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A new function space and applications

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Abstract

We define a new function space B , which contains in particular BMO, BV, and $W^{1/p,p}$, $1 < p < \infty$. We investigate its embedding into Lebesgue and Marcinkiewicz spaces. We present several inequalities involving L^p norms of integer-valued functions in B . We introduce a significant closed subspace, B_0 , of B , containing in particular VMO and $W^{1/p,p}$, $1 \leq p < \infty$. The above mentioned estimates imply in particular that integer-valued functions belonging to B_0 are necessarily constant. This framework provides a “common roof” to various, seemingly unrelated, statements asserting that integer-valued functions satisfying some kind of regularity condition must be constant.

1 Introduction

Let Ω be a connected domain in \mathbb{R}^n . Recall that if $f : \Omega \rightarrow \mathbb{Z}$ is a measurable function which satisfies one of the following regularity properties:

1. $f \in \text{VMO}(\Omega)$;
2. $f \in W^{1,1}(\Omega)$;
3. $f \in W^{1/p,p}(\Omega)$, with $1 < p < \infty$,

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then f is constant [3, Comment 2, pp. 223-224], [2, Theorem B.1]. The original motivation for this article was to provide a “common roof” to all these cases, and which yields in particular the following

Theorem 1. *Assume that $f : \Omega \rightarrow \mathbb{Z}$ is measurable and can be written as $f = f_1 + f_2 + f_3$, with $f_1 \in \text{VMO}(\Omega; \mathbb{R})$, $f_2 \in W^{1,1}(\Omega; \mathbb{R})$ and $f_3 \in W^{1/p,p}(\Omega; \mathbb{R})$ for some $1 < p < \infty$. Then f is constant.*

The proof of Theorem 1 relies heavily on the introduction of a new space of functions, which might be of interest well beyond the scope of Theorem 1.

In what follows we denote by Q the unit cube $(0, 1)^n$. For $0 < \varepsilon < 1$, $Q_\varepsilon(a)$ is the ε -cube centered at a .

Given $f \in L^1(Q; \mathbb{R})$ and an ε -cube $Q_\varepsilon \subset Q$, we set

$$M(f, Q_\varepsilon) = \int_{Q_\varepsilon} |f - f_{Q_\varepsilon}|, \text{ where } f_{Q_\varepsilon} = \int_{Q_\varepsilon} f, \quad (1)$$

and

$$M^*(f, Q_\varepsilon) = \int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(y) - f(z)| dy dz. \quad (2)$$

Clearly, we have

$$M(f, Q_\varepsilon) \leq M^*(f, Q_\varepsilon) \leq 2M(f, Q_\varepsilon). \quad (3)$$

Note that if $f = \mathbb{1}_A$, with $A \subset Q$ measurable, then

$$M(f, Q_\varepsilon) = M^*(f, Q_\varepsilon) = \frac{2|A \cap Q_\varepsilon|(|Q_\varepsilon| - |A \cap Q_\varepsilon|)}{|Q_\varepsilon|^2} \leq \frac{1}{2}. \quad (4)$$

The following quantity plays an important role:

$$[f]_\varepsilon = \sup_{\mathcal{F}} \left\{ \varepsilon^{n-1} \sum_{j \in J} M(f, Q_\varepsilon(a_j)) \right\}. \quad (5)$$

Here, \mathcal{F} denotes a collection of mutually disjoint ε -cubes, $\mathcal{F} = (Q_\varepsilon(a_j))_{j \in J}$, such that $\#J = \text{cardinality of } J \leq 1/\varepsilon^{n-1}$ (instead of $\#J$ we sometimes write $\#\mathcal{F}$) and the sup in (5) is taken over all such collections.

We then introduce the space

$$B = \left\{ f \in L^1(Q; \mathbb{R}); \sup_{0 < \varepsilon < 1} [f]_\varepsilon < \infty \right\},$$

and the corresponding norm (modulo constants)

$$\|f\|_B = \sup_{0 < \varepsilon < 1} [f]_\varepsilon. \quad (6)$$

The definition of B is inspired by the celebrated BMO space of John–Nirenberg [4] equipped with the norm (modulo constants)

$$\|f\|_{\text{BMO}} := \sup_{0 < \varepsilon < 1} \sup_{a \in Q} \{M(f, Q_\varepsilon(a)); Q_\varepsilon(a) \subset Q\}. \quad (7)$$

Here are several examples of functions in B .

Example 1. $\text{BMO} \subset B$ with continuous injection.

Indeed, using (7) we find that $\|f\|_B \leq \|f\|_{\text{BMO}}$.

When $n = 1$, we clearly have $B = \text{BMO}$; however, when $n \geq 2$, B is strictly bigger than BMO (see e.g. Example 2 below).

Example 2. $\text{BV} \subset B$ with continuous injection.

Indeed, by Poincaré's inequality

$$\int_{Q_\varepsilon} |f - f_{Q_\varepsilon}| \leq \frac{c_n}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\nabla f|,$$

so that

$$\sum_{j \in J} M(f, Q_\varepsilon(a_j)) \leq \frac{c_n}{\varepsilon^{n-1}} \int_{\cup_{j \in J} Q_\varepsilon(a_j)} |\nabla f| \quad (8)$$

and

$$[f]_\varepsilon \leq c_n \int_Q |\nabla f|. \quad (9)$$

Example 3. $W^{1/p,p} \subset B$, $1 < p < \infty$, with continuous injection.

Indeed, for every fixed $\alpha > 0$ we have

$$\int_{Q_\varepsilon} \int_{Q_\varepsilon} |f(y) - f(z)| dy dz \leq n^{\alpha/2} \varepsilon^\alpha \int_{Q_\varepsilon} \int_{Q_\varepsilon} \frac{|f(y) - f(z)|}{|y - z|^\alpha} dy dz.$$

Choosing $\alpha = (n + 1)/p$ and applying Hölder's inequality gives

$$M^*(f, Q_\varepsilon) \leq \frac{c_n}{\varepsilon^{(n-1)/p}} \left[\int_{Q_\varepsilon} \int_{Q_\varepsilon} \frac{|f(y) - f(z)|^p}{|y - z|^{n+1}} dy dz \right]^{1/p}, \text{ with } c_n = n^{(n+1)/2},$$

and since $\#J \leq 1/\varepsilon^{n-1}$ we obtain

$$\varepsilon^{n-1} \sum_{j \in J} M^*(f, Q_\varepsilon(a_j)) \leq c_n \left[\sum_{j \in J} \int_{Q_\varepsilon(a_j)} \int_{Q_\varepsilon(a_j)} \frac{|f(y) - f(z)|^p}{|y - z|^{n+1}} dy dz \right]^{1/p}. \quad (10)$$

Therefore

$$[f]_\varepsilon \leq c_n \|f\|_{W^{1/p,p}}.$$

An important quantity associated with B is defined by

$$[f] = \overline{\lim}_{\varepsilon \rightarrow 0} [f]_\varepsilon. \quad (11)$$

The subspace

$$B_0 = \{f \in B; [f] = 0\} \quad (12)$$

plays a key role in this article.

Example 1'. $\text{VMO} \subset B_0$.

This is clear, since VMO functions (see [5]) are characterized by

$$\limsup_{\varepsilon \rightarrow 0} \sup_{a \in Q} \{M(f, Q_\varepsilon(a)); Q_\varepsilon(a) \subset Q\} = 0.$$

Moreover, $\text{VMO} = B_0$ when $n = 1$.

Example 2'. $W^{1,1} \subset B_0$.

This is clear from (8) and the fact that $|\cup_{j \in J} Q_\varepsilon(a_j)| \leq \varepsilon$.

Example 3'. $W^{1/p,p} \subset B_0$, $1 < p < \infty$.

This is an immediate consequence of (10) and the fact that $|\cup_{j \in J} Q_\varepsilon(a_j) \times Q_\varepsilon(a_j)| \leq \varepsilon^{n+1}$.

In particular we see that

$$\text{VMO} + W^{1,1} + W^{1/p,p} \subset B_0. \quad (13)$$

2 Some properties of B

The main result of this section is

Theorem 2. *Let $n \geq 2$. Then we have $B \subset L^{n/(n-1),w}$, and*

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1),w}} \leq C_n \|f\|_B, \quad \forall f \in B. \quad (14)$$

In Theorem 2, the Marcinkiewicz space $L^{n/(n-1),w}$ cannot be replaced by $L^{n/(n-1)}$, as a consequence of the next result.

Proposition 3. *Let $n \geq 2$. There exists some $f \in B$ such that $f \notin L^{n/(n-1)}$.*

Proof of Theorem 2. We may assume that

$$\|f\|_B \leq 1 \text{ and } \int_Q f = 0. \quad (15)$$

We also temporarily make the additional assumption that $f \in L^\infty$.

Under these assumptions, we will prove that

$$\|f\|_{L^{n/(n-1),w}} \simeq \sup_{t>0} t |\{|f| > t\}|^{(n-1)/n} \leq C_n. \quad (16)$$

For this purpose it suffices to consider, in (16), only $t \gtrsim 1$.

We first note that, by (15), we have

$$\int_Q |f| \leq 1. \quad (17)$$

In view of (17) we may consider, for $t > 1$, a Calderón-Zygmund decomposition at height t , i.e., we consider families \mathcal{F}_j (with $j \geq 1$) of mutually disjoint cubes $Q_{2^{-j}} \subset Q$ of size 2^{-j} such that, if we set $\mathcal{F} = \cup_{j \geq 1} \mathcal{F}_j$, then

$$\int_{Q_*} |f| \simeq t \text{ for every } Q_* \in \mathcal{F} \quad (18)$$

and

$$|f| \leq t \text{ a.e. in } R := Q \setminus \cup_{Q_* \in \mathcal{F}} Q_*. \quad (19)$$

We next decompose $f = g + h$, with

$$g = f \mathbb{1}_R + \sum_{Q_* \in \mathcal{F}} \left(\int_{Q_*} f \right) \mathbb{1}_{Q_*},$$

$$h = \sum_{j \geq 1} h_j, \text{ and } h_j = \sum_{Q_* \in \mathcal{F}_j} \left(f - \int_{Q_*} f \right) \mathbb{1}_{Q_*}.$$

By (18) and (19), we have

$$|g| \leq Ct \text{ and thus } \{|f| > 2Ct\} \subset \{|h| > Ct\}. \quad (20)$$

Using (20), we see that (16) amounts to the following:

$$\sup_{t>1} t |\{|h| > Ct\}|^{(n-1)/n} \leq c. \quad (21)$$

We now proceed with the proof of (21). Since $\|f\|_B = 1$, for every family $\mathcal{G} \subset \mathcal{F}_j$ such that

$$\#\mathcal{G} \leq 1/(2^{-j})^{n-1} = 2^{j(n-1)},$$

we have

$$2^{-j(n-1)} \sum_{Q_* \in \mathcal{G}} \int_{Q_*} \left| f - \int_{Q_*} f \right| \leq 1.$$

By covering \mathcal{F}_j with mutually disjoint sets \mathcal{G} as above, we find that

$$\sum_{Q_* \in \mathcal{F}_j} \int_{Q_*} \left| f - \int_{Q_*} f \right| \leq 2^{j(n-1)} + \#\mathcal{F}_j, \quad (22)$$

and thus

$$\|h_j\|_{L^1} \leq 2^{-j} + 2^{-nj} \#\mathcal{F}_j. \quad (23)$$

On the other hand, we have (using (18))

$$1 \geq \|f\|_{L^1} \geq \sum_{j \geq 1} \sum_{Q_* \in \mathcal{F}_j} \int_{Q_*} |f| = \sum_{j \geq 1} \sum_{Q_* \in \mathcal{F}_j} 2^{-nj} \int_{Q_*} |f| \gtrsim \sum_{j \geq 1} 2^{-nj} t \#\mathcal{F}_j. \quad (24)$$

From (23) and (24), we deduce that

$$\sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \sum_{\mathcal{F}_j \neq \emptyset} 2^{-j}. \quad (25)$$

We next recall that

$$\|f\|_{L^{n/(n-1),w}} = \sup_{A \subset Q} |A|^{-1/n} \int_A |f|. \quad (26)$$

If $\mathcal{F}_j \neq \emptyset$ and $Q_* \in \mathcal{F}_j$, then (26) applied with $A = Q_*$, combined with (18), implies that

$$2^{-j} \lesssim \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \quad (27)$$

By (25) and (27), we have

$$\|h\|_{L^1} \leq \sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \quad (28)$$

In turn, (28) implies that (with C as in (21))

$$|\{|h| > Ct\}| \leq \frac{\|h\|_{L^1}}{Ct} \lesssim \frac{1}{t^2} + \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t^n} \right)^{1/(n-1)}, \quad (29)$$

and thus

$$t |\{|h| > Ct\}|^{(n-1)/n} \lesssim t^{(2-n)/n} + \|f\|_{L^{n/(n-1),w}}^{1/n} \leq 1 + \|f\|_{L^{n/(n-1),w}}^{1/n}. \quad (30)$$

By taking, in (30), the supremum over $t > 1$, we find that

$$\|f\|_{L^{n/(n-1),w}} \lesssim 1 + \|f\|_{L^{n/(n-1),w}}^{1/n},$$

and therefore $\|f\|_{L^{n/(n-1),w}} \lesssim 1$.

We complete the proof by removing the assumption that $f \in L^\infty$. Let

$$\Phi_N(s) = \begin{cases} s, & \text{if } |s| \leq N \\ N, & \text{if } s > N \\ -N, & \text{if } s < -N \end{cases}$$

and set $f_N := \Phi_N(f)$. By (3), we have $\|f_N\|_B \leq 2\|f\|_B$. In addition, f_N is bounded and thus satisfies (14), i.e.,

$$\left\| f_N - \int_Q f_N \right\|_{L^{n/(n-1),w}} \leq 2C_n \|f\|_B. \quad (31)$$

Using (26) and passing to the limit as $N \rightarrow \infty$ in (31) yields (14) for every $f \in B$. \square

Proof of Proposition 3. Set

$$\varphi(x) = (1 - |x|)^+, \quad \forall x \in \mathbb{R}^n$$

and

$$N_m = 2^{2^m}, \quad \forall m \geq 1.$$

Consider a sequence of points $(b_m)_{m \geq 1}$ such that the open balls $B(b_m, 2/N_m)$ are contained in Q and mutually disjoint. (We may e.g. choose the points b_m on a line segment parallel to the x_1 -axis.) Set

$$f_m(x) = N_m^{n-1} \varphi(N_m(x - b_m)), \quad \forall m \geq 1 \quad (32)$$

and

$$f(x) = \sum_{m \geq 1} f_m(x). \quad (33)$$

We will prove that $f \in B$ and $f \notin L^{n/(n-1)}$.

Note that

$$\text{supp } f_m = \overline{B}(b_m, 1/N_m),$$

and that the sets $\text{supp } f_m$, $m \geq 1$, are mutually disjoint.

Clearly,

$$\|f_m\|_{L^1(Q)} = \frac{C}{N_m}, \quad \forall m \geq 1, \quad (34)$$

and thus $f \in L^1(Q)$; here and in what follows we denote by C a generic constant depending only on n ,

We have

$$\|f_m\|_{L^{n/(n-1)}(Q)}^{n/(n-1)} = C, \quad \forall m \geq 1,$$

so that $f \notin L^{n/(n-1)}(Q)$.

Given $0 < \varepsilon < 1$ and integers $M_1 = M_1(\varepsilon) \geq 1$ and $M_2 = M_2(\varepsilon) > M_1(\varepsilon)$ to be defined later, we write

$$f = S_1 + S_2 + S_3, \quad (35)$$

with

$$S_1 = \sum_{m \leq M_1} f_m, \quad S_2 = \sum_{M_1 < m \leq M_2} f_m, \quad S_3 = \sum_{m > M_2} f_m. \quad (36)$$

We now estimate separately $[S_1]_\varepsilon$, $[S_2]_\varepsilon$ and $[S_3]_\varepsilon$.

Estimate of $[S_1]_\varepsilon$. Here we use the fact that if $h \in \text{Lip}(Q)$ then

$$M(h, Q_\varepsilon(a)) \leq \sqrt{n} \varepsilon \|h\|_{\text{Lip}}, \quad (37)$$

and thus

$$[h]_\varepsilon \leq \sqrt{n} \varepsilon \|h\|_{\text{Lip}}.$$

In particular,

$$[f_m]_\varepsilon \leq C \varepsilon (N_m)^n. \quad (38)$$

Using (38) and the fact that

$$\sum_{i=1}^p X^i \leq \frac{X^{p+1}}{X-1}, \quad \forall X > 1,$$

we deduce that

$$[S_1]_\varepsilon \leq C \varepsilon 2^{n 2^{M_1}}, \quad \forall \varepsilon \in (0, 1). \quad (39)$$

Estimate of $[S_2]_\varepsilon$. Applying (9) to f_m yields

$$[f_m]_\varepsilon \leq C, \quad \forall m \geq 1, \quad \forall \varepsilon \in (0, 1),$$

and in particular

$$[S_2]_\varepsilon \leq C(M_2 - M_1), \quad \forall \varepsilon \in (0, 1). \quad (40)$$

Estimate of $[S_3]_\varepsilon$. Clearly

$$[h]_\varepsilon \leq \frac{2}{\varepsilon} \|h\|_{L^1(Q)}, \quad \forall h \in L^1. \quad (41)$$

From (34) we deduce that

$$[f_m]_\varepsilon \leq \frac{C}{\varepsilon N_m}. \quad (42)$$

Using (42) and the fact that

$$\sum_{i=p}^{\infty} Y^i = \frac{Y^p}{1-Y}, \quad \forall Y \in [0, 1),$$

we see that

$$[S_3]_\varepsilon \leq \frac{C}{\varepsilon 2^{2^{M_2}}}. \quad (43)$$

We now explain how to choose $M_1(\varepsilon)$ and $M_2(\varepsilon)$. Given $0 < \varepsilon < 1$, we denote by $M_1 = M_1(\varepsilon)$ the largest integer $\ell \geq 1$ such that

$$2^{n 2^\ell} \leq \frac{2^{2n}}{\varepsilon}. \quad (44)$$

Equivalently, we have

$$2^{n 2^{M_1}} \leq \frac{2^{2n}}{\varepsilon} \quad (45)$$

and

$$2^{2n 2^{M_1}} > \frac{2^{2n}}{\varepsilon}. \quad (46)$$

Combining (39) and (45) yields

$$[S_1]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \quad (47)$$

From (45) and (46) we obtain

$$|M_1(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \leq C, \quad \forall \varepsilon \in (0, 1/2). \quad (48)$$

Next we denote by $M_2 = M_2(\varepsilon)$ the smallest integer $\ell \geq 1$ such that

$$2^{2^\ell} \geq \frac{4}{\varepsilon}.$$

(Note that $M_2 > M_1$ since $2^{2^{M_1}} < 4/\varepsilon$.)

Equivalently, we have

$$2^{2^{M_2}} \geq \frac{4}{\varepsilon} \quad (49)$$

and

$$2^{2^{M_2-1}} < \frac{4}{\varepsilon}. \quad (50)$$

Combining (43) and (49) yields

$$[S_3]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \quad (51)$$

From (49) and (50) we obtain

$$|M_2(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \leq C, \quad \forall \varepsilon \in (0, 1/2). \quad (52)$$

Therefore,

$$|M_2(\varepsilon) - M_1(\varepsilon)| \leq C, \quad \forall \varepsilon \in (0, 1). \quad (53)$$

(Inequality (53) is deduced from (48) and (52) when $\varepsilon \in (0, 1/2)$, and from (50) when $\varepsilon \in [1/2, 1)$.)

It follows from (40) and (53) that

$$[S_2]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \quad (54)$$

Putting together (47), (51) and (54) we conclude that

$$[f]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1),$$

and thus $f \in B$. □

3 Some properties of B_0 and $[f]$

Our first result is

Theorem 4. *Let f be a \mathbb{Z} -valued function on Q such that $f \in B_0$. Then f is constant.*

Combining Theorem 4 with (13) we obtain Theorem 1.

When $n = 1$ we have $B_0 = \text{VMO}$ and we may then invoke the fact that functions in $\text{VMO}(Q; \mathbb{Z})$ are constant (for any $n \geq 1$); see [3, Comment 2, p. 223–224]. Therefore it suffices to prove Theorem 4 when $n \geq 2$. Next, we observe that it suffices to prove Theorem 4 when $f = \mathbb{1}_A$ for some $A \subset Q$. Indeed, let $k \in \mathbb{Z}$ be such that $|f^{-1}(k)| > 0$. Set $A = f^{-1}(k)$ and $g = \mathbb{1}_A$. Clearly $M^*(f, Q_\varepsilon) \geq M^*(g, Q_\varepsilon)$ for every ε -cube Q_ε . Since $f \in B_0$, we deduce that $g \in B_0$. If Theorem 4 holds for g , then $g \equiv 1$, and thus $f \equiv k$.

Hence it remains to prove Theorem 4 when $n \geq 2$ and $f = \mathbb{1}_A$. In this case we have the following quantitative improvement of Theorem 4.

Theorem 5. *Let $n \geq 2$. There exists a constant C_n such that if $f = \mathbb{1}_A$ with $A \subset Q$ measurable, then*

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C_n [f]. \quad (55)$$

Remark 6. A much more precise result (see [1]) asserts that there exist two constants $0 < \underline{c}_n \leq \bar{c}_n < \infty$ such that if $f = \mathbb{1}_A$, then

$$\underline{c}_n \min \left\{ 1, \int_Q |\nabla f| \right\} \leq [f] \leq \bar{c}_n \min \left\{ 1, \int_Q |\nabla f| \right\}, \quad (56)$$

with the convention that $\int_Q |\nabla f| = \infty$ if $f \notin \text{BV}$.

Note that

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \int_Q |\nabla f| \quad (57)$$

by the Sobolev embedding, and that clearly

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \quad \text{when } f = \mathbb{1}_A. \quad (58)$$

Therefore

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \min \left\{ 1, \int_Q |\nabla f| \right\} \leq C' [f] \quad \text{by (56)}.$$

In fact, using a variant of the definition (5) involving ε -cubes of general orientation, one obtains a quantity $[f]_\varepsilon^*$ satisfying

$$[f]_\varepsilon \leq [f]_\varepsilon^* \leq C_1 [f]_{C_2 \varepsilon}$$

for some constants $C_1 > 1$, $C_2 > 1$ depending only on n (see [1]). The main result in [1] asserts that if $f = \mathbb{1}_A$, then

$$\lim_{\varepsilon \rightarrow 0} [f]_\varepsilon^* = \frac{1}{2} \min \left\{ 1, \int_Q |\nabla f| \right\}; \quad (59)$$

the ingredients of the proof of (59) are much more sophisticated than the arguments presented below. We acknowledge that it was Theorem 5 which prompted one of us to conjecture that (59) holds.

The main tool in the proof of Theorem 5 is

Lemma 7. *Let $n \geq 2$. Let $U = \cup_{j \in J} Q_\varepsilon(a_j)$ be a union of ε -cubes. Then $Q \setminus U$ contains a connected set S of measure $\geq 1 - \alpha_n (\#J)^{n/(n-1)} \varepsilon^n$, for some positive constant α_n depending only on n .*

Here, the ε -cubes are not necessarily mutually disjoint, and we do not assume that these cubes are completely contained in Q .

Remark 8. The conclusion of Lemma 7 is optimal. Indeed, consider a ball $B \subset Q$ of (small) radius R . We may cover the sphere $\Sigma = \partial B$ by a union of ε -cubes as above with $\#J \varepsilon^{n-1} \simeq R^{n-1}$. Then $|B| \simeq R^n \simeq (\#J)^{n/(n-1)} \varepsilon^n$.

Granted Lemma 7, we turn to the

Proof of Theorem 5. Let $f = \mathbb{1}_A$, with $A \subset Q$. Fix any $\lambda \in (0, 1/2)$, e.g. $\lambda = 1/4$.

In view of (58), we may assume that

$$0 \leq [f] < 2\lambda(1 - \lambda), \quad (60)$$

for otherwise the conclusion is clear with $C_n = \frac{1}{\lambda(1-\lambda)}$.

Note that, by (4),

$$M(f, Q_\varepsilon) = 2f_{Q_\varepsilon}(1 - f_{Q_\varepsilon}).$$

Therefore,

$$M(f, Q_\varepsilon) < 2\lambda(1-\lambda) \implies \text{either } f_{Q_\varepsilon} < \lambda, \text{ or } f_{Q_\varepsilon} > 1-\lambda. \quad (61)$$

With ε small and $\tilde{Q} = (\varepsilon, 1-\varepsilon)^n$, consider a maximal family $J = J_\varepsilon$ of points $a \in \tilde{Q}$ such that the cubes $Q_\varepsilon(a)$ are mutually disjoint and satisfy

$$M(f, Q_\varepsilon(a)) \geq 2\lambda(1-\lambda), \quad \forall a \in J. \quad (62)$$

Let $\nu > 0$ (to be chosen arbitrarily small later). We claim that for ε sufficiently small (depending on ν) we have

$$\#J \leq \delta/\varepsilon^{n-1}, \quad \text{with } \delta = \frac{[f] + \nu}{2\lambda(1-\lambda)}. \quad (63)$$

Indeed, we first see that, for ε sufficiently small,

$$\#J \leq 1/\varepsilon^{n-1}. \quad (64)$$

Otherwise, we may choose a subfamily \tilde{J} such that $\#\tilde{J} = I(1/\varepsilon^{n-1})$, where $I(t)$ denotes the integer part of t . Then

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#\tilde{J}) 2\lambda(1-\lambda) \geq \varepsilon^{n-1} \left(\frac{1}{\varepsilon^{n-1}} - 1 \right) 2\lambda(1-\lambda),$$

which, by (60), is impossible for ε small. From (64) and the definition of $[f]_\varepsilon$ we have

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#J) 2\lambda(1-\lambda),$$

which yields (63) for ε sufficiently small.

Set $U := \cup_{a \in J} Q_{2\varepsilon}(a)$. By Lemma 7 and a scaling argument, $\tilde{Q} \setminus U$ contains a connected set $S = S_\varepsilon$ such that

$$|S_\varepsilon| \geq (1-2\varepsilon)^n - \alpha'_n \delta^{n/(n-1)}, \quad (65)$$

where $\alpha'_n = 2^n \alpha_n$.

We next note that (by the maximality of J) U contains the set

$$V = V_\varepsilon := \{b \in \tilde{Q}; M(f, Q_\varepsilon(b)) \geq 2\lambda(1-\lambda)\}, \quad (66)$$

and thus $S \subset \tilde{Q} \setminus V$.

We consider the continuous function

$$f_\varepsilon : \tilde{Q} \rightarrow \mathbb{R}, \quad f_\varepsilon(a) = f_{Q_\varepsilon(a)}.$$

By (61) and (66), in the set $\tilde{Q} \setminus V$ the function f_ε takes values into $[0, \lambda) \cup (1 - \lambda, 1]$. $S \subset \tilde{Q} \setminus V$ being connected, we find that either $f_\varepsilon < \lambda$, or $f_\varepsilon > 1 - \lambda$ in S .

We assume e.g. that $f_\varepsilon < \lambda$ in S_ε along a sequence $\varepsilon_m \rightarrow 0$. Clearly,

$$\int_{A \cap \tilde{Q}} |1 - f_\varepsilon| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and thus

$$(1 - \lambda) |S_{\varepsilon_m} \cap A| \leq \int_{S_{\varepsilon_m} \cap A} (1 - f_{\varepsilon_m}) \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (67)$$

On the other hand, by (65) and (67) we have

$$|A| = |S_{\varepsilon_m} \cap A| + |(\tilde{Q} \setminus S_{\varepsilon_m}) \cap A| + |(Q \setminus \tilde{Q}) \cap A| \leq \alpha'_n \delta^{n/(n-1)} + o(1) \text{ as } m \rightarrow \infty,$$

and thus $|A| \leq \alpha'_n \delta^{n/(n-1)}$, so that

$$|A|^{(n-1)/n} \leq \alpha''_n \delta = \alpha''_n \frac{[f] + \nu}{2\lambda(1 - \lambda)}, \text{ with } \alpha''_n = (\alpha'_n)^{(n-1)/n}.$$

Since $\nu > 0$ can be chosen arbitrarily small, we deduce that

$$|A|^{(n-1)/n} \leq \frac{\alpha''_n [f]}{2\lambda(1 - \lambda)}. \quad (68)$$

Finally, we note that

$$\begin{aligned} \left\| f - \int f \right\|_{L^{n/(n-1)}} &= \left(|A|(1 - |A|)^{n/(n-1)} + (1 - |A|)|A|^{n/(n-1)} \right)^{(n-1)/n} \\ &\leq 2 \min \left\{ |A|^{(n-1)/n}, |A^c|^{(n-1)/n} \right\}. \end{aligned} \quad (69)$$

Combining (68) and (69) yields (55). \square

For further use, let us note that the proof of Theorem 5 leads to the following result.

Lemma 9. *Let $n \geq 2$ and $\lambda \in (0, 1/2)$. Let $A \subset Q$ be measurable and set $f := \mathbb{1}_A$. Assume that there exists a sequence $\varepsilon_m \rightarrow 0$ and families*

$$J_m \subset \tilde{Q}^m := (3\varepsilon_m, 1 - 3\varepsilon_m)^n$$

of points a with the following property:

If $b \in \tilde{Q}^m \setminus \cup_{a \in J_m} Q_{2\varepsilon_m}(a)$, then $M(f, Q_{\varepsilon_m}(b)) < 2\lambda(1 - \lambda)$.

Let

$$\delta := \liminf_{m \rightarrow \infty} \varepsilon_m^{n-1} \# J_m.$$

Then either $|A| \geq 1 - \tilde{c}_n \delta^{n/(n-1)}$, or $|A^c| \geq 1 - \tilde{c}_n \delta^{n/(n-1)}$.

Proof of Lemma 7. Recall a standard “relative” isoperimetric inequality. Let $B \subset Q$ satisfy $|B| \leq 1/2$. By (57) (applied with $f = \mathbb{1}_B$) and (69), we have

$$|B| \leq c_n \left(\int_Q |\nabla \mathbb{1}_B| \right)^{n/(n-1)} = c_n [P(B)]^{n/(n-1)}, \quad (70)$$

where $P(B)$ represents the perimeter of B relative to Q . When B is a Lipschitz domain (which will be the case in what follows), $P(B)$ is the (surface) measure of $\partial B \cap Q$.

We now turn to the proof of the lemma. Set $\delta = (\#J)\varepsilon^{n-1}$. Let $(A_i)_{i \in I}$ be the connected components of the open set $Q \setminus \cup_{j \in J} \overline{Q}_\varepsilon(a_j)$.

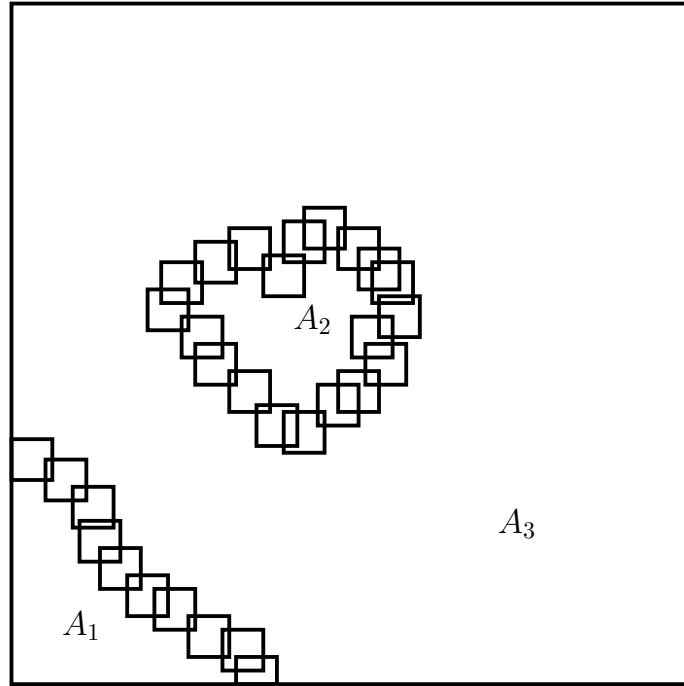


Figure 1: The components of $Q \setminus \cup_{j \in J} \overline{Q}_\varepsilon(a_j)$.

Note that each A_i is Lipschitz, and that

$$\cup_{i \in I} (\partial A_i \cap Q) \subset \cup_{j \in J} (\partial Q_\varepsilon(a_j) \cap Q). \quad (71)$$

Let

$$G_j := \{x \in \partial Q_\varepsilon(a_j) \cap Q; x \text{ does not belong to the } (n-2) \text{ skeleton of } \partial Q_\varepsilon(a_j)\}.$$

Note that

$$[\cup_{i \in I} (\partial A_i \cap Q)] \setminus [\cup_{j \in J} G_j] \text{ has zero } \mathcal{H}^{n-1} \text{ - measure.}$$

Since a point $x \in G_j$ belongs to at most one ∂A_i , we find, using (71), that

$$\sum_{i \in I} P(A_i) \leq \sum_{j \in J} P(Q_\varepsilon(a_j)) \leq c'_n \delta. \quad (72)$$

We claim that if $\delta < \delta_n$ (a positive number depending only on n), then there exists some $i_0 \in I$ such that $|A_{i_0}| > 1/2$. Indeed, argue by contradiction and assume that $|A_i| \leq 1/2, \forall i \in I$. By (70) and (72), we have

$$\begin{aligned} 1 - |U| = |Q \setminus U| &= \sum_{i \in I} |A_i| \leq c_n \sum_{i \in I} [P(A_i)]^{n/(n-1)} \leq c_n \left[\sum_{i \in I} P(A_i) \right]^{n/(n-1)} \\ &\leq c_n (c'_n \delta)^{n/(n-1)} = c''_n \delta^{n/(n-1)}. \end{aligned} \quad (73)$$

On the other hand

$$|U| \leq (\#J) \varepsilon^n = \delta \varepsilon < \delta. \quad (74)$$

Combining (73) and (74) we obtain

$$1 \leq \delta + c''_n \delta^{n/(n-1)};$$

this is impossible when $\delta < \delta_n$, where δ_n is the solution of

$$1 = \delta_n + c''_n (\delta_n)^{n/(n-1)},$$

and thus the claim is established when $\delta < \delta_n$.

Set $S = A_{i_0}$, which is clearly connected and contained in $Q \setminus U$. Applying (70) to $B = S^c$ we find (using (72))

$$1 - |S| \leq c_n [P(S^c)]^{n/(n-1)} = c_n [P(S)]^{n/(n-1)} \leq c''_n \delta^{n/(n-1)},$$

which is the desired conclusion when $\delta < \delta_n$.

Finally, we observe that

$$1 - \frac{1}{(\delta_n)^{n/(n-1)}} \delta^{n/(n-1)} \leq 0$$

when $\delta \geq \delta_n$ and therefore Lemma 7 holds with

$$\alpha_n = \max \left\{ c''_n, \frac{1}{(\delta_n)^{n/(n-1)}} \right\}. \quad \square$$

4 An extension of Theorem 5 to \mathbb{Z} -valued functions

Our main result in this section is

Theorem 10. *Let $n \geq 2$. There exists a positive constant c (independent of n) such that if f is a \mathbb{Z} -valued function in B and $[f] < c$, then $f \in L^{n/(n-1)}(\mathcal{Q})$ and*

$$\left\| f - \int_{\mathcal{Q}} f \right\|_{L^{n/(n-1)}(\mathcal{Q})} \leq C_n [f], \quad (75)$$

for some constant C_n depending only on n .

Theorem 5 can be deduced from Theorem 10. Indeed, let $f = \mathbb{1}_A$. Then either $[f] \leq c$, and Theorem 10 applies, or $[f] > c$, and then

$$\left\| f - \int_{\mathcal{Q}} f \right\|_{L^{n/(n-1)}(\mathcal{Q})} \leq 2 \leq (2/c)[f].$$

The smallness condition on $[f]$ in Theorem 10 is essential, as shown by the following improvement of Proposition 3.

Proposition 11. *Let $n \geq 2$. There exists a \mathbb{Z} -valued function $f \in B$ such that $f \notin L^{n/(n-1)}(\mathcal{Q})$.*

Proof of Theorem 10. Step 1. Decomposition of f as a sum of characteristic functions.

We temporarily assume that $f \geq 0$. Then f is a sum of characteristic functions. Indeed, set

$$A_k := \{x \in \mathcal{Q}; f(x) \geq k\}, \quad \forall k \in \mathbb{N}^*,$$

and let $g_k := \mathbb{1}_{A_k}$. Then we claim that

$$f = \sum_{k>0} g_k \quad (76)$$

and

$$|f(x) - f(y)| = \sum_{k>0} |g_k(x) - g_k(y)|, \quad \forall x, y \in \mathcal{Q}. \quad (77)$$

Indeed, on the one hand (76) follows from

$$\sum_{k>0} g_k(x) = \sum_{0 < k \leq f(x)} 1 = f(x).$$

On the other hand, assuming e.g. that $f(x) \geq f(y)$, we have $g_k(x) = g_k(y)$ provided either $k \leq f(y)$ or $k > f(x)$, and thus

$$\sum_{k>0} |g_k(x) - g_k(y)| = \sum_{f(y) < k \leq f(x)} |g_k(x) - g_k(y)| = \sum_{f(y) < k \leq f(x)} 1 = f(x) - f(y) = |f(x) - f(y)|;$$

that is, (77) holds.

We next note that (77) implies

$$M^*(f, Q_\varepsilon) = \sum_{k>0} M^*(g_k, Q_\varepsilon), \quad (78)$$

and in particular

$$M(g_k, Q_\varepsilon) \leq M^*(f, Q_\varepsilon), \quad \forall k > 0. \quad (79)$$

Step 2. Construction of maximal families of “bad” cubes.

Fix some $\lambda \in (0, 1/2)$ and consider a sequence $\varepsilon_m \rightarrow 0$. Let $\tilde{Q}^m := (3\varepsilon_m, 1 - 3\varepsilon_m)^n$. Let J_m be a maximal family of points $a \in \tilde{Q}^m$ such that the cubes $Q_{\varepsilon_m}(a)$, $a \in J_m$, are mutually disjoint and satisfy $M^*(f, Q_{\varepsilon_m}(a)) \geq 2\lambda(1 - \lambda)$.

By the maximality of J_m and by (79), we have

$$M(g_k, Q_{\varepsilon_m}(b)) \leq M^*(f, Q_{\varepsilon_m}(b)) < 2\lambda(1 - \lambda), \quad \forall b \in \tilde{Q}^m \setminus \cup_{a \in J_m} Q_{2\varepsilon_m}(a). \quad (80)$$

We next associate to each k an appropriate subfamily extracted from J_m . More specifically, let

$$J_m^k := \{a \in J_m; 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) \geq 2\lambda(1 - \lambda)\}. \quad (81)$$

We claim that

$$M(g_k, Q_{\varepsilon_m}(b)) < 2\lambda(1 - \lambda), \quad \forall b \in \tilde{Q}^m \setminus \cup_{a \in J_m^k} Q_{2\varepsilon_m}(a). \quad (82)$$

Indeed, (80) implies that (82) holds for $b \in \tilde{Q}^m \setminus \cup_{a \in J_m} Q_{2\varepsilon_m}(a)$.

It remains to establish (82) when $b \in Q_{2\varepsilon_m}(a)$ for some $a \in J_m \setminus J_m^k$. In this case, we have $Q_{\varepsilon_m}(b) \subset Q_{3\varepsilon_m}(a)$ and thus

$$M^*(g_k, Q_{\varepsilon_m}(b)) \leq 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) < 2\lambda(1 - \lambda).$$

This completes the proof of (82).

Step 3. A first estimate of $\|f - f_Q\|_{L^{n/(n-1)}}$.

By (69), (82), and Lemma 9, we have

$$\left\| g_k - \int_Q g_k \right\|_{L^{n/(n-1)}} \leq 2(\tilde{c}_n)^{(n-1)/n} \lim_{m \rightarrow \infty} (\varepsilon_m)^{n-1} \#J_m^k. \quad (83)$$

Thus

$$\sum_{k>0} \left\| g_k - \int_Q g_k \right\|_{L^{n/(n-1)}} \leq 2(\tilde{c}_n)^{(n-1)/n} \lim_{m \rightarrow \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k, \quad (84)$$

and therefore

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}} \leq 2(\tilde{c}_n)^{(n-1)/n} \liminf_{m \rightarrow \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k. \quad (85)$$

Step 4. A second estimate of $\|f - \int_Q f\|_{L^{n/(n-1)}}$.

In this step, we assume that

$$[f] < d := \lambda(1 - \lambda), \quad \text{with } \lambda \text{ chosen as in Step 2.} \quad (86)$$

Under this assumption, we will prove that

$$c'_n \liminf_{m \rightarrow \infty} (\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k \leq [f] \quad \text{for some constant } c'_n > 0. \quad (87)$$

Granted this estimate, we obtain (using (85)) that

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}} \leq \tilde{C}_n [f], \quad \text{with } \tilde{C}_n = 2(\tilde{c}_n)^{(n-1)/n} / c'_n. \quad (88)$$

We now proceed to the proof of (87). We first note that (by (3)) we have

$$M(f, Q_{\varepsilon_m}(a)) \geq \lambda(1 - \lambda), \quad \forall a \in J_m. \quad (89)$$

Repeating the proof of (64) (and using (86) and (89)), for large m we have

$$\#J_m \leq 1/(\varepsilon_m)^{n-1}. \quad (90)$$

We next rely on the following lemma, well-known to the experts, whose proof is omitted.

Lemma 12. *Let $\{Q_\varepsilon(a); a \in J\}$ be a family of mutually disjoint ε -cubes. Then there exists a constant $N = N(n)$ such that*

1. $J = J^1 \cup J^2 \cup \dots \cup J^N$.
2. For every j , the cubes $Q_{3\varepsilon}(a)$, $a \in J^j$, are mutually disjoint.
3. For every j , we have $\#J^j \leq \#J/3^{n-1}$.

By Lemma 12, for every family of mutually disjoint ε -cubes $Q_\varepsilon(a)$, $a \in J \subset (3\varepsilon, 1 - 3\varepsilon)^n$, such that $\#J \leq 1/\varepsilon^{n-1}$, we have

$$(3\varepsilon)^{n-1} \sum_{a \in J} M(h, Q_{3\varepsilon}(a)) \leq N[h]_{3\varepsilon}, \quad \forall h : Q \rightarrow \mathbb{R}. \quad (91)$$

In particular, for large m we have (using (90) and (91))

$$(\varepsilon_m)^{n-1} \sum_{a \in J_m} M(f, Q_{3\varepsilon_m}(a)) \leq N/3^{n-1} [f]_{3\varepsilon_m}. \quad (92)$$

Combining (92) with (3), we see that

$$(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) \leq 2N/3^{n-1} [f]_{3\varepsilon_m} \quad (93)$$

We now use successively (93), (78) and (81) and obtain that

$$\begin{aligned} [f]_{3\varepsilon_m} &\geq 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) \\ &= 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{a \in J_m} \sum_{k>0} M^*(g_k, Q_{3\varepsilon_m}(a)) \\ &\geq 3^{n-1}/(2N)(\varepsilon_m)^{n-1} \sum_{k>0} \sum_{a \in J_m^k} M^*(g_k, Q_{3\varepsilon_m}(a)) \\ &\geq \lambda(1-\lambda)/(3^{n+1}N)(\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k = c'_n(\varepsilon_m)^{n-1} \sum_{k>0} \#J_m^k, \end{aligned} \quad (94)$$

with $c'_n := \lambda(1-\lambda)/(3^{n+1}N)$.

We derive (87) by letting $m \rightarrow \infty$ in (94).

Step 5. We remove the assumption $f \geq 0$.

We note that $f = f^+ - f^-$, and that

$$|f^\pm(x) - f^\pm(y)| \leq |f(x) - f(y)|, \quad \forall x, y \in Q. \quad (95)$$

By (3) and (95), we have

$$M^*(f^\pm, Q_\varepsilon) \leq M^*(f, Q_\varepsilon) \leq 2M(f, Q_\varepsilon),$$

and thus $[f^\pm] \leq 2[f]$. By the first part of the proof of this theorem, we have

$$\left\| f^\pm - \int_Q f^\pm \right\|_{L^{n/(n-1)}} \leq \tilde{C}_n [f^\pm] \leq 2\tilde{C}_n [f], \quad (96)$$

provided $[f] < c := d/2$.

Finally, (96) implies that

$$\left\| f - \int_Q f \right\|_{L^{n/(n-1)}} \leq C_n [f] \quad \text{provided } [f] < c,$$

with $C_n := 4\tilde{C}_n$.

The proof of Theorem 10 is complete. \square

Proof of Proposition 11. We use the same notation and the same strategy as in the proof of Proposition 3, with some minor modifications.

Set

$$g_m(x) = I(f_m(x)), \quad \forall m \geq 1, \quad \text{where } I(t) \text{ denotes the integer part of } t,$$

and

$$g(x) = \sum_{m \geq 1} g_m(x).$$

Clearly,

$$\|g_m\|_{L^1(Q)} \leq \|f_m\|_{L^1(Q)} = \frac{C}{N_m} \quad (97)$$

(by (34)), so that $g \in L^1(Q)$. On the other hand

$$\|g_m\|_{L^{n/(n-1)}(Q)}^{n/(n-1)} \geq \|f_m - 1\|_{L^{n/(n-1)}(\{f_m > 1\})}^{n/(n-1)} \geq \alpha > 0, \quad \forall m \geq 1,$$

and thus $g \notin L^{n/(n-1)}(Q)$.

We will now prove that $g \in B$. Write

$$g = T_1 + T_2 + T_3,$$

with

$$T_1 = \sum_{m \leq M_1} g_m, \quad T_2 = \sum_{M_1 < m \leq M_2} g_m, \quad T_3 = \sum_{m > M_2} g_m,$$

where $M_1 = M_1(\varepsilon)$ and $M_2 = M_2(\varepsilon)$ are defined exactly as in the proof of Proposition 3.

Estimate of $[T_1]_\varepsilon$. Since $g_m \notin \text{Lip}(Q)$, we need to modify the argument. We claim that, for sufficiently small ε (depending only on n), given any cube $Q_\varepsilon(a)$ there exists at most one integer $m \leq M_1(\varepsilon)$ such that

$$Q_\varepsilon(a) \cap (\text{supp } g_m) \neq \emptyset. \quad (98)$$

Indeed, if (98) holds, then

$$Q_\varepsilon(a) \cap B(b_m, 1/N_m) \neq \emptyset,$$

and thus

$$Q_\varepsilon(a) \subset B(b_m, 2/N_m) \quad (99)$$

provided

$$\frac{1}{N_m} + \sqrt{n} \varepsilon \leq \frac{2}{N_m}, \quad \forall m \leq M_1. \quad (100)$$

On the other hand, (45) implies that

$$N_{M_1} \leq \frac{4}{\varepsilon^{1/n}},$$

and thus (100) holds when

$$\varepsilon \leq \varepsilon_0 := \frac{1}{4^{n/(n-1)} n^{n/[2(n-1)]}}.$$

We deduce the claim using (99) and the fact that the balls $B(b_m, 2/N_m)$ are mutually disjoint.

Therefore, for $\varepsilon \leq \varepsilon_0$ we have

$$M(T_1, Q_\varepsilon(a)) \leq \int_{Q_\varepsilon(a)} \int_{Q_\varepsilon(a)} |g_m(y) - g_m(z)| dy dz \quad (101)$$

for some $m \leq M_1(\varepsilon)$.

If $y, z \in Q_\varepsilon(a)$, we have

$$|f_m(y) - f_m(z)| \leq |y - z| \|f_m\|_{\text{Lip}} \leq (N_m)^n \sqrt{n} \varepsilon \leq C$$

(by (45)). Hence

$$|g_m(y) - g_m(z)| \leq C, \quad (102)$$

since

$$|I(t) - I(s)| \leq |t - s| + 1, \quad \forall t, s.$$

Combining (101) and (102) yields $M(T_1, Q_\varepsilon(a)) \leq C$ and therefore

$$[T_1]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (103)$$

For $\varepsilon \in [\varepsilon_0, 1)$, we use (41) to assert that

$$[T_1]_\varepsilon \leq \frac{2}{\varepsilon_0} \|T_1\|_{L^1(Q)} \leq \frac{2}{\varepsilon_0} \|g\|_{L^1(Q)}. \quad (104)$$

Combining (103) with (104) we deduce that

$$[T_1]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \quad (105)$$

Estimate of $[T_2]_\varepsilon$. We claim that

$$\int_Q |\nabla g_m| \leq C, \quad \forall m \geq 1, \quad (106)$$

and this implies via (9) that

$$[g_m]_\varepsilon \leq C, \quad \forall m \geq 1, \quad \forall \varepsilon \in (0, 1),$$

so that

$$[T_2]_\varepsilon \leq C(M_2 - M_1) \leq C, \quad \forall \varepsilon \in (0, 1) \quad (107)$$

(by (53)).

In order to prove (106), note that

$$\begin{aligned}\int_Q |\nabla g_m| &= \sum_{k=1}^{(N_m)^{n-1}} \mathcal{H}^{n-1}([f_m = k]) = C \sum_{k=1}^{(N_m)^{n-1}-1} \left(1 - \frac{k}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}} \\ &= C \sum_{\ell=1}^{(N_m)^{n-1}-1} \left(\frac{\ell}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}} \leq C.\end{aligned}$$

Estimate of $[T_3]_\varepsilon$. The technique for estimating $[S_3]_\varepsilon$ in the proof of Proposition 3 gives

$$[T_3]_\varepsilon \leq C, \quad \forall \varepsilon \in (0, 1). \quad (108)$$

Combining (105), (107) and (108) yields $g \in B$. \square

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