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Spectral monodromy of small non-selfadjoint perturbed operators: completely integrable or quasi-integrable case

¹QUANG SANG PHAN

Abstract

We build a combinatorial invariant, called the spectral monodromy from the spectrum of a non-selfadjoint h -pseudodifferential operator with two degrees of freedom in the semi-classical limit. We treat small non-selfadjoint perturbation of selfadjoint h -pseudodifferential operators in two case: in the first, we assume that the classical flow of the unperturbed part is integrable; the second case, more interesting when this flow is assumed to be quasi-integrable.

Keywords: Hamiltonian systems, non-selfadjoint, spectral analysis, pseudo-differential operators, Birkhoff normal form, KAM theory, asymptotic spectral.

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1 Introduction

This paper aims to understanding the structure of the spectrum of some classes of non-selfadjoint operators in the semi classical limit.

Locally, under some conditions, the spectrum of non-selfadjoint h -pseudo-differential operators is a deformed lattice. It is image by a pseudo-chart of a square lattice. A combinator invariant characterizes the global ability of the spectrum is spectral monodromy. If this monodromy is not trivial, then the spectrum is not a global lattice.

We treat in this paper small non-selfadjoint perturbation of selfadjoint h -pseudodifferential operators in dimension 2, assuming that the Hamiltonian flow of the unperturbed part possesses a invariant Lagrangian torus enjoying a uniformly Diophantine condition in every energy hyperplane. We propose two case: the first case when this flow is completely integrable, and the second case this flow is quasi-integrable.

2 Pseudodifferential operators

We will work throughout this article with pseudo-differentials operators obtained by the h -Weyl-quantization with standard classes of symbols on $T^*M = \mathbb{R}^{2n}_{(x,\xi)}$, here $M = \mathbb{R}^n$ or a manifold compact.

These operators admit the standard properties of pseudo differential operators. For more details, see the references [17], [38], [39].

Definition 2.1. *A function $m : \mathbb{R}^{2n} \rightarrow (0, +\infty)$ is called an order function (or tempered weight in the book of D. Robert [38]) if there are constants $C, N > 0$ such that*

$$m(X) \leq C \langle X - Y \rangle^N m(Y), \forall X, Y \in \mathbb{R}^{2n},$$

with notation $\langle Z \rangle = (1 + |Z|^2)^{1/2}$ for $Z \in \mathbb{R}^{2n}$.

One use often the order function $m(Z) \equiv 1$ or

$$m(Z) = \langle Z \rangle^{l/2} = (1 + |Z|^2)^{l/2},$$

with a given constant $l \in \mathbb{R}$.

Definition 2.2. *Let m be an order function and $k \in \mathbb{R}$, we define classes of symbols of h -order k , $S^k(m)$ (families of functions) of $(a(\cdot; h))_{h \in (0,1]}$ on $\mathbb{R}^{2n}_{(x,\xi)}$ by*

$$S^k(m) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid \forall \alpha \in \mathbb{N}^{2n}, \quad |\partial^\alpha a| \leq C_\alpha h^k m\}, \quad (2.1)$$

for some constant $C_\alpha > 0$, uniformly in $h \in (0, 1]$.

A symbol a is called $\mathcal{O}(h^\infty)$ if it's in $\cap_{k \in \mathbb{R}} S^k(m) := S^\infty(m)$.

Then $\Psi^k(m)(M)$ denotes the set of all (in general unbounded) linear operators A_h on $L^2(\mathbb{R}^n)$, obtained from the h -Weyl-quantization of symbols $a(\cdot; h) \in S^k(m)$ by the integral:

$$(A_h u)(x) = (Op_h^w(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi. \quad (2.2)$$

In this paper, we always assume that the symbols admit a classical asymptotic expansion in integer powers of h . The leading term in this expansion is called the principal symbol of the operator.

3 Spectral Asymptotic

3.1 Assumptions

We will first give the general assumptions of our operator as in the articles [27], [22], [23], [24], [25] and the assumptions on the classical flow of the principal symbol of the non-perturbed operator and some associated spectral results: the discrete spectrum, the localization of the spectrum, the expansion of asymptotic eigenvalues ...

M denotes \mathbb{R}^2 or a connected compact analytic real (riemannian) manifold of dimension 2 and we denote by \widetilde{M} the canonical complexification of M , which is either \mathbb{C}^2 in the Euclidean case or a Grauert tube in the case of manifold (see [10], [30]).

We consider a non-selfadjoint h - pseudodifferential operator P_ε on M and suppose that

$$P_{\varepsilon=0} := P \quad \text{is formally self-adjoint.} \quad (3.3)$$

Note that if $M = \mathbb{R}^2$, the volume form $\mu(dx)$ is naturally induced by the Lebesgue measure on \mathbb{R}^2 . If M is a compact Riemannian manifold, then the volume form $\mu(dx)$ is induced by the given Riemannian structure of M . Therefore in both cases the volume form is well defined and the operator P_ε may be seen as an (unbounded) operator on $L^2(M, \mu(dx))$.

We always denote the principal symbol of P_ε by p_ε which is defined on T^*M as we discussed in the previous section.

We will assume the ellipticity condition at infinity for P_ε at some energy level $E \in \mathbb{R}$ as follows:

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P(x, hD_x, \varepsilon; h) \quad (3.4)$$

be the Weyl quantification of a total symbol $P(x, \xi, \varepsilon; h)$ depending smoothly on ε in a neighborhood of $(0, \mathbb{R})$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of \mathbb{R}^4 in \mathbb{C}^4 on which we assume that:

$$|P(x, \xi, \varepsilon; h)| \leq \mathcal{O}(1)m(\text{Re}(x, \xi)). \quad (3.5)$$

Here m is an order function in the sense of definition 2.1. We assume moreover that $m > 1$ and P_ε is classical

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\varepsilon}(x, \xi) h^j, h \rightarrow 0, \quad (3.6)$$

in the selected space of symbols.

In this case, the main symbol is the first term of the above expansion, $p_\varepsilon = p_{0,\varepsilon}$ and the ellipticity condition at infinity is

$$|p_{0,\varepsilon}(x, \xi) - E| \geq \frac{1}{C} m(\text{Re}(x, \xi)), |x, \xi| \geq C, \quad (3.7)$$

for some $C > 0$ large enough.

When M is a compact manifold, we consider P_ε a differential operator on M such that in local coordinates x of M , it is of the form:

$$P_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha,\varepsilon}(x; h) (hD_x)^\alpha, \quad (3.8)$$

Where $D_x = \frac{1}{i} \frac{\partial}{\partial x}$ and $a_{\alpha,\varepsilon}$ are smooth functions of ε in a neighborhood of 0 with values in the space of holomorphic functions on a complex neighborhood of $x = 0$.

We assume that these $a_{\alpha,\varepsilon}$ are classical

$$a_{\alpha,\varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\varepsilon,j}(x) h^j, h \rightarrow 0, \quad (3.9)$$

in the selected space of symbols.

In this case, the principal symbol p_ε in the local canonical coordinates associated (x, ξ) on T^*M is

$$p_\varepsilon(x, \xi) = \sum_{|\alpha| \leq m} a_{\alpha,\varepsilon,0}(x) \xi^\alpha \quad (3.10)$$

and the ellipticity condition at infinity is

$$|p_\varepsilon(x, \xi) - E| \geq \frac{1}{C} \langle \xi \rangle^m, (x, \xi) \in T^*M, |\xi| \geq C, \quad (3.11)$$

for some $C > 0$ large enough.

Note here that M has a riemannian metric, then $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are defined.

It is known from articles [27], [23] that with the above conditions, the spectrum of P_ε in a small but fixed neighborhood of $E \in \mathbb{C}$ is discrete, when $h > 0, \varepsilon \geq 0$ are small enough. Moreover, this spectrum is contained in a horizontal band of size ε :

$$|\text{Im}(z)| \leq \mathcal{O}(\varepsilon). \quad (3.12)$$

Let $p = p_{\varepsilon=0}$, it is principal symbol of the selfadjoint unperturbed operator P and therefore real.

Let $q = \frac{1}{i}(\frac{\partial}{\partial \varepsilon})_{\varepsilon=0} p_{\varepsilon}$ and assume that q is a bounded analytic function on T^*M . We can write the principal symbol

$$p_{\varepsilon} = p + i\varepsilon q + \mathcal{O}(\varepsilon^2). \quad (3.13)$$

We assume that

$$p^{-1}(E) \cap T^*M \quad \text{is connected} \quad (3.14)$$

and the energy level E is regular for p , i.e $dp \neq 0$ on $p^{-1}(E) \cap T^*M$.

For $T > 0$, we introduce the symmetric average time T of q along the flow of p , defined near $p^{-1}(0) \cap T^*M$:

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) dt, \quad (3.15)$$

where H_p is the Hamiltonian vector field of p , defined by $\sigma(H_p, \cdot) = -dp$.

As explained in Ref. [27], the spectrum of P_{ε} in the neighborhood of $E \in \mathbb{C}$, $\{z \in \mathbb{C} : |z - E| < \delta\}$, when $\varepsilon, h, \delta \rightarrow 0$ is confined in the band (voir [40], [41]):

$$]-\delta + E, \delta + E[+ i\varepsilon \left[\liminf_{T \rightarrow \infty} \inf_{p^{-1}(E)} \operatorname{Re} \langle q \rangle_T - o(1), \limsup_{T \rightarrow \infty} \sup_{p^{-1}(E)} \operatorname{Re} \langle q \rangle_T + o(1) \right]. \quad (3.16)$$

With more assumptions about the dynamics of classical flow of the first term of the unperturbed symbol (in a certain energy level), one can obtain more detailed results on the asymptotic distribution of eigenvalues of P_{ε} in such a band.

Now, assume that p is completely integrable, i.e there exists an analytic real valued function f , differentially independent of p such that $\{p, f\} = 0$ with respect to the Poisson bracket on T^*M . That means

$$F = (p, f) : T^*M \rightarrow \mathbb{R}^2 \quad (3.17)$$

is a momentum map.

Then the space of regular leaves of F is foliated by Liouville Lagrangian invariant tori by Liouville-Arnold-Moser theorem 3.14.

As explained in ([27], p. 21-22 and 55), the energy space $p^{-1}(E)$ is decomposed into a singular foliation:

$$p^{-1}(E) \cap T^*M = \bigcup_{a \in J} \Lambda_a, \quad (3.18)$$

where J is assumed to be a compact interval, or, more generally, a connected graph with a finite number of vertices and of edges. We denote by S the set of vertices. For

each $a \in J$, Λ_a is a connected compact subset invariant with respect to H_p . Moreover, if $a \in J \setminus S$, Λ_a is a invariant Lagrangian torus depending analytically on a . These tori are regular leaves corresponding regular values of F .

Each edge of J can be identified with a bounded interval of \mathbb{R} and we have therefore a distance on J in the natural way. Next, we assume the continuity of tori: let $\Lambda_{a_0}, a_0 \in J \setminus S$, for all $\mu > 0, \exists \gamma > 0$, such that if $dist(a, a_0) < \gamma$, then $\Lambda_a \subset \{\rho \in p^{-1}(E) \cap T^*M : dist(\rho, \Lambda_0) < \mu\}$.

Note that this hypothesis holds for integrable systems with non-degenerate singularities.

We denote H_p the Hamiltonian vector field of p , defined by $\sigma(H_p, \cdot) = -dp(\cdot)$.

For each $a \in J$, we define a compact interval in \mathbb{R} :

$$Q_\infty(a) = \left[\liminf_{T \rightarrow \infty} Re \langle q \rangle_T, \limsup_{T \rightarrow \infty} Re \langle q \rangle_T \right], \quad (3.19)$$

where $\langle q \rangle_T$, for $T > 0$, is the symmetric average time T of q along the H_p -flow, defined by

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) dt. \quad (3.20)$$

Then the spectral localization (3.16) becomes

$$Im(\sigma(P_\varepsilon) \cap \{z \in \mathbb{C} : |Rez - E| \leq \delta\}) \subset \varepsilon \left[\inf_{a \in J} Q_\infty(a) - o(1), \sup_{a \in J} Q_\infty(a) + o(1) \right], \quad (3.21)$$

when $\varepsilon, h, \delta \rightarrow 0$.

Each Lagrangian invariant torus Λ_a , with $a \in J \setminus S$, locally can be embedded in a Lagrangian foliation of H_p -invariant tori. By the action-angle theorem 3.14, there are analytic action-angle local coordinates $\kappa = (x, \xi)$ on a neighborhood V of Λ_a in T^*M ,

$$\kappa = (x, \xi) : V \rightarrow \mathbb{T}^2 \times A, \quad (3.22)$$

with A is a neighborhood of $\xi_a \in \mathbb{R}^2$ such that Λ_a is symplectically identified with $\mathbb{T}^2 \times \{\xi_a\}$, and p becomes a function of action variables ξ ,

$$p = p(\xi) = p(\xi_1, \xi_2), \quad \xi \in A. \quad (3.23)$$

Then the frequency of an arbitrary Lagrangian invariant torus $\Lambda_\xi \subset V$, that is near Λ_a , is defined by

$$\omega(\xi) = \frac{\partial p}{\partial \xi}(\xi) = \left(\frac{\partial p}{\partial \xi_1}(\xi), \frac{\partial p}{\partial \xi_2}(\xi) \right), \quad \xi \in A. \quad (3.24)$$

In particular, the frequency of torus Λ_a is $\omega(\xi_a) = \frac{\partial p}{\partial \xi}(\xi_a)$. It knows that ω depends analytically of $\xi \in A$. In particular, the restriction $\omega(\xi_a)$ depends analytically of ξ_a , when $a \in J \setminus S$. We will assume that the function $a \mapsto \omega(\xi_a)$ is not identically constant on any connected component of $J \setminus S$.

From now, for simplicity, we will assume that q is real valued (in the general case, simply replace q by $Re\ q$).

We define the average of q on an arbitrary Lagrangian invariant torus $\Lambda_\xi \subset V$, with respect to the natural Liouville measure on Λ_ξ , denoted by $\langle q \rangle_{\Lambda}$, as following

$$\langle q \rangle_{\Lambda_\xi} = \int_{\Lambda_\xi} q. \quad (3.25)$$

Remark 3.1. In the above action-angle coordinates (x, ξ) near Λ_a , $a \in J \setminus S$, we have

$$\langle q \rangle_{\Lambda_\xi} = \langle q \rangle(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} q(x, \xi) dx, \quad \xi \in A. \quad (3.26)$$

In particular, $\langle q \rangle_{\Lambda_a} = \langle q \rangle(\xi_a)$.

Remark 3.2 ([27], p. 56-57). For $a \in J \setminus S$, if $\omega(a) \notin \mathbb{Q}$, that means the frequency $\omega(a)$ is non resonant, then along the torus Λ_a , the Hamiltonian flow of p is ergodic and the limit of $\langle q \rangle_T$, when $T \rightarrow \infty$ (the time average of q) exists, is equals to the space average of q over the torus, $\langle q \rangle_{\Lambda_a}$. Therefore we have

$$Q_\infty(a) = \{\langle q \rangle_{\Lambda_a}\}.$$

It is true that $\langle q \rangle_{\Lambda_a}$ depends analytically of $a \in J \setminus S$ and we assume it can be extended continuously on J . Furthermore, we assume that the function $a \mapsto \langle q \rangle(\xi_a) = \langle q \rangle_{\Lambda_a}$ is not identically constant on any connected component of $J \setminus S$.

Assume furthermore that the differential of the functions $p(\xi)$, given in (3.23), and of $\langle q \rangle$, given in (3.26) are \mathbb{R} -linearly independent when $\xi = \xi_a$.

Then $\langle q \rangle$ and p are in involution in the neighborhood V of Λ_a , due to $\langle q \rangle$ is invariant under the flow of p .

3.2 Asymptotic eigenvalues

The asymptotic spectral theory of Hitrik co-workers (see Refs [23]-[25] and especially Ref. [27]) allows us to give an asymptotic description of all the eigenvalues of P_ε in some small complex windows of the spectral band, which are associated with Diophantine tori in the phase space. In Ref. [27] the force of the perturbation ε is treated by several different case: it is small and can be dependent or independent of h .

In our work we present the result in the case when ε is sufficiently small, independent of h and moreover we will assume that

$$h \ll \varepsilon = \mathcal{O}(h^\delta),$$

where $\delta > 0$ is some number small enough but fixed. In this case, the spectral results are related to h -independent small window.

Definition 3.3. Let $\alpha > 0$, $d > 0$, and Λ_ξ be a H_p invariant Lagrangian torus. We say that Λ is (α, d) -Diophantine if its frequency $\omega(\xi)$, defined in (3.24), satisfies

$$\omega(\xi) \in D_{\alpha,d} = \left\{ \omega \in \mathbb{R}^2 \mid |\langle \omega, k \rangle| \geq \frac{\alpha}{|k|^{1+d}}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\} \right\}. \quad (3.27)$$

If (3.27) holds for some $\alpha > 0$, and $d > 0$, we say that the torus Λ_ξ or its frequency is uniformly Diophantine.

Note also that when $d > 0$ is fixed, the Diophantine property (for some α) of Λ_ξ is independent of the choice of action-angle coordinates. If Λ_ξ is (α, d) -Diophantine, then it must be irrational.

It knew that the set $D_{\alpha,d}$ is a closed set with closed half-line structure. When we take α to be sufficiently small, it is a nowhere dense set but with no isolated points, and its measure tends to full measure as α tends to 0: the measure of its complement is of order $\mathcal{O}(\alpha)$. On the other hand, the trace of $D_{\alpha,d}$ on the unit sphere is a Cantor set.

Definition 3.4. For some $\alpha > 0$ and some $d > 0$, we define the set of "good values" $\mathcal{G}(\alpha, d, E)$, associated with the energy level E , obtained from $\cup_{a \in J} Q_\infty(a)$ by removing the following set of "bad values" $\mathcal{B}(\alpha, d, E)$:

$$\mathcal{B}(\alpha, d, E) = \left(\bigcup_{\text{dist}(a,S) < \alpha} Q_\infty(a) \right) \cup \left(\bigcup_{a \in J \setminus S: |\omega'(\xi_a)| < \alpha} Q_\infty(a) \right) \cup \left(\bigcup_{a \in J \setminus S: |d\langle q \rangle_{\Lambda_a}| < \alpha} Q_\infty(a) \right) \\ \cup \left(\bigcup_{a \in J \setminus S: \omega(\xi_a) \text{ is not } (\alpha,d)\text{-Diophantine}} Q_\infty(a) \right).$$

Remark 3.5.

- When $d > 0$ is kept fixed, the measure of the set of bad values $\mathcal{B}(\alpha, d, E)$ in $\cup_{a \in J} Q_\infty(a)$ (and in $\langle q \rangle_{\Lambda_a}(J)$) is small together with α , is $\mathcal{O}(\alpha)$, when $\alpha > 0$ is small enough, provided that the measure of

$$\left(\bigcup_{a \in J \setminus S: \omega(\xi_a) \in \mathbb{Q}} Q_\infty(a) \right) \cup \left(\bigcup_{a \in S} Q_\infty(a) \right) \quad (3.28)$$

is sufficiently small, depending on α (see Ref. [27]).

- Let $G \in \mathcal{G}(\alpha, d, E)$ be a good value, then by definition of $\mathcal{B}(\alpha, d, E)$ and remark (3.2), there are a finite number of corresponding (α, d) -Diophantine tori $\Lambda_{a_1}, \dots, \Lambda_{a_L}$, with $L \in \mathbb{N}^*$ and $\{a_1, \dots, a_L\} \subset J \setminus S$, in the energy space $p^{-1}(E) \cap T^*M$, such that the pre-image

$$\langle q \rangle^{-1}(G) = \{\Lambda_{a_1}, \dots, \Lambda_{a_L}\}.$$

By this way, when G varies in $\mathcal{G}(\alpha, d, E)$, we obtain a Cantor family of (α, d) -Diophantine invariant tori in the phase space satisfying $\{p = E, \langle q \rangle = G\}$.

When $G \in \mathcal{G}(\alpha, d, E)$ is a good value, we define in the horizontal band of size ε of complex plan, given in (3.21), a adapted window of size $\mathcal{O}(h^\delta) \times \mathcal{O}(\varepsilon h^\delta)$, around the good center $E + iG$, called "good rectangle",

$$R^{(E,G)}(\varepsilon, h) = (E + i\varepsilon G) + \left[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)} \right] + i\varepsilon \left[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)} \right]. \quad (3.29)$$

Now let $G \in \mathcal{G}(\alpha, d, E)$ be a good value. As in Remark 3.5, there exists L elements in pre-image of G by $\langle q \rangle$. However for simplicity, we shall assume through this work that $L = 1$ and we write

$$\langle q \rangle^{-1}(G) = \Lambda_a \subset p^{-1}(E) \cap T^*M, \quad a \in J \setminus S, \quad (3.30)$$

Note that this hypothesis can be achieved if we assume that the function $\langle q \rangle$ is injective on $J \setminus S$.

Definition 3.6 (Refs. [44], [2], and [11]). *Let E be a symplectic space and let $\Lambda(E)$ be his Lagrangian Grassmannian (which is set of all Lagrangian subspaces of E). We consider a bundle B in E over the circle or a compact interval provided with a Lagrangian subbundle called vertical. Let $\lambda(t)$ be a section of $\Lambda(B)$ which is transverse to the vertical edges of the interval in the case where the base is an interval. The Maslov index of $\lambda(t)$ is the intersection number of this curve with the singular cycle of Lagrangians which do not cut transversely the vertical subbundle.*

Let $\Lambda_a, a \in J \setminus S$ be a some Lagrangian invariant torus and let κ be the action-angle local coordinates in (3.22). The fundamental cycles (γ_1, γ_2) of Λ_a are defined by

$$\gamma_j = \kappa^{-1}(\{(x, \xi) \in T^*\mathbb{T}^2 : x_j = 0, \xi = \xi_a\}), \quad j = 1, 2.$$

Then we note $\eta \in \mathbb{Z}^2$ the Maslov index and $S \in \mathbb{R}^2$ the integral action of these fundamental cycles,

$$S = (S_1, S_2) = \left(\int_{\gamma_1} \theta, \int_{\gamma_2} \theta \right), \quad (3.31)$$

where θ is locally a (primitive) Liouville 1-form of the closed form σ on (Λ_a, T^*M) , whose existence is guaranteed by the Poincaré lemma.

We recall here the result about the asymptotic spectrum treated for the stand at the energy level $E = 0$, given by [27]. However this result can be immediately generalized for any energy level E by a translation, that we will carry out after in Sec. 3.3 of this paper.

Theorem 3.7 ([27]). *For $E = 0$ and assume that action-angle coordinates κ in (3.22) send Λ_a to the section $\{\xi_a = 0\} \in T^*\mathbb{T}^2$. Suppose that P_ε is an operator with principal symbol (3.13) which satisfies Assumptions 3.1. Assume that $h \ll \varepsilon = \mathcal{O}(h^\delta)$ for $0 < \delta < 1$. Let $G \in \mathcal{G}(\alpha, d, 0)$ be a good value. With the above notations, then the eigenvalues*

μ of P_ε with multiplicity in a good rectangle $R^{(0,G)}(\varepsilon, h)$ of the form (3.29) have the following expression:

$$\mu = P^{(\infty)}\left(h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, \quad (3.32)$$

where $P^{(\infty)}(\xi; \varepsilon, h)$ is a smooth function of ξ evolving in a neighborhood of $(0, \mathbb{R}^2)$ and ε, h in neighborhoods of $(0, \mathbb{R})$. Moreover $P^{(\infty)}(\xi; \varepsilon, h)$ is real valued for $\varepsilon = 0$ and it admits the following polynomial asymptotic expansion in (ξ, ε, h) for the C^∞ topology:

$$P^{(\infty)}(\xi; \varepsilon, h) \sim \sum_{\alpha, j, k} C_{\alpha j k} \xi^\alpha \varepsilon^j h^k, \quad (3.33)$$

Particularly $P^{(\infty)}$ is classical in the space of symbols with h -leading term:

$$p_0^{(\infty)}(\xi, \varepsilon) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2). \quad (3.34)$$

Here p, q are the expressions of p, q in action-angle variables near Λ_a , given by (3.22), and $\langle q \rangle$ is the average of q on tori, defined in (3.26).

Remark 3.8. We can write the total symbol $P^{(\infty)}$ in the reduce form:

$$P^{(\infty)}(\xi; \varepsilon, h) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h), \quad (3.35)$$

uniformly for ε and h small. Note that $dp, d\langle q \rangle$ are linearly independent in $\xi = 0$, and in the regime $h \ll \varepsilon$ we can show that the function $P^{(\infty)}$ is a local diffeomorphism from a neighborhood $B(0, r)$, $r > 0$, of $\xi = 0 \in \mathbb{R}^2$ into its image which covers the good rectangle (see Ref. [51]).

3.3 The completely integrable case

In this section, we first apply the result of the stand case, spoken in Theorem 3.7 to given the asymptotic expansion of eigenvalues of P_ε in an any good rectangle $R^{(E,G)}(\varepsilon, h)$ of spectral band. After, we will give a brief construction to show that the spectrum satisfies the condition of an asymptotic pseudo-lattice. For more detail of this construction, we refer to [51].

We are considering the operator P_ε as in the preceding paragraph XXXX.

We note that $\langle q \rangle$, the average of q over the regular Lagrangian H_p -invariant torus Λ_ξ , given in (3.25), can be seen as a constant function on Λ_ξ . In this way $\langle q \rangle$ well defines a analytic function on the union of regular tori. It is true that the fibration of all regular Lagrangian tori Λ_ξ , given by the momentum map F , fills almost completely the phase space T^*M . So we assume that the function $\langle q \rangle$ can be extended smoothly on the whole phase space T^*M . Moreover we assume that the differentials dp and $d\langle q \rangle$ are linearly independent almost everywhere on T^*M . It is clear that p and $\langle q \rangle$ commute in

neighborhood of each regular torus Λ_a . So we can see $(p, \langle q \rangle)$ as a completely integrable system and we will take the momentum map $F = (p, f)$, with $f = \langle q \rangle$.

We assume moreover that the map F is proper and has connected fiber. This ensures all regular fibers of F are Lagrangian invariant tori.

Denote by U_r the set of its regular values and let U be a subset of U_r with compact closure.

We assume further all assumptions as in Sec. 3.1 for any energy level E , taken in a bounded interval of \mathbb{R} .

We can take E in the interval $p_x(U)$, where p_x is the projection on the real axe of variable x .

The fact that the horizontal spectral band of size $\mathcal{O}(\varepsilon)$ suggest us introducing the function

$$\begin{aligned} \chi : \mathbb{R}^2 \ni u = (u_1, u_2) \mapsto \chi_u &= (u_1, \varepsilon u_2) \in \mathbb{R}^2 \\ &\cong u_1 + i\varepsilon u_2 \in \mathbb{C}, \end{aligned} \quad (3.36)$$

in which we identify \mathbb{C} with \mathbb{R}^2 .

Let a point $c \in U$. We recall that by the action-angle theorem, we have action-angle coordinates in a neighborhood of torus $\Lambda_c := \Phi^{-1}(c)$ in M : there exists $r > 0$, a neighborhood $V := F^{-1}(B(c, r))$ of Λ_c , an small open $A \subset \mathbb{R}^2$ of center 0, a symplectomorphism $\kappa : V \rightarrow \mathbb{T}^2 \times A$ and a diffeomorphism $\varphi = (\varphi_1, \varphi_2) : A \rightarrow \varphi(A) = B(c, r)$ such that: $\kappa(\Lambda_c) = \mathbb{T}^2 \times \{\xi = 0\}$, $F \circ \kappa^{-1}(x, \xi) = \varphi(\xi)$, for all $x \in \mathbb{T}^2, \xi \in A$ and here we have $\varphi(0) = c$.

Denote

$$B(u, r; \varepsilon) := \chi(B(u, r))$$

for some ball $B(u, r)$ ($r > 0$) and

$$U(\varepsilon) := \chi(U).$$

Note that in previous action-angle coordinates the functions p and $\langle q \rangle$ become $p(\xi) = \varphi_1(\xi)$, $\langle q \rangle = \varphi_2(\xi)$, as in (3.23) and (3.26).

Now for any point $a = (E, G) \in B(c, r)$ such that G is a good value, $G \in \mathcal{G}(\alpha, d, E)$, as in Definition 3.4. Then the corresponding torus $\Lambda_a = F^{-1}(a)$, as told about in (3.30), is (α, d) -Diophantine (note that in our context $L = 1$). Suppose that by κ , the tours Λ_a is sent to the section $\mathbb{T}^2 \times \{\xi = \xi_a\}$, $\xi_a = \varphi^{-1}(a) \in D$.

Noting that $\sigma(P_\varepsilon) = \sigma(P_\varepsilon - \chi_a) + \chi_a$ and applying the asymptotic spectral result in the stand case for the operator $(P_\varepsilon - \chi_a)$ with respect to the good rectangle $R^{(0, G)}(\varepsilon, h)$, Theorem 3.7, we obtain the asymptotic eigenvalues of P_ε in the good rectangle $R^{(E, G)}(\varepsilon, h)$.

Then all the eigenvalues μ of P_ε in the good rectangle $R^{(E, G)}(\varepsilon, h)$, defined by 3.29, modulo $\mathcal{O}(h^\infty)$ are given by the asymptotic expansion of a smooth function $P(\xi, \varepsilon; h)$ (for ease the notation, writing P instead of $P^{(\infty)}$, given in Theorem 3.7) of ξ in a neighborhood of (ξ_a, \mathbb{R}^2) and ε, h in neighborhoods of $(0, \mathbb{R})$.

$$\sigma(P_\varepsilon) \cap R^{(E, G)}(\varepsilon, h) \ni \mu = P\left(\xi_a + h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, \quad (3.37)$$

Here $\eta \in \mathbb{Z}^2$ is the Maslov index and $S \in \mathbb{R}^2$ is the integral action of the fundamental cycles of Λ_a . Moreover $P(\xi; \varepsilon, h)$ admits the asymptotic expansion in (ξ, ε, h) of the form (3.33) and in particularity the h -leading term of P is of the form (3.34):

$$p_0(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2) \quad (3.38)$$

$$= \varphi_1(\xi) + i\varepsilon\varphi_2(\xi) + \mathcal{O}(\varepsilon^2). \quad (3.39)$$

Remark 3.9. The previous result shows that the eigenvalues of $P(\xi; \varepsilon, h)$ in a good rectangle form (micro-locally) a deformed lattice, it is image of a square lattice of $h\mathbb{Z}^2$ by a local diffeomorphism. Moreover we can show that the lattice has horizontal spacing $\mathcal{O}(h)$ and vertical spacing $\mathcal{O}(\varepsilon h)$.

In this paper, we use the vocabulary "micro" for some objects in the sense that they are linked with a domain depending of h .

In the case when the map F is not supposed to have connected fibre, then fiber $F^{-1}(a)$ can contain of L tori, with $L \geq 1$, and the eigenvalues of P_ε in the associated good rectangle are union of L deformed lattices.

As a property of classical system, it is true that the difference between the integral actions, with factor $\frac{1}{2\pi}$ and the action coordinates are locally constant on regular tori Λ_a (see Ref. [51]):

$$\frac{S}{2\pi} - \xi_a := \tau_c \in \mathbb{R}^2, \quad (3.40)$$

is locally constant in $c \in U_r$.

From Remark 3.8, we know that Equation (3.37) have a bijective correspondence between the eigenvalues in the good rectangle $R^{(E,G)}(\varepsilon, h)$ and hk in a part of $h\mathbb{Z}^2$. Moreover this correspondence is given by a smooth local diffeomorphism in $E \in \mathbb{C}$, denoted by $f = f(\mu; \varepsilon, h)$, which sends $R^{(E,G)}(\varepsilon, h)$ to its image, denoted by $E(a, \varepsilon, h)$ (which is close to $\frac{S}{2\pi}$), such that it send $\mu \in \sigma(P_\varepsilon) \cap R^{(E,G)}(\varepsilon, h)$ to $hk \in h\mathbb{Z}^2$ modulo $\mathcal{O}(h^\infty)$:

$$f = f(\mu, \varepsilon; h) = \tau_c + h\frac{\eta}{4} + P^{-1}(\mu) \quad (3.41)$$

$$\begin{aligned} f(R^{(E,G)}(\varepsilon, h)) &= E(a, \varepsilon, h) \\ \sigma(P_\varepsilon) \cap R^{(E,G)}(\varepsilon, h) \ni \mu &\mapsto f(\mu, \varepsilon; h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \end{aligned} \quad (3.42)$$

We say that the map f in (3.41) is a micro-chart of the spectrum of P_ε on the good rectangle $R^{(E,G)}(\varepsilon, h)$.

Let $\tilde{f} = f \circ \chi$, then we have

$$\tilde{f} = \tau_c + h\frac{\eta}{4} + P^{-1} \circ \chi. \quad (3.43)$$

To analysis $P^{-1} \circ \chi$, we first discuss about its inverse, $\widehat{P} := \chi^{-1} \circ P$. It is obtained from P by dividing the imaginary part of P by ε . As P admits an asymptotic expansion

in (ξ, ε, h) , so it is true that \widehat{P} admits an asymptotic expansion in $(\xi, \varepsilon, \frac{h}{\varepsilon})$ (here $h \ll \varepsilon$). Moreover we can write \widehat{P} in the reduce form as below:

$$\begin{aligned}\widehat{P}(\xi, \varepsilon, h) &= \widehat{P}_0(\xi) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{h}{\varepsilon}\right) \\ &= \widehat{P}_0(\xi) + \mathcal{O}\left(\varepsilon, \frac{h}{\varepsilon}\right),\end{aligned}\tag{3.44}$$

uniformly for h, ε small and $h \ll \varepsilon$, with $\widehat{P}_0(\xi) = \varphi_1(\xi) + i\varphi_2(\xi)$.

Proposition 3.10. (see [51]) *Let $\widehat{P} = \widehat{P}(\xi; X)$ a complex-valued smooth function of ξ near $0 \in \mathbb{R}^2$ and X near $0 \in \mathbb{R}^n$. Assume that \widehat{P} admits an asymptotic expansion in X near 0 of the form*

$$\widehat{P}(\xi; X) \sim \sum_{\alpha} C_{\alpha}(\xi) X^{\alpha}$$

with $C_{\alpha}(\xi)$ are smooth functions and $C_0(\xi) := \widehat{P}_0(\xi)$ is local diffeomorphism near $\xi = 0$.

Then, for $|X|$ small enough, \widehat{P} is also a smooth local diffeomorphism near $\xi = 0$ and its inverse admits an asymptotic expansion in X near 0 whose the first term is $(\widehat{P}_0)^{-1}$.

This proposition ensure that the map $\widehat{P}^{-1} = P^{-1} \circ \chi$ admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ whose first term is $(\widehat{P}_0)^{-1} = (\varphi)^{-1}$. So from (3.43), \widetilde{f} admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ with the leading term

$$\widetilde{f}_0 = \tau_c + (\varphi)^{-1}.\tag{3.45}$$

We note that the leading term \widetilde{f}_0 is a local diffeomorphism, completely defined on $B(c, r)$. It does n't depend of the chosen good rectangle, is valid for any good value $a \in B(c, r)$.

However, in the domain $B(c, r; \varepsilon)$, we have many good rectangles. They can be disjoint or no, and despite their density, not quite fill $B(c, r; \varepsilon)$. In fact we know moreover from Remark 3.5 that the set of good values is a Cantor set, and outside a set of small measure. We have therefore locally a Cantor family of micro-charts with common leading term of the spectrum of P_{ε} on the domain $B(c, r; \varepsilon)$, called spectral pseudo-chart.

The above construction ensures that the spectrum of P_{ε} on the domain $U(\varepsilon)$ satisfies all conditions of a particular discrete lattice - an asymptotic pseudo-lattice, given in [51], that we recall here:

Definition 3.11. *Let U be a subset of \mathbb{R}^2 with compact closure, $U(\varepsilon) = \chi(U)$, and let $\Sigma(\varepsilon, h)$ (which depends on small h and ε) be a discrete set of $U(\varepsilon)$. For h, ε small enough and in the regime $h \ll \varepsilon$, we say that $(\Sigma(\varepsilon, h), U(\varepsilon))$ is an asymptotic pseudo-lattice if: for any small parameter $\alpha > 0$, there exists a set of "good values" in \mathbb{R}^2 , denoted by $\mathcal{G}(\alpha)$, whose complement is of small measure together with α in the sense:*

$$|\mathcal{G}(\alpha) \cap I| \leq C\alpha |I|,$$

for any domain $I \subset \mathbb{R}^2$ with a constant $C > 0$.

For all $c \in U$, there exists a ball $B(c, r) \subset U$ around c ($r > 0$) such that for every "good value" $a \in B(c, r) \cap \mathcal{G}(\alpha)$, there is a good rectangle $R^{(a)}(\varepsilon, h) \subset \chi(B(c, r))$ of good center $\chi(a)$ of size $\mathcal{O}(h^\delta) \times \varepsilon \mathcal{O}(h^\delta)$, $0 < \delta < 1$ and a smooth local diffeomorphism $f = f(\cdot; \varepsilon, h)$ which sends $R^{(a)}(\varepsilon, h)$ on its image, denoted by $E(a, \varepsilon, h)$ satisfying

$$\begin{aligned} f : R^{(a)}(\varepsilon, h) &\rightarrow E(a, \varepsilon, h) \\ \Sigma(\varepsilon, h) \cap R^{(a)}(\varepsilon, h) \ni \mu &\mapsto f(\mu; \varepsilon, h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty) \end{aligned} \quad (3.46)$$

such that $\tilde{f} := f \circ \chi$ admits asymptotic expansions in $(\varepsilon, \frac{h}{\varepsilon})$ for the C^∞ topology in a neighborhood of a , uniformly with respect to the parameters h, ε , with the leading term \tilde{f}_0 is a diffeomorphism, independent of α , globally defined on $B(c, r)$ and independent of the selected good value $a \in B(c, r)$.

We also say that the couple $(f(\cdot; \varepsilon, h), R^{(a)}(\varepsilon, h))$ is a micro-chart, and the family of micro-charts $(f(\cdot; \varepsilon, h), R^{(a)}(\varepsilon, h))$, with $a \in B(c, r) \cap \mathcal{G}(\alpha)$, is a local pseudo-chart on $B(c, r, \varepsilon)$ of $(\Sigma(\varepsilon, h), U(\varepsilon))$.

Remark 3.12. There exists an other lattice with some similar properties but lighter, called asymptotic lattice, given in [45]. The last lattice is locally defined, while the asymptotic pseudo-lattice is very fine, it is micro-locally defined.

The introduction of this discrete lattice aims to show that a invariant-monodromy that we will define is directly built from the spectrum of a operator. If the different operators have the same spectrum, then they have the same monodromy.

Now we can define the spectral monodromy of the operator P_ε on the domain $U(\varepsilon)$, denoted by $[\mathcal{M}_{sp}]$, as the monodromy of the asymptotic pseudo-lattice $(\sigma(P_\varepsilon), U(\varepsilon))$, as the following.

Let $\{B_\alpha\}_\alpha$ be an arbitrary (small enough) locally finite covering of U . Then as in Definition 3.11, the asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$ is covered by the associated local pseudo-charts $\{((f_\alpha(\cdot; \varepsilon, h)), B_\alpha(\varepsilon))\}_\alpha$, here $B_\alpha(\varepsilon) := \chi(B_\alpha)$. Note that the leading terms $\tilde{f}_{\alpha,0}(\cdot; \varepsilon, h)$ are well defined on whole B_α and we can see them as the charts of U . It know from [51] that on each nonempty intersection $B_\alpha \cap B_\beta \neq \emptyset$, there exists a unique integer linear map $M_{\alpha\beta} \in GL(2, \mathbb{Z})$ (independent of h, ε) such that:

$$d(\tilde{f}_{\alpha,0} \circ (\tilde{f}_{\beta,0})^{-1}) = M_{\alpha\beta}. \quad (3.47)$$

The class, denoted by $[\mathcal{M}] \in \check{H}^1(U(\varepsilon), GL(2, \mathbb{Z}))$, defined from the 1-cocycle of $\{M_{\alpha\beta}\}$, in the Čech cohomology with value in the linear group $GL(2, \mathbb{Z})$, is called the (linear) monodromy of the asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$. It does n't depend on the selection of the covering $\{B_\alpha\}_\alpha$.

We can also associate the class $[\mathcal{M}]$ with its holonomy, a group morphism from the fundamental of $U(\varepsilon)$ into the group $GL(2, \mathbb{Z})$, modulo conjugation. Their trivial property is the same.

Definition 3.13. For $\varepsilon, h > 0$ small such that $h \ll \varepsilon \leq h^\delta, 0 < \delta < 1$, the spectral monodromy of the operator P_ε , discussed in, on the domain $U(\varepsilon)$, is the class $[\mathcal{M}_{sp}] \in \check{H}^1(U(\varepsilon), GL(2, \mathbb{Z}))$.

3.4 Relationship with the classical monodromy

Theorem 3.14 (Angle-action theorem). Let $F = (f_1, \dots, f_n) : W \rightarrow \mathbb{R}^n$ be completely integrable system on a connected smooth symplectic manifold (W, σ) in the sense that their component functions are in involution with respect to the Poisson bracket generated from σ .

Let U be a subset of the set regular values of F and let Λ_c be a compact regular sheet of F with $F|_{\Lambda_c} = c \in U$. Then there exists a neighborhood V^c of Λ_c in M such that $F|_{V^c}$ defines a locally trivial fibres bundle on a neighborhood $B^c \subset U$ of c , whose fibres are Lagrangian n -tori. Moreover, there exists a symplectic diffeomorphism

$$\kappa^c = (x, \xi) : V^c \rightarrow \mathbb{T}^n \times A^c,$$

with $A^c \subset \mathbb{R}^n$ is an open set, such that $F \circ (\kappa^c)^{-1}(x, \xi) = \varphi^c(\xi)$ for all $x = (x_1, \dots, x_n) \in \mathbb{T}^n$, all $\xi = (\xi_1, \dots, \xi_n) \in A^c$, and here $\varphi^c : A^c \rightarrow \varphi(A^c) = B^c$ is locally a diffeomorphism. We call (x, ξ) local angle-action variables near Λ_c and (V^c, κ^c) an angle-action chart on V^c .

Note that we can choose the local chart such that the torus Λ_c is sent by κ^c to the zero section $T^n \times \{0\}$. By this theorem, if $a \in B^c$, then $\Lambda_a := F^{-1}(a) \cap V^c$ is Lagrangian n -torus, and its image by κ^c is a n -torus $T^n \times \{\xi_a\}$, with $\xi_a \in A^c$. We often write

$$\Lambda_a \simeq T^n \times \{\xi_a\}.$$

4 The quasi-integrable case

In this section we will study the spectral monodromy of perturbed non-selfadjoint classical operator in a more general case when the unperturbed symbol is close to a completely integrable non-degenerate one. In this case, according to spectral asymptotics [27], the similar result about asymptotic spectrum of such operator is already known. We apply it to show that the spectrum should be an asymptotic pseudo-lattice on spectral band.

To get asymptotic expansions of eigenvalues from a h -dependent complex window, a radical general hypothesis, given in [27], is that the Hamiltonian flow of the unperturbed leading symbol, on the real energy surface at certain level, possesses several Diophantine invariant Lagrangian tori. However, this hypothesis is ensured when this symbol is not only completely integrable system, as in Sec. but also nearly integrable one, by classical Kolmogorov-Arnold-Moser (KAM) theory.

4.1 KAM theorem

The classical KAM theory allows to treat perturbations of a completely integrable Hamiltonian system. Under the isoenergetic condition, this theory proves the persistence of invariant Lagrangian tori, called KAM tori, on which the flow of perturbed system stills quasi-periodic with Diophantine frequencies. Moreover, the union of these KAM tori is a nowhere dense set, with complement of small measure in phase space.

We consider a perturbed Hamiltonian that is close to a completely integrable non-degenerate one:

$$p_\lambda = p + \lambda p_1, \quad 0 < \lambda \ll 1, \quad (4.48)$$

where p and p_1 are holomorphic bounded Hamiltonian in a tubular neighborhood of T^*M , real on T^*M and furthermore p is assumed to be completely integrable, as in Sec.

Let Λ_a be a H_p -invariant Diophantine Lagrangian torus in the energy space $p^{-1}(E)$. In action-angle coordinates (x, ξ) on a neighborhood Ω of Λ_a , given as in (3.22), the function p_λ becomes:

$$p_\lambda \circ \kappa^{-1} = p_\lambda(x, \xi) = p(\xi) + \lambda p_1(x, \xi).$$

The Hamiltonian flow of p on a H_p -invariant Lagrangian torus $\Lambda \subset \Omega$, close to Λ_a , $\Lambda \simeq \mathbb{T}^2 \times \{\xi\}$, denoted by Λ_ξ , is quasi-periodic of constant Hamiltonian vector field

$$H_p(x, \xi) = \omega_1(\xi) \frac{\partial}{\partial x_1} + \omega_2(\xi) \frac{\partial}{\partial x_2}, \quad x \in \mathbb{T}^2, \quad \xi \in A, \quad (4.49)$$

with the frequency $\omega(\xi)$, given by (3.24).

In particular, the frequency of $\Lambda_a \simeq \Lambda_{\xi_a}$ is $\omega(\xi_a) = \frac{\partial p}{\partial \xi}(\xi_a)$ satisfying the Diophantine condition in (3.27), for some $\alpha > 0$.

We say that p is (Kolmogorov) non-degenerate if the isoenergetic condition holds: the local frequency map $\omega : A \rightarrow \mathbb{R}^2$, defined by (3.24), is a diffeomorphism onto its image. In fact, this condition is equivalent to:

$$\det \frac{\partial \omega}{\partial \xi}(\xi_a) = \det \left(\frac{\partial^2 p}{\partial \xi^2} \right)(\xi_a) \neq 0.$$

It means that the H_p -invariant Lagrangian tori near Λ_a can (locally) be parametrized by their frequencies.

Let $A_{\alpha,d} = \omega^{-1}(\omega(A) \cap D_{\alpha,d})$, where $D_{\alpha,d}$ is defined in (3.27). The intersection of $p^{-1}(E)$ with $\mathbb{T}^2 \times A$ is of the form $\mathbb{T}^2 \times \Gamma_a$, with a certain curve Γ_a in A passing through ξ_a . Note that when we take α to be sufficiently small, the set $A_{\alpha,d}$ is nowhere dense of positive measure. Moreover, the measure of $A \setminus A_{\alpha,d}$ is $\mathcal{O}(\alpha)$.

Now for α small enough, we have the quasi-periodic stability of Diophantine invariant Lagrangian tori in $\mathbb{T}^2 \times A_{\alpha,d}$, as then following theorem.

This theorem is combined form the different known versions of classical KAM theorem as citeuyen 3, 17, 5, , 8, 33. In particular the paper cite 33 proved the smooth dependence, in the sense of Whitney, of the KAM tori on the frequencies.

Theorem 4.1. *Assume that p is non-degenerate. Let $d > 0$ fixed and $\alpha > 0$ be small enough. Assume that $0 < \lambda \ll \alpha^2$. Then there exists a map $\Phi_\lambda : \mathbb{T}^2 \times A \rightarrow \mathbb{T}^2 \times A$ with the following properties:*

1. Φ_λ , depending analytically on λ , is a C^∞ - diffeomorphism onto its image, close to the identity map in the C^∞ - topology.
2. For each $\xi \in A$, the Lagrangian invariant torus $\Lambda_\xi := \mathbb{T}^2 \times \{\xi\}$ is sent, by Φ_λ , to a Lagrangian invariant torus $\Phi_\lambda(\Lambda_\xi)$, close to Λ_ξ , denoted by Λ_{ξ_λ} , and of the form $\Lambda_{\xi_\lambda} = \mathbb{T}^2 \times \{\xi_\lambda\}$, with ξ_λ in a certain open set $A_\lambda \subset A$, induced by Φ_λ .

Moreover, if $\xi \in A_{\alpha,d}$, then Λ_{ξ_λ} , with ξ_λ in a certain set $A_{\alpha,d,\lambda} \subset A_\lambda$, are still uniformly Diophantine tori, called KAM tori. The restricted map on each Diophantine Lagrangian torus Λ_ξ , with $\xi \in A_{\alpha,d}$, $\Phi_\lambda|_{\Lambda_\xi}$ conjugates the Hamiltonian vector field $H_p|_{\Lambda_\xi} = H_p(x, \xi)$, as in (4.49), to the Hamiltonian vector field $H_{p_\lambda}|_{\Lambda_{\xi_\lambda}}$, i.e. $\Phi_\lambda|_{\Lambda_\xi} * H_p = H_{p_\lambda}$.

In particular, if $\xi \in \Gamma_a$, then the torus $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E) \cap \mathbb{T}^2 \times \Gamma_{a,\lambda}$, with a certain curve $\Gamma_{a,\lambda} \subset A_\lambda$. Moreover, when $\xi \in \Gamma_a \cap A_{\alpha,d}$, the Liouville measure of the complement of the union of the KAM tori Λ_{ξ_λ} , in $p_\lambda^{-1}(E)$, is small, $\mathcal{O}(\alpha)$.

Note that in the previous theorem, we can see ξ_λ as a smooth function of $\xi \in A$. From this theorem, we obtain a family of KAM invariant tori, $A_{\alpha,d,\lambda}$, of positive measure that is furthermore Cantor type, hence nowhere dense set, on which the Hamiltonian flow of the perturbed system p_λ is quasi-periodic of constant vector field.

We would to cite a very interesting paper Uni-kam, that attested the unicity of KAM tori on a subset of full measure of Diophantine tori. In the context of the above theorem, the latter paper asserts that conjugacy map Φ_λ , after restriction to $\mathbb{T}^2 \times A_{\alpha,d}^*$, is unique up to a torus translation. Here $A_{\alpha,d}^*$ is the subset of density points of $A_{\alpha,d}$, that has similar properties as $A_{\alpha,d}$.

4.2 Monodromy

In this section, we use also again all notations of Section 3.3.

Let p_1 be an analytic function in a tubular neighborhood of T^*M , real on the real domain, with $p_1(x, \xi) = \mathcal{O}(m(\text{Re}(x, \xi)))$ in the case when $M = \mathbb{R}^2$, and $p_1(x, \xi) = \mathcal{O}(\langle \xi \rangle^m)$ in the manifold case.

Let

$$P_{\varepsilon,\lambda}, \quad 0 < \lambda \ll 1, \quad (4.50)$$

be a classical operator that is perturbation of the self-adjoint operator $P_\lambda := P_{\varepsilon=0,\lambda}$, and with the h -leading symbol

$$p_{\varepsilon,\lambda} = p_\lambda + i\varepsilon q + \mathcal{O}(\varepsilon^2), \quad p_\lambda = p + \lambda p_1, \quad (4.51)$$

where p, q are symbols satisfying all assumptions of Sec. 3.3. Moreover, we assume that p is global non-degenerate, as following.

Let $X = F^{-1}(U)$, where the map F and the set U are given as in Sec. 3.3. Then the map $F|_X : X \rightarrow U$ defines a smooth locally trivial bundle, whose fibres are Lagrangian tori. From the angle-action theorem, we can cover X by an atlas of angle-action charts $\{(V^c, \kappa^c)\}_{c \in U}$, where $\kappa^c = (x^c, \xi^c) : V^c = F^{-1}(U^c) \rightarrow \mathbb{T}^2 \times A^c$, as in Theorem. As in 3.24, we define the local frequency map $\omega^c : A^c \rightarrow \mathbb{R}^2$, $\omega^c(\xi) = \frac{\partial(p \circ \kappa^{-1})}{\partial \xi^c}(\xi)$. We say that p is global non-degenerate (with respect to $F|_X$) if each local frequency map is a diffeomorphism onto its image, for every atlas of angle-action charts $\{V^c, \kappa^c\}_{c \in U}$ of X .

We will introduce a set, associated with the energy level E of p_λ , which is similar to the set of good values, given in the integrable case.

Let $a = (E, G)$ be a point in U such that G is a good value in $\mathcal{G}(\alpha, d, E)$, as in Definition 3.4, and $\Lambda_a \simeq \Lambda_{\xi_a}$, $\xi_a \in \Gamma_a$, be the corresponding (α, d) -Diophantine torus.

We assume moreover $0 < \lambda \ll \alpha^2$. Then by Theorem 4.1, there exists a smooth Cantor family of KAM tori, $\Lambda_{\xi_\lambda} = \mathbb{T}^2 \times \{\xi_\lambda\}$, close to Λ_{ξ_a} , with $\xi_\lambda \in A_{\alpha, d, \lambda}$, on which the H_{p_λ} -flow is quasi periodic of a constant uniformly Diophantine frequency, denoted by $\omega(\xi_\lambda)$. Therefore, over these KAM tori, p_λ become a function of only ξ_λ :

$$p_\lambda = p_\lambda(\xi_\lambda), \xi_\lambda \in A_{\alpha, d, \lambda}.$$

In particular, when $\xi \in \Gamma_a \cap A_{\alpha, d}$, then $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E) \cap \mathbb{T}^2 \times (\Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda})$.

Similarly as in (3.25), we define locally a smooth function $\langle q \rangle_{\Lambda_{\xi_\lambda}}$ of $\xi_\lambda \in A_\lambda$, denoted by $\langle q \rangle(\xi_\lambda)$, obtained by averaging q over the tori Λ_{ξ_λ} .

Then, we have, in C^1 -sense in $\xi_\lambda \in A_\lambda$, as $\lambda \rightarrow 0$,

$$\langle q \rangle(\xi_\lambda) = \langle q \rangle_{\Lambda_{\xi_\lambda}} \rightarrow \langle q \rangle_{\Lambda_{\xi_\lambda | \lambda=0}} = \langle q \rangle_{\Lambda_\xi}.$$

Therefore, for every $\xi_\lambda \in \Gamma_{a, \lambda}$ and λ small enough, we get

$$|d\langle q \rangle(\xi_\lambda)| \geq \frac{\alpha}{2}. \quad (4.52)$$

Note that we have also the differentials of $p_\lambda(\xi_\lambda)$ and $\langle q \rangle(\xi_\lambda)$ in every $\xi_\lambda \in \Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda}$ are \mathbb{R} -linearly independent:

$$\omega(\xi_\lambda) \wedge d\langle q \rangle(\xi_\lambda) \neq 0. \quad (4.53)$$

Let us define the set of good values

$$\mathcal{G}_\lambda(\alpha, E, G) = \bigcup_{\xi_\lambda \in \Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda}} \langle q \rangle(\xi_\lambda) = \bigcup \{ \langle q \rangle(\xi_\lambda) : \xi \in \Gamma_a \cap A_{\alpha, d} \}. \quad (4.54)$$

Then the measure of the complement of $\mathcal{G}_\lambda(\alpha, E, G)$ in $\bigcup_{\xi_\lambda \in \Gamma_{a, \lambda}} \langle q \rangle(\xi_\lambda)$ is small, is equal to $\mathcal{O}(\alpha)$, when α is small and d is kept fixed.

If $K \in \mathcal{G}_\lambda(\alpha, E, G)$, then there exists $\xi \in \Gamma_a \cap A_{\alpha, d}$ such that the corresponding KAM torus $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E)$ is still uniformly Diophantine Lagrangian torus, $\langle q \rangle(\xi_\lambda) = \langle q \rangle_{\Lambda_{\xi_\lambda}} =$

K . And moreover, the H_{p_λ} -flow on Λ_{ξ_λ} is quasi-periodic of the Diophantine constant frequency $\omega(\xi_\lambda)$, satisfying 4.52 and 4.53. Therefore, all these basis assumptions on the dynamic of H_{p_λ} allow us to carry out, microlocally near Λ_{ξ_λ} , a construction of the Birkhoff normal form for $P_{\varepsilon,\lambda}$. The principle of this normalization is use excessively canonical transformations near the torus Λ_{ξ_λ} , to conjugate $P_{\varepsilon,\lambda}$, by means of analytic Fourier integral operators, to a new operator, whose total symbol of the new operator is independent of angle variables x , and homogeneous polynomial to high order in (h, ξ, ε) . The Diophantine condition of Λ_{ξ_λ} is indispensable for this construction.

So as result of Ref. [27], one obtains asymptotic spectral results for the operator $P_{\varepsilon,\lambda}$, that are similar to those of the operator P_ε , in the integrable case. More explicitly:

Theorem 4.2 ([27]). *Let $P_{\varepsilon,\lambda}$ be the operator in (4.50), with leading term $p_{\varepsilon,\lambda}$ (4.51), satisfying all assumptions given in this section. We work in the regime $h \ll \varepsilon = \mathcal{O}(h^\delta)$, for $0 < \delta < 1$, and $0 < \lambda$ small enough. Let $a = (E, G)$ be any point in U such that $G \in \mathcal{G}(\alpha, d)$, with $0 < \alpha$ small enough, and $d > 0$ fixed, as in Definition 3.4. We assume that $\lambda \ll \alpha^2$, and define the set $\mathcal{G}_\lambda(\alpha, E, G)$ as in (4.54).*

*For each $K \in \mathcal{G}_\lambda(\alpha, E, G)$, there exists a KAM torus $\Lambda_{\xi_\lambda} = p_\lambda^{-1}(E) \cap \langle q \rangle^{-1}(G)$, as already discussed above, and canonical coordinates $\kappa = (x, \xi) : \text{neigh}(\Lambda_{\xi_\lambda}, T^*M) \rightarrow \text{neigh}(\xi = \xi_\lambda, T^*\mathbb{T}^2)$ mapping Λ_{ξ_λ} to $\mathbb{T}^2 \times \{\xi_\lambda\}$, with $\xi_\lambda \in \text{neigh}(0, \mathbb{R}^2)$, such that*

$$p_\lambda \circ \kappa^{-1} = p_\lambda(x, \xi) = p_{\lambda,\infty}(\xi) + \mathcal{O}(\xi - \xi_\lambda)^\infty,$$

and $p_{\lambda,\infty}(\xi) = E + \omega(\xi_\lambda) \cdot (\xi - \xi_\lambda) + \mathcal{O}(\xi - \xi_\lambda)^2$.

Then all the eigenvalues μ of $P_{\varepsilon,\lambda}$ in the rectangle $R^{(E,K)}(\varepsilon, h)$, defined by 3.29 with K instead of G , are given by the asymptotic expansion of a smooth function $P_\lambda(\xi, \varepsilon; h)$, of ξ in a neighborhood of $(\xi_\lambda, \mathbb{R}^2)$ and ε, h in neighborhoods of $(0, \mathbb{R})$, with modulo $\mathcal{O}(h^\infty)$:

$$\sigma(P_{\varepsilon,\lambda}) \cap R^{(E,K)}(\varepsilon, h) \ni \mu = P_\lambda\left(\xi_\lambda + h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbb{Z}^2, \quad (4.55)$$

Here $\eta \in \mathbb{Z}^2$ is the Maslov index and $S \in \mathbb{R}^2$ is the integral action of the fundamental cycles of Λ_{ξ_λ} . Moreover P_λ admits the asymptotic expansion in (ξ, ε, h) of the form (3.33) and particularity the h -leading term of P_λ is of the form:

$$p_{0,\lambda}(\xi, \varepsilon) = p_\lambda(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2), \quad (4.56)$$

where $p_\lambda(\xi)$ and $\langle q \rangle(\xi)$ are expression of p_λ and $\langle q \rangle$ in the previous angle-action coordinates near Λ_{ξ_λ} .

We can see that the formulas (4.55) and (4.56) are exactly the same as (3.37) and (3.38). So from the construction in Sec. 3.3 and the Definition 3.11, we can state that the spectrum of $P_{\varepsilon,\lambda}$ on the domain $U(\varepsilon)$, $(\sigma(P_{\varepsilon,\lambda}), U(\varepsilon))$ satisfies all hypothesis of an asymptotic pseudo-lattice once again. Therefore we can define the monodromy of the operator $P_{\varepsilon,\lambda}$ due to the monodromy of the asymptotic pseudo-lattice $(\sigma(P_{\varepsilon,\lambda}), U(\varepsilon))$, discussed in Sec. 3.3 (see also [51] for more detail).

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