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THE DYNAMICAL HIERARCHY FOR ROELCKE PRECOMPACT POLISH GROUPS

TOMÁS IBARLUCÍA

ABSTRACT. We study several distinguished function algebras on a Polish group G , under the assumption that G is Roelcke precompact. We do this by means of the model-theoretic translation initiated by Ben Yaacov and Tsankov: we investigate the dynamics of \aleph_0 -categorical metric structures under the action of their automorphism group. We show that, in this context, every strongly uniformly continuous function (in particular, every Asplund function) is weakly almost periodic. We also point out the correspondence between tame functions and NIP formulas, deducing that the isometry group of the Urysohn sphere is $\text{Tame} \cap \text{UC}$ -trivial.

In a series of recent papers, Glasner and Megrelishvili [GM06, GM08, Meg08, GM12, GM13a] have studied different classes of functions on topological dynamical systems, arising from compactifications with particular properties. Thus, for example, a real valued continuous function on a G -space X might be *almost periodic*, *Hilbert-representable*, *weakly almost periodic*, *Asplund-representable* or *tame*, and this classes form a hierarchy

$$\text{AP}(X) \subset \text{Hilb}(X) \subset \text{WAP}(X) \subset \text{Asp}(X) \subset \text{Tame}(X) \subset \text{RUC}(X)$$

of subalgebras of the class of *right uniformly continuous* functions. These algebras can be defined in different ways. The latter coincides with the class of functions that can be in some sense represented through a Banach space, and from this point of view the previous subalgebras can be identified, respectively, with the cases when the Banach space is asked to be euclidean, Hilbert, reflexive, Asplund or Rosenthal. When $X = G$ and the action is given by group multiplication, functions might also be *left uniformly continuous*, and if they are simultaneously in $\text{RUC}(G)$ then they form part of the algebra $\text{UC}(G)$ of *Roelcke uniformly continuous* functions.

We study these algebras for the case of Roelcke precompact Polish groups, by means of the model-theoretic translation developed by Ben Yaacov and Tsankov [BYT14]. As established in their work, Roelcke precompact Polish groups are exactly those arising as automorphism groups of \aleph_0 -categorical metric structures. Moreover, one might turn continuous functions on the group into definable predicates on the structure. Under this correlation, the authors showed, weakly almost periodic functions translate into *stable* formulas: a most studied concept of topological dynamics leads to one of crucial notions of model theory. This provides a unified understanding of several previously studied examples: the permutation group $S(\mathbb{N})$, the unitary group $\mathcal{U}(H)$, the group of measure preserving transformations of the unit interval $\text{Aut}(\mu)$, the group $\text{Aut}(RG)$ of automorphisms of the random graph or the isometry group $\text{Iso}(\mathbb{U}_1)$ of the Urysohn sphere, among many others, are automorphism groups of \aleph_0 -categorical structures, thus Roelcke precompact. In the first three cases the structures are stable, thus $\text{WAP}(G) = \text{UC}(G)$: their WAP and Roelcke compactifications coincide. Using model-theoretic insight, the authors were able to prove for example that, whenever the latter is the case, the group G is totally minimal.

The so-called *dynamical hierarchy* presented above has been partially described for some of the habitual examples. For the groups $S(\mathbb{N})$, $\mathcal{U}(H)$ or $\text{Aut}(\mu)$ we have in fact $\text{Hilb}(G) = \text{UC}(G)$; see [GM13b, Meg08]. From [BYT14] we know that the inclusion $\text{WAP}(G) \subset \text{UC}(G)$ is strict for the group $\text{Aut}(\mathbb{Q}, <)$ of monotone bijections of the rationals. More drastically, Megrelishvili [Meg01] had shown that the group $H_+[0, 1]$ of orientation preserving homeomorphisms of the unit interval, also Roelcke precompact, has a trivial WAP-compactification: $\text{WAP}(G)$ is the algebra of constants; in [GM08] this conclusion was extended to the algebra $\text{Asp}(G)$ (and indeed to the algebra $\text{SUC}(G)$ of *strongly uniformly continuous functions*, containing $\text{Asp}(G)$). The same was established for the group $\text{Iso}(\mathbb{U}_1)$. If one drops the requirement of Roelcke precompactness, all inclusions in the hierarchy are known to be strict (see [GM13b, GM06]).

We show that in fact $\text{WAP}(G) = \text{Asp}(G) = \text{SUC}(G)$ for every Roelcke precompact Polish group G . In addition, we observe that Roelcke uniformly continuous tame functions correspond to NIP formulas on the model-theoretic side. Thus, for instance: $\text{Asp}(G) \subsetneq \text{Tame}(G) \cap \text{UC}(G) = \text{UC}(G)$ for $G = \text{Aut}(\mathbb{Q}, <)$; $\text{WAP}(G) = \text{Tame}(G) \cap \text{UC}(G) \subsetneq \text{UC}(G)$ for $G = \text{Aut}(RG)$; and $\text{Asp}(G) \subsetneq \text{Tame}(G) \cap \text{UC}(G) \subsetneq \text{UC}(G)$ for $G = \text{Homeo}(2^\omega)$. We also deduce that the $\text{Tame} \cap \text{UC}$ -compactification of $\text{Iso}(\mathbb{U}_1)$ is trivial.

Our approach is model-theoretic, and we shall assume familiarity with continuous logic as presented in [BYU10] or [BYBHU08]. We will mainly study the dynamics of \aleph_0 -categorical structures, and then derive the corresponding conclusions for their automorphism groups.

The algebra $\text{Hilb}(G)$ will not be addressed in this paper. Unlike the properties of stability and dependence, which can be studied locally (that is, formula-by-formula), the model-theoretic interpretation of the algebra $\text{Hilb}(G)$ presents a different phenomenon, and will be considered in a future work.

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1. THE SETTING AND BASIC FACTS

Most of the material on topology in this and subsequent sections comes from the works of Glasner and Megrelishvili referred to in the introduction.

A G -space X is given by a continuous left action of a topological group G on a topological space X . Then G acts as well on the space $\mathcal{C}(X)$ of continuous bounded functions on X , by $gf(x) = f(g^{-1}x)$. If X is not compact, however, the action on $\mathcal{C}(X)$ need not be continuous, and the functions $f \in \mathcal{C}(X)$ for which the orbit map $g \in G \mapsto gf \in Gf$ is norm continuous are called right uniformly continuous (RUC). That is, $f \in \text{RUC}(X)$ if for all $\epsilon > 0$ there is a neighborhood U of the identity such that $|f(g^{-1}x) - f(x)| < \epsilon$ for all $x \in X$ and $g \in U$. When $X = G$ with the regular left action, we also have the algebra $\text{LUC}(G)$ of left uniformly continuous functions, where the condition is that $|f(xg) - f(x)|$ be small for all $x \in G$ and g close to the identity. The intersection $\text{UC}(G) = \text{RUC}(G) \cap \text{LUC}(G)$ forms the algebra of Roelcke uniformly continuous functions on G .

Note 1.1. Our spaces will be metric, and we will restrict our attention to those functions $f \in \text{RUC}(X)$ that are also uniformly continuous with respect to the metric on X ; we denote this algebra by $\text{RUC}_u(X)$. The same subscript might be added to the other function algebras in the dynamical hierarchy, in order to keep this restriction in mind, but will be omitted when there be no danger to it. The algebra $\text{RUC}_u(X)$ is G -invariant, since we presume that G acts on X by uniformly continuous transformations.

Our groups will be Polish. When we take $X = G$, we assume that a left-invariant compatible metric d_L on G has been fixed. One should notice that, then, $\text{RUC}_u(G) = \text{UC}(G)$. The algebra $\text{SUC}(G)$, containing $\text{Asp}(G)$ (both to be defined later), is always a subalgebra of $\text{UC}(G)$ (see [GM08]); in particular, $\text{SUC}_u(G) = \text{SUC}(G)$ and $\text{Asp}_u(G) = \text{Asp}(G)$. As pointed out to us by M. Megrelishvili, this is not the case for the algebra $\text{Tame}(G)$, so we will mind the distinction between $\text{Tame}(G)$ and $\text{Tame}_u(G) = \text{Tame}(G) \cap \text{UC}(G)$.

We turn to logic. By a *formula* we shall understand any \emptyset -definable predicate, usually in two partitioned sets of variables, x and y . If thought as functions on a metric structure M , recall that formulas are always $\text{Aut}(M)$ -invariant, uniformly continuous and bounded. Given a formula $f(x, y)$ and an x -parameter a on a structure M , we denote by f_a the continuous function given by $f_a(b) = f(a, b)$. Unless otherwise stated, the variable x on a formula $f(x, y)$ will denote a generalized tuple, i.e. a tuple of length ω , while y will denote a single variable (albeit one could also take it to denote a tuple, if we were to consider functions in more than one argument). By a pre-zeroset on a structure M we mean a subset whose closure is the zeroset of an \emptyset -definable

predicate; in an \aleph_0 -categorical structure, any $\text{Aut}(M)$ -invariant set, thus for example the orbit of a tuple, is a pre-zeroset.

Whenever we talk of a metric structure M as a G -space, we understand that $G = \text{Aut}(M)$, with the topology of pointwise convergence; this is always an action by isometries. In this case we have another special function algebra: the family of functions of the form f_a for a formula $f(x, y)$ and a parameter $a \in M^\omega$, which we will denote by $\text{Def}(M)$.

We aim to study the dynamics of \aleph_0 -categorical metric structures. Recall that an \aleph_0 -categorical structure is a separable structure such that any other separable model of its first-order theory is isomorphic to it. Ben Yaacov and Tsankov [BYT14] showed that their automorphism groups are exactly those Polish groups G such that, for every non-empty neighborhood U of the identity, there is a finite set $F \subset G$ satisfying $UFU = G$. Equivalently, the completion with respect to the Roelcke uniformity (the infimum of the left and right uniformities) is compact, whence their name: *Roelcke precompact*.

Our conclusions will translate easily from structures to groups. For any Roelcke precompact Polish G , we recall from [BYT14] that the completion \widehat{G}_L with respect to d_L admits a continuous first-order structure rendering it an \aleph_0 -categorical structure with automorphism group G ; see also Melleray [Mel10]. On the other hand, the restriction map $\text{RUC}_u(\widehat{G}_L) \rightarrow \text{UC}(G)$ is always a norm-preserving G -isomorphism, as it is easy to verify. By this means, some results about the dynamics of \aleph_0 -categorical structures will carry immediately to Roelcke precompact Polish groups. In other cases we shall use the more direct approach of [BYT14]; see Remark 3.4 below.

The starting point for our analysis is given by the following observation, based on the ideas from [BYT14].

Proposition 1.2. *For a metric structure M we have $\text{Def}(M) \subset \text{RUC}_u(M)$. If M is \aleph_0 -categorical, then moreover $\text{Def}(M) = \text{RUC}_u(M)$.*

Proof. For the first part consider a neighborhood U of the identity such that $d(a, ga) < \Delta_f(\epsilon)$ for $g \in U$, where Δ_f is a modulus of uniform continuity for $f(x, y)$. Thus $\|gf_a - f_a\| = \|f_{ga} - f_a\| < \epsilon$ whenever $g \in U$.

Now let $h \in \text{RUC}_u(M)$, and set $a \in M^\omega$ to enumerate a dense subset of M . We define $f : Ga \times M \rightarrow \mathbb{R}$ by

$$f(ga, b) = gh(b) = h(g^{-1}b).$$

This is well defined because a is dense in M ; note also that f is G -invariant and uniformly continuous. Indeed, we have

$$|f(ga, b) - f(g'a, b')| \leq |gh(b) - gh(b')| + |gh(b') - g'h(b')|.$$

The first term on the right side is small: simply observe that $d(g^{-1}b, g^{-1}b') = d(b, b')$, so we use the uniform continuity of h . For the second, given $\epsilon > 0$ there is a neighborhood U of the identity of G such that $\|gh - g'h\| < \epsilon$ whenever $g^{-1}g' \in U$, because h is RUC; since a is dense, there is $\delta > 0$ such that $d(ga, g'a) < \delta$ implies $g^{-1}g' \in U$; thus if $d(ga, g'a) < \delta$ we have $|gh(b') - g'h(b')| < \epsilon$.

This means that f can be extended continuously to $[a] \times M$ (where $[a] = \overline{Ga}$ is the closed orbit of a). The extension remains G -invariant, so we may regard f as defined on $[a] \times M // G$ (the space of closed orbits of $[a] \times M$ under G), and apply Tietze extension theorem to get a G -invariant continuous function

$$f : M^\omega \times M \rightarrow \mathbb{R}.$$

Finally, if M is \aleph_0 -categorical then every G -invariant continuous function on a countable power of M coincides with a formula in the appropriate number of variables. Thus we have $h = f_a$ for the formula $f(x, y)$ and the parameter $a \in M^\omega$. \square

A *compactification* of a G -space X is a continuous G -map $\nu : X \rightarrow Y$ into a compact Hausdorff G -space Y , whose range is dense in Y ; since we work with metric X , we will moreover ask that ν be uniformly continuous (with respect to the unique uniformity of the compact space Y). Then there is a canonical one-to-one correspondence between compactifications of X and G -invariant uniformly closed subalgebras of $\text{RUC}_u(X)$ (a subalgebra is always assumed to contain the constants). The compactification $X^{\mathcal{A}}$ corresponding to one such algebra $\mathcal{A} \subset \text{RUC}_u(X)$ can be thought

as the Gelfand space $Y_{\mathcal{A}}$ of \mathcal{A} together with the evaluation map $\nu_{\mathcal{A}} : X \rightarrow Y_{\mathcal{A}}$, $\nu_{\mathcal{A}}(x)(\varphi) = \varphi(x)$. Equivalently, $X^{\mathcal{A}}$ is the space of maximal ideals of \mathcal{A} .

Remark 1.3. If M is a metric structure, the compactification $M^{\text{Def}(M)}$ is the type space $S(M)$ (on the variable y) with the usual inclusion $b \in M \mapsto \text{tp}(b/M) \in S(M)$. If M is \aleph_0 -categorical, then Proposition 1.2 says that $S(M)$ coincides with $M^{\text{RUC}_u(M)}$, which we might call the *Roelcke compactification* of M .

A function $f \in \mathcal{C}(X)$ comes from a compactification $\nu : X \rightarrow Y$ if there is $\hat{f} \in \mathcal{C}(Y)$ such that $f = \hat{f} \circ \nu$; note that the extension \hat{f} is unique. The algebra $\mathcal{A}_\nu \subset \text{RUC}_u(X)$ associated to the compactification ν in the correspondence above is precisely the family of functions coming from ν . Evidently, every function $f \in \text{RUC}_u(X)$ comes from the Roelcke compactification; the minimal compactification from which f comes, corresponding to the closed algebra generated by the orbit Gf in $\mathcal{C}(X)$, is called the *cyclic G -space of f* , and denoted by X_f .

Remark 1.4. For a structure M and a function $f_a \in \text{Def}(M)$, the cyclic G -space of f_a should be identified with the local type space $S_f(A)$ of the formula $f(x, y)$ on the variable y , with parameters from the orbit $A = Ga$ (or equivalently from its closure $A = [a] = \overline{Ga}$). The function $\hat{f}_a \in \mathcal{C}(S_f(A))$ extending f_a is given simply by $\hat{f}_a(p) = f(a, p)$, the value of $f(a, y)$ in the type p .

N.B. The local type space $S_f(A)$ we consider here, which is determined by the predicates $f(a', y)$, $a' \in A$, coincides with the one used in [BYT14] or defined in [TZ12], and not with the finer one from [BYU10] or [Pil96], which is determined by the M -definable A -invariant f -predicates.

Remark 1.5. Let G be a Polish group. Since we ask compactifications to be uniformly continuous, every compactification of G factorizes through the left completion \widehat{G}_L , and we have a canonical one-to-one correspondence between compactifications of G and of \widehat{G}_L .

An important part of the project developed in [GM06, GM12, GM13b] has been to classify the dynamical systems (and particularly their compactifications) by the possibility of representing them as an isometric action on a «good» Banach space. Although we will not make use of it in the present work, the precise meaning of a *representation* of a G -space X on a Banach space V is given by a pair

$$h : G \rightarrow \text{Iso}(V), \quad \alpha : X \rightarrow V^*,$$

where h is a continuous homomorphism and α is a weak*-continuous bounded G -map with respect to the dual action $G \times V^* \rightarrow V^*$, $(g\phi)(v) = \phi(h(g)^{-1}(v))$. The topology on $\text{Iso}(V)$ is that of pointwise convergence. The representation is *faithful* if α is a topological embedding.

For a family \mathcal{K} of Banach spaces, a G -space X is \mathcal{K} -*representable* if it admits a faithful representation on a member $V \in \mathcal{K}$, and it is \mathcal{K} -*approximable* if for distinct $x, y \in X$ there is a representation (h, α) on some $V \in \mathcal{K}$ with $\alpha(x) \neq \alpha(y)$.

We end this section with a model-theoretic account of the smallest of the function algebras presented in the introduction. A uniformly continuous bounded function f on a metric G -space X is almost periodic (AP) if the orbit Gf is a precompact subset of $\mathcal{C}(X)$. From the point of view of Banach space representations, almost periodic functions are precisely those coming from euclidean-approximable compactifications of X ; see [GM13b]. The family $\text{AP}(X)$ of almost periodic functions on X is a subalgebra of $\text{RUC}_u(X)$.

Proposition 1.6. *Let M be an \aleph_0 -categorical structure. Let $f(x, y)$ be a formula, $A \subset M^\omega$ a pre-zero set. The following are equivalent, and in any of these cases we say that $f(x, y)$ is almost periodic on A :*

- (1) *the set $\{f_a : a \in A\}$ is norm precompact;*
- (2) *the set $\{f_a : a \in \overline{A}\}$ is norm precompact;*
- (3) *for every indiscernible sequence $(a_i)_{i < \omega} \subset \overline{A}$, the predicates $f(a_i, y)$ are all equivalent, i.e. we have $f(a_i, b) = f(a_j, b)$ for all i, j and b .*

Thus, a function $f_a \in \text{Def}(M)$ is in $\text{AP}(M)$ if and only if $f(x, y)$ is almost periodic on the orbit Ga .

Proof. (1) \Rightarrow (2). Note that $\{f_a : a \in \overline{A}\} \subset \overline{\{f_a : a \in A\}}$ due to uniform continuity of formulas.

(2) \Rightarrow (3). By precompactness, the sequence (f_{a_i}) has a Cauchy subsequence, so in particular there are i and j such that $\sup_y |f(a_i, y) - f(a_j, y)| < \epsilon$. By indiscernibility, this is true for all i, j , and the claim follows.

(3) \Rightarrow (1). Given $\epsilon > 0$, by compactness (and \aleph_0 -categoricity) there are a finite set Δ of formulas and $\delta > 0$ such that any Δ - δ -indiscernible sequence $(a_i)_{i < \omega} \subset A$ (i.e. one verifying $|\phi(a_{i_1}, \dots, a_{i_k}) - \phi(a_{j_1}, \dots, a_{j_k})| < \delta$ for every $i_1 < \dots < i_k, j_1 < \dots < j_k$ and every $\phi \in \Delta$) satisfies $\sup_y |f(a_i, y) - f(a_j, y)| < \epsilon$ for all i, j . On the other hand, given any sequence $(a_i)_{i < \omega}$ we can extract by Ramsey's theorem a Δ - δ -indiscernible subsequence; iterating this for smaller ϵ and taking the diagonal we get a subsequence $(a_{i_j})_{j < \omega}$ such that $(f_{a_{i_j}})$ is Cauchy. \square

In the terminology of Pillay [Pil96], the function f_a is almost periodic if and only if the predicate $f(a, y)$ is *almost \emptyset -definable*.

One could call a G -space X *almost periodic* if $\text{AP}(X) = \text{RUC}_u(X)$. This is a very strong condition. Indeed, for an action of a topological group G by isometries on a complete bounded metric space (X, d) , the function $d_a(y) = d(a, y)$ is AP if and only if the closed orbit $[a]$ is compact. If the space of closed orbits $X // G$ (with the induced metric) is compact, we can deduce that X is almost periodic if and only if X is compact. This is the case for \aleph_0 -categorical structures, but also, trivially, for the action of any Polish group on its left-completion: $\widehat{G}_L // G$ is just one point. So for any Polish group G , $X = \widehat{G}_L$ is almost periodic if and only if it is compact, and this leads to the same characterization for $X = G$. Indeed, note first that the natural isomorphism $\text{RUC}_u(\widehat{G}_L) \simeq \text{UC}(G)$ preserves AP functions, so G is almost periodic if and only if \widehat{G}_L is. Of course, if G is compact then $G = \widehat{G}_L$; on the other hand, if \widehat{G}_L is compact then $\text{Iso}(\widehat{G}_L)$ is compact, and since G is Polish and $G \leq \text{Iso}(\widehat{G}_L)$, we conclude that G is compact too.

Remark 1.7. For a Polish group G , $\text{AP}(G) = \text{UC}(G)$ if and only if G is compact.

2. WAP = Asp = SUC

Recall that a formula $f(x, y)$ has the *order property*, let us say, on a pre-zero set $A \subset M^\omega$, if there are $\epsilon > 0$ and sequences $(a_i)_{i < \omega}$ in A , $(b_j)_{j < \omega}$ in M , such that $|f(a_i, b_j) - f(a_j, b_i)| \geq \epsilon$ for all $i < j < \omega$. If $f(x, y)$ lacks the order property on A we say that it is *stable on A* . We invoke the following crucial result, essentially due to Grothendieck, as pointed out by Ben Yaacov in [BY13].

Fact 2.1. *The formula $f(x, y)$ is stable on A if and only if $\{f_a : a \in A\}$ is weakly precompact in $\mathcal{C}(M)$.*

On the other hand, a function $h \in \mathcal{C}(X)$ on a G -space X is weakly almost periodic (WAP) whenever the orbit $gh \subset \mathcal{C}(X)$ is weakly precompact. Thus, f_a is WAP if and only if $f(x, y)$ is stable on the orbit Ga . For an \aleph_0 -categorical structure one concludes, paraphrasing what has been established in [BYT14], that a uniformly continuous bounded function is WAP if and only if it is of the form f_a for a formula $f(x, y)$ stable on Ga .

The algebra $\text{WAP}(X)$ can also be characterized as the class of functions coming from a reflexively-representable compactification of X ; see [Meg03] or [Meg08]. We also point out the paper of Iovino [Iov99] for an earlier treatment of the connection between stability and reflexive Banach spaces.

A natural generalization of weak almost periodicity is thus to replace *reflexive* by *Asplund* in the latter characterization. Recall that a Banach is Asplund if the dual of every separable subspace is separable, and that every reflexive space has this property. In this way one gets the algebra of Asplund functions, $\text{Asp}(X)$.

For a *compact* G -space Y , a function $h \in \mathcal{C}(Y)$ is shown to be Asplund if and only if the orbit $gh \subset \mathcal{C}(Y)$ is a *fragmented family*; see for example Proposition 5.9 in [GM12]. This means that for any nonempty $B \subset Y$ and any $\epsilon > 0$ there exists an open $O \subset Y$ such that $B \cap O$ is nonempty and $gh(B \cap O)$ is ϵ -small for every $g \in G$. If X is an arbitrary G -space, then $f \in \mathcal{C}(X)$ belongs to $\text{Asp}(X)$ if and only if it comes from an Asplund function on some G -compactification of X . Equivalently, f is Asplund on X if and only if the orbit $G\hat{f}$ of its extension to X_f is fragmented.

It will be interesting to bring in a further weaker notion, introduced in [GM08]. A function $h \in \mathcal{C}(Y)$ on a compact G -system is said to be strongly uniformly continuous (SUC) if for every $y \in Y$ and $\epsilon > 0$ there exists a neighborhood U of the identity of G such that $|h(gy) - h(guy)| < \epsilon$ for all $g \in G$ and $u \in U$. A function f on an arbitrary G -space X is SUC if it comes from a SUC function on a G -compactification of X , or equivalently if its extension \hat{f} to X_f is SUC. The family of all strongly uniformly continuous functions on a G -space X forms a subalgebra $\text{SUC}(X)$ of $\text{RUC}(X)$, and in fact we have

$$\text{WAP}(X) \subset \text{Asp}(X) \subset \text{SUC}(X) \subset \text{RUC}(X),$$

and $\text{SUC}(G) \subset \text{UC}(G)$ for the regular left action of G on itself. See [GM08].

An important motivation for the algebra of SUC functions comes from the viewpoint of semigroup compactifications of G . The compactifications G^{AP} and G^{WAP} of G associated to the algebras $\text{AP}(G)$ and $\text{WAP}(G)$ are respectively the universal *topological group* and *semitopological semigroup* compactifications of G . For their part, G^{ASP} and G^{RUC} are *right topological semigroup* compactifications of G . In the paper referred above, the authors showed that the compactification G^{SUC} is also a right topological semigroup compactification of G , and that $\text{SUC}(G)$ is the biggest subalgebra of $\text{UC}(G)$ with this property. In particular, the Roelcke compactification G^{UC} has the structure of a right topological semigroup if and only if $\text{SUC}(G) = \text{UC}(G)$.

We aim to prove the equality $\text{WAP} = \text{SUC}$ for \aleph_0 -categorical structures and for their automorphism groups.

Switching to logic language, let us say that a formula $f(x, y)$ is SUC on a pre-zero set $A \subset M^\omega$ if for any b in any elementary extension of M and every $\epsilon > 0$ there are $\delta > 0$ and c_1, \dots, c_k in M such that for every $a \in A$ and every automorphism σ satisfying $d(c_i, \sigma c_i) < \delta$ we have

$$|f(a, b) - f(\sigma a, b)| < \epsilon.$$

Lemma 2.2. *For a metric structure M , a function $f_a \in \text{Def}(M)$ is SUC if and only if the formula $f(x, y)$ is SUC on Ga .*

Proof. This follows readily from Remark 1.4. □

From a model-theoretic point of view, the inclusion $\text{WAP}(M) \subset \text{SUC}(M)$ (restricted to $\text{Def}(M)$) is clear: the elements c_i for the condition of SUC can be taken from the parameters of an M -definition of the f -type of b over Ga . In the *classical* first-order \aleph_0 -categorical case, the converse inclusion can also be argued easily by definability of f -types, recalling that the addition of the finite constants c_i to the language of (the classical structure) M preserves \aleph_0 -categoricity, and using that invariant relations on \aleph_0 -categorical theories are definable. However, since we are concerned with the general metric case, we will take a different route. For this, let us bring out the easy case of $(\mathbb{Q}, <)$, which we will generalize to an argument based on the order property.

Remark 2.3. The order relation on the discrete structure $(\mathbb{Q}, <)$ is not SUC. Indeed, let $r \in \mathbb{R} \setminus \mathbb{Q}$, $c_1, \dots, c_k \in \mathbb{Q}$. Suppose $c_i < c_{i+1}$, and say $c_{i_0} < r < c_{i_0+1}$. Take $a \in \mathbb{Q}$, $c_{i_0} < a < r$. There is a monotone bijection σ fixing every c_i and such that $r < \sigma a < c_{i_0+1}$. The claim follows.

Recall that \aleph_0 -categorical structures are approximately \aleph_0 -homogeneous, that is, for every $a, b \in M^\omega$ and every $\epsilon > 0$ there is an automorphism σ of M such that $d(\sigma a, b) \leq d(\text{tp}(a), \text{tp}(b)) + \epsilon$. In the following proposition, for $\alpha \leq \omega$ we let the distance on M^α be $d(a, b) = \sum_{i < \alpha} 2^{-i} d(a_i, b_i)$.

Proposition 2.4. *Let M be \aleph_0 -categorical. If $f(x, y)$ has the order property on a pre-zero set A of M , then $f(x, y)$ is not SUC on A .*

Proof. From the order property and saturation we can find elements $(a_i)_{i \in \mathbb{Q}} \subset \bar{A}$, $(b_j)_{j \in \mathbb{R}}$ in some elementary extension of M , and reals $r \neq s$ such that $f(a_i, b_j) = r$ if $i < j$, $f(a_i, b_j) = s$ if $j \leq i$. Suppose that $f(x, y)$ has the SUC property for $\epsilon = |r - s|/2$; since G is separable and \mathbb{R} is uncountable, there is an open neighborhood U of the identity that witnesses the property for an infinite number of elements b_j , say for every b_j with j in an infinite $J \subset \mathbb{R}$. By passing to a subset we may assume that J is discrete, and so for each $j \in J$ we may take a rational $i(j) < j$ such that $j' < i(j)$ for every $j' < j$, $j' \in J$. Suppose U is the family of automorphisms moving a finite tuple

c at a distance less than δ ; say n is the length of the tuple c . Let $\eta = \Delta_f(|r-s|/2)$, where Δ_f is a modulus of uniform continuity for f .

Since M is \aleph_0 -categorical, the space of types over the empty set with the metric topology is compact. In particular, there must be a pair $j < j'$ in J such that $d(\text{tp}(ca_{i(j)}), \text{tp}(ca_{i(j')})) < \min(\delta, \eta/2^n)$. By homogeneity, there is an automorphism σ such that

$$d(\sigma ca_{i(j)}, ca_{i(j')}) < \min(\delta, \eta/2^n).$$

In particular $d(\sigma c, c) < \delta$, so $\sigma \in U$. In addition, since $d(\sigma a_{i(j)}, a_{i(j')}) < \eta$, $f(a_{i(j')}, b_j) = s$ and $f(a_{i(j)}, b_j) = r$, we have

$$|f(a_{i(j)}, b_j) - f(\sigma a_{i(j)}, b_j)| \geq |r-s|/2,$$

contradicting the fact that U witnesses the SUC property for $\epsilon = |r-s|/2$. \square

Corollary 2.5. *Let M be an \aleph_0 -categorical structure. Then $\text{WAP}(M) = \text{Asp}(M) = \text{SUC}(M)$.*

Theorem 2.6. *Let G be a Roelcke precompact Polish group. Then $\text{WAP}(G) = \text{Asp}(G) = \text{SUC}(G)$.*

Proof. From Remark 1.5 we can deduce that the isomorphism $\text{RUC}_u(\widehat{G}_L) \simeq \text{UC}(G)$ preserves WAP and SUC functions. The identity holds for \widehat{G}_L , by the previous corollary. \square

For the case of Asplund functions we can give a slight generalization to saturated structures. If c is an n -tuple (of tuples) and I is an n -tuple of intervals of \mathbb{R} , let us write $f(c, d) \in I$ instead of $(f(c_k, d))_{k < n} \in \prod_{k < n} I_k$. Let us call a formula $f(x, y)$ *Asplund* on a subset $A \subset M^\omega$ of a metric structure M if it *lacks* the following property: (SP) There exist $\epsilon > 0$ and a set B in some elementary extension of M such that, if $f(c, d) \in I$ for some $d \in B$, some tuple c from A and some tuple I of open intervals of \mathbb{R} , then there are $b, b' \in B$, $a \in A$ with $f(c, b), f(c, b') \in I$ and $\epsilon < |f(a, b) - f(a, b')|$. This makes a function $f_a \in \text{Def}(M)$ Asplund in the topological sense if and only if $f(x, y)$ is Asplund on the orbit Ga , or on its closure $[a]$.

Proposition 2.7. *Let M be a separable structure realizing every \emptyset -type in countably many variables. Let $f(x, y)$ be a formula and a a parameter whose orbit Ga is a pre-zeroset. If $f_a \in \text{Asp}(M)$, then $f_a \in \text{WAP}(M)$.*

Proof. Suppose that $f(x, y)$ has the order property on Ga . By the saturation hypothesis and since $[a]$ is a zeroset, we can find sequences $(a_i)_{i \in \mathbb{Q}}$ in $[a]$, $(b_j)_{j \in \mathbb{R}}$ in some elementary extension of M , and distinct reals r, s such that $f(a_i, b_j) = r$ if $i < j$ and $f(a_i, b_j) = s$ if $j \leq i$. Since M is separable, it is enough to check the condition SP for a countable family C of pairs (c, I) . There is at most a countable number of reals l such that, for some $(c, I) \in C$, we have $f(c, b_j) \in I$ if and only if $j = l$. So by throwing them away we may assume that, whenever $f(c, b_l) \in I$, $(c, I) \in C$, there is $j \neq l$ with $f(c, b_j) \in I$; if we then choose $i \in \mathbb{Q}$ lying between l and j , we have $|f(a_i, b_l) - f(a_i, b_j)| = |r-s|$. So $f(x, y)$ has SP for $B = \{b_j\}$ and any $\epsilon < |r-s|$. \square

3. $\text{TAME} \cap \text{UC} = \text{NIP} = \text{NULL} \cap \text{UC}$

Tame functions have been studied by Glasner and Megrelishvili in [GM12], after the introduction of tame dynamical systems by Köhler [Köh95] (who called them *regular systems*) and later by Glasner in [Gla06]. If the translation of Ben Yaacov and Tsankov for Roelcke precompact Polish groups identifies WAP functions with stable formulas, we remark in this section that tame functions correspond to *NIP* (or *dependent*) formulas. The study of this model-theoretic notion, a generalization of local stability introduced by Shelah [She71], is an active and important domain of research, mainly in the classical first-order setting —though, as the third item of the following proposition remarks, the notion has a very natural metric presentation.

Proposition 3.1. *Let M be an \aleph_0 -categorical metric structure, and let $A \subset M^\omega$ be a pre-zeroset. The following are equivalent for a formula $f(x, y)$; in any of these cases, we say that $f(x, y)$ is *NIP* on A .*

- (1) *There do not exist sequences $(a_i)_{i < \omega} \subset \overline{A}$, $(b_I)_{I \subset \omega}$ in some elementary extension, and real numbers $r \neq s$ such that, for all i, I ,*

$$f(a_i, b_I) = r \text{ if } i \in I \text{ and } f(a_i, b_I) = s \text{ if } i \notin I.$$

- (2) *For every indiscernible sequence $(a_i)_{i < \omega} \subset \overline{A}$ and every b in any elementary extension, the sequence $(f(a_i, b))_{i < \omega}$ converges in \mathbb{R} .*
 (3) *Every sequence $(a_i)_{i < \omega} \subset A$ admits a subsequence $(a_{i_j})_{j < \omega}$ such that $(f(a_{i_j}, b))_{j < \omega}$ converges in \mathbb{R} for any b in any elementary extension.*

Proof. (1) \Rightarrow (2). Let $(a_i)_{i < \omega} \subset \overline{A}$ be indiscernible, b arbitrary. If $(f(a_i, b))_{i < \omega}$ does not converge, there exist $r \neq s$ such that (replacing $(a_i)_{i < \omega}$ by a subsequence) $f(a_{2i}, b) \rightarrow r$, $f(a_{2i+1}, b) \rightarrow s$. By compactness we may assume that $f(a_{2i}, b) = r$ and $f(a_{2i+1}, b) = s$ for all i ; we can ensure that $(a_i)_{i < \omega} \subset \overline{A}$ because \overline{A} is definable and M is \aleph_0 -categorical. Given $I \subset \omega$, take a strictly increasing function $\tau : \omega \rightarrow \omega$ such that $\tau(i)$ is even if and only if i is in I . By indiscernibility, the set of conditions

$$\{f(a_i, y) = t : f(a_{\tau(i)}, b) = t\}$$

is approximately finitely realizable in M ; take b_I to be a realization in some model. Thus, for all i and I , $f(a_i, b_I) = r$ if $i \in I$ and $f(a_i, b_I) = s$ if $i \notin I$, contradicting (1).

(2) \Rightarrow (3). From (2) and compactness we see that for every $\epsilon > 0$ there are some $\delta > 0$ and a finite set of formulas Δ such that for any b and every Δ - δ -indiscernible sequence $(a_i)_{i < \omega} \subset A$ there exists $N < \omega$ with $|f(a_i, b) - f(a_j, b)| < \epsilon$ for all $i, j \geq N$. Suppose that Δ_n, δ_n correspond to $\epsilon = 2^{-n}$. Starting with an arbitrary sequence $(a_i)_{i < \omega}$, using Ramsey's theorem we can extract a Δ_1 - δ_1 -indiscernible subsequence, say $(a_i^1)_{i < \omega}$. Inductively, let $(a_i^{n+1})_{i < \omega}$ be a Δ_{n+1} - δ_{n+1} -indiscernible subsequence of $(a_i^n)_{i < \omega}$. If we take $a_j^\omega = a_j^j$, then $(f(a_j^\omega, b))_{j < \omega}$ converges in \mathbb{R} for any b .

(3) \Rightarrow (1). Assume we have $(a_i)_{i < \omega}$, $(b_I)_{I \subset \omega}$, and r, s contradicting (1). Say $r < u < v < s$. Modifying slightly the elements of the sequence, we may assume that $a_i \in A$ and

$$f(a_i, b_I) < u \text{ if } i \in I, \quad f(a_i, b_I) > v \text{ if } i \notin I$$

for all i, I . If $(a_{i_j})_{j < \omega}$ is as given by (3) and $J \subset \omega$ is infinite and coinfinite in $\{i_j : j < \omega\}$, then $(f(a_{i_j}, b_J))_{j < \omega}$ converges to a contradiction. \square

Note that $f(x, y)$ is NIP on A if and only if it is NIP on \overline{A} . The \aleph_0 -categoricity hypothesis is asked to preserve the elements a_i in M , but can be forgotten if we think of A as the \emptyset -predicate defining \overline{A} , i.e. as a first order condition rather than as a subset of M^ω (so the elements a_i might lie in an extension of M). In the literature, a formula in any given theory is said simply *NIP* if the conditions are satisfied with no restriction on the provenance of the elements a_i .

Tame dynamical systems were originally introduced in terms of the enveloping semigroup of a dynamical system, and admit several equivalent presentations. The common theme are certain dichotomy theorems that have their root in the fundamental result of Rosenthal [Ros74]: a Banach space either contains an isomorphic copy of ℓ_1 or has the property that every bounded sequence has a weak-Cauchy subsequence.

A Banach space is thus called *Rosenthal* if it contains no isomorphic copy of ℓ_1 , and a *compact* metric G -space Y is tame precisely if it admits a faithful representation on a Rosenthal Banach space. We refer to [Gla06, GM12] and the survey [GM13b] for equivalent definitions and for extensions to non-metric spaces. A continuous function $f \in \mathcal{C}(X)$ on an arbitrary G -space is tame if it comes from a tame G -compactification of X . The collection $\text{Tame}(X)$ of all tame functions on X forms a G -subalgebra of $\text{RUC}(X)$; for metric X we shall consider the restriction $\text{Tame}_u(X) = \text{Tame}(X) \cap \text{RUC}_u(X)$, as per Note 1.1.

From Proposition 5.6 and Fact 4.3 from [GM12] we have the following characterization of tame functions on compact systems.

Fact 3.2. *A function $f \in C(Y)$ on a compact G -space Y is tame if and only if every sequence of functions in the orbit Gf admits a weak-Cauchy subsequence, or equivalently if every sequence in Gf admits a pointwise convergent subsequence in \mathbb{R}^Y .*

The link with NIP formulas is then immediate.

Proposition 3.3. *Let M be an \aleph_0 -categorical structure, $a \in M^\omega$ a parameter. A formula $f(x, y)$ is NIP on the orbit Ga if and only if the function $f_a \in \text{Def}(M)$ is tame.*

Proof. The function f_a is tame if and only if its extension \hat{f}_a to the Roelcke compactification $S(M)$ is tame. Since $g\hat{f}_a(p) = f(ga, b)$ for any b realizing the type p , the claim follows directly from the third item in Proposition 3.1 and Fact 3.2. \square

Remark 3.4. An analogous conclusion can be derived for tame functions on the group G , as in the model-theoretic approach of [BYT14] to $\text{WAP}(G)$. Indeed, a function $h \in \text{UC}(G)$ is tame if and only if it can be written as $h(g) = f(a, ga)$ for a formula $f(x, y)$ in two ω -variables x, y that is NIP on $Ga \times Ga$ (meaning that the elements b or b_I in Proposition 3.1 are also taken from the \emptyset -type of a), where $a \in M^\omega$ is a parameter enumerating a dense subset of M .

The picture of the parallelism between these ideas in logic and topology is quite colorful. As we have already said, NIP formulas were introduced by Shelah [She71] in 1971, in the classical first order context. He defined them by the lack of an *independence property* (IP), whence the name *NIP*. This independence property is the condition negated in the first item of Proposition 3.1. In the classical first-order setting it can be read like this: a formula $\varphi(x, y)$ has IP if for some sequence of elements $(a_i)_{i < \omega}$ and every pair of non-empty finite disjoint subsets $I, J \subset \omega$, there is b in some model such that

$$b \models \bigwedge_{i \in I} \varphi(a_i, y) \wedge \bigwedge_{j \in J} \neg \varphi(a_j, y).$$

In other words, $\varphi(x, y)$ has IP if for some $(a_i)_{i < \omega}$ the sequence $(\varphi(a_i, N))_{i < \omega}$ of the sets defined by $\varphi(a_i, y)$ on some big enough model N is an *independent sequence* in the sense of mere sets: all boolean intersections are non-empty.

In the introductory section 1.5 of the survey [GM13b] on Banach representations of dynamical systems, Glasner and Megrelishvili write: «*In addition to those characterizations already mentioned, tameness can also be characterized by the lack of an “independence property” (section 8.6), where combinatorial Ramsey type arguments take a leading role [...]*». The characterization they allude to is Proposition 6.6 from Kerr and Li [KL07], and the independence property involved there can indeed be seen as a topological generalization of Shelah’s IP (see also Fact 3.5 below). But the notion of independence is already present in the seminal work of Rosenthal from 1974 [Ros74], where a crucial first step towards his dichotomy theorem implies showing that a sequence of sets with no pointwise-convergent subsequence admits a boolean independent subsequence.

This is not the first time that the concept of NIP is linked with a notion of another area. In 1992 Laskowski [Las92] noted that a formula $\varphi(x, y)$ has the independence property if and only if the family of definable sets of the form $\varphi(a, y)$ is a *Vapnik-Chervonenkis class*, a concept coming from probability theory, and also from the 70’s [VC71]. He then profited of the examples provided by model theory to exhibit new Vapnik-Chervonenkis classes. In the section that follows we shall do the same thing with respect to tame dynamical systems, complementing the analysis of the examples done by Ben Yaacov and Tsankov [BYT14].

Finally, let us mention that during the writing of this paper we came to know that, independently from us, Artem Chernikov and Pierre Simon also noticed the connection between topological tameness and logical dependence, in the somehow parallel context of definable dynamics.

We end this section by pointing out that $\text{Tame}_u(G)$ coincides, for Roelcke precompact Polish groups, with the restriction to $\text{UC}(G)$ of the algebra $\text{Null}(G)$ of *null functions* on G . Null functions arise from the study of topological sequence entropy of dynamical systems, initiated in [Goo74]. A compact G -space Y is null if its topological sequence entropy along any sequence is zero; we refer to [KL07] for the pertinent definitions. We may say that a function f on an arbitrary G -space is null if it comes from a null compactification of X . The resulting algebra $\text{Null}(X)$ is always a subalgebra of $\text{Tame}(X)$.

The following fact is just a rephrasing of a characterization due to Kerr and Li; see Proposition 5.8 of the previously referred paper.

Fact 3.5. *A function $f \in RUC(X)$ is null if and only if there are no real numbers $r < s$ such that for every n one can find $\{g_i\}_{i < n} \subset G$ and $\{x_i\}_{i < n} \subset X$ such that*

$$f(g_i x_i) < r \text{ if } i \in I \text{ and } f(g_i x_i) > s \text{ if } i \notin I.$$

When $X = M$ is an \aleph_0 -categorical structure, it is then immediate by compactness that a formula $f(x, y)$ is NIP on Ga if and only if f_a is null. Thus $\text{Null}_u(M) = \text{Tame}_u(M)$, and by considering $M = \widehat{G}_L$ one gets $\text{Null}_u(G) = \text{Tame}_u(G)$ for every Roelcke precompact Polish G .

4. EXAMPLES

We derive first some conclusions for the oligomorphic groups $\text{Aut}(\mathbb{Q}, <)$, $\text{Aut}(RG)$ and $\text{Homeo}(2^\omega)$.

As shown in [BYT14], the only stable formulas on the dense linear order $(\mathbb{Q}, <)$ are those expressible using only the identity relation. On the other hand, every formula on $M = (\mathbb{Q}, <)$ is NIP, whence $\text{Tame}_u(M) = \text{RUC}_u(M)$. For its automorphism group G we have also $\text{Tame}_u(G) = \text{UC}(G)$; this can be argued directly by representing the functions on G in the form $h(g) = f(a, ga)$ as in [BYT14] (recall Remark 3.4 above). Alternatively, since \aleph_0 -categorical structures are determined up to bi-interpretability by their automorphism groups (see [AZ86] and [BYK13]), we deduce that every formula on the structure \widehat{G}_L is NIP; the claim then follows from $\text{Tame}_u(\widehat{G}_L) = \text{RUC}_u(\widehat{G}_L)$ and the isomorphism $\text{UC}(G) \simeq \text{RUC}_u(\widehat{G}_L)$, which respects tame functions by Remark 1.5 and the definition of tame functions via tame compactifications. Combining with Theorem 2.6, we can state the following.

Corollary 4.1. *For $G = \text{Aut}(\mathbb{Q}, <)$ we have $\text{Asp}(G) \subsetneq \text{Tame}_u(G) = \text{UC}(G)$.*

The situation is different for the random graph RG . Stable formulas are again exactly those expressible in the reduct of RG to the identity relation ([BYT14]), but in this case no other formula is NIP.

Lemma 4.2. *On the random graph, every NIP formula is stable.*

Proof. The usual proof that every NIP theory without the *strict order property* is stable (see, for example, Theorem 2.67 in [Sim14]) shows in fact that NIP and stable formulas coincide in any theory without the strict order property. This is the case of the random graph, whose theory is even simple. \square

Corollary 4.3. *For $G = \text{Aut}(RG)$ we have $\text{WAP}(G) = \text{Tame}_u(G) \subsetneq \text{UC}(G)$.*

By the Stone duality, the group $\text{Homeo}(2^\omega)$ of homeomorphisms of the Cantor space coincides with the group of automorphisms of the countable atomless boolean algebra \mathbb{B} . The family of stable formulas on \mathbb{B} has also been described in [BYT14]. We remark here that: (a) the order relation $x \leq y$ on \mathcal{B} (defined, in the standard signature, by the formula $x \wedge y = x$) is unstable and NIP; and (b) the disjointness relation $x \wedge y = 0$ has the independence property. In this case we obtain thus the following.

Corollary 4.4. *For $G = \text{Homeo}(2^\omega)$ we have $\text{Asp}(G) \subsetneq \text{Tame}_u(G) \subsetneq \text{UC}(G)$.*

The previous examples come from classical structures; we consider now a purely metric one: the Urysohn sphere \mathbb{U}_1 .

Theorem 4.5. *The Tame_u -compactification of $\text{Iso}(\mathbb{U}_1)$ is trivial.*

Proof. We are to show that the algebra $\text{Tame}_u(\text{Iso}(\mathbb{U}_1))$ consists only of the constants, that is to say, as per Remark 3.4, that if $f_a \in \text{Def}((\mathbb{U}_1)^\omega)$ is not constant on Ga then $f(x, y)$ is not NIP on $Ga \times Ga$ (so, here, y is an ω -variable in the type of a).

Suppose that $f(a, c) \neq f(a', c')$, with $a \simeq a'$, $c \simeq c'$ as metric spaces. We will need to assume that the elements of a are separated enough from the elements of c , (the same for a', c'), and that the metric space ac is similar to $a'c'$; so we precise and justify this. Let $0 < \epsilon < 1$. By the universality and homogeneity of the Urysohn sphere, note first that for any tuples x, y in \mathbb{U}_1 we can find \tilde{y} in

\mathbb{U}_1 such that $y \simeq \tilde{y}$, $d(y_n, \tilde{y}_n) = \epsilon$ and $d(x_n, \tilde{y}_m) = (d(x_n, y_m) + \epsilon) \wedge 1$ for all coordinates n, m (where $r \wedge s$ denotes $\min(r, s)$). Note secondly that finitely many iterations of this process eventually end with $d(x_n, \tilde{y}_m) = 1$ for all n, m .

If we chose ϵ small enough and do one iteration of the previous process for $xy = ac$, by continuity of f we can assume (replacing ac by $a\tilde{c}$) that

$$d(a_n, a_m) \leq d(a_n, c_k) + d(c_k, a_m) - \epsilon, \quad d(c_n, c_m) \leq d(c_n, a_k) + d(a_k, c_m) - \epsilon$$

for all n, m, k —and still $f(a, c) \neq f(a', c')$. So we have separated the elements of a from those of c , and we do the same for a', c' .

Now we iterate the process described above for $\epsilon/2$, starting with $xy = ac$, thus producing a finite sequence of copies of c , the last copy \tilde{c} verifying $d(a_n, \tilde{c}_m) = 1$ for all n, m . We do the same starting with a', c' , finishing with a copy \tilde{c}' with the analogous property. Since the theory of \mathbb{U}_1 has quantifier elimination, we have $f(a, \tilde{c}) = f(a', \tilde{c}')$. So f differs in two consecutive steps of the process, and by replacing our tuples $ac, a'c'$ by these consecutive tuples we may assume also that $|d(a_n, c_m) - d(a'_n, c'_m)| \leq \epsilon/2$ for all n, m .

With the previous assumptions in mind, we now construct a metric space containing a sequence $(a^i)_{i < \omega}$ of different copies of $a \simeq a'$ and, for each $I \subset \omega$, a copy b^I of $c \simeq c'$ such that $a^i b^I \simeq ac$ if $i \in I$ and $a^i b^I \simeq a'c'$ if $i \notin I$. For $i \neq j, I \neq J$ and each n, m we define $d(a_n^i, a_m^j) = (d(a_n, a_m) + \epsilon/2) \wedge 1$, $d(b_n^I, b_m^J) = (d(c_n, c_m) + \epsilon/2) \wedge 1$. The triangle inequalities are satisfied; for example, for $a_n^i, a_m^j, b_k^I, i \neq j, i \in I$, we have $d(a_n^i, a_m^j) = (d(a_n, a_m) + \epsilon/2) \wedge 1 \leq (d(a_n, c_k) + d(c_k, a_m) - \epsilon/2) \wedge 1 \leq d(a_n^i, b_k^I) + d(b_k^I, a_m^j)$, and also $d(a_n^i, b_k^I) \leq (d(a_n^i, a_m^j) + \epsilon/2) \wedge 1 + (d(a_m^j, b_k^I) - \epsilon/2) \wedge 1 \leq d(a_n^i, a_m^j) + d(a_m^j, b_k^I)$. The other inequalities are proved similarly.

By the universality of the Urysohn sphere we can assume that the tuples a^i lie in \mathbb{U}_1 , the b^I in some extension. By quantifier elimination, $f(a^i, b^I) = f(a, c)$ if $i \in I$ and $f(a^i, b^I) = f(a', c')$ if $i \notin I$. This shows that $f(x, y)$ is not NIP on $Ga \times Ga$, and proves the theorem. \square

In Question 7.10 from [GM13a] it was asked whether the algebra $\text{Tame}(G)$ separates points and closed subsets of G for every Polish group G . Unfortunately, we do not know how big may the gap between $\text{Tame}(G)$ and $\text{Tame}_u(G)$ be.

Question 4.6. Is there a way to *regularize* a (non-constant) function $f \in \text{RUC}(G)$ to get (a non-constant) $\tilde{f} \in \text{UC}(G)$, in such a manner that tameness is preserved?

Finally, we consider the group $G = H_+[0, 1]$ of orientation preserving homeomorphisms of $[0, 1]$ with the compact-open topology—which coincides on G with those of pointwise or uniform convergence. In spite of not being naturally presented as an automorphism group of some \aleph_0 -categorical metric structure, this group is Roelcke precompact; see [RD81].

The following result was explained to us by M. Megrelishvili.

Theorem 4.7. $\text{UC}(H_+[0, 1]) \subset \text{Tame}(H_+[0, 1])$.

See [GM14], Theorem 8.1. As remarked there, the inclusion is strict: the function $f : G \rightarrow \mathbb{R}$ given by $f(g) = g(1/2)$, for example, is tame (it comes from the Helly space, which is a Rosenthal compactification of G) but not left uniformly continuous: $\sup_g |f(gh) - f(g)| = 1$ for any $h \in G$ with $h(1/2) \neq 1/2$. In fact, f is even null: it is clear that, for reals $r < s$, there are no increasing functions $g_0, g_1 \in [0, 1]^{[0, 1]}$ and elements $x_{\{0\}}, x_{\{1\}} \in [0, 1]$ such that $g_i(x_I) < r$ if $i \in I$ and $g_i(x_I) > s$ if $i \notin I$. One deduces that $\text{UC}(G) \subsetneq \text{Null}(G)$.

Additionally, as we have already recalled, the celebrated result of [Meg01] says that $\text{WAP}(H_+[0, 1])$ is trivial. This leads an interesting model-theoretic example, addressed in the following corollary.

If $f(x, y)$ is a formula in two partitioned sets of variables x, y (of arbitrary length), let us say that $f(x, y)$ is *separated* if it is equivalent to a continuous combination of formulas $f_i(z_i)$ where, for each i , $z_i = x$ or $z_i = y$. Equivalently, $f(x, y)$ is separated if it factors through the product of type spaces $S_x(\emptyset) \times S_y(\emptyset)$. Of course, separated formulas are stable. Let us say that a structure is *purely unstable* if every stable formula $f(x, y)$ is separated. No infinite classical structure can be purely unstable, since the identity relation $x = y$ is always stable and never separated.

Corollary 4.8. *The \aleph_0 -categorical structure $M = \widehat{G}_L$ associated to $G = H_+[0, 1]$ is purely unstable and NIP.*

Proof. Of course, WAP-triviality implies that M is purely unstable: if $f(x, y)$ is stable and a, b are parameters, then the function $g \mapsto f(a, gb)$ belongs to $\text{WAP}(G)$ and so is constant; it follows by \aleph_0 -categoricity that $f(x, y)$ is separated.

On the other hand, Theorem 4.7 and Proposition 3.3 imply that every formula $f(x, y)$ with $|y| = 1$ is NIP. A well-known argument (see for example Proposition 2.11 in [Sim14], which adapts easily to the metric setting), shows that then every formula is NIP. \square

We finish with a remark connecting sections 2 and 3 of this paper. Since reflexively-representable functions correspond to stable formulas and Rosenthal-representable functions correspond to NIP formulas, it is not surprising that, as we have seen, the natural intermediate subalgebra of Asplund-representable functions collapses to one of the other two: on the model-theoretic side, there is no known natural notion between stable and NIP. However, one might be slightly surprised to find that $\text{WAP} = \text{Asp}$ rather than $\text{Asp} = \text{Tame}_u$ (although, in fact, this was already known for $G = H_+[0, 1]$). Indeed, Asplund and Rosenthal Banach spaces were once difficult to distinguish, with the first examples coming in the mid-seventies from independent works of James and of Lindenstrauss and Stegall. It is thus worthy to remark that, via our results and the Banach space construction of Glasner and Megrelishvili [GM12], every NIP unstable \aleph_0 -categorical structure yields an example of a Rosenthal non-Asplund Banach space.

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