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KERNEL AND EIGENFUNCTION ESTIMATES FOR SOME SECOND ORDER ELLIPTIC OPERATORS

EL MAATI OUHABAZ AND ABDELAZIZ RHANDI

ABSTRACT. For a potential V such that $V(x) \geq |x|^\alpha$ with $\alpha > 2$ we prove that the heat kernel $k_t(x, y)$ associated to the uniformly elliptic operator $A = -\sum_{j,k=1}^n \partial_k(a_{jk}\partial_j) + V$ satisfies the estimate

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct^{-b}} \left(\frac{e^{-\frac{2\sqrt{\theta}}{\alpha+2}|x|^{1+\frac{\alpha}{2}}}}{|x|^{\frac{\alpha}{4} + \frac{n-1}{2}}} \right) \left(\frac{e^{-\frac{2\sqrt{\theta}}{\alpha+2}|y|^{1+\frac{\alpha}{2}}}}{|y|^{\frac{\alpha}{4} + \frac{n-1}{2}}} \right)$$

for large $x, y \in \mathbb{R}^n$ and all $t > 0$. Here $0 < \theta \leq 1$ is an appropriate constant, $b > \frac{\alpha+2}{\alpha-2}$ and μ_0 is the first eigenvalue of A . We also obtain an estimate for large $|x|$ of the eigenfunctions of A .

1. INTRODUCTION

In this paper we consider operators of the type

$$Au = -\sum_{k,j=1}^n \partial_k(a_{kj}\partial_j u) + Vu,$$

with potentials V that grow more than a polynomial. We study the behaviour for large $|x|$ of the associated heat kernel and associated eigenfunctions.

We make the following assumptions in which $\alpha > 2$, η and Λ are positive constants:

$$(H) \begin{cases} a_{kj} = a_{jk} \in W_{loc}^{1,\infty}(\mathbb{R}^n, \mathbb{R}), \partial_j a_{kj} = o(|x|^{\frac{\alpha}{2}}) \text{ as } |x| \rightarrow \infty, \\ \eta|\xi|^2 \leq \sum_{j,k=1}^n a_{kj}(x)\xi_k\xi_j \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \\ V \in L_{loc}^1(\mathbb{R}^n) \text{ such that } V(x) \geq |x|^\alpha. \end{cases}$$

We define the quadratic forms

$$\begin{aligned} \mathbf{a}(u, u) &:= \sum_{k,j=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j u + \int_{\mathbb{R}^n} V|u|^2 \\ \mathbf{b}(u, u) &:= \int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} V|u|^2 \end{aligned}$$

for $u \in D(\mathbf{a}) = D(\mathbf{b}) = \{u \in W^{1,2}(\mathbb{R}^n); \int_{\mathbb{R}^n} V|u|^2 < \infty\}$. It is easy to see that \mathbf{a} and \mathbf{b} are closed symmetric forms. Their associated self-adjoint operators are A and B , respectively.

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Formally,

$$Au = - \sum_{k,j=1}^n \partial_k(a_{kj}\partial_j u) + Vu \quad \text{and} \quad Bu = -\Delta u + Vu.$$

Since $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$ it follows that A and B have compact resolvents. Therefore their spectra are discrete

$$\begin{aligned} \sigma(A) &= \{\mu_i; i = 0, 1, \dots\} \\ \sigma(B) &= \{\lambda_i; i = 0, 1, \dots\}. \end{aligned}$$

We shall denote by $(\psi_i)_{i \geq 0}$ and $(\varphi_i)_{i \geq 0}$ the corresponding normalized eigenfunctions of A and B , respectively. In particular, ψ_0 and φ_0 will denote the normalized eigenfunctions corresponding to the first eigenvalues μ_0 for A and λ_0 for B .

Since V is non-negative, the semigroup generated by $-A$ is dominated by the Gaussian semigroup. Similarly, the semigroup of $-B$ is dominated by the semigroup corresponding to the operator with $V = 0$. Therefore, A and B have heat kernels $k_t(x, y)$ and $p_t(x, y)$ which satisfy Gaussian upper bounds

$$p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}, \quad k_t(x, y) \leq Ct^{-n/2} e^{-c\frac{|x-y|^2}{t}}$$

for $t > 0$ and some constants $c, C > 0$ (see [2], [9]). Better estimates for large time t are proved in [10]. Indeed, it is shown there that

$$p_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\lambda_0 t} e^{-\frac{|x-y|^2}{4t}} \left[1 + \lambda_0 t + \frac{|x-y|^2}{t} \right]^{\frac{n}{2}}$$

and the same arguments give

$$k_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\mu_0 t} e^{-\frac{\rho^2(x,y)}{4t}} \left[1 + \mu_0 t + \frac{\rho^2(x,y)}{t} \right]^{\frac{n}{2}}, \quad t > 0$$

where

$$\rho(x, y) := \sup\{\phi(x) - \phi(y) : \phi \in C_c^\infty(\mathbb{R}^n), \sum_{k,j=1}^n a_{kj}\partial_k\phi\partial_j\phi \leq 1 \text{ a.e. on } \mathbb{R}^n\}$$

is the metric associated to coefficients a_{kj} . By uniform ellipticity, ρ is equivalent to the Euclidean distance.

These estimates are sharp with respect to t but they do not reflect the behaviour of the heat kernel for large x and y . For the operator B , it is known that

$$p_t(x, y) \leq Ce^{ct-b} \varphi_0(x)\varphi_0(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}^n$, $0 < t \leq 1$, where C, c are positive constants and b is a constant such that $b > \frac{\alpha+2}{\alpha-2}$; see [2, Theorem 4.5.4]. Note that in [6, Theorem 6.2] (see also [5, Example 3.13]) this result is obtained with $b = \frac{\alpha+2}{\alpha-2}$. Note also that for large $|x|$ the ground state φ_0 satisfies

$$c_1 \frac{1}{|x|^{\frac{\alpha}{4} + \frac{n-1}{2}}} e^{-\frac{|x|^\gamma}{\gamma}} \leq \varphi_0(x) \leq c_2 \frac{1}{|x|^{\frac{\alpha}{4} + \frac{n-1}{2}}} e^{-\frac{|x|^\gamma}{\gamma}} \tag{1.2}$$

for some positive constants c_1, c_2 and $\gamma := 1 + \frac{\alpha}{2}$; see [2, Corollary 4.5.8]. From this and (1.1) one has for large x and y

$$p_t(x, y) \leq C e^{ct-b} \frac{1}{(|x||y|)^{\frac{\alpha}{4} + \frac{n-1}{2}}} e^{-\frac{|x|^\gamma}{\gamma}} e^{-\frac{|y|^\gamma}{\gamma}}. \quad (1.3)$$

Our contribution in this note is to show a similar estimate for the heat kernel $k_t(x, y)$ of the operator A . It is not clear how to adapt the proof in [2] to obtain (1.1) for $k_t(x, y)$ with ψ_0 instead of φ_0 . What we shall do is to prove that the estimate (1.3) carries over from $p_t(x, y)$ to $k_t(x, y)$. The idea is to use a log-Sobolev inequality for B from [2] and then by ellipticity the the same inequality holds for A . In order to conclude we will need an L^∞ estimate for a "weighted" semigroup $\varphi^{-1}e^{-tA}\varphi$ with an appropriate function φ . This estimate will be obtained by using the Beurling-Deny criterion. As a result we prove that if $\Lambda < 1$, then for large $|x|$ and $|y|$

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct-b} \frac{1}{(|x||y|)^{\frac{\alpha}{4} + \frac{n-1}{2}}} e^{-\frac{|x|^\gamma}{\gamma}} e^{-\frac{|y|^\gamma}{\gamma}}$$

for all $t > 0$. We also mention that some related results on heat kernel bounds for Schrödinger operators on manifolds are proved in [11].

As a consequence of our result we obtain the following estimate for the eigenfunctions ψ_j , $j \geq 0$

$$|\psi_j(x)| \leq C' \frac{1}{|x|^{\frac{\alpha}{4} + \frac{n-1}{2}}} e^{-\frac{|x|^\gamma}{\gamma}}$$

for large $|x|$. If the condition $\Lambda < 1$ is not satisfied we prove similar estimates with $e^{-\frac{|x|^\gamma}{\gamma}}$ replaced by $e^{-c\frac{|x|^\gamma}{\gamma}}$ for an appropriate constant $c > 0$.

2. HEAT KERNEL ESTIMATES

Before we state the main results of this section, let us recall that ψ_0 is strictly positive.

Lemma 2.1. *The ground state ψ_0 of A is strictly positive, i.e.*

$$\psi_0(x) > 0, \quad a.e. \ x \in \mathbb{R}^n.$$

Proof. By the Beurling-Deny criterion the semigroup (e^{-tA}) is positive. This follows from the fact

$$u \in D(\mathbf{a}) \Rightarrow u^+ \in D(\mathbf{a}) \text{ and } \mathbf{a}(u^+, u^-) = 0.$$

Since the Gaussian semigroup $(e^{t\Delta})$ is irreducible (as a convolution with the Gaussian function) we have from [9, Theorem 2.10] that for every measurable subset Ω of \mathbb{R}^n such that

$$\chi_\Omega u \in W^{1,2}(\mathbb{R}^n), \quad \forall u \in W^{1,2}(\mathbb{R}^n),$$

then $|\Omega| = 0$ or $|\mathbb{R}^n \setminus \Omega| = 0$, where $|\cdot|$ denotes the Lebesgue measure. From this and the definition of $D(\mathbf{a})$, we see that

$$\chi_\Omega u \in D(\mathbf{a}), \quad \forall u \in D(\mathbf{a}),$$

then $|\Omega| = 0$ or $|\mathbb{R}^n \setminus \Omega| = 0$, where

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{cases}$$

denotes the characteristic function of the set Ω . By [9, Corollary 2.11], (e^{-tA}) is irreducible. It is now a classical fact that this implies that μ_0 is a simple eigenvalue with $\psi_0 > 0$ (cf [2, Proposition 1.4.3]). \square

The following is the first main result of this paper.

Theorem 2.2. *Assume that (H) is satisfied with some constant $\Lambda < 1$. Then the heat kernel $k_t(x, y)$ satisfies*

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct-b} \frac{1}{(|x||y|)^\beta} e^{-\frac{|x|^\gamma}{\gamma} - \frac{|y|^\gamma}{\gamma}}$$

for large $x, y \in \mathbb{R}^n$ and all $t > 0$. Here $C, c > 0, b > \frac{\alpha+2}{\alpha-2}, \beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Proof. We note first that, by domination (cf. [9, Theorem 2.24]), one can assume that $V(x) = |x|^\alpha$.

Denote by φ a positive and C^∞ -function on \mathbb{R}^n such that

$$\varphi(x) = |x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

for $|x| \geq R$ large enough. By normalization, one can assume that $\|\varphi\|_{L^2} = 1$. Define on $L_\varphi^2 := L^2(\mathbb{R}^n, \varphi^2(x)dx)$ the forms

$$\begin{aligned} \tilde{\mathbf{a}}(u, v) &:= \mathbf{a}(\varphi u, \varphi v), \quad \tilde{\mathbf{b}}(u, v) := \mathbf{b}(\varphi u, \varphi v) \\ D(\tilde{\mathbf{a}}) = D(\tilde{\mathbf{b}}) &= \{u \in L^2(\mathbb{R}^n, \varphi^2 dx); \varphi u \in D(\mathbf{a}) = D(\mathbf{b})\}. \end{aligned}$$

By considering the unitary operator

$$\begin{aligned} \mathcal{U}_\varphi : L^2(\mathbb{R}^n, \varphi^2 dx) &\longrightarrow L^2(\mathbb{R}^n, dx) \\ f &\longmapsto \varphi f \end{aligned}$$

we see that the associated operators \tilde{A}, \tilde{B} (with $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$, respectively) are $\tilde{A} = \mathcal{U}_{\varphi^{-1}} A \mathcal{U}_\varphi, \tilde{B} = \mathcal{U}_{\varphi^{-1}} B \mathcal{U}_\varphi$. Hence,

$$e^{-t\tilde{A}} = \varphi^{-1} e^{-tA} \varphi, \quad e^{-t\tilde{B}} = \varphi^{-1} e^{-tB} \varphi$$

and their associated kernels are

$$\tilde{k}_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)} \quad \text{and} \quad \tilde{p}_t(x, y) = \frac{p_t(x, y)}{\varphi(x)\varphi(y)}.$$

We need the following lemmas.

Lemma 2.3. *For every non-negative $f \in D(\tilde{\mathbf{a}})$ we have*

$$1 \wedge f := \inf\{1, f\} \in D(\tilde{\mathbf{a}}).$$

Proof. Since $(\varphi_0^{-1}e^{-tB}\varphi_0)1 = \tilde{\varphi}_0^{-1}e^{-\lambda_0 t}\varphi_0 = e^{-\lambda_0 t} \leq 1$, the semigroup $(\varphi_0^{-1}e^{-tB}\varphi_0)$ is L^∞ -contractive. Therefore, we can apply the Beurling-Deny criterion (see e.g. [9, Corollary 2.17]) and obtain

$$1 \wedge f := \inf\{1, f\} \in D(\mathfrak{b}^\sharp) \text{ for all } 0 \leq f \in D(\mathfrak{b}^\sharp), \quad (2.1)$$

where $(\mathfrak{b}^\sharp, D(\mathfrak{b}^\sharp))$ is the form associated to the operator $\varphi_0^{-1}B\varphi_0$ on $L^2(\mathbb{R}^n, \varphi_0^2 dx)$ which is (minus) the generator of $\tilde{\varphi}_0^{-1}e^{-tB}\varphi_0$.

We claim now that

$$D(\mathfrak{b}^\sharp) = D(\tilde{\mathfrak{b}}).$$

By (1.2), φ and φ_0 are equivalent and hence $L^2(\mathbb{R}^n, \varphi^2 dx) = L^2(\mathbb{R}^n, \varphi_0^2 dx)$. In addition,

$$\int_{\mathbb{R}^n} V(\varphi u)^2 dx < \infty \Leftrightarrow \int_{\mathbb{R}^n} V(\varphi_0 u)^2 dx < \infty.$$

Using the fact that for large $|x|$ we have

$$a_1 |\nabla \varphi(x)| \leq |\nabla \varphi_0(x)| \leq a_2 |\nabla \varphi(x)| \quad (2.2)$$

where a_1 and a_2 are positive constants (see [6, Lemma 6.1], [8, Theorem 2.1, Chapter 6]), we obtain

$$\left| \frac{\nabla \varphi_0(x)}{\varphi_0(x)} \right|^2 \leq C_1 |x|^\alpha + C_2, \quad x \in \mathbb{R}^n,$$

where C_1 and C_2 are two positive constants.

Suppose now that $u \in D(\tilde{\mathfrak{b}})$. Then, $v := \varphi u \in W^{1,2}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} |x|^\alpha |v(x)|^2 dx < \infty$ and $g := \frac{\varphi_0}{\varphi}$ satisfies

$$|\nabla g(x)|^2 \leq C_1 |x|^\alpha + C_2, \quad x \in \mathbb{R}^n. \quad (2.3)$$

Since $C_c^\infty(\mathbb{R}^n)$ is a core for B (cf. [3]) and hence a core for \mathfrak{b} (cf. [9, Lemma 1.25]), it follows that there is a sequence (v_n) in $C_c^\infty(\mathbb{R}^n)$ such that $v_n \rightarrow v$, $|\nabla v_n| \rightarrow |\nabla v|$ and $|x|^{\frac{\alpha}{2}} v_n \rightarrow |x|^{\frac{\alpha}{2}} v$ in $L^2(\mathbb{R}^n)$. Take now $h \in \mathbb{R}^n$. Then for every x

$$\begin{aligned} g(x+h)v_n(x+h) - g(x)v_n(x) &= (g(x+h) - g(x))v_n(x) + g(x+h)(v_n(x+h) - v_n(x)) \\ &= v_n(x) \int_0^1 h \cdot \nabla g(x+th) dt + \\ &\quad g(x+h) \int_0^1 h \cdot \nabla v_n(x+th) dt. \end{aligned}$$

Hence, using (2.3), we deduce that there are positive constants C_1, C_2, C_3 such that

$$\|\tau_h(gv_n) - (gv_n)\|_{L^2} \leq |h| \left[C_1 \left(\int_{\mathbb{R}^n} |x|^\alpha |v_n(x)|^2 dx \right)^{\frac{1}{2}} + C_2 \|v_n\|_{L^2} + C_3 \|\nabla v_n\|_{L^2} \right]$$

holds for all $n \in \mathbb{N}$. Passing to the limit, we obtain

$$\|\tau_h(gv) - (gv)\|_{L^2} \leq |h| \left[C_1 \left(\int_{\mathbb{R}^n} |x|^\alpha |v(x)|^2 dx \right)^{\frac{1}{2}} + C_2 \|v\|_{L^2} + C_3 \|\nabla v\|_{L^2} \right].$$

Therefore, $\varphi_0 u = gv \in W^{1,2}(\mathbb{R}^n)$ (cf. [1, Proposition 9.3]) and hence $D(\tilde{\mathbf{b}}) \subset D(\mathbf{b}^\sharp)$. The reverse inclusion can be proved similarly.

Using now $D(\mathbf{b}^\sharp) = D(\tilde{\mathbf{b}})$, the lemma follows from (2.1). \square

Lemma 2.4. *There exists a constant $\lambda_{\mathbf{a}} \in \mathbb{R}$ such that*

$$\|e^{-t\tilde{A}}\|_{\mathcal{L}(L^\infty)} \leq e^{\lambda_{\mathbf{a}} t}$$

for all $t \geq 0$.

Proof. Starting from the expression of $\tilde{\mathbf{a}}$ we obtain

$$\begin{aligned} \tilde{\mathbf{a}}(u, v) &= \sum_{j,k=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k(\varphi u) \partial_j(\varphi v) dx + \int_{\mathbb{R}^n} Vuv\varphi^2 dx \\ &= \sum_{j,k=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j v \varphi^2 dx + \int_{\mathbb{R}^n} W_{\mathbf{a}} uv \varphi^2 dx \end{aligned}$$

with $W_{\mathbf{a}} := V - \sum_{j,k=1}^n \partial_j a_{kj} \frac{\partial_k \varphi}{\varphi} - \sum_{j,k=1}^n a_{kj} \frac{\partial_k \partial_j \varphi}{\varphi}$. For $|x|$ large, we have

$$\begin{aligned} W_{\mathbf{a}}(x) &= V(x) + \left(\sum_{j,k=1}^n \partial_j a_{kj}(x) x_k \right) (\beta|x|^{-2} + |x|^{\gamma-2}) \\ &\quad - \left(\sum_{j,k=1}^n a_{kj}(x) x_j x_k \right) [(\beta|x|^{-2} + |x|^{\gamma-2})^2 + 2\beta|x|^{-4} - (\gamma-2)|x|^{\gamma-4}] \\ &\quad + \left(\sum_{k=1}^n a_{kk}(x) \right) (\beta|x|^{-2} + |x|^{\gamma-2}) \\ &\geq |x|^\alpha - \sum_{k,j} |\partial_j a_{kj}| (\beta|x|^{-1} + |x|^{\gamma-1}) - \left\| \sum_k a_{kk} \right\|_\infty (\beta|x|^{-2} + |x|^{\gamma-2}) \\ &\quad - \Lambda [|x|^2(\beta|x|^{-2} + |x|^{\gamma-2})^2 + 2\beta|x|^{-2}] + \eta(\gamma-2)|x|^{\gamma-2} \\ &= (1-\Lambda)|x|^\alpha - \sum_{k,j} |\partial_j a_{kj}| (\beta|x|^{-1} + |x|^{\gamma-1}) + \eta(\gamma-2)|x|^{\gamma-2} \\ &\quad - \Lambda (\beta^2|x|^{-2} + 2\beta|x|^{\gamma-2} + 2\beta|x|^{-2}) - \left\| \sum_k a_{kk} \right\|_\infty (\beta|x|^{-2} + |x|^{\gamma-2}). \end{aligned}$$

Then, the assumptions $\Lambda < 1$ and $\partial_j a_{kj} = o(|x|^{\frac{\alpha}{2}})$ yield

$$W_{\mathbf{a}}(x) \geq 0 \quad \text{for } |x| \text{ large.}$$

Hence, there is $\lambda_{\mathbf{a}} \in \mathbb{R}$ such that

$$W_{\mathbf{a}}(x) \geq -\lambda_{\mathbf{a}} \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (2.4)$$

By the previous lemma and the obvious equality $D(\tilde{\mathbf{a}}) = D(\tilde{\mathbf{b}})$ we have $1 \wedge u \in D(\tilde{\mathbf{a}})$ for all non-negative $u \in D(\tilde{\mathbf{a}})$. Applying now (2.4) we obtain

$$\begin{aligned} \tilde{\mathbf{a}}(1 \wedge u, (u-1)^+) &= \int_{\mathbb{R}^n} W_{\mathbf{a}}(1 \wedge u)(u-1)^+ \varphi^2 dx \\ &\geq -\lambda_{\mathbf{a}} \int_{\mathbb{R}^n} (1 \wedge u)(u-1)^+ \varphi^2 dx. \end{aligned}$$

Applying the Beurling-Deny criterion (cf. [9, Corollary 2.17]), we obtain the lemma. \square

We continue the proof of the theorem. Similarly to Lemma 2.4 we have

$$\tilde{\mathbf{b}}(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \varphi^2 dx + \int_{\mathbb{R}^n} W_{\mathbf{b}} uv \varphi^2 dx,$$

where $W_{\mathbf{b}} = V - \frac{\Delta \varphi}{\varphi} \geq -|x|^{-2} \beta(\beta + 2 - n)$ for large $|x|$. Hence, as above, we obtain

$$\|e^{-t\tilde{B}}\|_{\mathcal{L}(L^\infty)} \leq e^{\lambda_{\mathbf{b}} t}, \quad t \geq 0, \quad (2.5)$$

for some constant $\lambda_{\mathbf{b}} \in \mathbb{R}$.

On the other hand, let $b > \frac{\alpha+2}{\alpha-2}$. Then, by (1.2) and [2, Corollary 4.5.5], we have

$$\begin{aligned} \tilde{p}_t(x, y) &= \frac{p_t(x, y)}{\varphi(x)\varphi(y)} \\ &\leq C \frac{p_t(x, y)}{\varphi_0(x)\varphi_0(y)} \\ &\leq M e^{ct-b}, \quad 0 < t \leq 1. \end{aligned}$$

For $t \geq 1$ we recall first that, by the semigroup law and the symmetry of p_t ,

$$p_{s+r}(x, y) = \int_{\mathbb{R}^n} p_s(x, z) p_r(y, z) dz, \quad s, r > 0. \quad (2.6)$$

Hence,

$$\begin{aligned} p_t(x, x) &= \int_{\mathbb{R}^n} |p_{\frac{t}{2}}(x, y)|^2 dy \\ &= \int_{\mathbb{R}^n} \left| e^{-(\frac{t}{2}-\frac{1}{2})B} p_{\frac{1}{2}}(x, \cdot)(y) \right|^2 dy \\ &= \left\| e^{-(\frac{t}{2}-\frac{1}{2})B} p_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \\ &\leq e^{-2\lambda_0(\frac{t}{2}-\frac{1}{2})} \left\| p_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \\ &\leq e^{-\lambda_0 t} p_1(x, x) \\ &\leq M e^{-\lambda_0 t} \varphi^2(x). \end{aligned}$$

Applying the inequality

$$p_t(x, y) \leq \sqrt{p_t(x, x)} \sqrt{p_t(y, y)}, \quad x, y \in \mathbb{R}^n,$$

which is a consequence of (2.6) and Hölder's inequality, we obtain the estimate

$$\tilde{p}_t(x, y) \leq M e^{-\lambda_0 t} e^{ct-b}, \quad t > 0.$$

Thus, by (2.5), the semigroup $e^{-t\tilde{B}}$ is ultracontractive. By [2, Theorem 2.2.3] we obtain that \tilde{b} satisfies a log-Sobolev inequality

$$\int_{\mathbb{R}^n} u^2 \log u \varphi^2 dx \leq \varepsilon \tilde{b}(u, u) + \beta(\varepsilon) \|u\|_{L_\varphi^2}^2 + \|u\|_{L_\varphi^2}^2 \log \|u\|_{L_\varphi^2} \quad (2.7)$$

for all $\varepsilon > 0$ and $0 \leq u \in L^1 \cap L^\infty \cap D(\tilde{b})$, with $\beta(\varepsilon) = b_1 - \lambda_0 \varepsilon + c\varepsilon^{-b}$, where b_1, c are positive constants.

Since, by ellipticity (see (H)), $\tilde{b}(u, u) \leq \max\{\frac{1}{\eta}, 1\} \tilde{\mathfrak{a}}(u, u)$ for all $u \in D(\tilde{\mathfrak{a}}) = D(\tilde{b})$, it follows that $\tilde{\mathfrak{a}}$ satisfies the same log-Sobolev inequality (2.7). Using Lemma 2.4 and [2, Corollary 2.2.8 and Example 2.3.4] we conclude that $e^{-t\tilde{A}}$ is ultracontractive and

$$\tilde{k}_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)} \leq C e^{ct-b}, \quad 0 < t \leq 1. \quad (2.8)$$

For $t \geq 1$ we proceed as above and obtain

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct-b} \varphi(x)\varphi(y), \quad x, y \in \mathbb{R}^n, t > 0.$$

This proves the theorem. \square

Corollary 2.5. *If the assumptions of Theorem 2.2 hold, then all the eigenfunctions $\psi_j, j = 0, 1, \dots$ of A satisfy*

$$|\psi_j(x)| \leq C |x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

for large $|x|$ and a constant $C > 0$, where $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Proof. For each $j \geq 0$ we have

$$\begin{aligned} |\psi_j(x)| e^{-\mu_j t} &= |e^{-tA} \psi_j(x)| \\ &= \left| \int_{\mathbb{R}^n} k_t(x, y) \psi_j(y) dy \right| \\ &\leq \left(\int_{\mathbb{R}^n} k_t(x, y)^2 dy \right)^{1/2} \|\psi_j\|_2 \\ &= (k_{2t}(x, x))^{1/2}. \end{aligned}$$

We apply the previous theorem and obtain the estimate of the corollary. \square

Remark 2.6. For the first eigenfunction ψ_0 we do not know if the reverse inequality

$$\psi_0(x) \geq \tilde{C} |x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

holds for large $|x|$ and some constant $\tilde{C} > 0$.

We consider now the general case where the condition $\Lambda < 1$ is not necessary satisfied. To this purpose we consider the operator $H := -\Delta + \theta|x|^\alpha$ on $L^2(\mathbb{R}^n)$ with $0 < \theta < 1$ and denotes by $h_t(x, y)$ its corresponding heat kernel. By [2, Theorem 4.5.4] and as in the above proof, we deduce that

$$h_t(x, y) \leq b_1 e^{-\xi_0 t} e^{b_2 t^{-b}} \phi_0(x) \phi_0(y)$$

for all $x, y \in \mathbb{R}^n$, $t > 0$, where $b_1, b_2 > 0$ and $b > \frac{\alpha+2}{\alpha-2}$. Here ϕ_0 and ξ_0 are the (normalized) ground state and the first eigenvalue of H respectively.

Take the radial function

$$f_\mu(x) := \left(|x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} \right) e^{-\frac{\mu}{\alpha-2}|x|^{1-\frac{\alpha}{2}}},$$

where $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$ and $\mu \in \mathbb{R}$. A direct computation yields

$$\frac{\Delta f_\mu}{f_\mu}(x) = \theta|x|^\alpha - \mu\sqrt{\theta} - (\beta^2 + 2\beta - n\beta)|x|^{-2} - \frac{\mu\alpha}{2}|x|^{-\frac{\alpha}{2}-1} + \frac{\mu^2}{4}|x|^{-\alpha}.$$

By the same proof as in [2, Lemma 4.5.7] we obtain that there exist two positive constants c_1, c_2 such that

$$c_1 f_0(x) \leq \phi_0(x) \leq c_2 f_{\frac{2\xi_0}{\sqrt{\theta}}}(x)$$

for all large $|x|$. Hence,

$$c_1 |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} \leq \phi_0(x) \leq c_2 |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} \quad (2.9)$$

for all large $|x|$. As in the proof of Theorem 2.2 we consider a positive and smooth function φ (with $\|\varphi\|_{L^2} = 1$) such that

$$\varphi(x) = |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma}$$

for large $|x|$.

By the same computation as in [6, Lemma 6.1], we obtain

$$a_1 |\nabla \varphi(x)| \leq |\nabla \phi_0(x)| \leq a_2 |\nabla \varphi(x)| \quad (2.10)$$

for all large $|x|$ and some constants $a_1, a_2 > 0$.

Theorem 2.7. *Assume that (H) is satisfied. Then for $\theta > 0$ such that $\theta\Lambda < 1$ we have*

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{c t^{-b}} \frac{1}{(|x||y|)^\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} e^{-\frac{\sqrt{\theta}}{\gamma}|y|^\gamma}$$

for large $x, y \in \mathbb{R}^n$ and all $t > 0$. Here $C, c > 0$, $b > \frac{\alpha+2}{\alpha-2}$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Proof. We proceed as in the proof of Theorem 2.2. We consider the function φ defined above and the form h associated to H instead of \mathfrak{h} . We have

$$\begin{aligned} \tilde{\mathfrak{a}}(u, v) &= \sum_{j,k=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k(\varphi u) \partial_j(\varphi v) dx + \int_{\mathbb{R}^n} |x|^\alpha uv \varphi^2 dx \\ &= \sum_{j,k=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j v \varphi^2 dx + \int_{\mathbb{R}^n} W_\alpha uv \varphi^2 dx, \end{aligned}$$

and for large $|x|$, a direct computation yields

$$W_\alpha(x) \geq (1 - \theta\Lambda)|x|^\alpha + o(|x|^\alpha).$$

The assumption $\theta\Lambda < 1$ gives $W_\alpha(x) \geq 0$ for large $|x|$ and hence $W_\alpha(x) \geq -\lambda$ for some $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. By Lemmas 2.3 and 2.4 we obtain

$$\|e^{-t\tilde{A}}\|_{\mathcal{L}(L^\infty)} \leq e^{\lambda t}, \quad t \geq 0.$$

We conclude exactly as in the proof of Theorem 2.2. \square

As a consequence we obtain an estimate for the eigenfunctions ψ_j of A .

Corollary 2.8. *Assume that (H) holds. Then for $\theta > 0$ such that $\theta\Lambda < 1$ we have for all $j \geq 0$*

$$|\psi_j(x)| \leq C|x|^{-\beta}e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma}$$

for large $|x|$ and some constant $C > 0$, where $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Remark 2.9. Take $\theta > 0$ such that $\theta\Lambda < 1$. Using Lyapunov functions techniques it is proved in [4, Corollary 3.4] that if the assumption (H) is satisfied and the coefficients $a_{jk} \in C^{1+\eta}(\mathbb{R}^n)$ with some $0 < \eta < 1$ then $b = \frac{\alpha+2}{\alpha-2}$ can be taken. More precisely one has the following estimate:

$$k_t(x, y) \leq Ce^{ct-\frac{\alpha+2}{\alpha-2}t} e^{-\frac{2\sqrt{\theta}}{\alpha+2}|x|^{1+\frac{\alpha}{2}}} e^{-\frac{2\sqrt{\theta}}{\alpha+2}|y|^{1+\frac{\alpha}{2}}}$$

for all $x, y \in \mathbb{R}^n$, $0 < t \leq 1$.

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