)/e^{n\gamma} = \alpha$$

for all  $x \in B(\gamma, c_1, c_2, N_1)$ . Hence, the lower bound follows directly from Lemma 2.3.

The upper bound is a modification of that of Theorem 1.1. Denote the set in question by  $E(\gamma, \alpha)$ . We consider the case  $\alpha = 1$  only, since for other  $\alpha > 0$ , the proofs are similar.

Notice that for any  $\varepsilon > 0$ , if  $x \in E(\gamma, 1)$ , then for  $n$  large enough,

$$(1 - \varepsilon)e^{n\gamma} \leq S_n(x) \leq n(1 + \varepsilon)e^{n\gamma}.$$

Take a subsequence  $n_k = k^{1/\gamma}(\log k)^{1/\gamma^2}$ . Then

$$(1 - \varepsilon)e^{k(\log k)^{1/\gamma}} \leq S_{n_k}(x) \leq k^{1/\gamma}(\log k)^{1/\gamma^2}(1 + \varepsilon)e^{k(\log k)^{1/\gamma}},$$

and

$$u_k \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq v_k,$$

with

$$u_k := (1 - \varepsilon)e^{k(\log k)^{1/\gamma}} - (k-1)^{1/\gamma}(\log(k-1))^{1/\gamma^2}(1 + \varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}},$$

and

$$v_k := k^{1/\gamma}(\log k)^{1/\gamma^2}(1 + \varepsilon)e^{k(\log k)^{1/\gamma}} - (1 - \varepsilon)e^{(k-1)(\log(k-1))^{1/\gamma}}.$$

We remark that

$$u_k > \frac{1}{2}e^{k(\log k)^{1/\gamma}}, \quad v_k < \frac{3}{2}k^{1/\gamma}(\log k)^{1/\gamma^2}e^{k(\log k)^{1/\gamma}}$$

when  $k$  is large enough.

Observe that

$$E(\gamma, 1) \subset \bigcup_N B(\gamma, N),$$

with  $B(\gamma, N)$  being the union of the intervals  $\{I_{n_k}(a_1, a_2, \dots, a_{n_k})\}_{k \geq N}$  such that

$$\sum_{j=n_{\ell-1}+1}^{n_\ell} a_j = m \quad \text{with} \quad m \in D_\ell, \quad N \leq \ell \leq k,$$

where  $D_\ell$  is the set of integers in the interval  $[u_\ell, v_\ell]$ .

As in the proof of Theorem 1.1, we need only study the set  $B(\gamma, 1)$ . We have for any  $s > 1/2$ , since

$$|I_{n_k}|^s \leq \prod_{\ell=1}^k (a_{n_{\ell-1}+1} a_{n_{\ell-1}+2} \cdots a_{n_\ell})^{-2s},$$

by Lemma 2.1,

$$\begin{aligned} \sum_{I_{n_k} \in \mathcal{A}} |I_{n_k}|^s &\leq \prod_{\ell=1}^k \sum_{m \in D_\ell} \left( \frac{9}{2}(1 + \zeta(2s)) \right)^{n_\ell - n_{\ell-1}} m^{-2s} \\ &\leq \prod_{\ell=1}^k \frac{3}{2} \cdot \ell^{1/\gamma} (\log \ell)^{1/\gamma^2} e^{\ell(\log \ell)^{1/\gamma}} \left( \frac{9}{2}(1 + \zeta(2s)) \right)^{n_\ell - n_{\ell-1}} 2^{2s} e^{-2s\ell(\log \ell)^{1/\gamma}}. \end{aligned}$$

Since  $n_\ell - n_{\ell-1} \approx \ell^{1/\gamma-1+o(\varepsilon)}$  and  $1/\gamma - 1 < 1$ , the main term in the above estimation is  $e^{(1-2s)\ell(\log \ell)^{1/\gamma}}$ . Thus for any  $s > 1/2$ , the product is uniformly bounded and we have the Hausdorff dimension of  $B(\gamma, 1)$  is not greater than  $1/2$ . Then we can conclude  $\dim_H E(\gamma, 1) \leq 1/2$  and the proof is completed.

#### 4. GENERALIZATIONS

In this section we consider after [4] certain infinite iterated function systems that are natural generalizations of the Gauss map. For each  $n \in \mathbb{N}$ , let  $f_n : [0, 1] \rightarrow [0, 1]$  be  $C^1$  maps such that

- (1) there exists  $m \in \mathbb{N}$  and  $0 < A < 1$  such that for all  $(a_1, \dots, a_m) \in \mathbb{N}^m$  and for all  $x \in [0, 1]$

$$0 < |(f_{a_1} \circ \dots \circ f_{a_m})'(x)| \leq A < 1,$$

- (2) for any  $i, j \in \mathbb{N}$   $f_i((0, 1)) \cap f_j((0, 1)) = \emptyset$ ,  
(3) there exists  $d > 1$  such that for any  $\varepsilon > 0$  there exist  $C_1(\varepsilon), C_2(\varepsilon) > 0$  such that for  $i \in \mathbb{N}$  there exist constants  $\xi_i, \lambda_i$  such that for all  $x \in [0, 1]$   $\xi_i \leq |f_i'(x)| \leq \lambda_i$  and

$$\frac{C_1}{i^{d+\varepsilon}} \leq \xi_i \leq \lambda_i \leq \frac{C_2}{i^{d-\varepsilon}}.$$

We will call such an iterated function system a *d-decaying system*. It will be further called *Gauss like* if

$$\bigcup_{i=1}^{\infty} f_i([0, 1]) = [0, 1]$$

and if for all  $x \in [0, 1]$  we have that  $f_i(x) < f_j(x)$  implies  $i < j$ .

We have a natural projection  $\Pi : \mathbb{N}^{\mathbb{N}} \rightarrow [0, 1]$  defined by

$$\Pi(\underline{a}) = \lim_{n \rightarrow \infty} f_{a_1} \circ \dots \circ f_{a_n}(1),$$

which gives for any point  $x \in [0, 1]$  its symbolic expansion  $(a_1(x), a_2(x), \dots)$ . This expansion is not uniquely defined, but there are only countably many points with more than one symbolic expansions.

For a *d-decaying Gauss like system* we consider  $S_n(x) = \sum_1^n a_i(x)$ . Given an increasing function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  we denote

$$E_d(\varphi) = \left\{ x \in (0, 1) : \lim_{n \rightarrow \infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\}.$$

**Theorem 4.1.** *Let  $\{f_i\}$  be a d-decaying Gauss like system. We have*

- i) if  $\varphi(n) = e^{n^\gamma}$  for  $\gamma < 1/d$ ,

$$\dim_H E_d(\varphi) = 1,$$

- ii) if  $\varphi(n) = e^{n^\gamma}$  for  $\gamma > 1/d$ ,

$$\dim_H E_d(\varphi) = \frac{1}{d},$$

- iii) if  $\varphi(n) = e^{\gamma n}$  for  $\gamma > 1$ ,

$$\dim_H E_d(\varphi) = \frac{1}{\gamma + d - 1}.$$

The proofs (both from Section 3 and from [9, 10]) go through without significant changes.

□

## REFERENCES

- [1] K. Falconer, *Fractal Geometry, Mathematical Foundations and Application*, Wiley, 1990.
- [2] A. H. Fan, L. M. Liao, B. W. Wang, and J. Wu, *On Kintchine exponents and Lyapunov exponents of continued fractions*, *Ergod. Th. Dynam. Sys.*, 29 (2009), 73-109.
- [3] G. Iommi and T. Jordan, *Multifractal analysis of Birkhoff averages for countable Markov maps*, <http://arxiv.org/abs/1003.2979>.
- [4] T. Jordan and M. Rams, *Increasing digit subsystems of infinite iterated function systems*. *Proc. Amer. Math. Soc.* 140 (2012), no. 4, 1267-1279.
- [5] A. Ya. Khintchine, *Metrische Kettenbruchprobleme*, *Compositio Math.* (1935) (1), 361-382.
- [6] W. Philipp, *A conjecture of Erdős on continued fractions*, *Acta Arith.* 28 (1975/76), no. 4, 379-386.
- [7] W. Philipp, *Limit theorems for sums of partial quotients of continued fractions*, *Monatshefte für Math.*, 105 (1988), 195-206.
- [8] J. Wu and J. Xu, *The distribution of the largest digit in continued fraction expansions*, *Math. Proc. Cambridge Philos. Soc.* 146 (2009), no. 1, 207-212.
- [9] J. Wu and J. Xu, *On the distribution for sums of partial quotients in continued fraction expansions*, *Nonlinearity* 24 (2011), no. 4, 1177-1187.
- [10] J. Xu, *On sums of partial quotients in continued fraction expansions*, *Nonlinearity* 21 (2008), no. 9, 2113-2120.

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