On tail dependence coefficients of transformed multivariate Archimedean copulas

Elena Di Bernardino, Didier Rullière

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Abstract

This paper presents the impact of a class of transformations of copulas in their upper and lower multivariate tail dependence coefficients. In particular we focus on multivariate Archimedean copulas. In the first part of this paper, we calculate multivariate tail dependence coefficients when the generator of the considered copula exhibits some regular variation properties, and we investigate the behaviour of these coefficients in cases that are close to tail independence. This first part exploits previous works of Charpentier and Segers [9] and extends some results of Juri and Wüthrich [36] and De Luca and Rivieccio [13]. We also introduce a new *Regular Index Function* (RIF) exhibiting some interesting properties. In the second part of the paper we analyse the impact in the upper and lower multivariate tail dependence coefficients of a large class of transformations of dependence structures. These results are based on the transformations exploited by Di Bernardino and Rullière [14] and [15]. We extend some bivariate results of Durante et al. [20] in a multivariate setting by calculating multivariate tail dependence coefficients for transformed copulas. We obtain new results under specific conditions involving regularly varying hazard rates of components of the transformation. In the third part, we show the utility of using transformed Archimedean copulas, as they permit to build Archimedean generators exhibiting any chosen couple of lower and upper tail dependence coefficients. The interest of such study is also illustrated through applications in bivariate settings. At last, we explain possible applications with Markov chains with specific dependence structure.

Keywords: Archimedean copulas, tail dependence coefficients, regular variation, transformations of Archimedean copulas, Regular Index Function.

Introduction

Tail problem. Depending upon targeted applications, understanding the tail behaviour of a copula is of great importance. In many practical problems, like hydrology, finance, insurance, etc. one needs to understand the risk of simultaneous threshold crossing for the considered random variables. Tail dependence measurements have been proposed in the literature to explain the asymptotic probability that all random variables in a given set become large, given that random variables of another set are also large. For example, the probability that losses of some financial derivatives are important, given that losses of other derivatives are also important. The whole field of Extreme Value Theory is especially interested in such coefficients. In particular some bivariate tail dependence coefficients have been introduced by Sibuya [53]. Multivariate dependence coefficients are discussed in De Luca and Rivieccio [13] and Li [43]. Statistical literature is also available for tail coefficients, see e.g. Joe [34]. More detailed studies are available in the Archimedean case, where some results on the conditional joint distribution of the considered random variables are given. The interested reader is referred to Charpentier and Segers [9] for a very precise analysis of the Archimedean copulas tail behaviour, including many developments in the (difficult) cases that are close to asymptotic independence.

Transformations of copulas. Transformations of copulas are based on an initial copula C_0 and on a transformation function T. The idea is to exploit characteristics of both functions, C_0 and T, to build new classes of copulas. The

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main interest of transformations is that, under suitable conditions, it becomes easy to get analytical expressions for level curves of the copula, as well as some properties that are useful for estimating parameters of the transformation, as detailed for example in Di Bernardino and Rullière [14] or [15].
Different types of transformations can be found in the literature, cf. Valdez and Xiao [56] or Michiels and De Schepper [46] for a review of some existing transforms. See also Durante and Sempi [21]. Klement et al. [37] and Klement et al. [38], for transformations in the bivariate case. For transformations based on mixtures, see e.g. Morillas [47].
The transformations of copulas that will be considered here are relying on transformations function that are described in Bienvenüe and Rullière [6]. Some adaptations of such transformations to copulas are given in Di Bernardino and Rullière [14] or [15], dealing mainly with statistical estimation of these transformations. In particular, we will consider transformations that permit to transform Archimedean copulas into other Archimedean copulas. Notice that non-parametric estimation of these transformations can be linked to estimation of the generator, and several non-parametric estimators of the generator of Archimedean copulas are available in the literature (see e.g. Dimitrova et al. [18], Genest et al. [28], Di Bernardino and Rullière [15]).
Despite their utility, there are however few works on tail transformations of Archimedean copulas. Such knowledge would help building new classes of Archimedean copulas exhibiting desired tail behaviour. As one will see in this paper, the flexibility of transformed Archimedean copula, coupled with a good knowledge of tail dependence coefficients, will be the starting point of the construction of copulas with given tail dependence coefficients.

Copulas with given tail dependence coefficients. Estimating the dependence structure of a multivariate data is not an easy task, because non-parametric estimation may lack good representations of the tail dependence, whereas parametric representations may lack good representation of the central part of the copula. In particular, it has been shown that most non-parametric estimators have difficulties to capture tail dependence. For the special case of an Archimedean copula C, some estimators are based on the diagonal section $\delta_C(u) := C(u, \ldots, u)$, $u \in [0, 1]$. In dimension d, conditions satisfied by a diagonal section are given in Erdely et al. [25] and existence of a copula with given diagonal section is recalled in their Theorem A. The uniqueness of a copula having a given diagonal section is only ensured for copulas having derivative $\delta'_C(1-) = d$, this condition being referred as *Frank's condition* in the above mentioned article. In the dimension $d = 2$, one can find counterexamples showing that for an Archimedean copula C presenting upper tail dependence, the diagonal does not characterize uniquely its generator ϕ (see Alsina et al. [2], Section 3.8, when $\phi^{'}(0) = -\infty$, or equivalently $\delta^{'}_C(1^-) < 2$). It results that in the general case, estimators based on diagonal section only use partial information about the dependence and thus might not be efficient to capture tail dependence (see Hofert et al. [33]). The estimation of dependence structure thus raises some problems for the tail. Coping with such problems requires a good knowledge of tail dependence behaviour. It also requires some specific ways to use tail dependence measures.
It thus seems that there is a need to propose a flexible parametric estimation of the generator of an Archimedean copula with given tail dependence coefficients. Such result would help adjusting parametrically both the tail and the central part of the considered copula. The first step is to be able to produce not only bivariate but multivariate tail dependence coefficients for transformed copulas. The second step is to be able to link these coefficients with parameters of a transformation, and to adapt the estimation using these coefficients. One can imagine for example starting from a copula exhibiting a good fit on the central part of the multivariate data, and applying a transformation to improve the fit of the tails. Or starting from a copula exhibiting a good fit on the tails and distorting it in order to improve its central part. Or more generally finding the best transformation T to fit both the tails and the central part of a given multivariate data-set, starting from a given copula C_0.
In the following, we will show that for some particular transformations and starting from some particular initial Archimedean copulas, it is possible to produce Archimedean copulas having tunable regular variation properties, and thus to get specific targeted lower and upper tail coefficients.

Organization of the paper. The paper is organized as follow: In Section 1, we present Multivariate tail dependence coefficients for copulas, starting from definition in De Luca and Rivieccio [13]. The particular cases of exchangeable copulas and Archimedean copulas are developed respectively in Section 1.1 and Section 1.2. In Section 2, we present some suitable Regular Variation properties for Archimedean copulas. Under these properties we introduce the *Regular Index Function* (RIF) in order to easily study the regular varying behaviour of a considered Archimedean generator. In Section 2.3 we provide the multivariate tail dependence coefficients introduced before, for these regular varying generators. Particular attention is devoted to the interesting asymptotic independent case. Indeed under some supplementary regular conditions, it is possible to quantify the rate of convergence toward 0 of the upper and lower tails dependence coefficients. The impact of the considered transformations on the regular generator of Archimedean copulas is considered in Sections 3.1 and 3.2. In particular we give tail dependence coefficients for these transformed copulas (Section 3.3). We obtain new results under specific conditions involving regularly varying hazard rates of components of the transformation. In Section 4, we propose a methodology to parametrically estimate a generator given tail dependence coefficients. We detail some applications in the bivariate settings (see Section 5.1). At last, a good knowledge on the tail behaviour of the generator of Archimedean copulas allows to get supplementary results on stationary Markov chains exhibiting specific dependence structure (Section 5.2).

1. Multivariate tail dependence coefficients

Assume that we have a $d-$dimensional non-negative real-valued random vector $\mathbf{X} = (X_1, \ldots, X_d)$. Denote its multivariate distribution function by $F : \mathbb{R}^d \to [0,1]$ with continuous univariate margins $F_i(x_i) = P(X_i \leq x_i)$, for $i = 1, \ldots, d$. Sklar's Theorem (1959) is a well-known result which states that for any random vector $\mathbf{X}$, its multivariate distribution function has the representation

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)),$$

where C is called the *copula*. Effectively, it is a distribution function on the $d-$cube $[0,1]^d$ with uniform margins and it links the univariate margins to their full multivariate distribution. In the case where we have a continuous random vector, we know that $U_i = F_i(X_i)$ is an uniform random variable so that we can write

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)),$$

to be the unique copula associated with $\mathbf{X}$, with quantile functions F_i^{-1} defined by:

$$F_i^{-1}(p) = \inf\{x \in \mathbb{R} : F_i(x) \geq p\}, \qquad \text{for } p \in (0,1).$$

Observe that, for continuous multivariate distributions F, the univariate margins and multivariate dependence structure can be separated, and the dependence structure can be represented by a copula C.

The tail dependence of bivariate copulas has been discussed extensively in recent statistics literature (see for instance Joe [34]). In particular the bivariate tail dependence coefficients were introduced by Sibuya [53] and play a role in the bivariate Extreme Value Theory.

The tail dependence coefficients in the general multivariate case can be expressed as follows (as defined in De Luca and Rivieccio [13], Li [43]).

Definition 1.1 (Multivariate tail dependence coefficients) *Assume that the considered copula C is the distribution of some random vector $\boldsymbol{U} := (U_1, \ldots, U_d)$. Denote $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinal $h \geq 1$ and $d - h \geq 1$. A multivariate version of classical bivariate tail dependence coefficients is given by*

$$\begin{aligned} \lambda_L^{I_h, \bar{I}_h} &= \lim_{u \to 0^+} \mathbb{P}\left[U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h\right], \\ \lambda_U^{I_h, \bar{I}_h} &= \lim_{u \to 1^-} \mathbb{P}\left[U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h\right]. \end{aligned}$$

If for all $I_h \subset I$, $\lambda_L^{I_h, \bar{I}_h} = 0$, (resp. $\lambda_U^{I_h, \bar{I}_h} = 0$) then we say $\boldsymbol{U}$ is lower tail independent (resp. upper tail independent).

From Definition 1.1, the multivariate tail dependence coefficients describe the relative deviation of upper- (or lower-) tail probabilities of a random vector from similar orthant tail probabilities of a subset of its components. They can be used in the study of dependence among extreme values (see, for instance, Li [43]).

One can easily check that Definition 1.1 corresponds to the classical definition in the bivariate case where $d = 2$, with necessarily $h = d - h = 1$. Indeed the classical bivariate upper and lower tail dependence coefficients, λ_U and λ_L, are defined as

$$\lambda_L = \lim_{u\to 0^+} \mathbb{P}[V \le u \,|\, U \le u] \quad \text{and} \quad \lambda_U = \lim_{u\to 1^-} \mathbb{P}[V > u \,|\, U > u],$$

see Sibuya [53]. Remark that for the bivariate independence copula $\Pi(u, v) = u\,v$ we have $\lambda_L = \lambda_U = 0$ (tail independence) and for the bivariate comonotonic copula $M(u, v) = \min\{u, v\}$ we have $\lambda_L = \lambda_U = 1$ (perfect tail dependence). Note that these tail dependence coefficients are a copula-based dependence measures. Unfortunately, they have some drawback. For example, they evaluate the copula C solely on its diagonal section. In other words, the limiting behaviour may be very different if we tend to the copula's lower left (resp. copula's upper right) corner on a different route than on the main diagonal. Regarding this drawback and other pitfalls, the reader may consult Schlather [51], Abdous et al. [1] and Frahm et al. [26].

Moreover, it is well known that the bivariate lower and upper tail coefficients can also be defined as (see, for example, Coles et al. [11]):

$$\begin{aligned}
\lambda_L^{(1,1)} &= \lim_{u\to 0^+} \frac{C(u,u)}{u} = 2 - \lim_{u\to 0^+} \frac{\ln(1-2u+C(u,u))}{\ln(1-u)},\\
\lambda_U^{(1,1)} &= \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} = 2 - \lim_{u\to 1^-} \frac{\ln C(u,u)}{\ln u}.
\end{aligned}$$

Furthermore, the bivariate lower and upper tail coefficients λ_U and λ_L can be written using the diagonal section $\delta_C(u) = C(u,\, u)$ of the associated copula C (see, e.g., Nelsen et al. [49], Nelsen [48]):

$$\lambda_L^{(1,1)} = \lim_{u\to 0^+} \frac{\mathrm{d}}{\mathrm{d}u}\delta(u) = \delta_C^{'}(0^+), \tag{1}$$

$$\lambda_U^{(1,1)} = 2 - \lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\delta(u) = 2 - \delta_C^{'}(1^-). \tag{2}$$

Remark that $\lambda_L^{I_h,\bar{I}_h}$ and $\lambda_U^{I_h,\bar{I}_h}$ in Definition (1.1) can be easily written as

$$\lambda_L^{I_h,\bar{I}_h} = \lim_{u\to 0^+} \frac{\mathbb{P}\left[\mathbf{U} \in \prod_{i=1}^{d}(0,u)\right]}{\mathbb{P}\left[\mathbf{U} \in \prod_{i\in\bar{I}_h}(0,u)\right]}, \tag{3}$$

$$\lambda_U^{I_h,\bar{I}_h} = \lim_{u\to 1^-} \frac{\mathbb{P}\left[\mathbf{U} \in \prod_{i=1}^{d}(u,1)\right]}{\mathbb{P}\left[\mathbf{U} \in \prod_{i\in\bar{I}_h}(u,1)\right]}, \tag{4}$$

where $\mathbf{U} := (U_1, \ldots, U_d)$ is the random vector with uniform margins. Numerators and denominators of (3)-(4) can be written using copula and survival copula of $\mathbf{U}$. In Section 1.1 some easy formulas for $\lambda_L^{I_h,\bar{I}_h}$ and $\lambda_U^{I_h,\bar{I}_h}$ are proposed in the exchangeable case. Section 1.2 focuses on the Archimedean case.

1.1. Tail dependence coefficients for d-exchangeable random variables

We consider here finite-exchangeable random variables $U_1, \ldots, U_d$, which means that the joint distribution of $(U_{\sigma(1)}, \ldots, U_{\sigma(d)})$ is identical to the one of $(U_1, \ldots, U_d)$, for any permutation σ of $\{1, \ldots, d\}$.

Remark 1 *A well-known specific example of exchangeability is the case where $U_1, \ldots, U_d$ is a subsequence of an infinite exchangeable sequence $U_1, \ldots, U_n$ where the exchangeable property holds for each $n \in \mathbb{N}$. In this particular case De Finetti's theorem implies that $U_1, \ldots, U_d$ can be seen as random variables that are conditionally independent given a common random factor. The finite exchangeability considered here is more general, as some finite exchangeable sequences can not be seen as conditional independent random variables (see Diaconis [17] for more details).*

The following result provides a multivariate version of tail dependence coefficients in Equations (1)-(2) in the exchangeable case. This can be seen as an extension of Definition 2 in De Luca and Rivieccio [13] in the more general case of exchangeable random variables.

Proposition 1.1 *If $U_1, \ldots, U_d$ are exchangeable random variables with copula C, then, for $h \geq 1$ and $d-h \geq 1$, it holds that*

$$\lambda_L^{(h,d-h)} = \lambda_L^{I_h,\bar{I}_h} = \lim_{u\to 0^+} \frac{[\delta^{(d)}]'(u)}{[\delta^{(d-h)}]'(u)},$$

$$\lambda_U^{(h,d-h)} = \lambda_U^{I_h,\bar{I}_h} = \lim_{u\to 1^-} \frac{\sum_{i=1}^{d}(-1)^i\, C_d^i\, [\delta^{(i)}]'(u)}{\sum_{i=1}^{d-h}(-1)^i\, C_{d-h}^i\, [\delta^{(i)}]'(u)},$$

where $\delta^{(i)}(u) = C_i(u, \ldots, u)$ is the diagonal section of the copula in the dimension i, and where C_d^i denotes the binomial coefficient $C_d^i = d!/(i!(d-i)!)$.

The proof is postponed to Appendix B. Remark that, contrarily to the bivariate case in Equations (1)-(2), in the general multivariate case the exchangeable assumption is a central tool to obtain the compact forms in Equations (B.1)-(B.2) respectively for $\lambda_L^{(h,d-h)}$ and $\lambda_U^{(h,d-h)}$. About the exchangeable copulas, the interested reader is, for instance, referred to Ricci [50] and references within.

1.2. Tail dependence coefficients for d-Archimedean copulas

In this section we consider a particular class of exchangeable copulas, i.e. the d-Archimedean copulas. Archimedean copulas play a central role in the understanding of dependencies of multivariate random vectors. A good introduction to copulas in general is given in Nelsen [48]. For a focus on Archimedean copulas in particular the reader is referred to McNeil and Nešlehová [44]. While Archimedean copulas are obviously exchangeable, the converse is not necessarily true. For instance the $d-$dimensional exchangeable Upper Fréchet Bound (i.e. the comonotonic copula) is not Archimedean (e.g., see Nelsen [48]). Then, since Archimedean copulas are cumulative distributions of exchangeable random vector with uniform margins, results of Section 1.1 can be applied to these copulas.

The d-Archimedean copulas are described by a real function ϕ, called the *generator* of the copula:

$$C(u_1, \ldots, u_d) = \phi\left(\phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d)\right), \text{ for } u_1, \ldots, u_d \in (0,1]. \tag{5}$$

The generator $\phi(t)$ is a continuous, decreasing and convex function of t, with $\phi(0) = 1$. From Theorem 2.2 in McNeil and Nešlehová [44], C in Equation (5) is a $d-$dimensional copula if and only if its generator ϕ is $d-$monotone on $[0, \infty)$ (for details the reader is referred to McNeil and Nešlehová [44]). Required proprieties on generator ϕ an its inverse ψ are gathered in Assumption 1.1 below.

Assumption 1.1 (Considered generators) *Consider an Archimedean generator ϕ. One assumes that ϕ is differentiable, that ϕ is $d-$monotone on $[0, \infty)$ and that ϕ is a strict generator, i.e. strictly positive with limit $\lim_{t\to+\infty} \phi(t) = 0$ (see Section 4 in Nelsen [48]). The generator ϕ thus has a proper inverse that will be denoted*

$$\psi(t) = \phi^{-1}(t), \quad t \in (0,1]. \tag{6}$$

Remark that ψ is convex and, if ψ is differentiable, then ψ' is increasing and in particular ultimately monotone. Then, Assumption 1.1 guarantees the application of the Monotone Density Theorem (see, e.g., Theorem 1.7.2 in Bingham et al. [7] and Theorem 1.20 in Soulier [54]).

Remark 2 *It is interesting to notice that if a particular generator is valid in any dimension $d \geq 2$, the Kimberling theorem states that this generator is completely monotone, and thus, using Bernstein Theorem, can be seen as a Laplace transform of some positive random variable V. It results that in this case, if $(U_1, \ldots, U_d)$ have the joint distribution of the Archimedean copula C, they can be seen as conditionally independent random variables, given the random factor V, as exploited in Marshall, Olkin algorithm for sampling Archimedean copulas (see Hofert [31]). As remarked previously for exchangeable random variables, the validity of a generator in any dimension implies exchangeability in any dimension and conditional independence given a common random factor (see Remark 1).*

For Archimedean copulas, we take back from De Luca and Rivieccio [13] the definition of multivariate lower and upper tail coefficients (see Definition 2 in De Luca and Rivieccio [13]). This definition naturally corresponds to previous Definition 1.1 is the particular Archimedean copulas case.

Definition 1.2 (Multivariate tail dependence coefficients for Archimedean copulas) *For Archimedean copulas the multivariate lower and upper tail dependence coefficients in Definition 1.1 are respectively:*

$$\begin{aligned}
\lambda_L^{(h,d-h)} &= \lim_{u \to 0^+} \frac{\psi^{-1}\left(d\,\psi(u)\right)}{\psi^{-1}\left((d-h)\,\psi(u)\right)}, \\
\lambda_U^{(h,d-h)} &= \lim_{u \to 1^-} \frac{\sum_{i=0}^{d} (-1)^i\, C_d^i\, \psi^{-1}\left(i\,\psi(u)\right)}{\sum_{i=0}^{d-h} (-1)^i\, C_{d-h}^i\, \psi^{-1}\left(i\,\psi(u)\right)}.
\end{aligned}$$

2. Regular Archimedean copulas

For the sake of simplicity, we define as *regular Archimedean copulas* the Archimedean copulas having a *lower and upper regularly varying generator* ψ (see Equation (6)) in zero and one respectively. We define in following paragraphs regularly varying function and considered generators of regular Archimedean copulas.

2.1. Regular Variation

We recall here some basic definitions of regular variation for a measurable function $f : (0, \infty) \to (0, \infty)$. We say that f is regularly varying at $+\infty$ with index α, and we denote $f \in \mathcal{RV}_\alpha(\infty)$ if

$$f \in \mathcal{RV}_\alpha(\infty) \quad \Leftrightarrow \quad \forall\ s > 0,\ \lim_{x \to +\infty} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

A survey of regular variations can be found for example in Embrechts et al. [24], de Haan and Ferreira [12], Soulier [54]. Main results about regular variations are presented by Bingham et al. [7].

For $0 < x < \infty$, define x^∞ by ∞, 1, or 0, according to whether x is larger than, equal to, or smaller than 1, respectively; similarly, define $x^{-\infty}$ by 0, 1, or ∞ according to whether x is larger than, equal to, or smaller than 1, respectively. This proper interpretation of the considered power functions, x^∞ and $x^{-\infty}$, will be used in the following.

Regular variations can also be defined at zero or at one by requiring that simple transformations of the function are regularly varying at infinity. Let us define, for example,

$$M : [0,1] \to [0,1],\ M(x) = 1 - x, \qquad I : (0,\infty) \to (0,\infty),\ I(x) = \frac{1}{x}, \qquad N : \mathbb{R} \to \mathbb{R},\ N(x) = -x.$$

One easily check that $M \circ M$, $I \circ I$ and $N \circ N$ are the identity function on considered intervals. For a function f defined on $(0, 1)$, we will say that f is regularly varying at zero with index α if

$$f \in \mathcal{RV}_\alpha(0) \ \Leftrightarrow\ f \circ M \in \mathcal{RV}_\alpha(1) \ \Leftrightarrow\ f \circ I \in \mathcal{RV}_{-\alpha}(\infty) \ \Leftrightarrow\ \forall\, s > 0,\ \lim_{x \to 0^+} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

We also define functions that are regularly varying at one with index α as

$$f \in \mathcal{RV}_\alpha(1) \ \Leftrightarrow\ f \circ M \circ I \in \mathcal{RV}_{-\alpha}(\infty) \ \Leftrightarrow\ \forall\, s > 0,\ \lim_{x \to 0^+} \frac{f(1 - s\,x)}{f(1 - x)} = s^\alpha.$$

Finally, one can say that f is regularly varying with index α at $-\infty$ if

$$f \in \mathcal{RV}_\alpha(-\infty) \ \Leftrightarrow\ f \circ N \in \mathcal{RV}_\alpha(\infty) \ \Leftrightarrow\ \forall\, s > 0,\ \lim_{x \to -\infty} \frac{f(s\,x)}{f(x)} = s^\alpha.$$

2.2. Upper and lower regular generators

As $\psi \circ M \circ I$ is a decreasing function, if $\psi \circ M \circ I$ is regularly varying, it can only be varying with some negative index $-\alpha$, $\alpha \in [0, +\infty]$, so that from (2.1), $\psi \in \mathcal{RV}_\alpha(1)$ requires that $\alpha \in [0, +\infty]$. We recall in Remark A that convexity assumptions restrict the range of possible regular variation indexes for Archimedean generators.

Table 4.1 in Nelsen [48] contains several examples of strict archimedean copulas whose inverse generators ψ in (6) are regularly varying at one and zero.

Remark A (Valid regularly varying generator) *Assume that the generator ϕ and its inverse ψ satisfy Assumption 1.1. It holds that:*

i) If an inverse generator ψ is regularly varying at zero with index $-r$, i.e. $\psi \in \mathcal{RV}_{-r}(0)$, then necessarily $r \in [0, +\infty]$.

ii) If an inverse generator ψ is regularly varying at one with index ρ, i.e. $\psi \in \mathcal{RV}_\rho(1)$, then necessarily $\rho \in [1, +\infty]$.

The proof is postponed to Appendix C.

Remark that the independent generator is $\psi^{\text{Indep}} \in \mathcal{RV}_0(0)$, with $r = 0$. Furthermore, for any $r \in (0, +\infty)$, there exists a valid inverse generator ψ with index $-r$ at zero. Indeed the strict Clayton generator is an example of generator valid in any dimension allowing to reach any regular variation index $-r$, with $r \in (0, +\infty)$ at zero (see Equation (A.2)). Furthermore for any $\rho \in [1, +\infty)$, there exists a valid inverse generator ψ with index ρ at one. Indeed the Gumbel generator is an example of generator valid in any dimension allowing to reach any regular variation index $\rho \in [1, +\infty)$ at one (see Equation (A.1)). Charpentier and Segers [9] listed different generators with $\rho = +\infty$ or $r = +\infty$ (see Table 1 in Charpentier and Segers [9]). Then the bounds of the intervals for ρ and r in Remark A can be reached.

For the sake of simplicity, we denote by *regular generator* any regularly varying Archimedean generator. We summarize here assumptions on these generators.

Assumption 2.1 (Regular varying generators) *Assume that the generator ϕ and its inverse ψ satisfy Assumption 1.1. Then*

- *if ψ is regularly varying at zero with index $-r$, with $r \in [0, +\infty]$, then we say that ψ is a lower regular generator.*
- *if ψ is regularly varying at one with index ρ, with $\rho \in [1, +\infty]$, then we say that ψ is an upper regular generator.*

A generator ψ which is both an upper and lower regular generator is simply called a (full) regular generator.

Identifying regular generator is easier if we use the following property, that will be useful when calculating tail dependence coefficients. The result, recalled in Remark B below, follows from the *Monotone Density theorem* (see Theorem 1.7.2 in Bingham et al. [7]). The interest reader is also referred to Theorem 1 in Charpentier and Segers [8].

Remark B (Identifying regular generators) *Let ψ be a lower regular generator satisfying Assumption 2.1 with $r \in [0, +\infty]$, then*

$$\psi \in \mathcal{RV}_{-r}(0) \;\Leftrightarrow\; \psi \circ I \in \mathcal{RV}_r(\infty) \;\Leftrightarrow\; \lim_{y \to 0^+} \frac{y\, \psi'(y)}{\psi(y)} = -r,$$

with proper interpretations for r equal to zero or infinity.
Consider an upper regular generator ψ satisfying Assumption 2.1 with $\rho \in [1, +\infty]$, then

$$\psi \in \mathcal{RV}_\rho(1) \;\Leftrightarrow\; \psi \circ M \circ I \in \mathcal{RV}_{-\rho}(\infty) \;\Leftrightarrow\; \lim_{z \to 1^-} \frac{(1-z)\, \psi'(z)}{\psi(z)} = -\rho,$$

with proper interpretation for ρ equal to infinity.

The proof is postponed to Appendix C.

Under Assumption 2.1 and using Remarks A-B, we will introduce in Lemma 2.1 a function (here called *Regular Index Function*) in order to easily study the regular varying behaviour of the inverse generator ψ.

Lemma 2.1 (Regular Index Function) *Consider a full regular generator ψ satisfying Assumption 2.1, with in particular $\psi \in \mathcal{RV}_{-r}(0)$, $r \in [0, +\infty]$ and $\psi \in \mathcal{RV}_{\rho}(1)$, $\rho \in [1, +\infty]$. Define the Regular Index Function as*

$$\mathrm{RIF}_{\psi}(x) = -\frac{x(1-x)\psi'(x)}{\psi(x)} + \frac{1-x}{\ln x}, \quad \text{for } x \in (0,1).$$

Then this function RIF_{ψ}, is such that:

i) $\lim\limits_{x \to 0^+} \mathrm{RIF}_{\psi} = r$, *with* $r \in [0, +\infty]$,

ii) $\lim\limits_{x \to 1^-} \mathrm{RIF}_{\psi} = \rho - 1$, *with* $\rho - 1 \in [0, +\infty]$,

iii) $\mathrm{RIF}_{\psi_\perp}(x) = 0$ *for all* $x \in (0,1)$ *in the independence case where* $\psi_\perp(x) = -\ln(x)$,

iv) $\mathrm{RIF}_{\psi_A} = \mathrm{RIF}_{\psi_B}$ *for any equivalent generators* $\psi_B = c\,\psi_A$, $c > 0$.

Note that $\mathrm{RIF}_{\psi}(x) = \frac{1-x}{\ln x}\left(\frac{\lambda(x)-\lambda_\perp(x)}{\lambda(x)}\right)$, *where* $\lambda(x) := \psi(x)/\psi'(x)$, *as introduced by Genest and Rivest [29] for statistical inference purposes, and where* $\lambda_\perp(x) = x\,\ln(x)$, *i.e. the lambda function in the independent case with* $\psi_\perp(x) = -\ln(x)$.

The proof is postponed to Appendix B. An illustration of Lemma 2.1 in the case of Gumbel and Clayton generator will be given in Appendix A. This function will be useful to understand regular variation properties of some specific generators in Section 4. Note that other functions having the same properties can be constructed.

Remark that from Lemma 2.1, if we have a consistent differentiable estimator of the *lambda function* $\lambda(x) := \psi(x)/\psi'(x)$ we can study the type of dependence in the upper and lower tails of the considered data. For some possible estimators of $\lambda(x)$ the interested reader is referred to Genest et al. [28], Di Bernardino and Rullière [15].

2.3. Tail dependence coefficients for regular Archimedean copulas

In accordance with Remark A, one only considers here inverse generators $\psi \in \mathcal{RV}_{-r}(0)$ with $r \in [0, +\infty]$ and $\psi \in \mathcal{RV}_{\rho}(1)$ with $\rho \in [1, +\infty]$, satisfying Assumption 2.1. In the following Remark C we recall the relationship in terms of regular variation between ϕ and its inverse ψ.

Remark C (Inverse of a regular generator) *Let ψ be an inverse generator satisfying Assumption 2.1, then, for $s > 0$,*

$$\psi \in \mathcal{RV}_{-r}(0),\, r \in [0, +\infty] \;\Leftrightarrow\; \phi \in \mathcal{RV}_{-1/r}(\infty) \;\Leftrightarrow\; \lim_{y \to \infty} \frac{\phi(s\,y)}{\phi(y)} = s^{-\frac{1}{r}},$$

with proper interpretations for r equal to zero or infinity, and

$$\psi \in \mathcal{RV}_{\rho}(1),\, \rho \in [1, +\infty] \;\Leftrightarrow\; M \circ \phi \in \mathcal{RV}_{1/\rho}(0) \;\Leftrightarrow\; \lim_{y \to 0} \frac{1-\phi(s\,y)}{1-\phi(y)} = s^{\frac{1}{\rho}},$$

with proper interpretation for ρ equal to infinity.

The proof is postponed to Appendix C. Generator ψ^{-1} may be seen as a survival function, so that this result can be linked to classical results on quantile functions, as in Soulier [54], page 37. Using Remark C we can obtain the following result.

Lemma 2.2 (Derivatives of copula diagonal sections) *Let ψ be an inverse generator satisfying Assumption 2.1, with in particular $\psi \in \mathcal{RV}_{-r}(0)$, with $r \in [0, +\infty]$, then*

$$\lim_{u \to 0^+} \frac{\delta^{(i)}(u)}{u} = i^{-1/r} \quad \text{and} \quad \lim_{u \to 0^+} [\delta^{(i)}]'(u) = i^{-1/r},$$

with proper interpretations for r equal to zero or infinity, where $\delta^{(i)}(u) = \psi^{-1}(i \cdot \psi(u))$ is the diagonal section of the copula in dimension i, $u \in (0, 1)$, and $[\delta^{(i)}]'$ its derivatives.
Let ψ be an inverse generator satisfying Assumption 2.1, with in particular $\psi \in \mathcal{RV}_{\rho}(1)$, with $\rho \in [1, +\infty]$, then

$$\lim_{u \to 1^-} \frac{1 - \delta^{(i)}(u)}{1 - u} = i^{1/\rho} \quad \text{and} \quad \lim_{u \to 1^-} [\delta^{(i)}]'(u) = i^{1/\rho},$$

with proper interpretation for ρ equal to infinity.

The proof is postponed to Appendix B. Note that when derivatives of the diagonal section δ tend to 0 or to 1, it is useful to get some results on the convergence speed toward 0 or 1. From Charpentier and Segers [9], the following Lemma 2.3 will be helpful in the case where $r = 0$ or $\rho = 1$.

Lemma 2.3 (Regular Variation of copula diagonal sections) *Recall that the diagonal of an Archimedean copula in dimension $i \in \mathbb{N} \setminus \{0\}$ is $\delta^{(i)}(u) = \phi\left(i \cdot \phi^{-1}(u)\right)$, for $u \in (0, 1)$. Assume that the generator of the copula satisfies Assumption 2.1, and notice that the generator ϕ of the copula can be seen as the survival function of a positive random variable.*
On the lower side, if the hazard rate $\mu_\phi = \phi'/\phi \in \mathcal{RV}_{k-1}(\infty)$, then for $i \in \mathbb{N} \setminus \{0\}$,

$$\delta^{(i)} \in \mathcal{RV}_{z_i}(0), \text{ with } z_i = i^k \text{ and } k \in [0, +\infty]. \tag{7}$$

On the upper side, if the rate $m_\phi = \phi'/(1 - \phi) \in \mathcal{RV}_{-\kappa-1}(0)$, then for $i \in \mathbb{N} \setminus \{0\}$,

$$M \circ \delta^{(i)} \in \mathcal{RV}_{\zeta_i}(1), \text{ with } \zeta_i = i^{-\kappa} \text{ and } \kappa \in [0, +\infty]. \tag{8}$$

The proof is postponed to Appendix B.

Remark 3 *In Charpentier and Segers [9] (Theorem 3.3) the authors required that the function $-\frac{1}{D(\ln \phi)} \in \mathcal{RV}_{k^*}$ with $k^* \leq 1$, where D is the derivative operator. We can write, using the notation of Lemma 2.3,*

$$-\frac{1}{D(\ln \phi)} = -\frac{\phi}{\phi'} = -\frac{1}{\mu_\phi}.$$

Then the assumption in Charpentier and Segers [9] is equivalent to $-\frac{1}{\mu_\phi} \in \mathcal{RV}_{k^*}$. Finally $\mu_\phi \in \mathcal{RV}_{\bar{k}}$, with $\bar{k} = -k^*$ and $\bar{k} \geq -1$. So it is exactly the assumption in Lemma 2.3, i.e., $\mu_\phi \in \mathcal{RV}_{k-1}$, with $k - 1 \geq -1$ i.e. $k \geq 0$.

Remark 4 *Result in Equation (7) in Lemma 2.3 (for the lower tails) will be useful in Theorem 2.2 to describe the behaviour of the lower tail coefficient in the case of lower asymptotic independence ($r = 0$). Conversely result in Equation (8) in Lemma 2.3 (for the upper tails) will not be sufficient to characterize the regular variation of upper tail coefficient in Theorem 2.3 in the case of upper asymptotic independence ($\rho = 1$). Indeed the linear combination of $\mathcal{RV}_1(1)$ functions can be a $\mathcal{RV}_j(1)$ function with $j > 1$, due to eventual compensations between the terms of the sum. This is consistent with the fact that in a upper asymptotic independent case the diagonal section of a copula does not characterize uniquely its generator (see for instance Embrechts and Hofert [23], Alsina et al. [2]). For this reason, we need to introduce some different assumptions in Theorem 2.3 below to obtain the desired result.*

Furthermore we can prove the following result for the range of index k and κ in Lemma 2.3.

Remark 5 *Assume that* $\psi \in \mathcal{RV}_{-r}(0)$, $r \in [0,+\infty]$, *and that* $\mu_\phi = \phi'/\phi \in \mathcal{RV}_{k-1}(\infty)$, $k \in [0,+\infty]$, *then*

$$r \in (0,+\infty) \;\Rightarrow\; k = 0.$$

Assume that $\psi \in \mathcal{RV}_\rho(1)$, $\rho \in [1,+\infty]$, *and that* $m_\phi = \phi'/(1-\phi) \in \mathcal{RV}_{-\kappa-1}(0)$, $\kappa \in [0,+\infty]$, *then*

$$\rho \in [1,+\infty) \;\Rightarrow\; \kappa = 0.$$

The proof is postponed to Appendix B. Remark for instance that in the Gumbel case $r = 0$ and $k \in (0,1]$ (see Appendix A). The case where $\rho = +\infty$ does not occur frequently. Charpentier and Segers [9] provide an example of generator such that $\rho = +\infty$ (see generator (18) in Table 1 in Charpentier and Segers [9]) but also in this case we find $\kappa = 0$. Then to the best of our knowledge $\kappa \in [0,+\infty]$ but this is in particular equal to zero for a large class of Archimedean copulas. Indeed we have not been able to find a known Archimedean copula family with $\kappa \neq 0$.

We now focus on the multivariate upper and lower tail coefficients.

Theorem 2.1 (Upper and lower tail coefficients) *Let* ψ *be an inverse generator satisfying Assumption 2.1, with in particular* $\psi \in \mathcal{RV}_{-r}(0)$, *with* $r \in [0,+\infty]$. *Then, from Definition 1.2 and Lemma 2.2, one have*

$$\lambda_L^{(h,d-h)} = \begin{cases} 0, & \text{if } r = 0, \\ d^{-1/r}\,(d-h)^{1/r}, & \text{if } r \in (0,+\infty), \\ 1, & \text{if } r = +\infty. \end{cases}$$

Let ψ *be an inverse generator satisfying Assumption 2.1, with in particular* $\psi \in \mathcal{RV}_\rho(1)$, *with* $\rho \in [1,+\infty]$, *then*

$$\lambda_U^{(h,d-h)} = \begin{cases} 0, & \text{if } \rho = 1, \\ \frac{\sum_{i=1}^{d}(-1)^i\, C_d^i \cdot i^{1/\rho}}{\sum_{i=1}^{d-h}(-1)^i\, C_{d-h}^i \cdot i^{1/\rho}}, & \text{if } \rho \in (1,+\infty), \\ 1, & \text{if } \rho = +\infty. \end{cases}$$

The proof is postponed in Appendix B. Note that in the case where $r \in (0,+\infty)$ or $\rho \in (1,+\infty)$ this extends results that are obtained by De Luca and Rivieccio [13] for some particular multivariate bi-parametric (MB) copulas (namely MB1 and MB7 copulas).

From Theorem 2.1, there are essentially two categories: if $r \in (0,+\infty]$ (*resp.* $\rho \in (1,+\infty]$), then the lower (*resp.* upper) tail exhibits asymptotic dependence, while if $r = 0$ (*resp.* $\rho = 1$), then there is asymptotic independence. However in these asymptotic independent case, under some regular conditions, it is possible to quantify the rate of convergence toward 0 of $\lambda_L^{(h,d-h)}(u)$ and $\lambda_U^{(h,d-h)}(u)$ (see Theorems 2.2 and 2.3 below).

Theorem 2.2 (case $r = 0$) *Assume that* $\psi \in \mathcal{RV}_{-r}(0)$, *with* $r = 0$. *Denote by* $\lambda_L^{(h,d-h)}(u) = \frac{\delta^{(d)}(u)}{\delta^{(d-h)}(u)}$, *such that*

$$\lambda_L^{(h,d-h)} = \lim_{u\to 0^+} \lambda_L^{(h,d-h)}(u) = 0.$$

As ϕ *can be seen as a survival function, define the hazard rate* $\mu_\phi = \frac{\phi'}{\phi}$. *If one assumes that* $\mu_\phi \in \mathcal{RV}_{k-1}(\infty)$, *for* $k \in [0,+\infty)$, *then*

$$\lambda_L^{(h,d-h)}(u) = \frac{\delta^{(d)}(u)}{\delta^{(d-h)}(u)} \in \mathcal{RV}_z(0), \quad \text{with } z = d^k - (d-h)^k.$$

The proof is postponed in Appendix B.

Under assumptions of Theorem 4.3 and Corollary 4.7 in Charpentier and Segers [9] we can obtain the following result.

Theorem 2.3 (case $\rho = 1$) *Assume that $\psi \in \mathcal{RV}_\rho(1)$, with $\rho = 1$, and that the associated ϕ is a d times continuously differentiable generator.*

- *If $(-D)^d\phi(0)$ is finite and not zero, where D is the derivative operator, then we get*
$$\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1).$$
- *If $\psi'(1) = 0$ and the function $L(s) := s\,\frac{\mathrm{d}}{\mathrm{d}s}\{\frac{\psi(1-s)}{s}\}$ is positive and $L \in \mathcal{RV}_0(0)$, then we get*
$$\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1).$$

The proof is postponed in Appendix B. Remark that $(-D)^d\phi(0) < +\infty$ implies $\psi'(1) < 0$ (see Theorem 4.3 in Charpentier and Segers [9]). In the first case of Theorem 2.3, there is upper asymptotic independence in a rather strong sense, a case which is called *near independence* in Ledford and Tawn [42] (Section 4.2). Indeed in this case the upper tail coefficient goes to zero as a regular variation function of index $h \geq 1$. In the second case we are on the boundary between asymptotic independence and asymptotic dependence. Charpentier and Segers [9] called this second case *near asymptotic dependence* and they provided an example of generator such that $\rho = 1$ and $\psi'(1) = 0$ (see generator (23) in Table 1 in Charpentier and Segers [9]). From Theorem 2.3, in the near asymptotic dependence case the upper tail coefficient goes to zero as a slowly variation function.

Remark 6 (Case $\rho = 1$) *Notice that the limit case $\rho = 1$ and $d - h \geq 2$ is not considered in De Luca and Rivieccio [13]. Indeed if $\rho = 1$ and $d - h \geq 2$, then the denominator $\sum_{i=1}^{d-h} C_{d-h}^i(-1)^i \cdot i^{1/\rho} = 0$ in Theorem 2.1. Indeed when $\rho = 1$, for any $n \geq 2$,*
$$\textstyle\sum_{i=1}^{n} C_n^i(-1)^i \cdot i^{1/\rho} = n\sum_{i=1}^{n} C_{n-1}^{i-1}(-1)^i = -n\sum_{j=0}^{n-1} C_{n-1}^{j}(-1)^j = -n(1-1)^{n-1} = 0.$$

Theorem 2.1 in the particular bivariate case, when $d = 2$ and $h = 1$, is analysed in the following result. Remark that Corollary 2.1 below is exactly Theorem 4.4 in Juri and Wüthrich [36]. In this sense Theorem 2.1 can be seen as a multivariate extension of Theorem 4.4 in Juri and Wüthrich [36] and Theorem 3.9 in Juri and Wüthrich [35].

Corollary 2.1 (Bivariate upper and lower tail coefficients) *Let ψ be an inverse generator satisfying Assumption 2.1, with $\psi \in \mathcal{RV}_{-r}(0)$, for $r \in [0, +\infty]$. Then, when $d = 2$ and $h = 1$,*
$$\lambda_L^{(1,1)} = \begin{cases} 0, & \text{if } r = 0, \\ 2^{-1/r}, & \text{if } r \in (0, +\infty), \\ 1, & \text{if } r = +\infty. \end{cases}$$

Let ψ be an inverse generator satisfying Assumption 2.1, with $\psi \in \mathcal{RV}_\rho(1)$, for $\rho \in [1, +\infty]$, then when $d = 2$ and $h = 1$,
$$\lambda_U^{(1,1)} = \begin{cases} 0, & \text{if } \rho = 1, \\ 2 - 2^{1/\rho}, & \text{if } \rho \in (1, +\infty), \\ 1, & \text{if } \rho = +\infty. \end{cases}$$

The proof comes down trivially from Theorem 2.1.

In Appendix A we illustrate results obtained in Section 2 for some usual Archimedean copulas.

3. Transformed regular Archimedean copulas

3.1. Considered transformations

We consider transformations $T_f : [0,1] \to [0,1]$ such that
$$T_f(u) = \begin{cases} 0 & \text{if } u = 0, \\ G \circ f \circ G^{-1}(u) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1, \end{cases} \qquad (9)$$

where f is any continuous bijective increasing function, $f:\mathbb{R}\to\mathbb{R}$, and is said to be a *conversion function.* This kind of transformations are presented in the univariate case in Bienvenüe and Rullière [6]. The transformation T_f have support $[0,1]$, and the function G aims at transferring this support on the whole real line $\mathbb{R}$, in order to allow f to be defined on the whole set $\mathbb{R}$, without bounding constraints. The function G is thus chosen as a continuous and invertible c.d.f with support $\mathbb{R}$, i.e. such that $\forall\, x\in\mathbb{R}$, $G(x)\in(0,1)$. The distribution G is a proper non-defective distribution, so that $\lim_{x\to-\infty} G(x)=0$ and $\lim_{x\to\infty} G(x)=1$, where these bounds are never reached due to the chosen support of G.

We consider here a transformed copula $\widetilde{C}_{T_f,C_0}$, which is transformed from an initial copula C_0 using the transformation T_f as in Equation (9), i.e.,

$$\widetilde{C}_{T_f,C_0}(u_1,\ldots,u_d)=T_f\circ C_0\left(T_f^{-1}(u_1),\ldots,T_f^{-1}(u_d)\right). \tag{10}$$

The presence of the transformation T_f both inside and outside the copula C_0 in Equation (10) is necessary to ensure that the copula preserves uniform margins, and in particular that $\widetilde{C}_{T_f,C_0}(u,1,\ldots,1)=u$ as required.

Transformations of copulas are used in the recent literature as a simple way to generate new copulas from initial ones. Many types of transformations of copulas have been considered, see for example Valdez and Xiao [56] or Michiels and De Schepper [46] for a review of some existing transforms. Transformations of bivariate copula, semicopulas and quasi-copulas are studied in Durante and Sempi [21]. Klement et al. [37] and Klement et al. [38] focused on transformations of bivariate Archimax copulas.

Transformations in Equation (10) have been considered for example in Durrleman et al. [22], Valdez and Xiao [56] (Definitions 3.6, in dimension $d=2$), Hofert [32]. If we focus on the two-dimensional setting, the transformation considered in this paper corresponds to the Right Composition (RC, see Lemma 5 in Michiels and De Schepper [46]), initially defined in Genest et al. [27].

As remarked in Di Bernardino and Rullière [15], if we restrict ourselves to the case where C_0 in Equation (10) is an Archimedean copula (see Equation (5)) then, under supplementary assumptions (see Assumption 3.1 below) the transformed copula $\widetilde{C}_{T_f,C_0}$ will be Archimedean. This implies that the obtained transformed copula is still symmetric, for example. Remark that, in general, $\widetilde{C}_{T_f,C_0}$ is not necessarily a copula. Determination of sufficient and necessary conditions in order to obtain admissible transformations T_f is fundamental to propose tractable transformations in operational problems (see for instance Di Bernardino and Rullière [15]). In Assumption 3.1, using Theorem 2.2 in McNeil and Nešlehová [44], we give supplementary assumptions to guarantee that $\widetilde{C}_{T_f,C_0}$ is also an Archimedean copula.

Assumption 3.1 (Considered transformed generators) *Consider an initial Archimedean copula C_0 as in Equation* (5)*, with generator ϕ_0, and the associated transformed one, $\widetilde{C}_{T_f,C_0}$, with generator $\widetilde{\phi}=T_f\circ\phi_0$ where T_f as in Equation* (9)*. One assumes that both generators ϕ_0 and $\widetilde{\phi}$ satisfy Assumption 1.1.*

Remark that under Assumption 3.1 $\widetilde{C}_{T_f,C_0}$ is a proper $d-$dimensional Archimedean copula and $\widetilde{\phi}$ is an admissible generator with differentiable inverse $\widetilde{\psi}$. The interested reader is referred to McNeil and Nešlehová [44], Di Bernardino and Rullière [15]. Furthermore, Di Bernardino and Rullière [15] (Proposition 2.5) proved that, when the initial copula C_0 is the independent one, and when T_f in Equation (9) is d-times differentiable, one can find tractable necessary and sufficient admissibility conditions for the transformation T_f.

3.2. Regularly varying transformed generator

In Assumption 3.2 (*resp.* 3.3) below, we summarize the *assumption setting* used in the following, for the study of the transformed multivariate lower (*resp.* upper) dependence coefficients.

Assumption 3.2 (Lower-tails: assumptions on f, ϕ_0, G) *Assume that f, ϕ_0 and G are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted f^{-1}, $\psi_0 = \phi_0^{-1}$ and G^{-1}. Furthermore,*

i) The function f has an asymptote $\overline{f}(x) = ax + b$ as x tends to $-\infty$, for $a \in (0, +\infty)$ and $b \in (-\infty, +\infty)$.

ii) The inverse initial generator ψ_0 is regularly varying at 0 with some index $-r_0$, that is $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in [0, +\infty]$.

iii) The function G is a non-defective continuous c.d.f. with support $\mathbb{R}$. The following rate of G is regularly varying with some index $g - 1$: $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, with $g \in (0, +\infty)$.

Assumption 3.3 (Upper-tails: assumptions on f, ϕ_0, G) *Assume that f, ϕ_0 and G are continuous and differentiable functions, strictly monotone with respective proper inverse functions denoted f^{-1}, $\psi_0 = \phi_0^{-1}$ and G^{-1}. Furthermore,*

i) The function f has an asymptote $\overline{f}(x) = \alpha x + \beta$ as x tends to $+\infty$, for $\alpha \in (0, +\infty)$ and $\beta \in (-\infty, +\infty)$.

ii) The inverse initial generator ψ_0 is regularly varying at 1 with some index ρ_0, i.e., $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty]$.

iii) The function G is a non-defective continuous c.d.f. with support $\mathbb{R}$. The hazard rate of G is regularly varying with some index $\gamma - 1$, that is $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, with $\bar{G} = 1 - G$ and $\gamma \in (0, +\infty)$.

In the following result we provide an easy way to construct the distribution G such that $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, with $g \in (0, +\infty)$ (see Lower-tail Assumption 3.2) or $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$ with $\gamma \in (0, +\infty)$ (see Upper-tail Assumption 3.3).

Lemma 3.1 *Let F and G be two non-defective cumulative distribution functions with support $\mathbb{R}$, with proper inverse functions, and let P be an increasing differentiable bijection from $\mathbb{R}$ to $\mathbb{R}$. If $G = F \circ P$ then the following relation holds between respective hazard rates of F and G:*

$$m_G = m_F \cdot P' \quad \text{and} \quad \mu_G = \mu_F \cdot P',$$

where $m_F = F'/F$, $m_G = G'/G$ and $\mu_F = F'/\bar{F}$, $\mu_G = G'/\bar{G}$. In particular, if

$$P_{g,\gamma}(x) = \begin{cases} -|x|^g, & x < 0, \\ +|x|^\gamma, & x \geq 0, \end{cases}$$

for γ, $g \in (0, +\infty)$, and if $m_F \in \mathcal{RV}_{g_0}(-\infty)$ and $\mu_F \in \mathcal{RV}_{\gamma_0}(\infty)$, then

$$m_G \in \mathcal{RV}_{g_0+g-1}(-\infty) \quad \text{and} \quad \mu_G \in \mathcal{RV}_{\gamma_0+\gamma-1}(\infty).$$

If furthermore $g_0 + g \in (0, +\infty)$ and $\gamma_0 + \gamma \in (0, +\infty)$, since G is by assumption a valid distribution, then G satisfies the point iii) in respective Assumptions 3.2 and 3.3.

The proof is postponed in Appendix B. It would be possible to extend the assumptions of Lemma 3.1. However we will focus here on invertible distributions on the whole support $\mathbb{R}$. Note that we focus here on distributions constructed by using the function $P_{g,\gamma}$, where g and γ must be positive in order to get an increasing cdf for induced distribution G.

It is worth mentioning that for distributions F having a symmetric density, such that $F'(-x) = F'(x)$, for all $x \in \mathbb{R}$, one easily shows that $m_F(-x) = \mu_F(x)$, for all $x \in \mathbb{R}$. In this symmetric case, when the hazard rate is regularly varying, both rates $m_F = F'/F$ and $\mu_F = F'/\bar{F}$ have the same respective regular variation index at $-\infty$ and $+\infty$.

Remark that many distributions having regularly varying hazard rate can be proposed. For instance, a study of some distributions having regularly varying hazard rate is given in Asmussen and Kortschak [3]. Lemma 3.1 may help building such distributions. Interesting relations can be found between the class of subexponential distributions (denoted by $\mathcal{S}$) and regularly varying hazard rates. The interested reader is referred for instance to Klüppelberg [40], Klüppelberg [41], Su and Tang [55]. Recall that subexponential distributions have unbounded support and may belong to Fréchet and Gumbel maximum domain of attraction (see Section 4 in Goldie and Klüppelberg [30]). In particular Goldie and Klüppelberg [30] proved that if $\lim_{x\to\infty}\mu_G(x)=\lim_{x\to\infty}\frac{G'(x)}{\bar{G}(x)}=0$, $\lim_{x\to\infty}x\,\mu_G(x)=\infty$, and $\mu_G\in RV_{\gamma-1}(\infty)$, with $\gamma\in(0,1)$, then $G\in\mathcal{S}^*$, where "the class $\mathcal{S}^*$ is almost $\mathcal{S}\cap\{G:\mathbb{E}[G]<\infty\}$." A precise formulation can be found in Klüppelberg [39] (see Remark 1 and Corollary 3.9 in Goldie and Klüppelberg [30]). Remark that this type of distribution functions above satisfies Assumption 3.3.

In the following we illustrate Lemma 3.1 through two examples of transformations. Furthermore, we discuss if the considered transformations satisfy Lower-tail Assumption 3.2 and Upper-tail Assumption 3.3. These transformations will be useful in Section 5.1.

Examples of transformations. Using the idea of Lemma 3.1, we now present two particular distributions, defined on the whole real line, with regularly varying hazard rates. In particular, as explained in the beginning of Section 3.1, we exclude distributions having support $[x_0,\infty)$ or $(-\infty,x_0]$ for some finite given real x_0. The considered distributions rely on the following exponentiation function, with parameters $g\in(0,\infty)$ and $\gamma\in(0,\infty)$,

$$P_{g,\gamma}(x)=\begin{cases}-|x|^g & \text{if } x<0,\\ +|x|^\gamma & \text{if } x\geq 0.\end{cases}$$

One can check that $P'_{g,\gamma}(x)=g\,|x|^{g-1}$ when $x<0$ and $P'_{g,\gamma}(x)=\gamma\,|x|^{\gamma-1}$ when $x>0$ so that the derivative $P'_{g,\gamma}(x)$ is positive and well defined on $\mathbb{R}\setminus\{0\}$, with finite identical right and left derivatives at $x=0$. Indeed, when $(g-1)(\gamma-1)\geq 0$, we have

$$P'_{g,\gamma}(0)=\begin{cases}0, & \text{if } g>1,\gamma>1,\\ 1, & \text{if } g=1,\gamma=1,\\ \infty, & \text{if } g<1,\gamma<1.\end{cases}$$

The function $P_{g,\gamma}$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$, with inverse $P^{-1}_{g,\gamma}=P_{\frac{1}{g},\frac{1}{\gamma}}$. In the particular case where $g=\gamma$, $P_{\gamma,\gamma}(x)=\text{sign}(x)\,|x|^\gamma$.

- Case power logit: Remark that $F(x)=\text{logit}^{-1}(x)$ has a symmetric density and a slowly varying hazard rate, so that
$$m_F\in\mathcal{RV}_{g_0}(-\infty)\quad\text{and}\quad \mu_F\in\mathcal{RV}_{\gamma_0}(+\infty),\ \text{with } g_0=\gamma_0=0.$$
As an application of Lemma 3.1, one then defines $G=F\circ P_{g,\gamma}$ for $g,\gamma,\sigma\in(0,\infty)$ and $\mu\in\mathbb{R}$, as
$$G(x)=\frac{1}{1+\exp\left(-P_{g,\gamma}\left(\frac{x-\mu}{\sigma}\right)\right)},\quad x\in\mathbb{R}.$$
One easily checks that G is a continuous c.d.f such that $\forall\,x\in\mathbb{R}$, $G(x)\in(0,1)$ and $\lim_{x\to-\infty}G(x)=0$, $\lim_{x\to\infty}G(x)=1$. Setting $R(x)=\frac{x-\mu}{\sigma}$, $G=\text{logit}^{-1}\circ P_{g,\gamma}\circ R$ and the inverse function of G is $G^{-1}=R^{-1}\circ P_{\frac{1}{g},\frac{1}{\gamma}}\circ\text{logit}$, i.e.
$$G^{-1}(x)=\mu+\sigma\cdot P_{\frac{1}{g},\frac{1}{\gamma}}\left(\text{logit}(x)\right),\quad x\in(0,1).$$
From derivatives of $P_{g,\gamma}$, one can check that $G'(x)$ is positive and well defined on $\mathbb{R}\setminus\{\mu\}$, with finite (*resp.* infinite) identical right and left derivatives at $x=\mu$ when $g>1$ and $\gamma>1$, or when $g=\gamma=1$ (*resp.* when $g<1$ and $\gamma<1$). Since $G'(x)=\frac{1}{\sigma}G(x)\cdot\bar{G}(x)\cdot P'_{g,\gamma}\circ R(x)$. One easily shows that the rates $m_G=G'/G$ and $\mu_G=G'/\bar{G}$ are such that
$$m_G\in\mathcal{RV}_{g-1}(-\infty)\quad\text{and}\quad \mu_G\in\mathcal{RV}_{\gamma-1}(\infty).$$
This power-logit case gives an example of a function G that can exhibit any chosen coefficient $g,\gamma\in(0,+\infty)$, thus satisfying both Assumptions 3.2 and 3.3.

- Case power Gumbel: Considering $F(x) = \exp(-\exp(-x))$, one easily shows that the lower rate m_F is rapidly varying at $-\infty$ (denoted by $m_F \in \mathcal{RV}_\infty(-\infty)$; see Section 2.4 in Bingham et al. [7] for a formal definition of rapid variation), and that the upper hazard rate μ_F is slowly varying i.e.,

$$m_F \in \mathcal{RV}_\infty(-\infty), \text{ with } g_0 = +\infty \quad \text{ and } \quad \mu_F \in \mathcal{RV}_{\gamma_0}(+\infty), \text{ with } \gamma_0 = 0.$$

As an application of Lemma 3.1, one then defines $G = F \circ P_{g,\gamma}$ for $g, \gamma, \sigma \in (0, \infty)$ and $\mu \in \mathbb{R}$, as

$$G(x) = \exp\left(-\exp\left(-P_{g,\gamma}\left(\frac{x-\mu}{\sigma}\right)\right)\right), \quad x \in \mathbb{R}.$$

One recognizes Gumbel (or Type I Extreme Value Distribution) when $g = \gamma = 1$ (other two types of extreme value distributions are not considered since they are not such that $G(x) \in (0, 1)$ for all $x \in \mathbb{R}$). One easily checks that G is a continuous c.d.f such that $\forall\, x \in \mathbb{R}$, $G(x) \in (0, 1)$ and $\lim_{x\to-\infty} G(x) = 0$, $\lim_{x\to\infty} G(x) = 1$. Setting $R(x) = \frac{x-\mu}{\sigma}$, $G = \exp \circ - \exp \circ - P_{g,\gamma} \circ R$ and the inverse function of G is $G^{-1} = R^{-1} \circ P_{\frac{1}{g},\frac{1}{\gamma}} \circ - \ln \circ - \ln$, i.e.

$$G^{-1}(x) = \mu + \sigma \cdot P_{\frac{1}{g},\frac{1}{\gamma}}\left(-\ln(-\ln x)\right), \quad x \in (0, 1).$$

From derivatives of $P_{g,\gamma}$, one can check that $G'(x)$ is positive and well defined on $\mathbb{R} \setminus \{\mu\}$, with finite (*resp.* infinite) identical right and left derivatives at $x = \mu$ when $g > 1$ and $\gamma > 1$, or when $g = \gamma = 1$ (*resp.* when $g < 1$ and $\gamma < 1$). Since $G'(x) = \frac{1}{\sigma} G(x) \cdot P'_{g,\gamma} \circ R(x) \cdot \exp\left(-P_\gamma \circ R(x)\right)$. The hazard rate can be written when x is large enough, $\mu_G(x) = \frac{\gamma}{\sigma}\left|\frac{x-\mu}{\sigma}\right|^{\gamma-1} G(x) \cdot H \circ \exp \circ - P_\gamma \circ R(x)$, with $H(x) = \frac{x}{1-\exp(-x)}$, $x \in \mathbb{R}^+$. As $\lim_{x\to0^+} H(x) = 1$, one finally have

$$m_G \in \mathcal{RV}_\infty(-\infty) \quad \text{ and } \quad \mu_G \in \mathcal{RV}_{\gamma-1}(\infty).$$

As one can see, this power-Gumbel case gives an example of a function G that can exhibit any chosen coefficient $\gamma \in (0, +\infty)$ but where $m_G \in \mathcal{RV}_\infty(-\infty)$, thus satisfying Assumption 3.3 but not Assumption 3.2. Using such a power-Gumbel distribution would require a separated treatment for the lower tail case.

Under Assumptions 3.2 and 3.3 our main following result characterizes the regularly varying property for the inverse transformed generator $\widetilde{\psi}$.

Theorem 3.1 (Regularly varying transformed generator) *Assume that the initial generator ϕ_0 and the transformed one $\widetilde{\phi}$ satisfy Assumption 3.1. Assume that f, ϕ_0 and G satisfy Lower-tail Assumptions 3.2, so that in particular when $x \to -\infty$, f has an asymptote $a\,x + b$, $a \in (0, +\infty)$, $b \in (-\infty, +\infty)$, $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, $r_0 \in [0, +\infty]$, and the rate $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, for $g \in (0, +\infty)$. Then the inverse transformed generator $\widetilde{\psi}$ is such that*

$$\widetilde{\psi} \in \mathcal{RV}_{-\tilde{r}}(0) \;\; \textit{with} \;\; \tilde{r} = r_0 \cdot a^{-g}. \tag{11}$$

Assume now that f, ϕ_0 and G satisfy Upper-tail Assumption 3.3, so that in particular when $x \to +\infty$, f has an asymptote $\alpha\, x + \beta$, $\alpha \in (0, +\infty)$, $\beta \in (-\infty, +\infty)$, $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, $\rho_0 \in [1, +\infty]$, and the hazard rate $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, for $\gamma \in (0, +\infty)$. Then, the inverse transformed generator $\widetilde{\psi}$ is such that

$$\widetilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1) \;\; \textit{with} \;\; \tilde{\rho} = \rho_0 \cdot \alpha^{-\gamma}. \tag{12}$$

Furthermore, $\tilde{\rho} = \rho_0 \cdot \alpha^{-\gamma} \geq 1$.

The proof is postponed to Appendix B.

Remark 7 *Notice that Theorem 3.1 provides an implicit admissible range of the slope parameter α of the upper asymptote of the conversion function f. Indeed if the transformed generator $\widetilde{\phi}$ satisfies Assumption 3.1, then, from Remark A, the regular variation index of the transformed inverse generator $\widetilde{\psi}$ in Equation* (12) *have to be $\tilde{\rho} = \rho_0 \cdot \alpha^{-\gamma} \geq 1$. In particular this implies that, given a regular variation index ρ_0 for the initial inverse generator ψ_0 and a regular variation index $\gamma - 1$ for the hazard rate μ_G, we obtain an admissible range for parameter α. An illustration of Theorem 3.1 in a bivariate setting will be given in Corollary 5.1.*

Lemma 3.2 (Regular Variation of transformation T_f) *Under lower regular Assumption 3.2, when in particular G is such that $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$ for $g \in (0,+\infty)$ and f has an asymptote $ax+b$ for $a \in (0,+\infty)$, $b \in (-\infty,+\infty)$ as x tends to $-\infty$, then*

$$T_f \in \mathcal{RV}_{\widetilde{a}}(0) \ \text{with} \ \widetilde{a} = a^g.$$

Under upper regular Assumption 3.3, when in particular G is such that $\mu_G = G'/(1-G) \in \mathcal{RV}_{\gamma-1}(+\infty)$ for $\gamma \in (0,+\infty)$ and f has an asymptote $\alpha x + \beta$ for $\alpha \in (0,+\infty)$, $\beta \in (-\infty,+\infty)$ as x tends to $+\infty$, then

$$M \circ T_f \in \mathcal{RV}_{\widetilde{\alpha}}(1) \ \text{with} \ \widetilde{\alpha} = \alpha^\gamma.$$

The proof is postponed to Appendix B.

Theorem 3.2 *Assume that $\mu_{\phi_0} = \phi_0'/\phi_0 \in \mathcal{RV}_{k_0-1}(\infty)$, with $k_0 \in [0,+\infty]$, then if furthermore $T_f \in \mathcal{RV}_{\widetilde{a}}(0)$ for some index $\widetilde{a} \in (0,+\infty)$, then*

$$\mu_{\tilde{\phi}} = \tilde{\phi}'/\tilde{\phi} \in \mathcal{RV}_{\widetilde{k}-1}(\infty) \ \text{with} \ \widetilde{k} = k_0.$$

Assume that $m_{\phi_0} = \phi_0'/(1-\phi_0) \in \mathcal{RV}_{-\kappa_0-1}(0)$, with $\kappa_0 \in [0,+\infty]$, then if furthermore $M \circ T_f \in \mathcal{RV}_{\widetilde{\alpha}}(1)$ for some index $\widetilde{\alpha} \in (0,+\infty)$, then

$$m_{\tilde{\phi}} = \tilde{\phi}'/(1-\tilde{\phi}) \in \mathcal{RV}_{\widetilde{\kappa}-1}(0) \ \text{with} \ \widetilde{\kappa} = \kappa_0.$$

The proof is postponed to Appendix B.

Using Theorem 3.1 and Theorem 3.2, we provide in Section 3.3 below, the expression of the transformed multivariate tail coefficients $\widetilde{\lambda}_L^{(h,d-h)}$ and $\widetilde{\lambda}_U^{(h,d-h)}$ associated to the transformed Archimedean copula $\widetilde{C}_{T_f,C_0}$ with generator $\tilde{\phi} = T_f \circ \phi_0$, satisfying Assumption 3.1.

3.3. Multivariate transformed tail dependence coefficients

From Theorem 3.1, the following result provides the form of the multivariate tail coefficients $\widetilde{\lambda}_L^{(h,d-h)}$ and $\widetilde{\lambda}_U^{(h,d-h)}$ for a transformed Archimedean copula $\widetilde{C}_{T_f,C_0}$ with generator $\widetilde{\phi} = T_f \circ \phi_0$, with T_f as in Equation (9). This result is a multivariate extension of Proposition 4.2 and 4.3 in Durante et al. [20] for transformations as in Equation (9). Some results about the tail dependence coefficients of bivariate Archimedean copulas are given by Juri and Wüthrich [35], Juri and Wüthrich [36]. Furthermore, Charpentier and Segers [9] in their Table 2 analysed the tail of some particular types of transformed generator.

Theorem 3.3 (Multivariate tail coefficients of transformed Archimedean copula) *Assume that the initial generator ϕ_0 and the transformed one $\widetilde{\phi}$ satisfy Assumption 3.1. Assume that f, ϕ_0 and G satisfy Lower-tail Assumption 3.2, so that in particular when $x \to -\infty$, f has an asymptote $ax+b$, $a \in (0,+\infty)$, $b \in (-\infty,+\infty)$, $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, $r_0 \in [0,+\infty]$, and the hazard rate $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, for $g \in (0,+\infty)$. Then, the transformed multivariate lower tail dependence coefficient associated to $\tilde{\phi}$ is given by:*

$$\widetilde{\lambda}_L^{(h,d-h)} = \begin{cases} \text{see Theorem 3.4}, & \text{if } r_0 = 0, \\ d^{-a^g\, r_0^{-1}}\,(d-h)^{a^g\, r_0^{-1}}, & \text{if } r_0 \in (0,+\infty), \\ 1, & \text{if } r_0 = +\infty. \end{cases} \tag{13}$$

Assume now that f, ϕ_0 and G satisfy Upper-tail Assumption 3.3, so that in particular f has an asymptote $\alpha\, x + \beta$, $\alpha \in (0,+\infty)$, $\beta \in (-\infty,+\infty)$, $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, $\rho_0 \in [1,+\infty]$, and the hazard rate $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, $\gamma \in (0,+\infty)$. Then, when $\widetilde{\rho} = \rho_0\, \alpha^{-\gamma} \neq 1$, the transformed multivariate upper tail dependence coefficient associated to $\widetilde{\phi}$ is given by:

$$\widetilde{\lambda}_U^{(h,d-h)} = \begin{cases} \text{see Theorem 3.5}, & \text{if } \rho_0 = 1, \\ \dfrac{\sum_{i=1}^{d} C_d^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}{\sum_{i=1}^{d-h} C_{d-h}^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}, & \text{if } \rho_0 \in (1,+\infty), \\ 1, & \text{if } \rho_0 = +\infty. \end{cases} \tag{14}$$

The proof is postponed to Appendix B. The case $\widetilde{\rho} = 1$ is discussed in Theorem 3.5. As one can see in previous Theorem 3.3, the impact of initial generator ϕ_0, conversion function f and distribution G are clearly separated via respective coefficients couples (r_0, ρ_0), (a, α), (g, γ). It is thus possible to modify any of these parts of the global distortion $T_f = G \circ f \circ G^{-1}$ to fit a desired tail dependence. Only some few cases do not allow to modify a given tail dependence, as showed in following Remark 8.

Remark 8 *Notice that if $r_0 = 0$ in Theorem 3.3 then in particular $\widetilde{\lambda}_L^{(h,d-h)} = 0$, for all h and d. This means for instance that if one considers as initial generator $\psi_0(t) = -\ln(t)$, i.e., the independent copula, it is not possible, with this kind of transformations, to generate some dependency in the multivariate transformed lower tails.*

For sake of clarity we provide in the following the bivariate version of Theorem 3.3.

Corollary 3.1 (Bivariate tail coefficients of transformed Archimedean copula) *Under assumptions of Theorem 3.3, in the bivariate setting, with $d = 2$ and $h = 1$, the transformed bivariate upper and lower tail dependence coefficients are given by:*

$$\widetilde{\lambda}_L^{(1,1)} = 2^{-a^g\, r_0^{-1}}, \qquad \text{and} \qquad \widetilde{\lambda}_U^{(1,1)} = 2 - 2^{\alpha^\gamma \cdot \rho_0^{-1}}.$$

Also for the transformed copulas in Equation (10), there are essentially two categories: if $\widetilde{r} \in (0, +\infty]$ in Theorem 3.1 (*resp.* $\widetilde{\rho} \in (1, +\infty]$), then the transformed lower (*resp.* upper) tail exhibits asymptotic dependence, while if $\widetilde{r} = 0$ (*resp.* $\widetilde{\rho} = 1$), then there is asymptotic independence. In the asymptotic independent case, Theorems 3.4 and 3.5 below quantify the rate of convergence toward 0 of $\widetilde{\lambda}_L^{(h,d-h)}(u)$ and $\widetilde{\lambda}_U^{(h,d-h)}(u)$.

Theorem 3.4 (Transformed case with $r_0 = 0$) *Assume that $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 = 0$. Denote by $\widetilde{\lambda}_L^{(h,d-h)}(u) = \frac{\tilde{\delta}^{(d)}(u)}{\tilde{\delta}^{(d-h)}(u)}$, with $\tilde{\delta}^{(d)}$ the transformed diagonal section associated to the $d-$dimensional copula $\tilde{C}$ in Equation (10). Then one gets,*

$$\widetilde{\lambda}_L^{(h,d-h)} = \lim_{u \to 0^+} \widetilde{\lambda}_L^{(h,d-h)}(u) = 0.$$

Assume that $\mu_{\phi_0} = \phi_0'/\phi_0 \in \mathcal{RV}_{k_0-1}(\infty)$, with $k_0 \in [0, +\infty)$, then if furthermore $T_f \in \mathcal{RV}_{\tilde{a}}(0)$ for some index $\tilde{a} \in (0, +\infty)$, then

$$\widetilde{\lambda}_L^{(h,d-h)}(u) = \frac{\tilde{\delta}^{(d)}(u)}{\tilde{\delta}^{(d-h)}(u)} \in \mathcal{RV}_{\widetilde{z}}(0) \;\; \text{with } \widetilde{z} = d^{k_0} - (d-h)^{k_0}.$$

In particular from Lemma 3.2, if T_f satisfies Assumption 3.2, then $T_f \in \mathcal{RV}_{\tilde{a}}(0)$ for some index $\tilde{a} \in (0, +\infty)$ as required.

Using Theorems 3.2 and 2.2, Theorem 3.4 comes down trivially.

Theorem 3.5 (Transformed case with $\tilde{\rho} = 1$) *Assume $\tilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1)$, with $\tilde{\rho} = 1$ and that the associated $\widetilde{\phi}$ is a d times continuously differentiable generator. Denote by $\widetilde{\lambda}_U^{(h,d-h)}(u) = \frac{\tilde{r}_d(u)}{\tilde{r}_{d-h}(u)}$, with $\tilde{r}_d(u) = \sum_{i=1}^d (-1)^i\, C_d^i\, \tilde{\delta}^{(i)}(u)$ and $\tilde{\delta}^{(i)}$ the transformed diagonal section associated to the $i-$dimensional copula $\widetilde{C}$ in Equation (10). Then one gets,*

$$\widetilde{\lambda}_U^{(h,d-h)} = \lim_{u \to 1^-} \widetilde{\lambda}_U^{(h,d-h)}(u) = 0.$$

Furthermore,

- *if $(-D)^d \tilde{\phi}(0)$ is finite and not zero, where D is the derivative operator, then we get*

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1);$$

- *if $\tilde{\psi}'(1) = 0$ and the function $\widetilde{L}(s) := s \,\frac{\mathrm{d}}{\mathrm{d}s}\{\frac{\tilde{\psi}(1-s)}{s}\}$ is positive and $\widetilde{L} \in \mathcal{RV}_0(0)$, then we get*

$$\widetilde{\lambda}_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1).$$

The proof of Theorem 3.5 comes down directly by Theorem 2.3.

Remark 9 *From Theorem 3.1, we know that if* $\psi_0 \in \mathcal{RV}_{-r_0}(0)$*, with* $r_0 = 0$*, then* $\tilde{r} = r_0 \cdot a^{-g} = 0$ *for all a and g (see Equation* (11)*). Then automatically* $\widetilde{\lambda}_L^{(h,d-h)} = 0$ *in Theorem 3.4 (see also discussion in Remark 8). Conversely in the upper case* $\tilde{\rho} = \rho_0 \cdot \alpha^{-\gamma}$ *(see Equation* (12)*) then* $\rho_0 = 1$ *does not guarantee in Theorem 3.5 that the transformed copula is also asymptotic independent in the upper tails, i.e.,* $\tilde{\rho} = 1$.

4. Transformed generators with given tail dependence

4.1. Hyperbolic conversion functions and tail dependence

In practice, one can estimate lower and upper tail coefficients. A large literature has been developed in order to propose consistent non-parametric tail dependence coefficients estimators, essentially based on the non-parametric copula estimator and Extreme Value Theory (see for instance de Haan and Ferreira [12], Schmidt and Stadtmüller [52], Dobric and Schmid [19]).

However, even when such tail coefficients are perfectly known, using usual Archimedean generators does not allow to get both lower and upper tail coefficients in the general case where their values belong to $(0, 1)$. For example, in Table 1 of Charpentier and Segers [8], only two generators among 23 exhibit both lower tail dependence and upper tail dependence (see generators numbered 12 and 14 in their table). Unfortunately none of these two allow to choose separately lower and upper tail coefficients as they both depend on one single parameter. Even for other kinds of copulas, like elliptical copulas, it is difficult to find simple parametric expressions with both lower and upper tail dependencies that are parametrized by more than one parameter.

Here we propose a generic way to construct families of Archimedean generators presenting a chosen couple of lower and upper tail coefficients. This construction relies directly on theoretical results of the previous Section 3.

Under assumptions of Theorem 3.3, the transformation $T_f = G \circ f \circ G^{-1}$ is a transformation with a particular conversion function f having asymptotes at $-\infty$ or $+\infty$. This is exactly the framework of Di Bernardino and Rullière [14], [15] and [16] where conversion functions f are compositions of hyperbolas, with asymptotes at both $-\infty$ or $+\infty$. In these papers, slopes a and α of each asymptote of f are tractable given functions of some isolated parameters of f. Parametric estimation of copulas within this framework is thus possible following the methodology of these articles, even with given tail coefficients. The only change is that some parameters will be fixed to some values instead of being estimated.

Consider for example a transformed Archimedean copula, having generator $\tilde{\phi} = T_f \circ \phi_0$, where ϕ_0 is an initial given generator, and where the transformation is as in Equation (9), with in particular for any $x \in (0, 1)$,

$$T_f(x) = G \circ f \circ G^{-1}(x).$$

Choose for example a simple hyperbolic conversion function $f(x) = H(x)$, $x \in \mathbb{R}$, as defined in Bienvenüe and Rullière [6], with

$$H_{m,\,h,\,p_1,\,p_2,\,\eta}(x) = m - h + (\mathrm{e}^{p_1} + \mathrm{e}^{p_2})\frac{x - m - h}{2} - (\mathrm{e}^{p_1} - \mathrm{e}^{p_2})\sqrt{\left(\frac{x - m - h}{2}\right)^2 + \mathrm{e}^{\eta - \frac{p_1 + p_2}{2}}}, \tag{15}$$

with m, h, p_1, $p_2 \in \mathbb{R}$, and one smoothing parameter $\eta \in \mathbb{R}$. Functions H has been chosen in order to have unbounded real parameters, and to be readily invertible: a simple change of the sign of some parameters leads to the inverse function. After some calculations, one can check indeed that

$$H^{-1}_{m,\,h,\,p_1,\,p_2,\,\eta}(x) \quad = \quad H_{m,-h,-p_1,-p_2,\eta}(x). \tag{16}$$

As a consequence, transformations T_H based on conversion functions H will also be readily invertible since $T_H^{-1} = T_{H^{-1}}$.

When the smoothing parameter η tends to $-\infty$, the hyperbole H tends to the angle function

$$A_{m,h,p_1,p_2}(x) = m - h + (x - m - h)\left(\mathrm{e}^{p_1} 1_{\{x<m+h\}} + \mathrm{e}^{p_2} 1_{\{x>m+h\}}\right).$$

It is clear that in this framework, $f = H$ satisfies respective Assumptions 3.2 and 3.3. In particular, f has an asymptote $ax+b$ at $-\infty$ with $a = \mathrm{e}^{p_1}$, and an asymptote $\alpha x + \beta$ at $+\infty$ with $\alpha = \mathrm{e}^{p_2}$. More complex conversion functions could be used by composing such hyperbolas, as explained in Bienvenüe and Rullière [6]. For the sake of clarity, we consider here a single hyperbola.

Assume that the inverse initial generator ψ_0 satisfies Assumptions 3.2 and 3.3, with in particular $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in (0, +\infty)$, and $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty)$.

Assume that the distribution G also satisfies Assumptions 3.2 and 3.3, with in particular $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$, with $g \in (0, +\infty)$ and $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, where $\bar{G} = 1 - G$ and $\gamma \in (0, +\infty)$. For example, if one set $G = \mathrm{logit}^{-1}$, then $g = \gamma = 1$.

From Theorem 3.3, the transformed multivariate tail coefficients can be written as:

$$\begin{aligned}
\widetilde{\lambda}_L^{(h,d-h)} &= d^{-a^g\, r_0^{-1}}\,(d-h)^{a^g\, r_0^{-1}}, \\
\widetilde{\lambda}_U^{(h,d-h)} &= \frac{\sum_{i=1}^{d} C_d^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}{\sum_{i=1}^{d-h} C_{d-h}^i (-1)^i \cdot i^{\alpha^\gamma \rho_0^{-1}}}.
\end{aligned}$$

Let us illustrate the bivariate case, for which expressing a and α as functions of $\widetilde{\lambda}_L^{(h,d-h)}$ and $\widetilde{\lambda}_U^{(h,d-h)}$ is straightforward. In the bivariate case, $\widetilde{\lambda}_L^{(1,1)} = 2^{-a^g\, r_0^{-1}}$ and $\widetilde{\lambda}_U^{(1,1)} = 2 - 2^{\alpha^\gamma \cdot \rho_0^{-1}}$ (see Corollary 3.1), so that if these tail coefficients are given, then we can easily find $a = \mathrm{e}^{p_1}$ and $\alpha = \mathrm{e}^{p_2}$ as functions of $\widetilde{\lambda}_L^{(1,1)}$ and $\widetilde{\lambda}_U^{(1,1)}$.
On can check that with the chosen assumptions, $\widetilde{r} = r_0\, a^{-g} \in (0, \infty)$ so that $\widetilde{\lambda}_L^{(1,1)} \in (0, 1)$. For the upper tail, $\widetilde{\rho} = \rho_0\, \alpha^{-\gamma}$ must belong to $[1, \infty]$ if the transformed generator is valid, and due to chosen assumptions, $\widetilde{\rho} < +\infty$ so that $\widetilde{\lambda}_U^{(1,1)} \in [0, 1)$. We get in this case

$$p_1 = \frac{1}{g} \ln\left(-r_0\, \frac{\ln \widetilde{\lambda}_L^{(1,1)}}{\ln 2}\right) \quad \text{and} \quad p_2 = \frac{1}{\gamma} \ln\left(\rho_0\, \frac{\ln(2 - \widetilde{\lambda}_U^{(1,1)})}{\ln 2}\right). \tag{17}$$

Finally, only m and h and η remain to be estimated and the detailed methodology of Di Bernardino and Rullière [16] can be adapted to this purpose. For the sake of simplicity, we have illustrated here the case where f is a single hyperbola, in the dimension $d = 2$, when we focus in intermediate cases for lower and upper tail dependence, excluding perfect tail dependence or perfect tail independence. One can easily consider composite hyperbolas, with more parameters and the ability to fit both the tails and the central part of the copula. One can also consider the multivariate case, or special case of non intermediate tail dependence, for example when $r_0 = 0$, when $\widetilde{\lambda}_L^{(1,1)} = 0$, or when $\widetilde{\lambda}_U^{(1,1)} = 1$. Other interesting cases are the tail behaviour when G is rapidly varying, as for lower tail dependence when G is Gumbel distributed, i.e., $G(x) = \exp(-\exp(-x))$ (see Section 2.4 in Bingham et al. [7]). These extensions would be out of the scope of the present section, which just illustrates the methodology given tail dependence coefficients. Note that for one single hyperbola, this gives a class of Archimedean copulas exhibiting given (estimated) upper and lower tail dependence coefficients.

4.2. Illustration with logit transformations

In the following we provide an illustration of the procedure described above. Let the initial copula C_0 be a Clayton copula with parameter $\theta > 0$ with $\psi_0(t) = \frac{1}{\theta}\,(t^{-\theta} - 1)$. Assume that we want to transform the tails of the bivariate distribution (X, Y) in order to obtain an arbitrarily chosen couple of target tail coefficients: $\widetilde{\lambda}_L^{(1,1)} = 1/4$ and $\widetilde{\lambda}_U^{(1,1)} = 3/4$.

	chosen parameters				deduced parameters		tails coefficients			
	m	h	η	θ	p_1	p_2	$\lambda_{0,L}^{(1,1)}$	$\lambda_{0,U}^{(1,1)}$	$\widetilde{\lambda}_L^{(1,1)}$	$\widetilde{\lambda}_U^{(1,1)}$
A	0.5	0.9	-1	2	1.386	-1.133	0.707	0	0.25	0.75
B	0.5	-0.9	-1	2	1.386	-1.133	0.707	0	0.25	0.75
C	0.5	0.9	-2	4	2.079	-1.133	0.841	0	0.25	0.75
D	2	-0.9	1	0.2	-0.916	-1.133	0.031	0	0.25	0.75

Table 1: Considered parameters settings.

Then the transformed generator is such that $\tilde{\psi} = \psi_0 \circ T_f^{-1}$, where $T_f^{-1} = G \circ T_H^{-1} \circ G^{-1} = G \circ T_{H^{-1}} \circ G^{-1}$ (see Equation (16)). In the following, we consider $G = \text{logit}^{-1}$, then $g = \gamma = 1$. Using Equation (17) we can obtain the parameters p_1 and p_2 of the conversion function H given in (15).

In Table 1 we summarize the parameters setting considered. In particular we take arbitrarily chosen different values of parameters m, h, η and θ, in order to illustrate the variety of generators that can exhibit the same tail dependence coefficients. Only p_1 and p_2 are deduced from chosen values $\widetilde{\lambda}_L^{(1,1)} = 1/4$ and $\widetilde{\lambda}_U^{(1,1)} = 3/4$ using Equation (17). The set of parameters m, h, p_1, p_2, η fully characterize the transformation T_f, and θ characterize the initial non-transformed copula C_0. Then the transformed copula $\widetilde{C}_{T_f,C_0}$ is fully characterized by the parameters in Table 1.

First, one can check that proposed transformed generators are valid Archimedean generators. By construction, $\widetilde{\psi}$ is an increasing transformation of a decreasing initial generator, so that $\widetilde{\psi}$ is a positive decreasing function, with $\widetilde{\psi}(1) = 0$. To be valid, $\widetilde{\psi}$ must be a d-monotone function (see McNeil and Nešlehová [44]), i.e. here a convex function. For all cases presented in Table 1, we check that the derivatives of $\widetilde{\psi}(t)$ are increasing functions of t, thus leading to a valid convex generator. The increasing derivatives of $\widetilde{\psi}(t)$ are illustrated in Figure 1 (left).

Figure 1 (right) presents the RIF function (see Lemma 2.1) for these different parameters settings. As one can see, the indexes of regular variation $\widetilde{r}$ and $\widetilde{\rho}$ are obtained here by considering the limit $\lim_{x\to 0^+} \text{RIF}_{\widetilde{\psi}}(x)$ and $\lim_{x\to 1^-} \text{RIF}_{\widetilde{\psi}}(x)$.

By construction, the regular variation indexes $\widetilde{r}$ and $\widetilde{\rho}$ correspond to the vertical position of black dots in Figure 1 (right) (see Lemma 2.1 and Corollary 2.1):

$$\widetilde{r} = -\frac{\ln 2}{\ln(\widetilde{\lambda}_L^{(1,1)})} = 0.5 \quad \text{and} \quad \widetilde{\rho} - 1 = \frac{\ln 2}{\ln(2 - \widetilde{\lambda}_U^{(1,1)})} - 1 \simeq 2.1$$

One can also notice the variety of generators behaviours despite their common tail dependence coefficients, some generators being closer to the theoretical independence horizontal line $\text{RIF}_{\text{indep}}(x) = 0$, for all $x \in (0,1)$.
These illustrations show that one can propose many generators with given tail dependence coefficients. When fitting some data, it is thus possible to propose a fit that respects some estimated tail dependence coefficients, by deducing parameters p_1 and p_2 from tail coefficients and by estimating other parameters m, h, η, θ. It is also possible to improve the global quality of the fit by adding some parameters: this can be done easily by compositing more than one hyperbola, as detailed in Di Bernardino and Rullière [16] for example.
Recall that in the bivariate case, for a given copula C, cdf of a random vector (U_1, U_2), one defines *lower and upper tail dependence functions* for $u \in (0,1)$ as

$$\lambda_L(u) = \mathbb{P}[U_2 < u \mid U_1 < u] \quad \text{and} \quad \lambda_U(u) = \mathbb{P}[U_2 > u \mid U_1 > u] .$$

In Figure 2, we have drawn the function $\lambda_{LU}(u) = \mathbf{1}_{\{u \le 1/2\}}\lambda_L(u) + \mathbf{1}_{\{u > 1/2\}}\lambda_U(u)$, for some Clayton and Gumbel copulas (left and center panel), and for considered models A, B, C, D in Table 1 (right panel). One can notice that left limit at 0 and right limit at 1 of the function λ_{LU} gives the respective lower and upper tail dependence coefficients. One retrieves here the chosen tail coefficient targets 1/4 and 3/4 for models A, B, C, D (black dots in

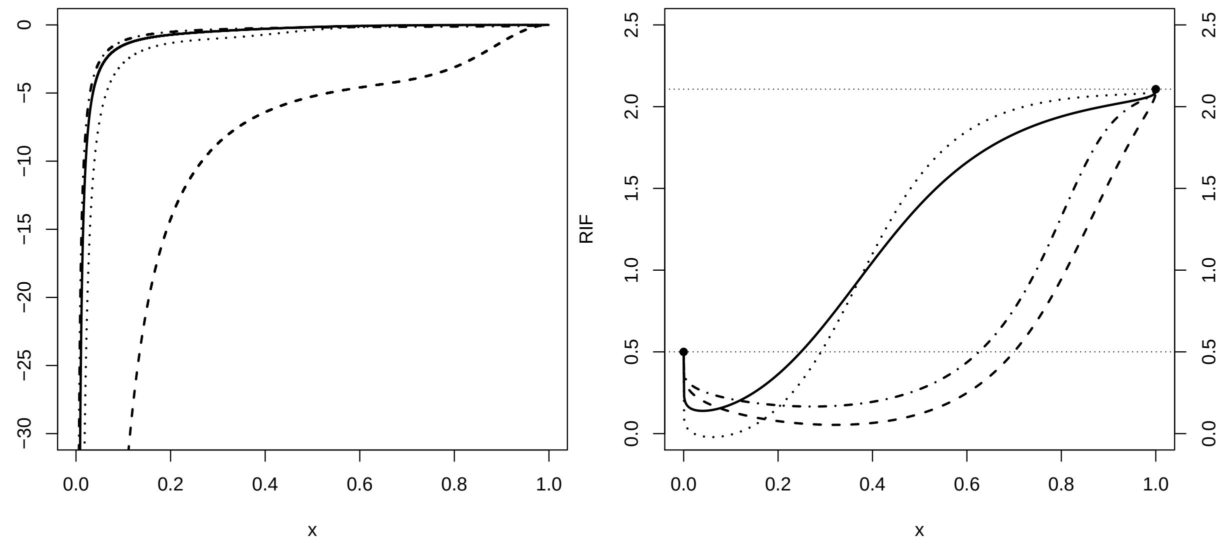

Figure 1: Derivatives of $\widetilde{\psi}$ (left) and $\mathrm{RIF}_{\widetilde{\psi}}$ functions (right) obtained using the parameter setting gathered in Table 1. Full line corresponds to parameters A in Table 1; dashed line to B; dotted line to C; dashed-dotted line to D.

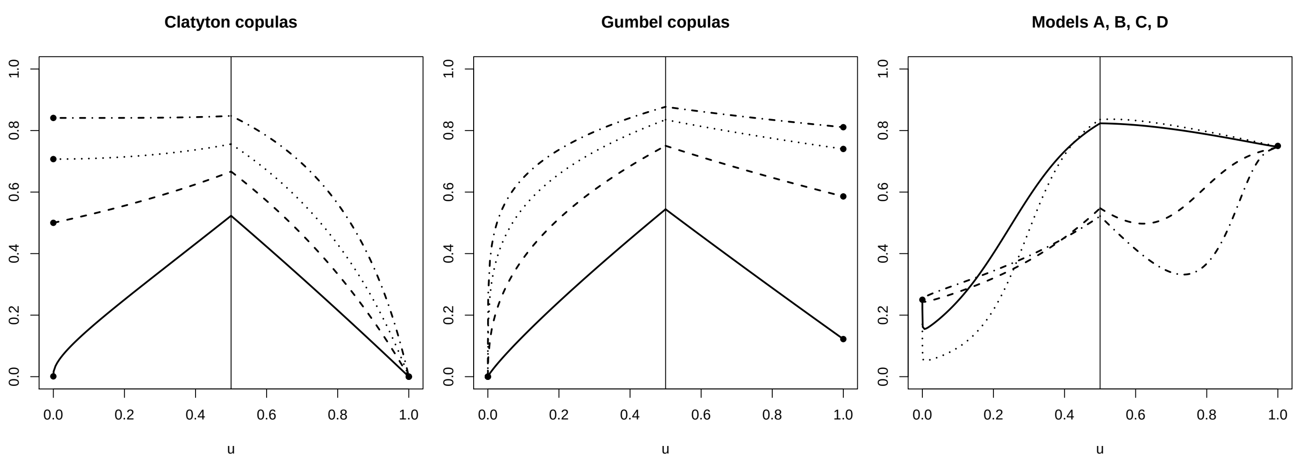

Figure 2: Shape of the function $\lambda_{LU}(u) = \mathbf{1}_{\{u\leq 1/2\}}\lambda_L(u) + \mathbf{1}_{\{u>1/2\}}\lambda_U(u)$ for some Clayton copulas (left panel), Gumbel copulas (center panel), and for models A,B,C,D in Table 1 (right panel). In the right panel, full line corresponds to parameters A in Table 1; dashed line to B; dotted line to C; dashed-dotted line to D.

right panel). Furthermore, one can see that among models A, B, C, D, in Table 1 the ways of converging toward fixed tail coefficients are very different.

As noticed in the bivariate case by Avérous and Dortet-Bernadet [4], "*many of the most commonly used parametric families of Archimedean copulas (such as the Clayton, Gumbel, Frank or Ali–Mikhail–Haq systems) possess strong dependence properties: they have the SI or the SD property and are ordered at least by* $<_{LTD}$", where Stochastic Increasingness (SI), Stochastically Decreasingness (SD) and Left-Tail Decreasingness (LTD) definitions are recalled in Avérous and Dortet-Bernadet [4], Joe [34]. It implies in particular that for many classical Archimedean copulas (and in particular for Clayton, Gumbel, Frank or Ali-Mikhail-Haq copulas) λ_{LU} is non-decreasing on $(0, 1/2)$ and non-increasing on $(1/2, 1)$. This can be seen on the left and center panel of Figure 2, but this is obviously not the case here in the right panel for models A, B, C, D. This surprising shape of λ_{LU} function emphasis the large diversity of tail behaviours that can be reached by proposed transformed Archimedean copulas.

5. Further applications and illustrations

In this section we propose some illustrations of possible applications of results developed above in this paper. In Section 5.1 we present a particular bivariate model. Finally in Section 5.2 we analyse the ergodic property of the stationary Markov chain $\{U_t : t \in \mathbb{Z}\}$ with transformed joint distribution $\widetilde{C}$ of (U_0, U_1).

5.1. Some applications in the bivariate setting

We now aim at illustrating the general previous results by considering a particular bivariate model ($d = 2$). In particular we are interested in the upper and lower tails of the obtained bivariate transformed copula. We consider the transformation T_f in Equation (9) with c.d.f. $G(x) = \text{logit}^{-1}(x)$, with $g = \gamma = 1$. The function $\text{logit}(x) = \ln(x/(1-x))$ has the advantage of having a symmetric density, with a very simple expression for the inverse function, $G(x) = \text{logit}^{-1}(x) = (1 + \exp(-x))^{-1}$. In this case $m_G \in \mathcal{RV}_0(-\infty)$ and $\mu_G \in \mathcal{RV}_0(+\infty)$. Furthermore, we consider the particular case where the conversion function f has affine asymptotes.

The bivariate version of model in Equation (10) is

$$\widetilde{C}_{T_f,C_0}(u,v) = T_f \circ C_0\left(T_f^{-1}(u),\, T_f^{-1}(v)\right), \tag{18}$$

where C_0 is an initial Archimedean copula transformed using T_f in Equation (9).
From Theorem 3.3, in the bivariate setting (i.e., $d = 2$ and $h = 1$) with $G(x) = \text{logit}^{-1}(x)$ ($g = \gamma = 1$), we obtain the following result.

Proposition 5.1 (Bivariate upper and lower tail coefficients for logit-linear transformed copulas) *Let C_0 be the initial bivariate Archimedean copula with associated generator ϕ. Assume that the conversion function f in Equation (9) has an asymptote $\overline{f}(x) = a\,x + b$ as x tends to $-\infty$ and $\overline{f}(x) = \alpha x + \beta$ as x tends to $+\infty$, with $a, \alpha \in (0, +\infty)$ and $b, \beta \in (-\infty, +\infty)$. We consider the transformation $T_{\overline{f}} = \text{logit}^{-1} \circ \overline{f} \circ \text{logit}(x)$, and the associated transformed copula $\widetilde{C}_{T_{\overline{f}},C_0}$ as in Equation (18). It holds that*

i) if $\lambda_L(C_0)$ in (1) exists, then $\widetilde{\lambda}_L(\widetilde{C}_{T_{\overline{f}},C_0}) = (\lambda_L(C_0))^a$, with $a \in (0, +\infty)$,

ii) if $\lambda_U(C_0)$ in (2) exists, then $\widetilde{\lambda}_U(\widetilde{C}_{T_{\overline{f}},C_0}) = 2 - (2 - \lambda_U(C_0))^\alpha$, with $\alpha \in \left(0, \frac{\ln(2)}{\ln(2-\lambda_U(C_0))}\right]$.

The proof is postponed to Appendix B. Remark that in this setting for $d = 2$, from the bivariate version of Theorem 3.3, i.e., Corollary 3.1, we get trivially

$$\widetilde{\lambda}_L^{(1,1)} = \left(2^{-\frac{1}{r_0}}\right)^{\widetilde{a}} = (\lambda_L(C_0))^{\widetilde{a}} \text{ and } \widetilde{\lambda}_U^{(1,1)} = \left(2 - \left(2^{-\frac{1}{\rho_0}}\right)^{\widetilde{\alpha}}\right) = 2 - (2 - \lambda_U(C_0))^{\widetilde{\alpha}}.$$

Since here $g = \gamma = 1$, then $\widetilde{\alpha} = \alpha^\gamma = \alpha$ and $\widetilde{a} = a^g = a$ and we obtain the result in Proposition 5.1. However, in this very specific framework where $d = 2$ and where $g = \gamma = 1$, Proposition 5.1 can be also seen as an application of Propositions 4.2 and 4.3 in Durante et al. [20].

By direct application of Proposition 5.1 we get the following easy restriction of the lower and upper asymptote slopes a and α when the initial copula C_0 is in the domain of attraction of the independence (i.e., $\lambda_L(C_0) = \lambda_U(C_0) = 0$).

Corollary 5.1 (Admissible values of asymptote slopes) *Let $\lambda_L(C_0) = \lambda_U(C_0) = 0$. For the lower tail coefficient it holds that:*

$$\widetilde{\lambda}_L(\widetilde{C}_{T_{\overline{f}},C_0}) = 0, \quad \textit{for all values of slope } a \in (0, +\infty).$$

For the upper tail coefficient, it holds that:

$$\widetilde{\lambda}_U(\widetilde{C}_{T_{\overline{f}},C_0}) = \begin{cases} 0, & \textit{if } \alpha = 1, \\ c \in (0,1), & \textit{if } \alpha \in (0,1). \end{cases}$$

Remark that Corollary 5.1 is consistent with Theorem 3.1 and Remark 7.

From Proposition 5.1, we prove that there exists a particular function that satisfies the assumptions of our Theorem 3.3 and Propositions 4.2-4.3 in Durante et al. [20], i.e. $T_{\overline{f}} = \text{logit}^{-1} \circ \overline{f} \circ \text{logit}(x)$. Furthermore we obtain the value of the link-coefficient between the initial and the transformed tail dependence coefficients, i.e. the slopes α and a of the asymptotes of the conversion function f. Finally Proposition 5.1 restricts the range of values for the parameter α.

5.2. Transformed geometrically ergodic Markov chain

Theorem 3.1 can be an important tool to obtain the ergodic property of the stationary Markov chain $\{U_t : t \in \mathbb{Z}\}$ with transformed joint distribution $\widetilde{C}$ of (U_0, U_1). For sake of clarity, in the following we firstly recall the definition of geometric ergodicity (see Definition 5.1) and the geometric ergodic theorem for Archimedean copulas (see Theorem 5.1).

Definition 5.1 (Definition 3.1. in Beare [5]) *The stationary Markov chain $\{U_t : t \in \mathbb{Z}\}$ is said to be geometrically ergodic if, for a.e. $u \in (0,1)$, there exists a real number $l > 1$ such that*

$$\sum_{j=1}^{\infty} l^j \sup_{B \in \mathcal{B}} |\, \mathbb{P}[U_j \in B \,|\, U_0 = u] - \mathbb{P}[U_j \in B] \,| < \infty,$$

where $\mathcal{B}$ denote the σ-field of Borel subsets of $(0,1)$.

For a stationary real valued Markov chain, geometric ergodicity is equivalent to exponentially fast β-mixing.

Theorem 5.1 (Theorem 3.1. in Beare [5]) *Suppose $\{U_t : t \in \mathbb{Z}\}$ is a stationary Markov chain whose invariant distribution is uniform on $(0,1)$. Let C denote the joint distribution function of (U_0, U_1). Assume that the copula C is strictly Archimedean, with an inverse generator ψ satisfying the following conditions:*

i) $\psi \in \mathcal{RV}_{-r}(0)$ *with* $r \in [0, +\infty)$, *and* $\psi \in \mathcal{RV}_{\rho}(1)$, *with* $\rho \in [1, +\infty)$;

ii) ψ *is twice continuously differentiable on* $(0,1)$;

iii) $\psi^{''}$ *is monotone in a right-neighborhood of zero and in a left-neighborhood of one;*

iv) $\psi^{''}$ *is strictly positive on* $(0,1)$;

v) If $r = 0$, *then*

a) $-\psi^{'} \in \mathcal{RV}_{-1}(0)$, *and*

b) $u\,\psi^{'}(u)$ *is bounded away from zero for u in a right-neighborhood of zero.*

vi) If $\rho = 1$, *then* $\psi^{'}$ *and* $\psi^{''}$ *are bounded away from zero in a left-neighborhood of one.*

Then $\{U_t : t \in \mathbb{Z}\}$ is geometrically ergodic.

Remark that Clayton, Ali-Mikhail-Haq, Gumbel, Frank and Joe copulas among others, with suitable parameter ranges, satisfy assumptions of Theorem 5.1 (see Examples 3.1-3.11 in Beare [5]). We present here a generator that does not satisfy Assumptions of Theorem 5.1. Indeed, the generator below has a regular variation index $r = +\infty$.

Remark 10 *Consider the family of Archimedean generators:*

$$\psi(t) = \exp\{u^{-\theta}\} - \mathrm{e}, \quad \text{for } \theta \in (0, \infty).$$

The corresponding family of copulas forms the twentieth entry in Table 4.1 of Nelsen [48]. In this case log $\psi \in \mathcal{RV}_{-\theta}(0)$, and ψ is said to be rapidly varying at zero, $\psi \in \mathcal{RV}_{\infty}(0)$ (see for instance Section 2.4 in Bingham et al. [7] for a formal definition of rapid variation). As $r = +\infty$, the Theorem 5.1 does not apply for such family of generators.

In particular Theorem 5.1 is an application of the Geometric Ergodic Theorem, discussed in detail in the text of Meyn and Tweedie [45]. The proof involves verifying that the one-step dependence characterized by copula C satisfies a Foster-Lyapunov drift condition. Chen et al. [10] used precisely this approach to prove geometric ergodicity for the Clayton and Gumbel families. Then Theorem 5.1 provides geometric ergodicity for a large class of Archimedean copulas.

Using Theorem 3.1 we obtain the regular varying properties of the transformed inverse generator $\widetilde{\psi}$. In Corollary 5.2 below, we underline that these properties can be used to get geometrically ergodic stationary Markov chain with transformed joint distribution $\widetilde{C}$ of (U_0, U_1). Remark that obviously differentiable and monotonic conditions of Theorem 5.1 have to be also verified.

Corollary 5.2 *Let C_0 be the initial Archimedean copula with inverse associated generator ψ_0, satisfying assumptions of Theorem 5.1. Under assumptions of Theorem 3.1 also the transformed inverse generator $\widetilde{\psi} = \psi_0 \circ T_f^{-1}$ satisfies assumption i) in Theorem 5.1. If $\widetilde{\psi}$ satisfies also the remaining assumptions of Theorem 5.1, then $\{U_t : t \in \mathbb{Z}\}$ with transformed joint distribution $\widetilde{C}$ of (U_0, U_1) is geometrically ergodic.*

The proof is postponed to Appendix B.

Conclusion

In this paper we deal with the study of the tails of certain transformed Archimedean copulas. We consider the class of transformations previously proposed for instance in Di Bernardino and Rullière [14], Di Bernardino and Rullière [15], Durante et al. [20]. In particular the relationship between the asymptote of the parametric transformation T and the regular variation of the transformed tails is investigated. These results extend some bivariate results of Durante et al. [20] using the definition of upper and lower multivariate tail dependence coefficients previously proposed by De Luca and Rivieccio [13]. The first part of the paper exploits previous works of Charpentier and Segers [8], and extends some bivariate results of Juri and Wüthrich [36]. Indeed in this part, we calculate tail dependence coefficients in this Archimedean setting when the generator of considered copula exhibits some regular variation properties. In the second part of the paper we analyse the impact in the upper and lower multivariate tail dependence coefficients of a large class of transformations of dependence structures. These results are based on the transformations proposed by Di Bernardino and Rullière [14], Di Bernardino and Rullière [15]. In this setting we obtain new results under specific conditions involving regularly varying hazard rates of components of the transformation. Finally we investigate the importance of using transformed Archimedean copulas. Indeed they permit to construct Archimedean generator exhibiting any chosen couple of lower and upper multivariate transformed tail dependence coefficients. Some perspectives are the derivation of a complete estimation procedure that uses these generators with given tail coefficients, and further developments in a dynamical setting.

Appendix A. Illustration for some usual copulas

Using Remark B, we now describe some Archimedean generators that are *lower and upper regularly varying.* A complete survey of Archimedean regularly varying generators would be out of the scope of the present paper. However a large list of common parametric families of Archimedean generators and associated regular varying properties is given in Table 1 in Charpentier and Segers [9]. In particular we now consider Gumbel and Clayton generators.

Gumbel (and independent) generator. Let us consider the Gumbel inverse generator $\psi(t) = (-\ln t)^\theta$ and $\psi^{-1}(x) = \exp(-x^{1/\theta})$, with $\theta \geq 1$. Calculating derivatives, one gets $\psi'(t) = -\frac{\theta}{t}(-\ln t)^{\theta-1}$. One easily remarks that

$$\lim_{y \to 0^+} \frac{y\,\psi'(y)}{\psi(y)} = \lim_{y \to 0^+} \frac{-\theta}{\ln(y)} = 0.$$

Furthermore,

$$\lim_{z\to 1^-}\frac{(1-z)\psi'(z)}{\psi(z)}=\lim_{z\to 1^-}\frac{-\theta\, z+\theta}{z\,\ln(z)}=-\theta.$$

So the regular variation index for the Gumbel generator of parameter θ at one is $\rho=\theta$. Then the Gumbel generator is such that:

$$\psi^{\text{Gumbel}(\theta)}\in\mathcal{RV}_0(0),\quad\text{and}\quad \psi^{\text{Gumbel}(\theta)}\in\mathcal{RV}_\theta(1),\ \text{ for }\ \theta\in[1,+\infty).\tag{A.1}$$

Remark that the comonotonic copula is not an Archimedean one, then the case $\theta=+\infty$ is excluded in Equation (A.1). By the lower and upper regularly variation of $\psi^{\text{Gumbel}(\theta)}=(-\ln t)^\theta$, for $\theta\in[1,+\infty)$, and Theorem 2.1 we get for $h\geq 1$ and $d-h\geq 1$,

$$\lambda_L^{(h,d-h)}=0\quad\text{and}\quad \lambda_U^{(h,d-h)}=\frac{\sum_{i=1}^d(-1)^i\,C_d^i\ \cdot i^{1/\theta}}{\sum_{i=1}^{d-h}(-1)^i\,C_{d-h}^i\ \cdot i^{1/\theta}}.$$

Remark that the Gumbel copula for $\theta=1$ is the independent one. Then the regular variation index at one and at zero for the inverse independence generator $\psi(t)=-\ln(t)$ is given by:

$$\psi^{\text{Indep}}\in\mathcal{RV}_0(0)\quad\text{and}\quad\psi^{\text{Indep}}\in\mathcal{RV}_1(1).$$

Obviously, in the independent case, for $h\geq 1$ and $d-h\geq 1$,

$$\lambda_L^{(h,d-h)}=0\quad\text{and}\quad\lambda_U^{(h,d-h)}=0.$$

An illustration of the generator $\phi^{\text{Gumbel}(\theta)}$ and its inverse $\psi^{\text{Gumbel}(\theta)}$ is presented in Figure A.3, both for the lower and upper regularly variation, for different values of parameter θ. Some indications *lower* and *upper* are given in order to illustrate the fact that right derivatives of $\psi^{\text{Gumbel}(\theta)}$ at zero are linked to lower-tail coefficients, whereas left derivatives at one are linked to upper-tail coefficients. In every case, upper-tail behaviour of the copula is linked to the generator behaviour around the attachment point $(t,\psi(t))=(1,0)$.

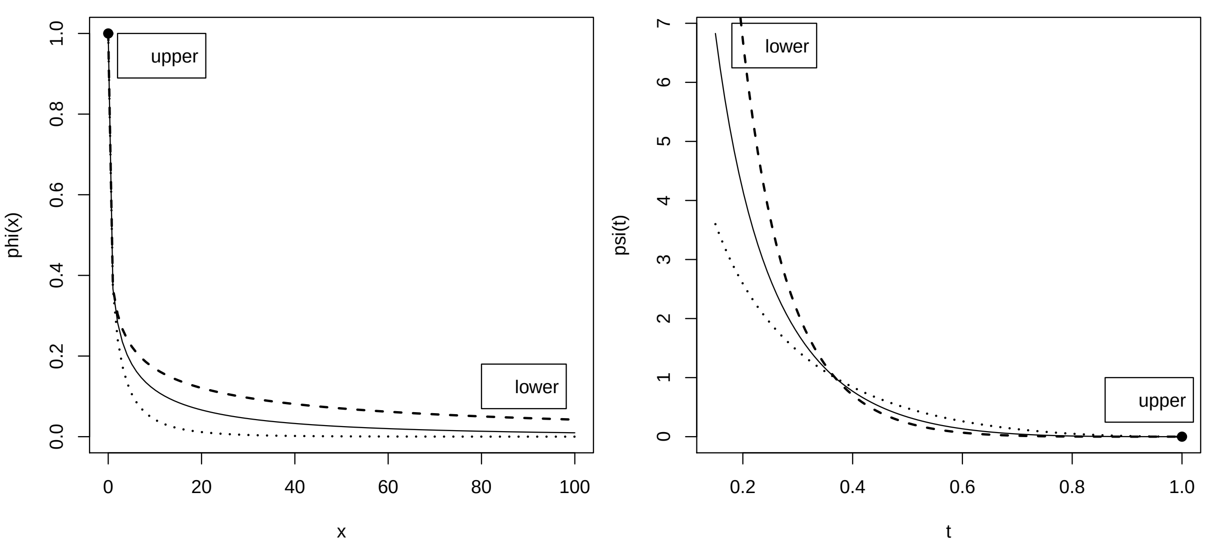

Figure A.3: Generators $\phi^{\text{Gumbel}(\theta)}=\exp(-x^{1/\theta})$ (left) and its inverse $\psi^{\text{Gumbel}(\theta)}=(-\ln t)^\theta$ (right) for a Gumbel copula with parameters $\theta=4$ (dashed lines), $\theta=3$ (full lines) and $\theta=2$ (dotted lines). As given in Equation (A.1), we indicate the upper and lower parts of the generator associated to the regular varying property of Remark B.

In the Gumbel case, one easily shows that $\mu_\phi=\phi'/\phi\in\mathcal{RV}_{k-1}(\infty)$ with $k=\frac{1}{\theta}$, so that in this case

$$\lambda_L^{(h,d-h)}(u)=\frac{\delta^{(d)}(u)}{\delta^{(d-h)}(u)}\in\mathcal{RV}_{d^{\frac{1}{\theta}}-(d-h)^{\frac{1}{\theta}}}(0).$$

In the independence case, when $\theta = 1$, then

$$\lambda_L^{(h,d-h)}(u) = \frac{\delta^{(d)}(u)}{\delta^{(d-h)}(u)} \in \mathcal{RV}_h(0)\,.$$

<u>*Clayton generator*</u>. Let us consider the Clayton inverse generator $\psi(t) = \frac{1}{\theta}\,(t^{-\theta} - 1)$ and $\psi^{-1}(x) = (1 + \theta\, x)^{-1/\theta}$, with $\theta \in (0, +\infty)$. Calculating derivatives, one gets $\psi'(t) = -t^{-(\theta+1)}$. Then the generator of Clayton's copula is a regular varying generator at zero with index $-\theta$ and at one with index 1, i.e.,

$$\psi^{\text{Clayton}(\theta)} \in \mathcal{RV}_{-\theta}(0), \quad \text{and} \quad \psi^{\text{Clayton}(\theta)} \in \mathcal{RV}_1(1),\ \forall\, \theta \in (0, +\infty). \tag{A.2}$$

Similarly, by application of Theorem 2.1, we obtain for $h \geq 1$ and $d - h \geq 1$,

$$\lambda_L^{(h,d-h)} = d^{1/\theta}\,(d-h)^{-1/\theta} \quad \text{and} \quad \lambda_U^{(h,d-h)} = 0.$$

An illustration of the generator $\phi^{\text{Clayton}(\theta)}$ and its inverse $\psi^{\text{Clayton}(\theta)}$ is presented in Figure A.4, both for the lower and upper regularly variation, for different values of parameter θ. Even in this case some indications *lower* and *upper* are given.

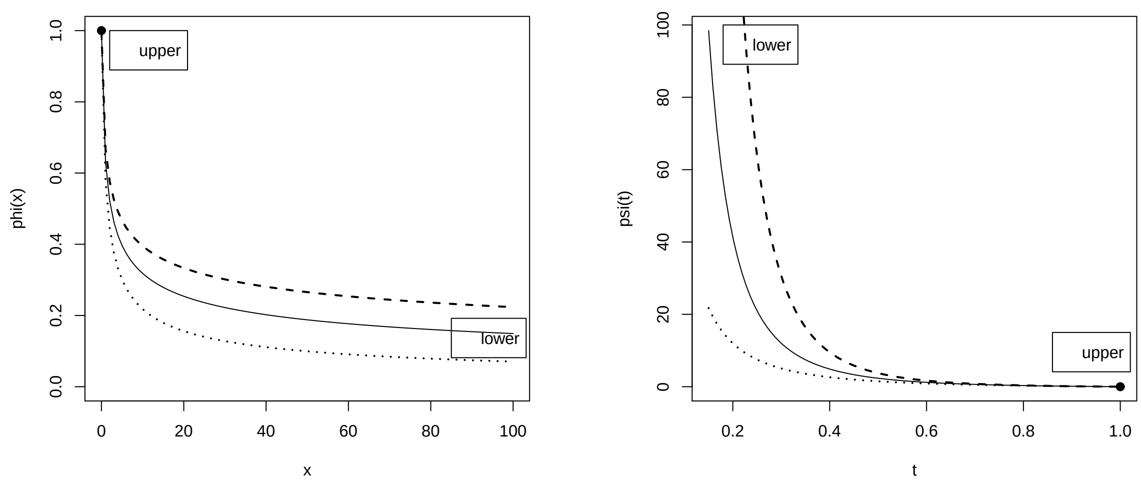

Figure A.4: Generators $\phi^{\text{Clayton}(\theta)}(x) = (1 + \theta\, x)^{-1/\theta}$ (left) and its inverse $\psi^{\text{Clayton}(\theta)}(t) = \frac{1}{\theta}\,(t^{-\theta} - 1)$ (right) for a Clayton copula with parameters $\theta = 4$ (dashed lines), $\theta = 3$ (full lines) and $\theta = 2$ (dotted lines). As given in Equation (A.2), we indicate the upper and lower parts of the generator associated to the regular varying property of Remark B.

In the Clayton case, one easily shows that $m_\phi = \phi'/(1-\phi) \in \mathcal{RV}_{-\kappa-1}(0)$ with $\kappa = 0$. Furthermore from Theorem 2.3 (case $\rho = 1$) one can check that $(-D)^d\phi(0)$ is finite and not zero, where D is the derivative operator, then we get

$$\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_h(1).$$

The asymptotic regular varying behaviour of a generator is difficult to interpret graphically on a plot of ψ or ϕ. We first recall that for any Archimedean copula, there exists a whole family of equivalent generators that are different but leading to the same copula, see Di Bernardino and Rullière [15]. Furthermore, the regular variation of ψ at 0 is difficult to compare to the regular variation at 1 (see Figures A.3-A.4). This is first due to the fact that the scale is not the same as ψ tends to $+\infty$ at 0 whereas it tends to 0 at 1. One can of course re-scale these figures, by drawing for example $1 - \exp(-\psi(t))$ as a function of t so that the curve decreases from $(0, 1)$ to $(1, 0)$. But even after a re-scaling, differences between upper-tail and lower-tail behaviour would be hard to distinguish.

This is a reason why we present in Figure A.5 the *Regular Index Function* RIF_ψ, as defined in Lemma 2.1, for Gumbel and Clayton generators. In these cases, we get the analytical expressions:

	Lower r	Lower k	Upper ρ	Upper κ
Gumbel, $\theta \in [1, +\infty)$	0	$\frac{1}{\theta}$	θ	0
Clayton, $\theta \in (0, +\infty)$	θ	0	1	0
Independence	0	1	1	0

Table A.2: Considered indexes for Gumbel and Clayton generators. On the lower side, $r \in [0, +\infty]$ and $k \in [0, +\infty)$ are such that $\psi \in \mathcal{RV}_{-r}(0)$ and $\mu_\phi = \phi'/\phi \in \mathcal{RV}_{k-1}(\infty)$. On the upper side, $\rho \in [1, +\infty]$ and $\kappa \in [0, +\infty)$ are such that $\psi \in \mathcal{RV}_\rho(1)$ and $m_\phi = \phi'/(1-\phi) \in \mathcal{RV}_{-\kappa-1}(0)$.

$$\mathrm{RIF}(x)^{\mathrm{Gumbel}} = -(\theta-1)\frac{1-x}{\ln x}, \text{ for } \theta \in [1, +\infty) \qquad \mathrm{RIF}^{\mathrm{Clayton}}(x) = \frac{(-x+1)\,(\theta\, x^{-\theta}\,\ln(x) + x^{-\theta} - 1)}{\ln(x)\,(x^{-\theta}-1)}, \text{ for } \theta \in (0, +\infty).$$

Regular variation indexes of ψ, for $\psi \in \mathcal{RV}_{-r}(0)$ and $\psi \in \mathcal{RV}_\rho(1)$, can be easily retrieved from Figure A.5. The lower tail regular variation index r is the limit of $\mathrm{RIF}_\psi(t)$ as t tends to 0^+, and the upper tail regular variation index $\rho - 1$ is the limit of $\mathrm{RIF}_\psi(t)$ as t tends to 1^-. From Figure A.5 (left), it is clear that Gumbel generator has no lower tail dependence since $r = 0$, whereas upper tail dependence depends on the parameter θ of the generator since $\rho = \theta$ and $\mathrm{RIF}_\psi(t)$ tends to $\theta - 1$ as t tends to 1^-. The independence generator is a special case of the Gumbel generator where $\theta = 1$, having the function $\mathrm{RIF}_{\psi^{\mathrm{Indep}}}(x) = 0$, for all $x \in (0, 1)$. From Figure A.5 (right), it is clear that the Clayton generator has lower tail dependence since $r = \theta$, whereas it has no upper tail dependence since $\rho = 1$.

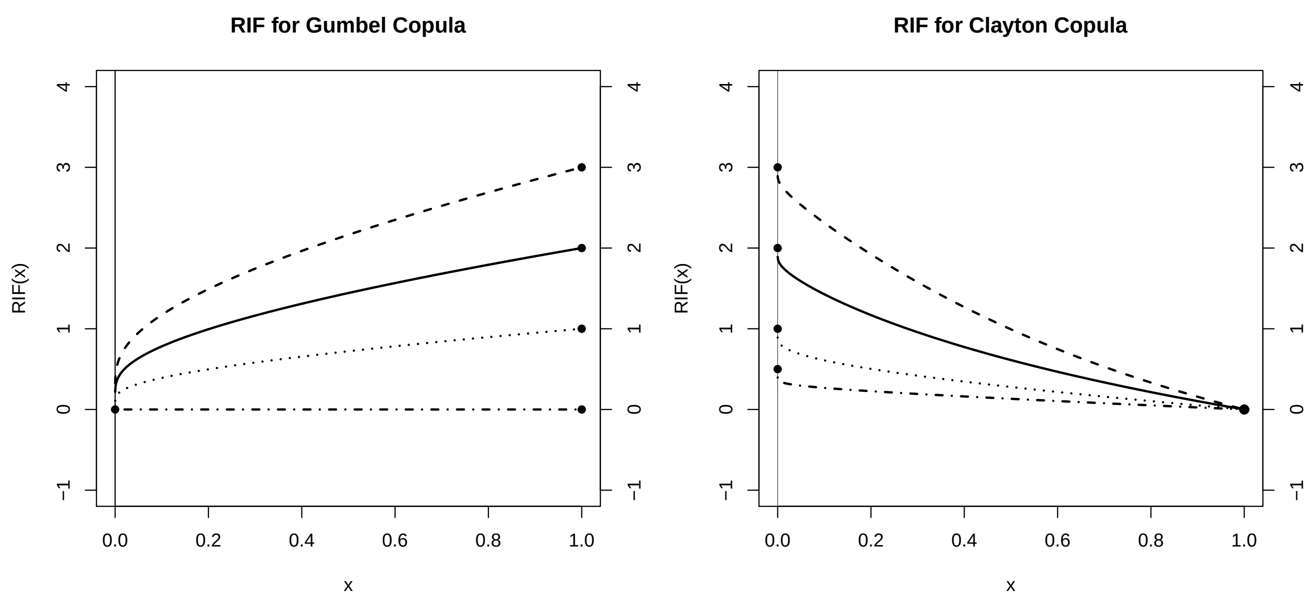

Figure A.5: Regular Index Function $\mathrm{RIF}_\psi(x)$ as a function of $x \in (0, 1)$, for Gumbel generator with parameter $\theta = 1, 2, 3, 4$ (left) and for Clayton generator with parameter $\theta = 0.5, 1, 2, 3$ (right). In both cases, lower curves are always corresponding to a lower value of the parameter θ.

Finally, in Table A.2 we gathered the considered indexes for Gumbel and Clayton generators.

Appendix B. Proofs

Proof of Proposition 1.1 In the case where $U_1, \ldots, U_d$ are exchangeable random variables, one easily shows that these quantities do only depend on the respective cardinals h and $d-h$, so that

$$\lambda_L^{(h,d-h)} = \lambda_L^{I_h,\bar{I}_h} = \lim_{u\to 0^+} \frac{C_d(u,\ldots,u)}{C_{d-h}(u,\ldots,u)}, \tag{B.1}$$

$$\lambda_U^{(h,d-h)} = \lambda_U^{I_h,\bar{I}_h} = \lim_{u\to 1^-} \frac{\bar{C}_d(1-u,\ldots,1-u)}{\bar{C}_{d-h}(1-u,\ldots,1-u)}, \tag{B.2}$$

where C_d and $\bar{C}_d$ are respectively the copula and the survival copula in the dimension d, with by convention $C_1(u) = u$ and $\bar{C}_1(u) = 1-u$. In the exchangeable case, using inclusion-exclusion principle,

$$\bar{C}_d(u,\ldots,u) = 1 - \sum_{i=1}^{d} (-1)^{i-1}\, C_d^i\, \delta^{(i)}(u),$$

where $\delta^{(i)}(u) = C_i(u,\ldots,u)$ is the diagonal section of the copula in the dimension i. By the Hôpital's rule, one gets the result. □

Proof of Lemma 2.1 The last point concerning equivalent generators is straightforward. Denote by $R_\psi(x) = -\frac{x(1-x)\psi'(x)}{\psi(x)}$, $x \in (0,1)$. From Remark B, it is clear that $\lim_{x\to 0^+} R_\psi(x) = r$ and that $\lim_{x\to 1^-} R_\psi(x) = \rho$. For the independence inverse generator $R_{\psi_\perp}(x) = -\frac{1-x}{\ln x}$, with $\psi_\perp(x) = -\ln(x)$. By construction, $\mathrm{RIF}_\psi(x) = R_\psi(x) - R_{\psi_\perp}(x) = 0$ if $\psi = \psi_\perp$, and one easily checks that $\lim_{x\to 0^+} R_{\psi_\perp}(x) = 0$ and $\lim_{x\to 1^-} R_{\psi_\perp}(x) = 1$, so that the result holds. □

Proof of Lemma 2.2 Let $\psi \in \mathcal{RV}_{-r}(0)$, with $r \in [0,+\infty]$ and $y = \psi(u)$. Then we obtain

$$\lim_{u\to 0^+} [\delta^{(i)}]'(u) = \lim_{u\to 0^+} \frac{\delta^{(i)}(u)}{u} = \lim_{u\to 0^+} \frac{\psi^{-1}(i\,\psi(u))}{u} = \lim_{u\to 0^+} \frac{\psi^{-1}(i\,\psi(u))}{\psi^{-1}(\psi(u))} = \lim_{y\to\infty} \frac{\psi^{-1}(i\,y)}{\psi^{-1}(y)} = i^{-1/r},$$

with proper interpretations for r equal to zero or infinity. Hence the first result.
Let $\psi \in \mathcal{RV}_\rho(1)$, $\rho \in [1,+\infty]$, by direct application of Remark C with $y = \psi \circ M \circ I(x)$. For the diagonal, one gets

$$\lim_{u\to 1^-} [\delta^{(i)}]'(u) = \lim_{x\to\infty} \left(1 - \psi^{-1}(i \cdot \psi \circ M \circ I(x))\right) \cdot x = \lim_{x\to\infty} \left(M \circ \delta^{(i)} \circ M \circ I(x)\right) \cdot x = i^{1/\rho},$$

i.e. setting $u = M \circ I(x)$ and $x = I \circ M(u) = 1/(1-u)$,

$$\lim_{u\to 1^-} \frac{1-\delta^{(i)}(u)}{1-u} = i^{1/\rho}.$$

By application of Hôpital's rule we obtain the second result. □

Proof of Lemma 2.3 On the lower side, this result can be seen as a direct consequence of Theorem 3.3. in Charpentier and Segers [9]. As $\delta^{(i)}(\cdot)$ is a decreasing function of i, if $\delta^{(i)} \in \mathcal{RV}_{z_i}(0)$ then $z_i = i^k$ must be an increasing function of i, with $k \geq 0$, which correspond to the index domain in Charpentier and Segers [9]. On the upper side, recall $M(x) = 1-x$ and $I(x) = 1/x$, and denote by $K = M \circ \delta^{(i)} \circ M \circ I$. As $x[M \circ I]'(x) = I(x) = M \circ \phi \circ \phi^{-1} \circ M \circ I$, one easily shows that

$$\frac{x\,K'(x)}{K(x)} = -i \times \frac{\phi' \circ i \circ \phi^{-1} \circ M \circ I}{M \circ \phi \circ i \circ \phi^{-1} \circ M \circ I} \times \frac{M \circ \phi \circ \phi^{-1} \circ M \circ I}{\phi' \circ \phi^{-1} \circ M \circ I}(x).$$

Setting $y = \phi^{-1} \circ M \circ I(x)$, one sees that $y \to 0^+$ as $x \to +\infty$, and

$$\lim_{x\to+\infty} \frac{x\,K'(x)}{K(x)} = -i \times \lim_{y\to 0^+} \frac{m_\phi(i\,y)}{m_\phi(y)}.$$

So that if $m_\phi \in \mathcal{RV}_{-\kappa-1}(0)$, $\lim_{x\to+\infty} \frac{x\,K'(x)}{K(x)} = -i^{-\kappa}$, thus $K = M\circ\delta^{(i)}\circ M\circ I \in \mathcal{RV}_{-i^{-\kappa}}(\infty)$, and $M\circ\delta^{(i)} \in \mathcal{RV}_{i^{-\kappa}}(1)$. Hence the result. As $M\circ\delta^{(i)}(\cdot)$ is an increasing function of i, if $M\circ\delta^{(i)} \in \mathcal{RV}_{\zeta_i}(1)$ then $\zeta_i = i^{-\kappa}$ must be a decreasing function of i, with $\kappa \geq 0$. □

Proof of Remark 5 Since $\psi \in \mathcal{RV}_{-r}(0)$ is equivalent to $\lim_{x\to\infty} x\,\frac{\phi'(x)}{\phi(x)} = -\frac{1}{r}$, $r>0$, and the ratio $\lim_{x\to\infty} s\,\frac{\mu_\phi(s\,x)}{\mu_\phi(x)} = 1$, for $s>0$, then $\mu_\phi \in \mathcal{RV}_{-1}(\infty)$. For the second part of the result, $\psi \in \mathcal{RV}_\rho(1) \Leftrightarrow M\circ\phi \in \mathcal{RV}_{\frac{1}{\rho}}(0)$ is equivalent to $\lim_{x\to0^+} -x\,\frac{\phi'(x)}{1-\phi(x)} = \frac{1}{\rho}$ and the ratio $\lim_{x\to0^+} s\,\frac{m_\phi(sx)}{m_\phi(x)} = 1$ and $m_\phi \in \mathcal{RV}_{-1}(0)$. Hence the result. □

Proof of Theorem 2.1 From Definition 1.2 and Lemma 2.2, we get

$$\lambda_L^{(h,d-h)} = \lim_{u\to0^+} \frac{\psi^{-1}(d\,\psi(u))}{\psi^{-1}((d-h)\,\psi(u))} = \lim_{u\to0^+} \frac{\psi^{-1}(d\,\psi(u))}{\psi^{-1}(\psi(u))}\frac{\psi^{-1}(\psi(u))}{\psi^{-1}((d-h)\,\psi(u))} = d^{-1/r}\,(d-h)^{1/r},$$

with proper interpretations for r equal to zero or infinity. Hence the first result. Write $r_d(u) = \sum_{i=0}^{d}(-1)^i\,C_d^i\,\psi^{-1}(i\,\psi(u))$. From Definition 1.2 (see also De Luca and Rivieccio [13]), one have

$$\lambda_U^{(h,d-h)} = \lim_{u\to1^-} \frac{r_d(u)}{r_{d-h}(u)}.$$

For $d\geq1$, $\sum_{i=0}^{d}(-1)^i\,C_d^i = (1-1)^d = 0$ and $r_d(u) = -\sum_{i=0}^{d}(-1)^i\,C_d^i\left(1-\psi^{-1}(i\psi(u))\right)$,

$$\lambda_U^{(h,d-h)} = \lim_{x\to\infty} \frac{r_d(M\circ I(x))}{r_{d-h}(M\circ I(x))}.$$

Finally,

$$\lambda_U^{(h,d-h)} = \lim_{x\to\infty} \frac{-x^{-1}\sum_{i=0}^{d}(-1)^i\,C_d^i\left(1-\psi^{-1}(i\cdot\psi\circ M\circ I(x))\right)\cdot x}{-x^{-1}\sum_{i=0}^{d-h}(-1)^i\,C_{d-h}^i\left(1-\psi^{-1}(i\cdot\psi\circ M\circ I(x))\right)\cdot x},$$

or equivalently

$$\lambda_U^{(h,d-h)} = \lim_{x\to\infty} \frac{\sum_{i=0}^{d}(-1)^i\,C_d^i\cdot\left(1-\psi^{-1}(i\cdot\psi\circ M\circ I(u))\right)\cdot x}{\sum_{i=0}^{d-h}(-1)^i\,C_{d-h}^i\cdot\left(1-\psi^{-1}(i\cdot\psi\circ M\circ I(u))\right)\cdot x}.$$

When the denominator is not zero, results comes from application of Lemma 2.2. Hence the result. □

Proof of Theorem 2.2 The first part of the theorem has already been stated, when $r=0$, $\lambda_L^{(h,d-h)} = 0$. Notice that $\mu_\phi = +\frac{\phi'}{\phi} \in \mathcal{RV}_{k-1}(\infty)$ implies that $-\frac{\phi}{\phi'} \in \mathcal{RV}_{-k+1}(\infty)$. Recall that the copula diagonal section in the dimension i is $\delta^{(i)}(u) = \psi^{-1}(i\,\psi(u))$, then from Theorem 3.3 in Charpentier and Segers [9] one gets for $i \in \mathbb{N}\setminus\{0\}$ that

$$\delta^{(i)} \in \mathcal{RV}_{z_i}(0) \text{ with } z_i = i^{-k},$$

see also Lemma 2.3. It can then be shown that if there exist $z_d, z_{d-h} \geq 0$ such that $\delta^{(d)} \in \mathcal{RV}_{z_d}(0)$ and $\delta^{(d-h)} \in \mathcal{RV}_{z_{d-h}}(0)$, then

$$\lambda_L^{(h,d-h)}(u) = \frac{\delta^{(d)}(u)}{\delta^{(d-h)}(u)} \in \mathcal{RV}_{z_d-z_{d-h}}(0).$$

This last result comes down easily by calculating $\lambda_L^{(h,d-h)}(s\,u)/\lambda_L^{(h,d-h)}(u)$ for a real $s>0$. □

Proof of Theorem 2.3 We start from the first result. Under these assumptions, from Theorem 4.3 in Charpentier and Segers [9], we get

$$\lim_{u\to0} u^{-d}\,\mathbb{P}[U_1 \geq 1-ux_1,\ldots,U_d \geq 1-ux_d] = |\psi'(1)|^d(-D)^d\phi(0)\prod_{i\in I} x_i.$$

If $\forall\, i,\ x_i = s$, we obtain

$$\lim_{u\to 0} u^{-d}\mathbb{P}[U_1 \geq 1 - u\, s, \ldots, U_d \geq 1 - u\, s] = |\psi'(1)|^d (-D)^d \phi(0)\, s^d.$$

Remark that $\lambda_U^{(h,d-h)}(u) = \frac{r_d(u)}{r_{d-h}(u)}$, with $r_d(u) = \sum_{i=0}^{d} (-1)^i\, C_d^i\, \delta^{(i)}(u)$. Then

$$\lim_{u\to 0} u^{-d} r_d(1-us) = |\psi'(1)|^d (-D)^d \phi(0)\, s^d.$$

Analogously for r_{d-h},

$$\lim_{u\to 0} u^{-(d-h)} r_{d-h}(1-us) = |\psi'(1)|^{d-h} (-D)^{d-h} \phi(0)\, s^{d-h}.$$

So

$$\lim_{u\to 0} \frac{u^{-d} r_d(1-us)}{u^{-(d-h)} r_{d-h}(1-us)} = \frac{|\psi'(1)|^d (-D)^d \phi(0)\, s^d}{|\psi'(1)|^{d-h} (-D)^{d-h} \phi(0)\, s^{d-h}} = \alpha_{d,d-h} s^h.$$

Finally

$$\lim_{u\to 0} u^{-h} \lambda_U^{(h,d-h)}(1-us) = \alpha_{d,d-h} s^h.$$

Since by assumptions, $\alpha_{d,d-h}$ is finite and not zero, then

$$\lim_{u\to 0} \frac{\lambda_U^{(h,d-h)}(1-us)}{\lambda_U^{(h,d-h)}(1-u)} = s^h.$$

Hence the first result. For the second result, from Corollary 4.7 in Charpentier and Segers [9], we get

$$\lim_{u\to 0} \mathbb{P}[i \in I,\, U_i \geq 1 - us \mid \forall i \in I_{d-h},\, U_i \geq 1 - us] = \frac{\mathfrak{r}_d(s,\ldots,s)}{\mathfrak{r}_{d-h}(s,\ldots,s)},$$

where

$$\mathfrak{r}_d(u_1,\ldots,u_d) = \sum_{J\subset\{1,\ldots,d\}} (-1)^{|J|} \left(\sum_{j\in J} u_j\right) \ln \left(\sum_{j\in J} u_j\right).$$

Then

$$\lim_{u\to 0} \lambda_U^{(h,d-h)}(1-us) = \frac{\mathfrak{r}_d(s,\ldots,s)}{\mathfrak{r}_{d-h}(s,\ldots,s)}.$$

One can obtain that

$$\mathfrak{r}_d(s,\ldots,s) = s \sum_{k=1}^{d} C_d^k (-1)^k k \ln(k),$$

and if $d-h \geq 2$,

$$\mathfrak{r}_{d-h}(s,\ldots,s) = s \sum_{k=1}^{d-h} C_{d-h}^k (-1)^k k \ln(k).$$

Then finally, for $d-h \geq 2$,

$$\lim_{u\to 0} \lambda_U^{(h,d-h)}(1-us) = \frac{\sum_{k=1}^{d} C_d^k (-1)^k k \ln k}{\sum_{k=1}^{d-h} C_{d-h}^k (-1)^k k \ln(k)},$$

and

$$\lim_{u\to 0} \frac{\lambda_U^{(h,d-h)}(1-us)}{\lambda_U^{(h,d-h)}(1-u)} = 1.$$

So $\lambda_U^{(h,d-h)}(u) \circ M \in \mathcal{RV}_0(0)$ and equivalently $\lambda_U^{(h,d-h)}(u) \in \mathcal{RV}_0(1)$. Hence the second result. $\square$

Proof of Lemma 3.1 The first result comes by checking that $P^{'}_{g,\gamma} \in \mathcal{RV}_{\gamma-1}(\infty)$ and $P^{'}_{g,\gamma} \in \mathcal{RV}_{g-1}(-\infty)$, for γ, $g > 0$. The second result comes directly by writing $\mu_G = \frac{G'}{\bar{G}} = \frac{F' \circ P}{M \circ F \circ P} \cdot P' = \mu_F \cdot P'$ and $m_g = \frac{G'}{G} = \frac{F' \circ P}{F \circ P} \cdot P' = m_F \cdot P'$. □

Proof of Theorem 3.1 To prove (11), one defines

$$K(x) = \widetilde{\psi} \circ I(x) = \psi_0 \circ G \circ f^{-1} \circ G^{-1} \circ I(x),$$

after some calculations, using $m_G = G'/G$, we get

$$\lim_{x\to\infty} \frac{x\,K'(x)}{K(x)} = \lim_{y\to-\infty} -\frac{m_G(f^{-1}(y))}{m_G(y)} \cdot [f^{-1}]'(y) \cdot \frac{\left(G \circ f^{-1}(y)\right) \cdot \psi_0' \circ G \circ f^{-1}(y)}{\psi_0 \circ G \circ f^{-1}(y)}.$$

Setting $z = G \circ f^{-1}(y)$, z tends to 0^+ as y tends to $-\infty$, under Lower-tail Assumptions 3.2, if $\psi_0 \in \mathcal{RV}_{-r_0}(0)$, with $r_0 \in [0, +\infty]$, then from, Remark B,

$$\lim_{z\to 0^+} \frac{z \cdot \psi_0'(z)}{\psi_0(z)} = -r_0.$$

If $m_G = G'/G \in \mathcal{RV}_{g-1}(-\infty)$ for $g > 0$, and if f has an asymptote $ax + b$ at $-\infty$, then finally

$$\lim_{x\to\infty} \frac{xK'(x)}{K(x)} = r_0 \cdot a^{-g},$$

and $\tilde{\phi}^{-1} = \tilde{\psi} \in \mathcal{RV}_{-\tilde{r}}(0)$, with $\tilde{r} = r_0 \cdot a^{-g} \geq 0$. Hence the result in (11).

To prove (12), one defines

$$H(x) = \widetilde{\psi} \circ M \circ I(x) = \psi_0 \circ G \circ f^{-1} \circ G^{-1} \circ M \circ I(x)\,,$$

after some calculations, we obtain

$$\frac{x\,H'(x)}{H(x)} = \frac{\psi_0' \circ G \circ f^{-1} \circ G^{-1} \circ M \circ I(x) \cdot G' \circ f^{-1} \circ G^{-1} \circ M \circ I(x) \cdot [f^{-1}]' \circ G^{-1} \circ M \circ I(x)}{x\,G' \circ G^{-1} \circ M \circ I(x) \cdot \psi_0 \circ G \circ f^{-1} \circ G^{-1} \circ M \circ I(x)}.$$

Now set $y = G^{-1} \circ M \circ I(x)$, i.e. $x = I \circ M \circ G(y) = 1/\bar{G}(y)$. When x tends to infinity, $M \circ I(x)$ tends to 1^- and y tends to infinity. Or reciprocally as y tends to infinity, $M \circ G(y)$ tends to 0^+ and $x = I \circ M \circ G(y)$ tends to infinity. We get

$$\frac{x\,H'(x)}{H(x)} = \frac{\psi_0' \circ G \circ f^{-1}(y) \cdot G' \circ f^{-1}(y) \cdot [f^{-1}]'(y)}{x\,G'(y) \cdot \psi_0 \circ G \circ f^{-1}(y)},$$

$$\frac{xH'(x)}{H(x)} = \frac{\bar{G}(y)}{G'(y)} \cdot \frac{G' \circ f^{-1}(y)}{\bar{G} \circ f^{-1}(y)} \cdot \frac{\bar{G} \circ f^{-1}(y) \cdot \psi_0' \circ G \circ f^{-1}(y) \cdot [f^{-1}]'(y)}{\psi_0 \circ G \circ f^{-1}(y)}.$$

Setting $\mu_G = G'/\bar{G}$,

$$\frac{x\,H'(x)}{H(x)} = \frac{\mu_G(f^{-1}(y))}{\mu_G(y)} \cdot [f^{-1}]'(y) \cdot \frac{\bar{G} \circ f^{-1}(y) \cdot \psi_0' \circ G \circ f^{-1}(y)}{\psi_0 \circ G \circ f^{-1}(y)}.$$

One remark that $y = G^{-1} \circ M \circ I(x)$ is an increasing function of x that tends to infinity, and

$$\lim_{x\to\infty} \frac{xH'(x)}{H(x)} = \lim_{y\to\infty} \frac{\mu_G(f^{-1}(y))}{\mu_G(y)} \cdot [f^{-1}]'(y) \cdot \frac{\left(1 - G \circ f^{-1}(y)\right) \cdot \psi_0' \circ G \circ f^{-1}(y)}{\psi_0 \circ G \circ f^{-1}(y)}.$$

Functions f^{-1} and G are increasing, and from Assumption 3.3, using Remark B, one obtain:

- from the assumed asymptotic behaviour of f, one have $\lim_{y\to\infty} \frac{f^{-1}(y)}{y} = \frac{1}{\alpha} = \lim_{y\to\infty} [f^{-1}]'(y)$ with $\alpha > 0$;

- from Remark B, as $\psi_0 \in \mathcal{RV}_{\rho_0}(1)$ with $\rho_0 \geq 1$, then $\lim_{z\to 1^-} \frac{(1-z)\psi_0'(z)}{\psi_0(z)} = -\rho_0$ with $\rho_0 \geq 1$;
- from $\mu_G = G'/\bar{G} \in \mathcal{RV}_{\gamma-1}(\infty)$, with $\gamma > 0$, $\lim_{y\to\infty} \frac{\mu_G(f^{-1}(y))}{\mu_G(y)} = \left(\frac{1}{\alpha}\right)^{\gamma-1}$.

At last one gets

$$\lim_{x\to\infty} \frac{x\,H'(x)}{H(x)} = \lim_{y\to\infty} \left(\frac{1}{\alpha}\right)^{\gamma-1} \cdot \frac{1}{\alpha} \cdot (-\rho_0).$$

and finally $\tilde{\psi} \circ M \circ I \in \mathcal{RV}_{-(\frac{1}{\alpha})^{\gamma}\cdot\rho_0}(\infty)$, so that $\tilde{\phi}^{-1} = \tilde{\psi} \in \mathcal{RV}_{\rho_0\cdot\alpha^{-\gamma}}(1)$. Hence the result in (12). Furthermore, since $\tilde{\psi}$ satisfies Assumption 3.1, from Remark A, $\rho_0 \cdot \alpha^{-\gamma} \geq 1$. □

Proof of Lemma 3.2 For the first part of the proof, $T_f \in \mathcal{RV}_{\tilde{a}}(0)$ if $T_f \circ I \in \mathcal{RV}_{-\tilde{a}}(\infty)$. One thus considers the function $H = T_f \circ I = G \circ f \circ G^{-1} \circ I$, and the limit $\lim_{x\to\infty} \frac{xH'(x)}{H(x)}$ which tends to the regular variation index $-\tilde{a}$ of T_f if this transformation is regularly varying at 0.

$$[T_f \circ I]' = (G' \circ f \circ G^{-1} \circ I) \times (f' \circ G^{-1} \circ I) \times ([G^{-1}]' \circ I) \times I'$$

so that, as $x\,I'(x) = -I(x) = -G \circ G^{-1} \circ I(x)$,

$$\frac{xH'(x)}{H(x)} = -\frac{(G' \circ f \circ G^{-1} \circ I)}{(G \circ f \circ G^{-1} \circ I)} \times (f' \circ G^{-1} \circ I) \times \frac{G \circ G^{-1} \circ I}{G' \circ G^{-1} \circ I}(x)$$

Setting $y = G^{-1} \circ I(x)$, y tends to $-\infty$ as x tends to $+\infty$ and under Assumption 3.2,

$$\lim_{x\to+\infty} \frac{xH'(x)}{H(x)} = -\lim_{y\to-\infty} \frac{m_G(f(y))}{m_G(y)} \cdot f'(y) = -\lim_{y\to-\infty} \left(\frac{f(y)}{y}\right)^{g-1} \cdot f'(y) = -a^g.$$

For the second part of the proof, $M \circ T_f \in \mathcal{RV}_{\tilde{\alpha}}(1)$ is equivalent to $M \circ T_f \circ M \circ I \in \mathcal{RV}_{-\tilde{\alpha}}(\infty)$. One thus considers now the function $K = M \circ T_f \circ M \circ I = M \circ G \circ f \circ G^{-1} \circ M \circ I$, and the limit $\lim_{x\to\infty} \frac{x\,K'(x)}{K(x)}$ which tends to the regular variation index $-\tilde{\alpha}$ of T_f if this transformation is regularly varying at 1.

$$[M \circ T_f \circ M \circ I]' = -(G' \circ f \circ G^{-1} \circ M \circ I) \times (f' \circ G^{-1} \circ M \circ I) \times ([G^{-1}]' \circ M \circ I) \times (M \circ I)'$$

so that, as $x \cdot (M \circ I)'(x) = I(x) = M \circ G \circ G^{-1} \circ M \circ I(x)$,

$$\frac{x\,K'(x)}{K(x)} = -\frac{(G' \circ f \circ G^{-1} \circ M \circ I)}{(M \circ G \circ f \circ G^{-1} \circ M \circ I)} \times (f' \circ G^{-1} \circ M \circ I) \times \frac{M \circ G \circ G^{-1} \circ M \circ I}{G' \circ G^{-1} \circ M \circ I}(x).$$

Setting $y = G^{-1} \circ M \circ I(x)$, y tends to $+\infty$ as x tends to $+\infty$ and under Assumption 3.3,

$$\lim_{x\to+\infty} \frac{x\,K'(x)}{K(x)} = -\lim_{y\to+\infty} \frac{\mu_G(f(y))}{\mu_G(y)} \cdot f'(y) = -\lim_{y\to+\infty} \left(\frac{f(y)}{y}\right)^{\gamma-1} \cdot f'(y) = -\alpha^{\gamma}.$$

Hence the result. □

Proof of Theorem 3.3 These results come down from a direct application of Theorem 3.1 and Theorem 2.1 for multivariate tail coefficients with a generator with regular variation at one or at zero. For (13) the index of the transformed regular variation at zero is $\widetilde{r} = r_0\, a^{-g} \geq 0$. For result in (14) the index of the transformed regular variation at one is $\widetilde{\rho} = \rho_0 \cdot \alpha^{-\gamma} \geq 1$. □

Proof of Theorem 3.2 On the lower side, one easily shows that the hazard rate for the transformed generator $\tilde{\phi} = T_f \circ \phi_0$ is

$$\mu_{\tilde{\phi}}(x) = \frac{\phi_0(x) \cdot T_f' \circ \phi_0(x)}{T_f \circ \phi_0(x)}\, \mu_{\phi_0}(x)$$

Assume that $T_f \in \mathcal{RV}_{\widetilde{a}}(0)$ with $\widetilde{a} \in (0,+\infty)$, then $u\,T_f'(u)/T_f(u) \to \widetilde{a}$ as u tends to 0. As a consequence, since $\widetilde{a} \neq 0$ and $\mu_{\phi_0} \in \mathcal{RV}_{k_0-1}(\infty)$ with $k_0 \geq 0$,

$$\lim_{x\to+\infty} \frac{\mu_{\tilde{\phi}}(s\,x)}{\mu_{\tilde{\phi}}(x)} = \lim_{x\to+\infty} \frac{\mu_{\phi_0}(sx)}{\mu_{\phi_0}(x)} = s^{(k_0-1)}.$$

On the upper side, one easily shows that the rate for the transformed generator $\tilde{\phi} = T_f \circ \phi_0$ is

$$m_{\tilde{\phi}}(x) = \frac{(1-\phi_0(x)) \cdot T_f' \circ \phi_0(x)}{1 - T_f \circ \phi_0(x)}\, m_{\phi_0}(x)$$

Assume that $M \circ T_f \in \mathcal{RV}_{\widetilde{\alpha}}(1)$ with $\widetilde{\alpha} \in (0,+\infty)$, then $(1-z)\,T_f'(z)/T_f(z) \to \widetilde{\alpha}$ as z tends to 1. As a consequence, since since $\widetilde{\alpha} \neq 0$ and $m_{\phi_0} \in \mathcal{RV}_{-\kappa_0-1}(0)$ with $\kappa_0 \geq 0$,

$$\lim_{x\to 0} \frac{m_{\tilde{\phi}}(s\,x)}{m_{\tilde{\phi}}(x)} = \lim_{x\to 0} \frac{m_{\phi_0}(sx)}{m_{\phi_0}(x)} = s^{(-\kappa_0-1)}.$$

Then we obtain the result. □

Proof of Proposition 5.1 Proposition 5.1 comes down trivially by Theorem 3.3 for $d = 2$ and $g = \gamma = 1$, (see also Corollary 3.1). Furthermore, Proposition 5.1 can be also proved using Propositions 4.2 and 4.3 in Durante et al. [20]. Since $T_{\overline{f}} = \left(1 + (\frac{x}{1-x})^{-a}\,\mathrm{e}^{-b}\right)^{-1}$, then $\lim_{x\to 0^+} \frac{T_{\overline{f}}(x)}{x^a} = c$, with $c > 0$. From Proposition 4.2 in Durante et al. [20], $\widetilde{\lambda}_L(\widetilde{C}_{T_{\overline{f}},C_0}) = (\lambda_L(C_0))^a$. Furthermore, since $T_{\overline{f}} = \left(1 + (\frac{x}{1-x})^{-\alpha}\,\mathrm{e}^{-\beta}\right)^{-1}$, then $\lim_{x\to 1^-} \frac{1-T_{\overline{f}}(x)}{(1-x)^\alpha} = c$, with $c > 0$. From Proposition 4.3 in Durante et al. [20], we obtain that $\widetilde{\lambda}_U(\widetilde{C}_{T_{\overline{f}},C_0}) = 2 - (2-\lambda_U(C_0))^\alpha$. Since $\widetilde{\lambda}_U(\widetilde{C}_{T_{\overline{f}},C_0}) \in [0,1]$, then we obtain the result. □

Proof of Corollary 5.2 Since ψ_0 satisfies assumptions of Theorem 5.1, under assumptions of Theorem 3.1, the transformed generator $\widetilde{\psi}$ is such that $\widetilde{\psi} \in \mathcal{RV}_{\tilde{\rho}}(1)$ with $\tilde{\rho} = \rho_0 \cdot \alpha^{-\gamma}$, with $\rho_0 \in [1,+\infty)$, $\alpha \in (0,+\infty)$ and $\gamma \in (0,+\infty)$, then $\tilde{\rho} \in [1,+\infty)$ (see (12)). Furthermore $\widetilde{\psi} \in \mathcal{RV}_{-\tilde{r}}(0)$ with $\tilde{r} = r_0 \cdot a^{-g}$, with $r_0 \in [0,+\infty)$, $a \in (0,+\infty)$ and $g \in (0,+\infty)$, then $\tilde{r} \in [0,+\infty)$ (see (11)).
Then $\widetilde{\psi}$ satisfies assumption *i)* of Theorem 5.1. If the obtained transformed generator $\widetilde{\psi}$ satisfies also the others assumptions of Theorem 5.1 we obtain the result. □

Appendix C. Proofs of auxiliary results

Proof of Remark A Result i) comes from Lemma 2.7 in Juri and Wüthrich [35].
ii) Since ψ is a strictly generator, and by applying the Hôpital's rule,

$$\lim_{x\to 0^+} \frac{\psi(1-s\,x)}{\psi(1-x)} = \lim_{x\to 0^+} \frac{-s\,\psi'(1-s\,x)}{-\psi'(1-x)} = s^\rho. \tag{C.1}$$

If the inverse generator is valid, it must be in particular decreasing and convex, so that ψ is decreasing and convex. For $s \in (0,1)$, $-\psi'(1-x) \geq -\psi'(1-s\,x)$ and Equation (C.1) can only be valid if $s^{\rho-1} \leq 1$, i.e. $\rho - 1 \geq 0$. □

Proof of Remark B If $\psi \in \mathcal{RV}_{-r}(0)$, with $r \in [0,+\infty]$, then

$$r = \lim_{x\to\infty} \frac{x \cdot (\psi \circ I)'(x)}{\psi \circ I(x)} = \lim_{x\to\infty} \frac{\psi'(I(x))}{-x\,\psi(I(x))} = \lim_{y\to 0^+} \frac{-y\,\psi'(y)}{\psi(y)},$$

hence the result (see also Theorem 1 in Charpentier and Segers [8]).
If $\psi \in \mathcal{RV}_\rho(1)$ then $\psi \circ M \circ I \in \mathcal{RV}_{-\rho}(\infty)$. Using $y = I(x)$ and $z = M(y)$,

$$-\rho = \lim_{x\to\infty} \frac{x \cdot (\psi \circ M \circ I)'(x)}{\psi \circ M \circ I(x)} = \lim_{x\to\infty} \frac{\psi'(M \circ I(x))}{x \cdot \psi \circ M \circ I(x)} = \lim_{y\to 0} \frac{y\psi'(M \circ y)}{\psi \circ M(y)} = \lim_{z\to 1} \frac{(1-z)\psi'(z)}{\psi(z)}.$$

Hence the result. $\square$

Proof of Remark C For the first result, one have to prove that $\phi \in \mathcal{RV}_{-1/r}(\infty)$, or equivalently that $\phi \circ I \circ M \in \mathcal{RV}_{1/r}(1)$. Using $h(x) = \phi(x)$, one gets

$$\lim_{x\to\infty} \frac{xh'(x)}{h(x)} = \lim_{x\to\infty} \frac{x}{\psi'(h(x))\,h(x)}.$$

Now setting $y = h(x)$, i.e. $x = \psi(y)$, and using Remark B for $\psi \in \mathcal{RV}_r(0)$,

$$\lim_{x\to\infty} \frac{x\,h'(x)}{h(x)} = \lim_{y\to 0^+} \frac{\psi(y)}{y\,\psi'(y)} = -\frac{1}{r},$$

finally $h \in \mathcal{RV}_{-1/r}(\infty)$ and the first result holds.
For the second result, one have to show that $M \circ \phi \in \mathcal{RV}_{1/\rho}(0)$, or equivalently that $M \circ \phi \circ M \in \mathcal{RV}_{1/\rho}(1)$, i.e. $M \circ \phi \circ I \in \mathcal{RV}_{-1/\rho}(\infty)$. Using $g(x) = M \circ \phi \circ I(x) = 1 - \phi(1/x)$, one gets

$$\lim_{x\to\infty} \frac{x\,g'(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{x\,\psi'(1-g(x))\,g(x)}\,.$$

Now setting $y = 1 - g(x)$, i.e. $x = I \circ \psi(y)$, and using Remark B with $\psi \in \mathcal{RV}_\rho(1)$,

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \lim_{y\to 1^-} \frac{\psi(y)}{\psi'(y)(1-y)} = -\frac{1}{\rho}.$$

Finally $g \in \mathcal{RV}_{-1/\rho}(\infty)$, $M \circ \phi \circ M \in \mathcal{RV}_{1/\rho}(1)$, $M \circ \phi \in \mathcal{RV}_{1/\rho}(0)$ and the second result holds. $\square$

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