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POTTS PARTITION FUNCTION AND ISOMORPHISM OF WEIGHTED TREES

MARTIN LOEBL AND JEAN-SÉBASTIEN SERENI

ABSTRACT. We consider a question pertaining to the topic of graph reconstruction, which entertains links with the W -polynomial and (theoretical) statistical physics. Motivated by several open questions, we slightly deviate from the usual approaches to study, in the context of weighted trees, whether a given data (which can be obtained from the W -polynomial) distinguishes non-isomorphic weighted trees. We prove that this is the case if one restricts to any *good class* of vertex-weighted trees. Good classes are rich: letting \mathcal{C} be the class of all vertex-weighted trees, one can obtain for each weighted tree (T, w) a weighted tree (T', w') in polynomial time, so that $\mathcal{C}' := \{(T', w') : (T, w) \in \mathcal{C}\}$ is good and two elements (A, b) and (X, y) of \mathcal{C} are isomorphic if and only if (A', b') and (X', y') are.

1. INTRODUCTION

As far back as 1941, Kelly and Ulam formulated a conjecture known today as the *reconstruction conjecture* (see [13, 30]). They affirmed that the family of the vertex-deleted subgraphs of a graph determines the graph up to isomorphism. To be more precise, for a graph $G = (V, E)$ let $\mathcal{D}(G)$ be the multiset $\{G - v : v \in V\}$. The reconstruction conjecture states that for every graph G with at least 3 vertices, if H is a graph such that $\mathcal{D}(G) = \mathcal{D}(H)$ then G and H are isomorphic.

This conjecture, which seems so true, stays stubbornly open to this day. It spawned a plethora of research: one can consult a survey was written by Bondy and Hemminger [6] in the late 1970s. The conjecture has been verified for graphs with at most 10 vertices by McKay [17]. Weaker versions have been studied, in particular by considering only some particular class of graphs: restricting to the class of regular graphs, or that of disconnected graphs or that of trees leads to a true statement [5, 21].

A variant of the conjecture is obtained by considering the multiset of edge-deleted subgraphs, instead of that of vertex-deleted subgraphs. The corresponding conjecture, in which the graph is required to have at least four edges, is known as the *edge-reconstruction conjecture*. This weaker conjecture — that it is a weaker form is not obvious — was formulated by Harary [12]. This problem is somehow more understood, partly thanks to some tools that can be applied. For instance, using the Möbius inversion formula, the edge-reconstruction conjecture was established for n -vertex graphs with more than $\frac{1}{2}\binom{n}{2}$ edges [16], and for n -vertex graphs with more than $\log_2(n!) + 1$ edges [20]. However, even this edge version remains rather widely open.

Seemingly a purely combinatorial problem, it turns out that graph reconstruction is linked to various topics. Algebraic aspects of the problem appear, for instance, in the work by Statman [28], which provides an equivalent conjecture in terms of algebraic properties of certain directed

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trees and their homomorphic images. Another deep aspect is the links of this conjecture with the graph isomorphism question and to graph polynomials in general. It is these relations with graph polynomials and statistical physics that motivate us to investigate a variant of the edge-reconstruction conjecture.

As we have seen, the reconstruction conjectures are known to be true over the class of trees. We investigate a natural stronger statement. Before explaining its roots in the field of graph polynomials, let us state the problem in a graph-reconstruction fashion. Consider the following data $D(T)$ associated with a tree T : for every integer n and every partition P of n , we are given the number of subsets X of edges of T such that P is equal to the multiset formed by the orders of the components of $T - X$. Note that this number is 0 if n is not the number of vertices of T . Note also that if P is composed of t integers, the corresponding subsets X , if any, all have cardinality $t - 1$. For instance, one can determine the number of vertices of T by checking, for each positive integer n , whether the trivial partition $\{n\}$ returns a non-zero value (which, necessarily, will be 1). Once the number of vertices of T is known, the number of leaves of T is precisely the number returned by the partition $\{|V(T)| - 1, 1\}$, which corresponds to the number of edges e such that $T - e$ has one component of order 1. The problem is to know whether this information distinguishes non-isomorphic trees. In other words, if T and T' are two trees such that $D(T) = D(T')$, is it true that necessarily T and T' are isomorphic? That such a reconstruction is always possible was suggested by different authors. Another important remark we shall make is that, similarly as for the reconstruction conjecture, there could be non-constructive proofs of the statement. Thus it is a different (harder) problem to be able to effectively recover the tree T from the knowledge of $D(T)$ (or the graph G from the knowledge of $\mathcal{D}(G)$, for that matter).

In this work, we investigate a weighted version of this problem, bearing in mind its connections with graph polynomials, graph colouring and the Potts model.

1.1. The Noble and Welsh conjecture. Motivated by the combinatorial aspects of the relationship between chord diagrams and Vassiliev invariants of knots, Noble and Welsh [22] introduced a polynomial of weighted graphs, the W -polynomial, which includes several specialisations in combinatorics, such as the Tutte polynomial, the matching polynomial (of ordinary graphs) and the polymatroid polynomial of Oxley and Whittle [23]. We need to introduce some terminology to define W .

A *weighted graph* is a graph $G = (V, E)$ together with a function $w: V \rightarrow \mathbf{Z}^+$. The *weight* of a subset V' of vertices is $w(V') := \sum_{v \in V'} w(v)$. If $A \subseteq E$, we let $c_V(A)$ be the number of components of the graph (V, A) , where we may omit the subscript when there is no risk of confusion. Further, let $n_1, \dots, n_{c(A)}$ be the weights of the vertex sets of these components, listed in decreasing order: $n_1 \geq \dots \geq n_{c(A)}$. We write $x(A)$ to mean $\prod_{i=1}^{c(A)} x_{n_i}$. Let

$$W_G(z, x_1, x_2, \dots) := \sum_{A \subseteq E} x(A)(z-1)^{|A|-|V|+c(A)}.$$

In particular, W_G depends on z if and only if G contains a cycle [22, Proposition 5.1-1)]. Unlike the Tutte polynomial, the W -polynomial is $\#P$ -hard to compute even for trees [22, Theorems 7.3 and 7.12] and for complete graphs [22, Theorems 7.11 and 7.14].

In the case of unweighted graphs, which corresponds here to the weight function w being identically 1, Noble and Welsh refers to the W -polynomial as to the *U -polynomial*. While computing W is hard for complete graphs, Annan [1] proved that $U_{K_n}(z, x_1, x_2, \dots)$ can be computed in polynomial time, which is also the case for the Tutte polynomial, for instance. However, U also exhibits differences with the Tutte polynomial: while finding two non-isomorphic graphs with the same Tutte polynomial is easy, the same problem is harder for U . Sarmiento [24] managed to achieve such a construction, but the question remains open for trees: does the

U -polynomial distinguishes non-isomorphic trees? That this is the case became known as the *Noble and Welsh conjecture*. As we shall see, this problem is very much related to the partition question stated earlier.

Noble and Welsh also discovered a very interesting specialization of U : they demonstrated the U -polynomial to be equivalent to the *symmetric function generalisation of the bad colouring polynomial*, a function introduced by Stanley [26].

1.2. The Stanley Conjecture. To introduce Stanley's isomorphism conjecture let us first define graph colouring. A *colouring* of a graph $G = (V, E)$ is a mapping $s: V \rightarrow \mathbf{N}^+$. We define $b(s)$ to be the number of *monochromatic edges* in s , that is, the number of edges uv such that $s(u) = s(v)$. The mapping s is a k -*colouring* if $s(V) \subseteq \{1, \dots, k\}$ and s is *proper* if $b(s) = 0$, that is, $s(u) \neq s(v)$ whenever u and v are two adjacent vertices of G . We let $\text{Col}(G; k)$ be the set of proper k -colourings of G and $\text{Col}(G)$ be the set of all proper colourings of G , that is, $\text{Col}(G) := \cup_{k \in \mathbf{N}^+} \text{Col}(G; k)$.

As was discovered by Birkhoff [3] while working on the Four colour problem, the number of proper k -colourings of a graph G is polynomial in k . More precisely, defining $p_\ell(G)$ to be the number of partitions of $V(G)$ into ℓ non-empty parts, each inducing an edgeless graph, it follows that

$$\text{Col}(G; k) = \sum_{\ell \geq 0} p_\ell(G) \cdot (k)_\ell$$

where $(k)_\ell$ is the ℓ^{th} -*falling factorial* of k , that is, $(k)_\ell := k(k-1) \dots (k-\ell+1)$, so $(k)_\ell = 0$ if $k < \ell$. In particular, only a finite number of summands of the sum are non-zero. The univariate polynomial $\sum_{\ell \geq 0} p_\ell(G) \cdot (X)_\ell$ is the *chromatic polynomial* of the graph G , a topic of extensive study.

In the mid 1990s, Stanley [26] introduced the *symmetric function generalization of the chromatic polynomial*, defined to be

$$X_G(x_1, x_2, \dots) := \sum_{S \in \text{Col}(G)} \prod_{v \in V} x_{s(v)}.$$

This is a homogeneous symmetric function in (x_1, x_2, \dots) of degree $|V|$. As is expectable, X_G does not distinguish non-isomorphic graphs: there exist two non-isomorphic graphs on 5 vertices with the same function X . However, Stanley [26] asked whether the polynomial X_G distinguishes non-isomorphic trees. The assertion that it does became known as *Stanley's isomorphism conjecture*. A step towards Stanley's conjecture has been made recently by Aliste-Prieto and Zamora [2], who established the statement restricted to the class of proper caterpillars: a *caterpillar* is a tree where all edges not incident with a leaf form a path; a caterpillar is *proper* if every vertex is a leaf or adjacent to a leaf. Prior to that, partial results had been obtained by Martin, Morin and Wagner [18] who had established the statement for a subclass of proper caterpillars (where no two non-leaf vertices are adjacent to the same number of leaves) and also to the class of spiders, which is composed of all trees with a unique vertex of degree greater than 2.¹ To obtain these results, these authors built upon Tutte polynomials for trees introduced by Chaudhary and Gordon [8]. Other related results can be found in the undergraduate thesis by Fougere [11] and the MSc thesis by Morin [19]. Finally, it is reported that Tan checked by computer that Stanley's conjecture is true for trees with at most 23 vertices (see [18, p. 238]).

Further, Stanley [27] later initiated the study of a common generalisation of X and the Tutte polynomial, namely the *symmetric function generalisation of the bad colouring polynomial*, defined

¹It is worth mentioning that Martin, Morin and Wagner [18] also demonstrated that the polynomial X_G distinguishes non-isomorphic graphs in the class of all graphs with a unique cycle and exactly one vertex of degree greater than 2.

for every graph $G = (V, E)$ by

$$X_G(t, x_1, x_2, \dots) := \sum_{s: V \rightarrow \mathbf{N}^+} (1+t)^{b(s)} \prod_{v \in V} x_{s(v)}.$$

Note that the sum runs over all colourings of G , not only the proper ones. Noble and Welsh [22, Theorem 6.2] proved $X_G(t, x_1, x_2, \dots)$ to be a specialisation of the U -polynomial of G .

A first hint as to why Stanley's conjecture could be related to our question is given by a theorem of Whitney [31], stating that for every (multi)graph $G = (V, E)$, the univariate chromatic polynomial of G is equal to

$$\sum_{S \subseteq E} (-1)^{|S|} X^{c_V(S)}.$$

In fact, the strong link between Stanley's conjecture and our question was clarified by Thatte. Let us introduce the following notation: given a tree $T = (V, E)$ and two integer vectors $\mathbf{v} = (v_1, \dots, v_r)$ and $\mathbf{e} = (e_1, \dots, e_r)$, let $\theta(T, \mathbf{v}, \mathbf{e})$ be the number of (ordered) partitions (V_1, \dots, V_r) of V such that for each $i \in \{1, \dots, r\}$, the subgraph of T induced by V_i has precisely v_i vertices and e_i edges. Thatte [29, Lemma 4.11] established that θ can be computed from X_T .

In other words, and summing-up: as reported earlier, Noble and Welsh [22, Theorem 6.2] showed that $X(t, x_1, x_2, \dots)$ is a specialization of the U -polynomial in general. For trees, it turns out that these two functions are actually equivalent. Thus Noble and Welsh's conjecture amounts to saying that $X(t, x_1, x_2, \dots)$ distinguishes non-isomorphic trees. So *a priori* it seems that Noble and Welsh's conjecture is weaker than Stanley's. However, what Thatte [29, Theorem 4.12] found out is that for trees, $X(t, x_1, x_2, \dots)$ and $X(x_1, x_2, \dots)$ are equivalent, thereby implying the equivalence of the two conjectures (that of Noble and Welsh and that of Stanley).

1.3. Further Links and Conjectures. It turns out that the fundamental question raised by Noble and Welsh and by Stanley is linked to further polynomials, related to some aspects of knot theory and (theoretical) statistical physics. Indeed, a connection between the Potts model and the U -polynomial was given by Ellis-Monaghan and Moffatt [10]. We review below another (slightly more recent) such connection.

We consider a standard model where magnetic materials are represented as lattices: vertices are atoms and weighted edges are nearest-neighbourhood interactions. We assume that each atom has one out of k possible magnetic moments, for a fixed positive integer k . Thus we let $S := \{0, \dots, k-1\}$. Every element of S is called a *spin*. A *state* of a graph $G = (V, E)$ is then an assignment of a single spin to each vertex of G , that is a function $s: V \rightarrow S$. We assume that all the coupling constants (nearest-neighbourhood interactions) are equal to a constant J . For each state s , the *Potts model energy of the state s* is then $E(P^k)(s) := \sum_{uv \in E} J\delta(s(u), s(v))$ where, as is customary, δ is the Kronecker delta function defined by $\delta(a, b) := 1$ if $a = b$ and $\delta(a, b) := 0$ otherwise. The *k -state Potts model partition function* is then

$$\sum_{s: V \rightarrow S} M(s, J) e^{E(P^k)(s)}.$$

Loebl [15] introduced the q -chromatic functions. Let $k \in \mathbf{N}$. The *q -chromatic function* of a graph $G = (V, E)$ is

$$(1.1) \quad M_G(k, q) := \sum_{s \in \text{Col}(G; k)} q^{\sum_{v \in V} s(v)}.$$

It is known [15] that

$$M_G(k, q) = \sum_{A \subseteq E} (-1)^{|A|} \prod_{C \in \mathcal{C}(A)} (k)_{q^{|C|}},$$

where $\mathcal{C}(A)$ is the set of components of the spanning subgraph (V, A) and $|C|$ is the number of vertices in the component C . Moreover Loeb1 also introduced the q -dichromate, defined as

$$B_G(x, y, q) := \sum_{A \subseteq E} x^{|A|} \prod_{C \in \mathcal{C}(A)} (y)_{q^{|C|}}.$$

He proved that for each real J ,

$$(1.2) \quad B_G(e^J - 1, k, q) = \sum_{s: V \rightarrow S} q^{\sum_{v \in V} s(v)} e^{E(P^k)(s)}.$$

Hence $B_G(x, y, q)$ specializes to the k -state Potts model partition function (with certain magnetic field contributions), as given above.

Recently a variant of the q -dichromate, $B_{r,G}(x, k, q)$, was proposed by Klazar, Loeb1 and Moffatt [14]:

$$B_{r,G}(x, k, q) := \sum_{A \subseteq E} x^{|A|} \prod_{C \in \mathcal{C}(A)} \sum_{i=0}^{k-1} r^{|C|q^i}.$$

They established that if $(k, r) \in \mathbf{N}^2$ with $r > 1$ and $x := e^{\beta J} - 1$, then

$$(1.3) \quad B_{r,G}(x, k, q) = \sum_{\sigma: V \rightarrow S} e^{\beta \sum_{uv \in E(G)} J \delta(\sigma(u), \sigma(v))} r^{\sum_{v \in V} q^{\sigma(v)}}.$$

Hence $B_{r,G}(x, k, q)$ is the k -state Potts model partition function with magnetic field contribution $r^{\sum_{v \in V} q^{\sigma(v)}}$. They also proved that $B_{r,G}$ is equivalent to U_G , thereby showing a link with our setting.

In addition, Loeb1 conjectured that the q -dichromate is equivalent to the U -polynomial. Further, he conjectured also that the q -dichromate actually distinguishes non-isomorphic chordal graphs. An incomplete argument against the former conjecture was presented recently [14].

Arguably, there could be a close link between the latter conjecture and that of Stanley: chordal graphs have a very distinguished tree structure — in graph theory, they are often referred to as ‘blown-up trees’. Indeed, a folklore theorem [4] states that the class of chordal graphs is precisely the class of intersection graphs of subtrees of a tree, that is, for each chordal graph G , there exists a tree T and a mapping f that assigns to each vertex of G a subtree T such that: two vertices u and v of G are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$.

On the other hand, another well-known fact is that the isomorphism problem for general graphs is equivalent to the isomorphism problem restricted to chordal graphs: given a graph $G = (V, E)$, consider the chordal graph $G' = (V', E')$ so that $V' := V \cup E$ and $E' = \binom{V}{2} \cup \{\{u, e\}, \{v, e\} : \{u, v\} = e \in E\}$. It clearly holds that G and H are isomorphic if and only if G' and H' are isomorphic. It thus seems particularly interesting to determine whether the U -polynomial does distinguish non-isomorphic chordal graphs.

In that respect, it seems natural to study weighted trees, rather than unweighted trees. (Thus bringing us to the setting of Noble and Welsh more than to that of Stanley.) Indeed, the tree mentioned in the characterisation of the class of chordal graphs can be chosen to be a *clique-tree*, where the vertices of the tree are the maximal cliques of the graph. Now, if v is a vertex of a weighted tree with weight $w(v)$, one can think of v as a clique of order $w(v)$, thus obtaining an unweighted chordal graph. This is what motivates to work in the (seemingly harder) setting of weighted trees and maybe deviate from the original conjectures of Noble and Welsh and Stanley.

1.4. Main Results. Two weighted graphs are isomorphic if there is an isomorphism of the graphs that preserves the vertex weights. The purpose of this work is to prove that the W -polynomial distinguishes non-isomorphic *weighted trees* when restricting to collections of weighted trees satisfying some properties made precise later. We call any such collection a *good family*. Good

families are rich: we provide examples of good families F so that the isomorphism problem for general weighted trees is equivalent to the isomorphism problem in F . We hope that eventually it will be possible to introduce an analogous notion restricted to chordal graphs, relying on the extensively studied tree-decomposition theory.

Let (T, w) be a weighted tree. We write $V(T)$ and $E(T)$ for the vertex set and the edge set of T , respectively. We define $\text{Ex}(T)$ to be the multi-set composed of all the vertex weights (with multiplicities) of T . If $e \in E(T)$, then $T - e$ is the disjoint union of two trees, which we consider to be weighted and rooted at the endvertex of e that they contain. A rooted weighted tree (S, w_S) is a *shape* of (T, w) if $2 \leq |V(S)| \leq |V(T)| - 2$ and there exists an edge $e \in E(T)$ such that S is one of the two components of $T - e$; moreover w_S is the restriction of w to the vertex set of S . We consider S rooted at the end-vertex of e . We usually shorten the notation and write S for the shape (S, w_S) . In a tree, a vertex of degree one is called a *leaf*.

Definition 1.1. A set \mathcal{T} of weighted trees (T, w) is *good* if it satisfies the following properties.

- (1) If a vertex of T is adjacent to a leaf, then all its neighbours but one are leaves.
- (2) If v is a leaf or has a neighbour that is a leaf, then $w(v) = 1$.
- (3) Let $(T, w), (T', w') \in \mathcal{T}$ and let S be a shape of T and such that $w(S) \leq w(T)/2$. Let S' be a shape of T' such that $\text{Ex}(S') = \text{Ex}(S)$. Then S' is isomorphic to S .

Example 1.2. Many good families \mathcal{F} can be defined inductively. For example, let \mathcal{F} initially consist of some stars with more than one leaf. In each step we consider a tree $T \in \mathcal{F}$ and a vertex v of T that is adjacent to a leaf. Let ℓ_1, \dots, ℓ_s be the leaves adjacent to v . We execute the following steps:

- replace each ℓ_i by a star of \mathcal{F} ;
- change $w(v)$ from being equal to one to a value that does not violate property (3); and
- add the new tree to \mathcal{F} .

Definition 1.3. A set \mathcal{Z} of weighted trees is *relevant* if there is an efficient algorithm that computes, for each weighted tree (T, w) , a weighted tree $(T', w') \in \mathcal{Z}$ so that whenever (T_1, w_1) and (T_2, w_2) are weighted trees, (T_1, w_1) is isomorphic to (T_2, w_2) if and only if (T'_1, w'_1) is isomorphic to (T'_2, w'_2) .

An example of a good relevant set of weighted trees is that composed of all weighted single vertices. Indeed, the classical isomorphism test for trees can easily be extended to the weighted trees; it amounts to coding n -vertex-weighted trees by 0-1 vectors of length polynomial in n . This can be seen as assigning a non-negative integer weight to a single vertex. Such weighted single vertices are of course distinguished by the W -polynomial. Our purpose is to provide a wide generalisation of this fact, by proving the next statement.

Theorem 1. *The W -polynomial distinguishes non-isomorphic weighted trees in any good set.*

Our proof of Theorem 1 is not constructive in the sense that we are not able to reconstruct the weighted tree (T, w) from $W_{(T, w)}$. The difficulty in proving the theorem is that while the main defining property of a good family is about shapes, the W -polynomial does not “see” shapes.

We illustrate the richness of good sets of weighted trees by our next theorem. Let \mathcal{C} be the class of all vertex-weighted trees (T, w) with $\max\{w(v) : v \in V(T)\} \leq 2^{|V(T)|}$, where we assume for convenience that each weight is given as a binary number with exactly $|V(T)|$ bits. For each tree T we define T' to be the tree obtained from T by identifying each leaf h with the root of one new star S_h with three vertices.

Theorem 2. *For each $(T, w) \in \mathcal{C}$, one can construct a weight function w' for the vertices of T' such that two elements (T_1, w_1) and (T_2, w_2) of \mathcal{C} are isomorphic if and only if the corresponding*

elements (T'_1, w'_1) and (T'_2, w'_2) are isomorphic. Furthermore, the set $\mathcal{C}' := \{(T', w') : (T, w) \in \mathcal{C}\}$ is good.

Proof. Let $(T, w) \in \mathcal{C}$. We start by showing how to construct a weight function w' for T' . First, we let the weight of each leaf of T' , that is, each vertex of $T' \setminus T$, be 1. Next we perform, on $(T, w) \subset (T', w)$, a variant of the coding for the classical isomorphism test for weighted trees mentioned above. The coding gradually assigns a binary vector $w''(v)$ to each vertex v , starting with the neighbours of the leaves of T' , i.e., the leaves of T , which are all assigned (01) in the first step. We assume binary vectors to be ordered according to the positive integer they code.

In each further step $t > 1$, we let A_t be the set of the vertices v with exactly one neighbour that has not been assigned its w'' -code before step t . For each $v \in A_t$, we define $c(v)$ to be obtained from $w(v)$ and from the constructed w'' -codes $z_1 \geq z_2 \geq \dots z_m$ of all but one of the neighbours of v , as follows:

$$c(v) := (0z_1 \dots z_m 1w(v)).$$

We choose one vertex x in A_t such that $c(x) = \min \{c(v) : v \in A_t\}$ and we set $w''(x) := c(x)$. This finishes step t .

We observe that the following hold after each step t .

- (i) If a vertex x is assigned its code $w''(x)$ in step t and its neighbour y is assigned its code $w''(y)$ in an earlier step, then $w''(x) > w''(y)$.
- (ii) If a vertex x is assigned its code $w''(x)$ in step t , then there is exactly one shape $S(x)$ rooted at x such that each vertex v of $S(x) \setminus x$ has been assigned its w'' -code before step t .

The coding terminates if there is a vertex r such that all of its neighbours have been assigned a w'' -code. We observe that when the coding terminates, each vertex but r has been assigned its w'' -code. Finally, if $v \neq r$, then $w'(v)$ is the positive integer coded by $w''(v)$ and we set $w'(r) := w(r) + \sum_{v \neq r} w'(v)$. The vertex r is the *root* of (T', w') . This finishes the construction of (T', w') .

Let us observe that (T, w) can be reconstructed from (T', w') : first, notice that T is obtained from T' by deleting all the leaves. In particular we thus know $|V(T)|$. Next, the vertex r is the unique vertex of T' with the largest weight. Moreover, $w(r) = w'(r) - \sum_{v \in V(T') \setminus \{r\}} w'(v)$. In addition, for each vertex $v \neq r$ the weight $w(v)$ can be deduced from $w'(v)$ as follows: the binary representation of $w(v)$ is equal to the last $|V(T)|$ digits of the binary representation of $w'(v)$. This proves the first part of the statement of Theorem 2.

It remains to show that \mathcal{C}' is a good class of weighted trees. Clearly, Properties (1) and (2) of Definition 1.1 hold and so it remains to show that so does (3). Let (A', b') be an element of \mathcal{C}' and let S be a shape of A' such that $b'(S) \leq b'(A)/2$. Then the root of (A', b') does not belong to S . Hence, letting s be the root of S we infer that $S = S(s)$ thanks to Property (ii) above. It follows that s is the unique vertex of S with maximum b' -weight and S can be uniquely reconstructed from $b''(s)$. This shows the second part of the statement of Theorem 2, thereby ending the proof. \square

2. THE STRUCTURE OF THE PROOF OF THEOREM 1.

We fix a good set of weighted trees and, from now on, we say that a weighted tree is *good* if it belongs to this set. We write down a procedure and with its help prove Theorem 1. The rest of the paper then describes our realisation of the procedure.

A j -form is an isomorphism class of rooted weighted trees with total weight j . Thus a j -form F is a collection of rooted weighted trees and, viewing a shape of a tree T as a rooted weighted tree, a shape can belong to a j -form. We start with an observation.

Observation 2.1. *Let (T, w) be a weighted tree. Assume that for each $j \leq w(T)/2$ and each j -form F , we know the number of shapes of (T, w) that belong to F . Then we know T .*

Proof. We use an easy but important observation that if two shapes of T have a common vertex then one is contained in the other. We order the shapes of (T, w) of weight at most $w(T)/2$ decreasingly according to their weights. Let m be the maximum weight of such a shape of T and let S_1, \dots, S_a be the shapes with weight m . Note that we know precisely these a trees. In addition, either the shapes S_1, \dots, S_a are joined in T to the same vertex, or $a = 2$ and $m = w(T)/2$. In the latter case ($m = w(T)/2$) we know that T consists of the two weighted rooted trees S_1 and S_2 (each of weight m) with an edge between their roots. In the first case, let r be the vertex of T to which each of S_1, \dots, S_a is joined. We show by descending induction on $j \in \{1, \dots, m\}$ that we know the subtree of T induced by all shapes of T with weight in $\{j, \dots, \lfloor W(T)/2 \rfloor\}$. The induction has thus been initialized above, so assume that $j \leq m - 1$. Let S_1, \dots, S_t be the shapes of T with weight in $\{j + 1, \dots, \lfloor W(T)/2 \rfloor\}$. Note that we know, in particular, each of these t trees. The shapes of T of weight equal to j , if any, are either shapes of S_1, \dots, S_t or joined to r by an edge from their root. Fix a j -form F . Since we do know the total number of shapes belonging to F and contained in each of S_1, \dots, S_t (because we know precisely those subtrees), we can deduce the number of shapes that belong to F and are attached to r . As this argument applies to all j -forms F , we infer that we know the subtree of T formed by all shapes with weight contained in $\{j, \dots, \lfloor w(T)/2 \rfloor\}$. This concludes the proof. \square

3. ISOMORPHISM OF GOOD WEIGHTED TREES IS W -RECOGNISABLE

Let (T, w) be a good weighted tree. Let $\alpha(T) = (\alpha_1, \dots, \alpha_n)$ be the weights of the shapes of T , with $\alpha_1 < \dots < \alpha_n$. The definition of a shape implies that $\alpha_1 \geq 2$. We shall consider both partitions of the integer $w(T)$ and partitions of the tree T . To distinguish between them clearly, partitions of an integer are referred to as *expressions*. For each partition P of T , the weights of the parts of T form an expression of $w(T)$, which we call the *characteristic* of P .

- A *j -expression* of an integer m is a partition of m where one of the parts is equal to $m - \alpha_j$. In particular, if S is a shape of T with weight α_j , then $(\text{Ex}(S), w(T) - \alpha_j)$ is a j -expression of $w(T)$.
- A *j -partition* of T is a partition of T whose characteristic is a j -expression of $w(T)$.
- A j -partition (T_0, \dots, T_k) of T with $w(T_0) = w(T) - \alpha_j$ is *shaped* if there exists an edge e of T such that T_0 is one of the components of $T - e$.
- If S is a shape of T with weight α_j and vertex set $V(S) = \{v_1, \dots, v_s\}$, we define $P(S)$ to be $(V(T) \setminus V(S), \{v_1\}, \dots, \{v_s\})$, which is a shaped j -partition of T .

For an expression E of a positive integer, we let $\theta(T, w, E)$ be the number of partitions of (T, w) with characteristic E . Note that this number is 0 if E is not an expression of $w(T)$. We notice that, for each expression E , the polynomial $W_{(T, w)}$ determines $\theta(T, w, E)$. We note that among the partitions of T corresponding to a given expression, some are shaped and others are not.

The proof of Theorem 1 relies on the following procedure.

Procedure 1.

INPUT: The polynomial $W_{(T, w)}$, an integer $j \in \{\alpha_1 + 1, \dots, w(T)/2\}$, a j -expression E and, for each $j' < j$ and each j' -form F , the number of shapes of T that belong to F .

OUTPUT: The number of shaped j -partitions of T with characteristic E .

Let us see how this procedure allows us to establish Theorem 1.

Proof of Theorem 1. Let \mathcal{T} be a good family of trees. Fix two weighted trees (T, w) and (T', w') in \mathcal{T} with $W_{(T, w)} = W_{(T', w')}$. By Observation 2.1, (T, w) and (T', w') are isomorphic if $w(T) = w'(T')$ and for each j -form F where $j \leq w(T)/2$, the numbers of shapes of T and of T' that

belong to F are equal. To establish this, first note that the vector $\alpha(T) = (\alpha_1, \dots, \alpha_n)$ can be computed from $W_{(T,w)}$, since the coordinates correspond to the partitions of T into two subtrees (each with at least two vertices). Thus $\alpha(T') = \alpha(T)$.

We prove by induction on $j \in \{\alpha_1, \dots, \lfloor w(T)/2 \rfloor\}$ that for every j -form F , the numbers of shapes of T and of T' that belong to F are the same. So suppose first that $j = \alpha_1$. Recall that $\alpha_1 \geq 2$. Furthermore, a shape S of T or T' belongs to an α_1 -form if and only if S is the star on α_1 vertices rooted at its centre. This is because the leaves and their neighbours have weight 1. It follows that the number of shapes of T of weight α_1 can be calculated from $W_{(T,w)}$ and thus this number is the same for (T', w') .

Now let $j \in \{\alpha_1 + 1, \dots, \lfloor w(T)/2 \rfloor\}$. We assume that the statement is true for $j' \in \{\alpha_1, \dots, j - 1\}$ and we establish it for j . We do so by setting a partial order on the j -forms, which allows us to link tree partitions with j -forms. Given a j -form F , we define $\text{Ex}(F)$ to be $\text{Ex}(f)$ for an arbitrary representative f of F . (This definition is valid, since all representatives of a j -form are isomorphic rooted weighted trees.) A j -form F' is *smaller than* a j -form F if $\text{Ex}(F')$ is a proper refinement of $\text{Ex}(F)$. If $P = (T_0, \dots, T_k)$ is a shaped j -partition of T where $w(T_0) = w(T) - \alpha_j$, we define $S(P)$ to be the shape of T formed by the union of all parts of T different from T_0 , that is, $S(P) := \cup_{i=1}^k T_i$.

A key observation is that if P is a shaped j -partition of T with characteristic $\text{Ex}(F)$ for some j -form F , then $\text{Ex}(S(P))$ is a refinement of $\text{Ex}(F)$, possibly equal to $\text{Ex}(F)$: actually, there is equality if and only if S belongs to F . We are now ready to argue the final step of the proof. Fix an arbitrary j -form F : our goal is to prove that the numbers of shapes of T and of T' that belong to F are the same. By applying Procedure 1 both in (T, w) and in (T', w') for the j -expression $(\text{Ex}(F), w(T) - \alpha_j)$, our induction hypothesis ensures that T and T' have the same number of shaped j -partitions with characteristic $(\text{Ex}(F), w(T) - \alpha_j)$. If P is not one of these partitions, then $S(P) \notin F$. Otherwise, $S(P)$ may belong to F or not. Consequently, it is enough to prove that for every j -form F , the number of shaped j -partitions P of T such that $S(P) \notin F$ is equal to the analogous number for T' . If F is a j -form, let $n_T(F)$ be the number of shapes of T that belong to F ; we use a similar notation for T' .

We proceed by induction on the j -form F considered (with respect to the partial order defined above). So we first deal with the case where T has no shape that belongs to a j -form F' such that $\text{Ex}(F')$ is a proper refinement of $\text{Ex}(F)$. Suppose that $n_T(F) = 0$. Then, as explained above, T has no shaped j -partition with characteristic $\text{Ex}(F)$. Hence neither has T' , so $n_{T'}(F) = 0$. We may now assume that both T and T' contain a shape belonging to F . If P is a shaped j -partition of T with characteristic $\text{Ex}(F)$, then $\text{Ex}(S(P))$ is a refinement of $\text{Ex}(F)$, which by our hypothesis on F must be equal to $\text{Ex}(F)$. Therefore we only need to prove that every shaped j -partition P' of T' with characteristic $\text{Ex}(F)$ satisfies that $S(P') \in F$. As before, if that were not the case then T' would contain a shape S' such that $\text{Ex}(S')$ is a proper refinement of $\text{Ex}(F)$, which would imply that T contains a shaped j -partition with characteristic $(\text{Ex}(S'), w(T) - \alpha_j)$, hence a shape S'' such that $\text{Ex}(S'')$ is a proper refinement of $\text{Ex}(F)$, a contradiction.

For the general case, we may now assume that $n_T(F') = n_{T'}(F')$ for every j -form F' such that $\text{Ex}(F')$ is a proper refinement of $\text{Ex}(F)$. Observe that for each j -form F' with $F' < F$, each shape of T that belongs to F' gives rise to a certain number of shaped j -partition of T with characteristic $\text{Ex}(F)$, and this number depends only on F' . Thus the number of shaped j -partitions of T with characteristic $(\text{Ex}(F), w(T) - \alpha_j)$ such that $S(P) \notin F$ depends only on the multi-set $\{n_T(F') : F' < F\}$. As $\{n_T(F') : F' < F\} = \{n_{T'}(F') : F' < F\}$, the conclusion follows. \square

4. DESIGNING PROCEDURE 1

An α_j -situation σ is a tuple $((\sigma_1, w_1), \dots, (\sigma_{t(\sigma)}, w_{t(\sigma)}))$ of disjoint weighted rooted trees with $t(\sigma) \geq 2$ such that $w_1(\sigma_1) \leq \dots \leq w_{t(\sigma)}(\sigma_{t(\sigma)})$ and $\sum_{i=1}^{t(\sigma)} w_i(\sigma_i) = \alpha_j$. An α_j -situation σ is said to *occur* in a tree T if there exists a subtree T' of T and $t(\sigma)$ distinct edges $e_1, \dots, e_{t(\sigma)}$ with exactly one end in $V(T')$ such that, for each $i \in \{1, \dots, t(\sigma)\}$, there is an isomorphism preserving the root and the weights between σ_i and the component of $T - e_i$ different from T' . Note that if σ occurs in T , then for each $i \in \{1, \dots, t(\sigma)\}$ the tree T has a shape isomorphic to σ_i .

We proceed in steps, the first one being an exhaustive listing that depends only on α_j .

Step 1. Explicitly list all α_j -situations for $\alpha_j \leq w(T)/2$.

Step 2. For each $\alpha_j \leq w(T)/2$ and each α_j -situation σ from Step 1, compute the number $m_T(\sigma)$ of times σ occurs in T .

Before designing Step 2, we show how Steps 1 and 2 accomplish Procedure 1. Suppose that the two steps are completed. Let $E = \{w(T) - \alpha_j, E_1, \dots, E_k\}$ be a j -expression of $w(T)$.

For each α_j -situation $\sigma = ((\sigma_1, w_1), \dots, (\sigma_{t(\sigma)}, w_{t(\sigma)}))$, let Ψ_σ be the collection of all surjections from the expression $\{E_1, \dots, E_k\}$ to $\{\sigma_1, \dots, \sigma_{t(\sigma)}\}$. Two elements f and g of Ψ_σ are *equivalent* if the multi-set $f^{-1}(\sigma_i)$ is equal to the multi-set $g^{-1}(\sigma_i)$ for every $i \in \{1, \dots, k\}$. We consider the equivalence classes for this relation on Ψ_σ and we form Ψ'_σ by arbitrarily choosing one representative in each equivalent class. We observe that the number X of non-shaped j -partitions of T with characteristic E is

$$(4.1) \quad \sum_{\alpha_j\text{-situation } \sigma} m_T(\sigma) \sum_{f \in \Psi'_\sigma} \sum_{i=1}^{t(\sigma)} \theta(\sigma_i, w_i, f^{-1}(\sigma_i)),$$

where the multi-set $f^{-1}(\sigma_i)$ is naturally interpreted as an expression. Indeed, a non-shaped partition of T with characteristic E corresponds precisely to the occurrence of some α_j -situation $\sigma = ((\sigma_1, w_1), \dots, (\sigma_{t(\sigma)}, w_{t(\sigma)}))$ where the trees $\sigma_1, \dots, \sigma_{t(\sigma)}$ are also partitioned (possibly trivially). Recalling that $\theta(\sigma_i, w_i, E')$ is zero if E' is not an expression of $w_i(\sigma_i)$, the formula (4.1) follows. Notice that (4.1) does allow us to compute X : for each $i \in \{1, \dots, t(\sigma)\}$, the tree σ_i is given by σ , hence the W -polynomial of σ_i can be computed. Consequently, we can compute the number of shaped j -partitions of T with characteristic E , which is

$$\theta(T, w, E) - X.$$

This accomplishes Procedure 1.

It remains to design Step 2. Fix an α_j -situation $\sigma = ((\sigma_1, w_1), \dots, (\sigma_t, w_t))$.

Observation 4.1. *For every pair $(i, j) \in \{1, \dots, t\}^2$, if T_i and T_j are two shapes of a tree T that are isomorphic to σ_i and σ_j , respectively, then either $T_i \subseteq T_j$ or $T_j \subseteq T_i$ or $T_i \cap T_j = \emptyset$.*

To see this, let e_k be the edge of T such that T_k is a component of $T - e_k$ for $k \in \{i, j\}$. Then, either $e_j \in E(T_i)$ or $e_j \in E(T - T_i)$. If $e_j \in E(T - T_i)$, then either $T_j \subseteq T_i$ or $T_j \subseteq T - T_i$, in which case $T_j \cap T_i = \emptyset$. If $e_j \in E(T_i)$, then $T_j \subseteq T_i$: otherwise, $T_j \cap T_i \neq \emptyset$ and $T - T_i \subset T_j$, so that $w(T_i) + w(T_j) > w(T)$. This would contradict the hypothesis that $\sum_{k=1}^{t(\sigma)} w_k(\sigma_k) = \alpha_j$, since $\alpha_j \leq w(T)/2$. This concludes the proof of Observation 4.1.

Define Λ to be the set of all t -tuples (T_1, \dots, T_t) such that for each $i \in \{1, \dots, t\}$,

- T_i is a shape of T that is isomorphic to (σ_i, w_i) , and
- if $j \in \{1, \dots, t\} \setminus \{i\}$, then T_i is not a subtree of T_j .

Observation 4.2. *The number of times that σ occurs in T is equal to $|\Lambda|$.*

Proof. We prove that the elements of Λ are exactly occurrences of σ in T . By the definition, each occurrence of σ gives rise to an element of Λ .

Conversely, let (T_1, \dots, T_t) be an element of Λ . Observation 4.1 implies that the shapes T_i are mutually disjoint. For each $k \in \{1, \dots, t\}$, let e_k be the edge of T associated to the shape T_k and let v_k be the endvertex of e_k that does not belong to T_k . Note that $v_k \notin \cup_{j=1}^t T_j$ since no tree T_i is a subtree of another tree T_j and $\alpha_j \leq w(T)/2$. Set $T'_0 := T$ and $T'_k := T'_{k-1} - T_k$ for $k \geq 1$.

Observe that each of T_{k+1}, \dots, T_t is a shape of T'_k . Hence T'_k is connected and contains all vertices v_1, \dots, v_t . Therefore setting $T' := T'_t$ shows that (T_1, \dots, T_t) occurs in T . \square

Our goal is to compute $|\Lambda|$. For a weighted tree (T', w') , define $\Lambda_0(T', w')$ to be the set of all t -tuples (T_1, \dots, T_t) such that T_i is a shape of T' that is isomorphic to (σ_i, w_i) for each $i \in \{1, \dots, t\}$. Set $\Lambda_0 := \Lambda_0(T, w)$. In this notation, the weight shall be omitted when there is no risk of confusion. The advantage of Λ_0 is that its size can be computed. Indeed,

$$|\Lambda_0| = \prod_{i=1}^t \sharp((\sigma_i, w_i) \hookrightarrow (T, w)),$$

where $\sharp((\sigma_i, w_i) \hookrightarrow (T, w))$ is the number of shapes of T that are isomorphic to (σ_i, w_i) . This number is given in the input of Procedure 1, since $w_i(\sigma_i) < \alpha_j$.

Next, we compute $|\Lambda|$ using the principle of inclusion and exclusion. Setting $I := \{1, \dots, t\}^2 \setminus \{(i, i) : 1 \leq i \leq t\}$, we have

$$|\Lambda| = |\Lambda_0| - \left| \bigcup_{(i,j) \in I} \Lambda_{(i,j)} \right|,$$

where $\Lambda_{(i,j)}$ is the subset of Λ_0 composed of the elements (T_1, \dots, T_t) with $T_i \subseteq T_j$.

By the principle of inclusion-exclusion, we deduce that the output of Step 2 is equal to

$$|\Lambda_0| - \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} \left| \bigcap_{(i,j) \in J} \Lambda_{(i,j)} \right|.$$

It remains to compute $\left| \bigcap_{(i,j) \in J} \Lambda_{(i,j)} \right|$ for each non-empty subset J of I . We start with an observation, which characterises the sets J for which the considered intersection is not empty.

Observation 4.3. *Let $J \subseteq I$. Then, $\bigcap_{(i,j) \in J} \Lambda_{(i,j)} \neq \emptyset$ if and only if for every $(i, j) \in J$, either σ_i is isomorphic to σ_j , or σ_j has a shape that is isomorphic to σ_i .*

From now on, we consider only contributing sets J . We construct four directed graphs A_0, A_1, A_2 and A_3 that depend on J . Each vertex x of A_k is labeled by a subset $\ell(x)$ of $\{(\sigma_1, w_1), \dots, (\sigma_t, w_t)\}$. These labels will have the following properties.

- (1) $(\ell(x))_{x \in V(A_k)}$ is a partition of $\{(\sigma_1, w_1), \dots, (\sigma_t, w_t)\}$.
- (2) For each vertex x of A_k , all weighted trees in $\ell(x)$ are isomorphic.
- (3) $\left| \bigcap_{(i,j) \in J} \Lambda_{(i,j)} \right|$ is equal to the number of elements (T_1, \dots, T_t) of Λ_0 such that
 - for each vertex x of A_k , if $(\sigma_i, w_i), (\sigma_j, w_j) \in \ell(x)$ then $T_i = T_j$; and
 - for every arc (x, y) of A_k , if $((\sigma_i, w_i), (\sigma_j, w_j)) \in \ell(x) \times \ell(y)$, then $T_i \subseteq T_j$.

The directed graph A_0 is obtained as follows. We start from the vertex set $\{z_1, \dots, z_t\}$. For each $i \in \{1, \dots, t\}$, the label $\ell(z_i)$ of z_i is set to be $\{(\sigma_i, w_i)\}$. For each $(i, j) \in J$, we add an arc from z_i to z_j . Thus A_0 satisfies properties (1)–(3). Note that A_0 may contain directed cycles, but by Observation 4.3, if C is a directed cycle then all elements in $\cup_{x \in V(C)} \ell(x)$ are isomorphic.

Now, A_1 is obtained from A_0 by the following recursive operation. Let (x, y, z) be a triple of vertices such that (x, y) and (x, z) are arcs, but neither (y, z) nor (z, y) are arcs. Let $(\sigma_y, w_y) \in \ell(y)$ and $(\sigma_z, w_z) \in \ell(z)$. We add the arc (y, z) if $|V(\sigma_y)| \leq |V(\sigma_z)|$, and the arc (z, y) if $|V(\sigma_z)| \leq |V(\sigma_y)|$. (In particular, if $|V(\sigma_y)| = |V(\sigma_z)|$, then both arcs are added.)

We observe that A_1 satisfies (1)–(3). Since neither the vertices nor the labels were changed, the only thing that we need to show is that if the arc (y, z) was added, then for all tuples $(T_1, \dots, T_t) \in \cap_{(i,j) \in J} \Lambda_{(i,j)}$ and all $((\sigma_i, w_i), (\sigma_j, w_j)) \in \ell(y) \times \ell(z)$, it holds that $T_i \subseteq T_j$. This follows from Observation 4.1: since (y, z) was added, there exists $s \in \{1, \dots, t\}$ such that T_s is contained in both T_i and T_j .

The directed graph A_2 is obtained from A_1 by recursively contracting all directed cycles of A_1 . Specifically, for each directed cycle C , all the vertices of C are contracted into a vertex z_C (parallel arcs are removed, but not directed cycles of length 2), and $\ell(z_C) := \cup_{x \in V(C)} \ell(x)$. We again observe that A_2 satisfies properties (1)–(3).

Finally, A_3 is obtained from A_2 by recursively deleting transitivity arcs, that is, the arc (y, z) is removed if there exists a directed path of length greater than 1 from y to z . Note that A_2 and A_3 have the same vertex-set, and every arc of A_3 is also an arc in A_2 . Again, A_3 readily satisfies properties (1)–(3).

Now, let us prove that each component of A_3 is an arborescence, that is a directed acyclic graph with each out-degree at most one. We only need to show that every vertex of A_3 has outdegree at most 1. Assume that (x, y) and (x, z) are two arcs of A_3 . First, note that, in A_2 , there is no directed path from y to z or from z to y , for otherwise the arc (x, y) or the arc (x, z) would not belong to A_2 , respectively. Therefore, regardless whether y and z arose from contractions of directed cycles in A_1 , there exist three vertices x' , y' and z' in A_1 such that both (x', y') and (x', z') are arcs but neither (y', z') nor (z', y') is an arc. This contradicts the definition of A_1 . Consequently, every vertex of A_3 has outdegree at most 1, as wanted.

We define τ_i to be the ordered $(t+1)$ -tuple

$$(\sharp((\sigma_i, w_i) \hookrightarrow (T, w)), \sharp((\sigma_i, w_i) \hookrightarrow (\sigma_1, w_1)), \dots, \sharp((\sigma_i, w_i) \hookrightarrow (\sigma_t, w_t)))$$

We recall that τ_1, \dots, τ_t are known from the assumptions of Procedure 1. Step 2 is completed by the following procedure.

Procedure 2.

INPUT: A labeled directed forest A of arborescences and the $(t+1)$ -tuples τ_1, \dots, τ_t .

OUTPUT: For each $H \in \{(T, w), (\sigma_1, w_1), \dots, (\sigma_t, w_t)\}$, the number $\mathcal{P}_3(H, A, \tau(T))$ of elements (T_1, \dots, T_t) of $\Lambda_0(H)$ such that

- for each vertex x of A , if $(\sigma_i, w_i), (\sigma_j, w_j) \in \ell(x)$ then $T_i = T_j$; and
- for every arc (x, y) of A , if $((\sigma_i, w_i), (\sigma_j, w_j)) \in \ell(x) \times \ell(y)$, then $T_i \subseteq T_j$.

The output of Procedure 2 can be recursively computed as follows. Let V_{\max} be the set of vertices of A with outdegree 0. For each vertex x of A , let (σ^x, w^x) be a representative of $\ell(x)$.

$$\mathcal{P}_3(H, A, \tau(T)) = \prod_{x \in V_{\max}} (\sharp((\sigma^x, w^x) \hookrightarrow H)) \cdot \mathcal{P}_3((\sigma^w, w^w), \tilde{A}(w), \tau(T)),$$

where $\tilde{A}(w)$ is obtained from the component of A that contains x by removing x .

By property (3) of the labels, the output $\mathcal{P}_3(T, A_3, \tau(T))$ is equal to $|\cap_{(i,j) \in J} \Lambda_{(i,j)}|$. This concludes the design of Procedure 1.

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