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Nonsmooth optimization algorithm for mixed $H_2/H_\infty$ synthesis

Pierre Apkarian, Dominikus Noll, Aude Rondepierre

Abstract—The mixed $H_2/H_\infty$ synthesis problem is addressed via nonsmooth mathematical programming. The proposed algorithm is of first order and can handle any controller structure of practical interest. Since computations are carried out in the frequency domain, the method does not suffer dimensional restrictions like LMI or BMI methods. Global convergence is established and several numerical tests are presented.

Index Terms—Mixed $H_2/H_\infty$ synthesis, multi-objective control, structured controllers design, nonsmooth optimization.

I. INTRODUCTION

Mixed $H_2/H_\infty$ output feedback control is a multi-objective design problem, where the feedback controller has to respond favorably to two concurrent performance specifications. Typically in $H_2/H_\infty$ synthesis, the $H_\infty$-channel is used to enhance the robustness of the design, whereas the $H_2$-channel guarantees the performance of the system.

Due to its importance in practice, mixed $H_2/H_\infty$ control has been addressed in various ways. First approaches are based on coupled Riccati equations in tandem with homotopy methods, but the numerical success of these strategies remains to be established. With the rise of LMIs in the later 1990s, different strategies which convexify the problem became increasingly popular. The price to pay for convexifying the problem is either a considerable conservatism, or that controllers have large state dimension [11], [10].

In [15], [16], [17], Scherer develops characterizations for the $H_2/H_\infty$ synthesis problem with full-order or Youla parameterized controllers. The problem is reduced to LMIs involving Lyapunov and controller matrix variables together with multipliers. The drawback of this approach is the presence of Lyapunov variables, which grow quadratically in the system size. The consequence is that current BMI and LMI solvers quickly succumb when plants get sizable.

Following [2], [3], [5], [4], we address $H_2/H_\infty$ synthesis by a new strategy which avoids Lyapunov variables. This leads to a nonsmooth and semi-infinite optimization program.

The paper is organized as follows. The $H_2/H_\infty$ synthesis problem is introduced in section II. In sections III and IV we successively present our method and a nonsmooth algorithm for solving the $H_2/H_\infty$ problem. After detailing some technical elements in section V, we discuss numerical examples to validate our algorithm in the last section.

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II. PROBLEM SETTINGS

We consider a plant in state space form

$$
P: \begin{bmatrix} \dot{x} \\ z_{\infty} \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{1m} & D_{12} \\ C_2 & 0 & D_{20} \end{bmatrix} \begin{bmatrix} x \\ w_{\infty} \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} u 
$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^n$ the control, $y \in \mathbb{R}^{n_y}$ is the measured output, $w_{\infty} \rightarrow z_{\infty}$ is the $H_\infty$ channel, $w_2 \rightarrow z_2$ the $H_2$ channel. We seek an output feedback controller

$$
K: \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix}
$$

with state $x_K \in \mathbb{R}^{nk}$ such that the closed-loop system (1)-(2) satisfies the following properties:

1) Internal stability. $K$ stabilizes $P$ exponentially in closed-loop.

2) Fixed $H_\infty$ performance. The $H_\infty$ channel has a pre-specified performance level $\|T_{w_{\infty} \rightarrow z_{\infty}}(K)\|_{H_\infty} \leq \gamma$.

3) Optimal $H_2$ performance. The $H_2$ performance $\|T_{w_2 \rightarrow z_2}(K)\|_2$ is minimized among all $K$ satisfying 1. and 2.

We will solve the $H_2/H_\infty$ synthesis problem by way of the following mathematical program

$$
\begin{align*}
\min_{K} & \quad f(K) := \|T_{w_2 \rightarrow z_2}(K)\|_2^2 \\
\text{subject to} & \quad g(K) := \|T_{w_{\infty} \rightarrow z_{\infty}}(K)\|_{H_\infty}^2 \leq \gamma^2
\end{align*}
$$

where $T_{w_2 \rightarrow z_2}(K,s)$ denotes the transfer function of the $H_2$ closed-loop performance channel, while $T_{w_{\infty} \rightarrow z_{\infty}}(K,s)$ stands for the $H_\infty$ robustness channel.

Notice that $f(K)$ is a smooth function, whereas $g(K)$ is not, being an infinite maximum of maximum eigenvalue functions. The unknown $K$ is in the space $\mathbb{R}^{(nK+n_u)(nK+n_y)}$, so the dimension $n = (nK+n_u)(nK+n_y)$ of (3) is usually small, which is particularly attractive when small or medium size controllers for large systems are sought.

For brevity, we set $T_2 := T_{w_2 \rightarrow z_2}$ and $T_\infty := T_{w_{\infty} \rightarrow z_{\infty}}$ in (1). The performance measures $H_2$ and $H_\infty$ are defined as:

$$
\begin{align*}
f(K) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr} \left[ T_2(K,j\omega)H_2 T_2(K,j\omega)^H \right] d\omega \\
g(K) &= \max_{\omega \in [0,\infty]} g(K,\omega) = \max_{\omega \in [0,\infty]} \lambda_1(T_\infty(K,j\omega)HT_\infty(K,j\omega))
\end{align*}
$$

where the transfer matrices $T_2$ and $T_\infty$ are stable and $T_2$ has to be strictly proper to ensure finiteness of the $H_2$ norm. For later use we define the set $\mathbb{B}_m = \{Y \in \mathbb{S}_m : Y \succeq 0, \text{Tr}(Y) = 1\}$ and the spectraplex $\mathbb{P}_m = \{Y_1, \ldots, Y_q : Y_i \in \mathbb{S}_m, Y_i \succeq 0, \sum_{i=1}^q \text{Tr}(Y_i) = 1\}$ where $\mathbb{S}_m$ denotes the space of $m \times m$ Hermitian matrices.
III. NONSMOOTH OPTIMIZATION METHOD

In this section, we present our main result, a nonsmooth optimization method for the mixed program (3).

A. Local model and optimality conditions

Following an idea in [14], we address the mixed program (3) by introducing the progress function: for \( (K, \tilde{K}) \in (\mathbb{R}^n)^2 \),

\[
F(\tilde{K}; K) = \max \left\{ f(\tilde{K}) - f(K) - \mu [g(K) - \gamma^2]_+ ; \ g(\tilde{K}) - \gamma^2 - [g(K) - \gamma^2]_+ \right\}
\]

where \( \mu > 0 \) is fixed. Its relation with (3) is given by

\[
\tilde{F}(K; \Omega) = \min \left\{ \min_{y \in \Omega} \min_{\omega \in \Omega(K)} \left\{ \begin{array}{l}
\tau_0 \mu [g(K) - \gamma^2]_+ \\
+ \sum_{\omega \in \Omega(K)} [g(K) - \gamma^2]_+ - [g(K, \omega) - \gamma^2]
\end{array} \right\} + \frac{1}{2\delta} \| \tau_0 f(K) + \sum_{\omega \in \Omega(K)} \tau_\omega \Phi_{\omega} \|^2 \right\}
\]

where \( \tau_0, (\tau_\omega)_{\omega \in \Omega(K)}, (Y_\omega)_{\omega \in \Omega(K)} \) are solution to (4).

The solution \( H(K) \) attaining \( \theta_e(K) = \tilde{F}(K + H(K); K) + \frac{1}{2\delta} \| H(K) \|^2 \) is given by:

\[
H(K) = -\frac{1}{\delta} \left[ \tau_0 f(K) + \sum_{\omega \in \Omega(K)} \tau_\omega \Phi_{\omega} \right]
\]

B. Optimality function and tangent program

We introduce the set \( \Omega(K) = \{ \omega \in [0, \infty] : g(K) = g(K, \omega) \} \) of active frequencies, or peaks. It can be shown [8] that \( \Omega(K) \) is either finite, or coincides with [0, \infty]. Since the latter never occurs in practice, we consider the finite case from now on. Consider a finite extension \( \Omega_e(K) \) of \( \Omega(K) \), which is built in such a way that it depends continuously on K (see [3] for more details). Procedures based on thresholding and discretization as in [3] guarantee this property. Using \( \Omega_e(K) \), we build a first order estimation of the progress function \( F \):

\[
\tilde{F}(K + H; K) = \max \left\{ f'(K)H - \mu [g(K) - \gamma^2]_+ ; \ \max_{\omega \in \Omega_e(K)} \left( g(K, \omega) - \gamma^2 - [g(K) - \gamma^2]_+ + \langle \Phi_{\omega}, H \rangle \right) \right\}
\]

where \( \Phi_{\omega} \) stands for the subgradients of \( g(K, \omega) \) as obtained in [3]. We observe that \( \partial F(K; \tilde{K}) = \partial F(F; K) \). Now for some fixed \( \delta > 0 \), we introduce the optimality function:

\[
\theta_e(K) = \min_{H \in \mathbb{R}^n} \tilde{F}(K + H; K) + \frac{1}{2} \delta \| H \|^2.
\]

The concept of optimality functions was introduced by E. Polak [14] for finite and infinite families of smooth functions. Its interest stems from the fact that for any stabilizing \( K, \theta_e(K) \leq 0 \), and that \( \theta_e(K) = 0 \) implies that \( K \) satisfies \( 0 \in \partial H(K) \). As we know from Lemma 2, in all cases of practical interest, this implies that \( K \) is a critical point of (3).

Proposition 1 (Dual form of \( \theta_e \)):

\[
\theta_e(K) = \min_{\tau_0, \tau_{\omega} \geq 0} \min_{Y_\omega \geq 0, \tau(Y_\omega) = 1} \left\{ \begin{array}{l}
\tau_0 \mu [g(K) - \gamma^2]_+ \\
+ \sum_{\omega \in \Omega(K)} [g(K) - \gamma^2]_+ - [g(K, \omega) - \gamma^2]
\end{array} \right\} + \frac{1}{2\delta} \| \tau_0 f(K) + \sum_{\omega \in \Omega(K)} \tau_\omega \Phi_{\omega} \|^2
\]

Proposition 2: For all stabilizing \( K \in \mathbb{R}^{(nk+mn_0) \times (nk+mn_0)} \),

i. \( \theta_e(K) \leq 0 \).

ii. \( d_1 F(K; H; K) \leq \theta_e(K) - \frac{1}{2} \delta \| H(K) \|^2 \leq \theta_e(K) \).
where
\[
\min_{H \in \mathbb{R}^n} \widetilde{F}(K_j + H; K_j) + \frac{\delta}{2} \|H\|^2.
\]
whose solution \(H(K_j)\) is (5) and is a qualified descent direction for \(F(\cdot; K_j)\) at \(K_j\). Performing a backtracking line search, we compute a step \(s\) such that \(K_j + sH(K_j)\) remains stabilizing and satisfies:
\[
F(K_j + sH(K_j); K_j) \leq s\alpha \theta(K_j)
\]
where \(\alpha \in (0, 1)\) is the minimum fraction required of the directional derivative along \(H_j\) at \(K_j\). The algorithm stops as soon as the optimality condition \(0 \in \partial F(K_j; K_j)\) is satisfied.

Algorithm 1 Nonsmooth algorithm for \(H_2/H_{\infty}\) synthesis

Require: \(\gamma\) the performance level, \(n_K\) the controller order, \(\mu > 0\), \(\delta > 0\) and \(\alpha \in (0, 0.25]\).

1: **Initialization.** Find initial closed loop stabilizing controller \(K_0\). Put main loop counter to \(j = 0\).
2: while \(K_j\) does not satisfy the optimality condition do
3: **Frequency generation.** Construct finite extension \(\Omega_j(K_j)\) of the set of active frequencies \(\Omega(K_j)\) at \(K_j\).
4: **Tangent program.** Solve tangent program:
\[
\min_{H \in \mathbb{R}^n} \widetilde{F}(K_j + H; K_j) + \frac{\delta}{2} \|H\|^2.
\]
Solution is \(H_j = H(K_j)\). Compute \(\theta_j = \theta(K_j)\).
5: **Line search.** Backtrack to compute a step \(s\) such that:
\[
F(K_j + sH_j; K_j) \leq s\alpha \theta_j
\]
and \(K_j + sH_j\) remains stabilizing.
6: **Update.** \(K_{j+1} := K_j + sH_j; j := j + 1\).
7: end while

We now prove global convergence of algorithm 1 in the sense that every accumulation point of a sequence of iterates generated by the algorithm is a critical point of the mixed \(H_2/H_{\infty}\) program. Consider:
\begin{align*}
\langle H_1 \rangle & \quad \text{The set } \{K \in \mathbb{R}^n : g(K) \leq g(K_0)\} \text{ is bounded.} \\
\langle H_2 \rangle & \quad f \text{ is weakly coercive on the level set } \{K \in \mathbb{R}^n : g(K) \leq \gamma_e^2\} \text{ in the following sense: if } K_j \text{ is a sequence of feasible iterates with } \limsup_{j \to \infty} \|K_j\| = \infty, \text{ then } f(K_j) \text{ is not monotonically decreasing.}
\end{align*}

Under these assumptions, any sequence of steps generated by our algorithm is bounded (see [6] for details) and we are now ready to show the convergence of our algorithm:

**Theorem 2:** Assume \(\langle H_1 \rangle, \langle H_2 \rangle\) at \(K_0\), and let \(K_j\) the sequence generated by algorithm 1. Then every accumulation point \(\hat{K}\) of \(K_j\) is either a F. John critical point of the mixed \(H_2/H_{\infty}\) problem, or a critical point of the constraint violation.

**Proof:** We have to show that \(0 \in \partial F(\hat{K}; \hat{K})\). There are two cases to be discussed. Either \(K_j\) are feasible from some index onwards, or \(K_j\) remain unfeasible all the time. Let us discuss the first case. Assume contrary to the statement that \(\theta_j(\hat{K}) < 0\). Then \(H(\hat{K})\) gives qualified descent at \(\hat{K}\).
in the sense that $F(\hat{K} + tH(\hat{K}); \hat{K}) \leq \alpha t \theta_e(\hat{K})$ for all $0 < t \leq t(\hat{K})$, where $t(\hat{K})$ is the largest step such that every $t \in (0, t(\hat{K}))$ satisfies the Armijo condition. Now observe that a practical backtracking line search does not compute $t(\hat{K})$, but some $t^j(\hat{K}) \in (0, t(\hat{K}))$. For instance Polak [14] advocates $t^j(\hat{K}) = \max(\beta^v : v \in \mathbb{N}, F(\hat{K} + \beta^v H(\hat{K}); \hat{K}) \leq \alpha \beta^v \theta_e(\hat{K})$ with some fixed $0 < \beta < 1$. Then $K_{j+1} = K_j + t^j(\hat{K}) H(\hat{K})$.

Now recall that the $\Omega_e(\hat{K})$ depend continuously on $\hat{K}$, hence $\theta_e(\hat{K})$ and $H(\hat{K})$ also depend continuously on $\hat{K}$. Suppose $t^j(\hat{K}_j) \rightarrow t^j$, then $t^j \in \{\beta^v t^j(\hat{K}), t^j(\hat{K})\}$ and $t^j \leq t(\hat{K})$. Since $K_j \rightarrow \hat{K}$ for a subsequence, we have $t^j(\hat{K}_j) H(\hat{K}_j) \rightarrow t^j H(\hat{K})$, hence $F(K_j + t^j(\hat{K}_j) H(\hat{K}_j); K_j) \leq \frac{1}{2} \alpha \beta^j t^j(\hat{K}) \theta_e(\hat{K}) \leq \frac{1}{2} \alpha \beta^j t^j(\hat{K}) \theta_e(\hat{K}) \rightarrow 0$ for $j \geq j_0$. This contradicts the fact that $F(K_{j+1}; K_j) \rightarrow 0$ and settles the first case. The proof of the second case is similar.

V. SOME PRACTICAL ASPECTS

Algorithm 1 has been implemented for both structured and unstructured $H_2/H_\infty$ synthesis. In practice it is often required that some controller gains be put to zero, while others can be freely assigned. This is e.g. the case when the controller has to be strictly proper to ensure finiteness of the $H_2$ norm.

A. Stopping criteria

Since our algorithm is a first order method, it may be slow in the neighborhood of a local solution of (3). As in [3], we have therefore implemented termination criteria which ensure that unnecessary iteration with marginal progress near the local optimum can be avoided.

Our first stopping test is based on $0 \in \partial_1 F(K; \hat{K})$ and checks whether the algorithm has reached a critical point of (3) by computing

$$\inf \{ \| \Phi \| : \Phi \in \partial_1 F(K; \hat{K}) \} < \epsilon_1.$$  

We also define two additional tests that compare the relative progress of the local model around the current iterate and the step length to the controller gains:

$$|F(\hat{K}^+; K)| \leq \epsilon_2 \quad \| K^+ - K \| \leq \epsilon_3 (1 + \| K \|).$$

For stopping, either the first or the last two tests are required.

B. Performance level

For all test examples, we compute the locally optimal $H_2$ controller $K^*_2$ for channel $T_2$, the locally optimal $H_\infty$ controllers $K^*_\infty$ for channel $T_\infty$ and then: $\gamma_2 := \| T_\infty(K^*_2) \|_\infty$, and $\gamma^*_\infty := \| T_\infty(K^*_\infty) \|_\infty$. It is now trivial (see e.g. [7]) that the performance level $\gamma$ in (3) has to satisfy

$$\gamma^*_\infty \leq \gamma < \gamma_2.$$  

Indeed the $H_2/H_\infty$ problem is unfeasible for $\gamma < \gamma^*_\infty$, while for $\gamma \geq \gamma_2$, the optimal $H_2$ controller $K^*_2$ is also optimal for (3).

Disregarding complications due to (multiple) local minima, it would make sense in a specific case study, to consider the entire one parameter family $K(\gamma)$ of solutions of (3) as a function of the gain value $\gamma$ in the range (8), as this transforms $K^*_\gamma$ continuously into $K^*_2$ (see Fig. 1).

In our tests we only compute $K(\gamma)$ for a fixed value $\gamma$ in order to compare our method to existing approaches.

VI. NUMERICAL TESTS

In this section we present numerical tests of algorithm 1 on a variety of $H_2/H_\infty$ synthesis problems. In all tests, we use the techniques in [8] to compute an initial stabilizing $K_0$, which is not necessarily feasible for (3). This allows to test phase I of the method. In some cases $K_\infty$ might be chosen as a feasible initial iterate, so that phase I can be avoided. We choose $\gamma \in [\gamma_\gamma, \gamma_2]$ as shown in Tab. I.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n_x$, $n_y$, $n_u$</th>
<th>$n_k$</th>
<th>$\alpha^2_2 / \gamma_2$</th>
<th>$\gamma_\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Academic ex [7]</td>
<td>(2, 1, 1)</td>
<td>0</td>
<td>$6^4 / \sqrt{5}$</td>
<td>1</td>
</tr>
<tr>
<td>Academic ex</td>
<td>(3, 1, 1)</td>
<td>3</td>
<td>7.748 / 23.586</td>
<td>9.5237</td>
</tr>
<tr>
<td>Vehicular [19]</td>
<td>(4, 2, 1)</td>
<td>0</td>
<td>32.416 / 6.3287</td>
<td>4.8602</td>
</tr>
<tr>
<td>suspension pb</td>
<td>2</td>
<td>2</td>
<td>32.299 / 6.1828</td>
<td>4.8573</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>2.267 / 6.3260</td>
<td>4.6797</td>
</tr>
<tr>
<td>From COMPl. [ib ]</td>
<td>(82, 4, 4)</td>
<td>0</td>
<td>0.79389 / 1.3167</td>
<td>0.67421</td>
</tr>
<tr>
<td>'BDT2'</td>
<td></td>
<td>10</td>
<td>0.78877 / 1.1386</td>
<td>0.72423</td>
</tr>
<tr>
<td></td>
<td></td>
<td>41</td>
<td>0.77867 / 1.1302</td>
<td>0.77405</td>
</tr>
<tr>
<td>'HFI'</td>
<td>(130, 1, 2)</td>
<td>0</td>
<td>5.8193e-2 / 0.4611</td>
<td>0.44721</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>5.8193e-2 / 0.4600</td>
<td>0.44721</td>
</tr>
<tr>
<td></td>
<td></td>
<td>25</td>
<td>5.8174e-2 / 0.4605</td>
<td>0.44721</td>
</tr>
<tr>
<td>'CM4'</td>
<td>(240, 1, 2)</td>
<td>0</td>
<td>9.2645e-1 / 1.6546</td>
<td>0.81650</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>9.3844e-1 / 2.2541</td>
<td>0.81746</td>
</tr>
</tbody>
</table>

TABLE I

RESULTS OF NON-CONSTRAINED $H_2$ AND $H_\infty$ SYNTHESIS WHERE $\alpha^2_2 = \| T_2(K^*_2) \|_2$; $\gamma_2 = \| T_\infty(K^*_2) \|_\infty$ and $\gamma^*_\infty = \| T_\infty(K^*_\infty) \|_\infty$.

Next, the parameter $\delta$ is arbitrarily choosen as 0.1. Inspired from trust region techniques [6], a way to improve the approximation of the progress function $F(\cdot; K)$ by the model $\hat{F}(\cdot; K) + \frac{\delta}{2} \| \cdot - K \|^2$, would be to evaluate the progress of the descent algorithm at each iteration and then to readjust $\delta$.

A. Two academic examples

We start with two academic examples whose models are given in [7] and [18, example 1]. The first one is simple enough to allow explicit computation of static output feedback controllers for $H_2$, $H_\infty$ and $H_2/H_\infty$ synthesis.
For the purpose of testing, we first apply our algorithm for a performance level $\gamma > \gamma^*_2 = \|T_\omega(K^*_2)\|_\omega$, so that it finds the optimal $H_2$ controller $K^*_2$. See Table I.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Academic ex. [7]</th>
<th>(n_L, n_R)</th>
<th>n_K</th>
<th>$\gamma$</th>
<th>Iter</th>
<th>$H_2$ norm</th>
<th>$H_\infty$ norm</th>
<th>Final $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n_L, n_R)$</td>
<td>(2, 1, 1)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>1.5651</td>
<td>1.5735</td>
</tr>
<tr>
<td>$(n_L, n_R)$</td>
<td>(3, 1, 1)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2.56</td>
<td>10</td>
<td>7.7484</td>
<td>10.4552</td>
</tr>
</tbody>
</table>

Table II

MIXED $H_2/H_\infty$ SYNTHESIS FOR TWO ACADEMIC EXAMPLES

We then perform the $H_2/H_\infty$ synthesis on the two considered examples (see Table II). We not only improve the results computed by LMI approaches in [7] and [18], but also obtain the theoretically best values of the $H_2$ and $H_\infty$ norms.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Academic ex. [18]</th>
<th>(n_L, n_R)</th>
<th>n_K</th>
<th>$\gamma$</th>
<th>Iter</th>
<th>$H_2$ norm</th>
<th>$H_\infty$ norm</th>
<th>Final $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n_L, n_R)$</td>
<td>(5, 1, 1)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>25.6</td>
<td>12</td>
<td>8.07</td>
<td>7.7484</td>
</tr>
<tr>
<td>$(n_L, n_R)$</td>
<td>(5, 1, 1)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>25.6</td>
<td>12</td>
<td>8.07</td>
<td>7.7484</td>
</tr>
</tbody>
</table>

B. Vehicular suspension controller design

The model of the vehicular suspension is described in [9] and [19]. We first focus on static $H_2/H_\infty$-synthesis. The $H_\infty$ performance level in (3) is chosen as $\gamma = 5.225$ and the optimal solution we obtain is

$$K^* = \begin{bmatrix} 4.1586 & 0.2393 \end{bmatrix}$$

The $H_2$ norm computed by our algorithm is $\|T_\omega(K^*)\|_2 = 34.446$ instead of 35.8065 obtained by [19] and the related $H_\infty$ performance is $\|T_\omega(K^*)\|_\infty = 5.2250$ instead of 5.0506 in [19]. This highlights the conservatism of the LMI approach in [19]. In contrast our algorithm attains the $H_\infty$ performance constraint, as it should. Results are given in Table III.

We also present numerical results of the $H_2/H_\infty$ synthesis for dynamic order controllers of orders $n_K = 2, 4$.

C. COMPl_iib examples

Models in this section are from the COMPl_iib collection [13]: distillation tower 'BDT2’, heat flow in a thin rod 'HF1’ and cable mass model 'CM4’. These problems are originally $H_\infty$ synthesis problems. As proposed by F. Leibfritz in [12], we have added a $H_2$ channel by setting $B_2 = B_\omega$ and $D_2 = 0$.

In each example, the $H_\infty$ performance constraint is first chosen as $\gamma > \gamma^*_2$ to obtain an upper bound of the optimal $H_2$ performance and an approximation of the related $H_\infty$ performance $\gamma^*$. Our results are presented in Tab. IV and V.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Vehicular suspension controller design [19]</th>
<th>(n_K, n_L, n_R)</th>
<th>$n_K$</th>
<th>$\gamma$</th>
<th>Iter</th>
<th>$H_2$ norm</th>
<th>$H_\infty$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n_K, n_L, n_R)$</td>
<td>(4, 2, 1)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
<td>192</td>
<td>8.0516-10-1</td>
</tr>
<tr>
<td>$(n_K, n_L, n_R)$</td>
<td>(3, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>25.6</td>
<td>10</td>
<td>1543</td>
</tr>
<tr>
<td>$(n_K, n_L, n_R)$</td>
<td>(2, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>25.6</td>
<td>10</td>
<td>1543</td>
</tr>
</tbody>
</table>

As an illustration, Figs. 2 and 3 show the evolution of the $H_2$ and $H_\infty$ norms for ‘BDT2’ example during first iterations. In Fig. 2, phases I and II clearly appear: while the current iterate is unfeasible, descent steps to minimize constraint violation are generated. When the $H_\infty$ constraint is met, the technique privileges minimization of the $H_2$ objective. Fig. 3 shows the evolution of the max singular value associated with the $H_\infty$ constraint in the first 5 iterations. Stars indicate frequencies selected to build the extension $\Omega_c(K)$. We observe that max singular values are simple at selected frequencies which seems valid as a rule in most applications.

VII. CONCLUSION

Mixed $H_2/H_\infty$ is a practically important problem for which successful numerical methods are lacking. In response we...
have proposed an algorithm based on non-smooth optimization, which improves systematically over numerical results from the literature, and in particular, over conservative results obtained by LMI techniques. Our approach seems promising since it is capable to handle large size problems with up to 240 states. Extensions to problems involving a mixture of time- and frequency-domain constraints as well as to nonlinear systems are currently under investigation.

REFERENCES