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# Optimal Switching in Finite Horizon under State Constraints

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## Abstract

We study an optimal switching problem with a state constraint: the controller is only allowed to choose strategies that keep the controlled diffusion in a closed domain. We prove that the value function associated to this problem is the limit of value functions associated with unconstrained switching problems with penalized coefficients, as the penalization parameter goes to infinity. This convergence allows to set a dynamic programming principle for the constrained switching problem. We then prove that the value function is a solution to a system of variational inequalities (SVI for short) in the constrained viscosity sense. We finally prove that uniqueness for our SVI cannot hold and we give a weaker characterization of the value function as the maximal solution to this SVI. All our results are obtained without any regularity assumption on the constraint domain.

**Key words:** Optimal switching, state constraints, dynamic programming, variational inequalities, energy management.

**Mathematics Subject Classification (2010):** 60H10, 60H30, 91G80, 93E20.

## 1 Introduction

Optimal control of multiples switching regimes consists in looking for the value of an optimization problem where the allowed strategies are sequences of interventions. It naturally arises in many applied disciplines where it is not realistic to assume that the involved quantities can be continuously controlled. More precisely, the optimal switching problem supposes that the control strategies are sequences  $\alpha = (\tau_k, \zeta_k)_k$  where the sequence  $(\tau_k)_k$  represents the intervention times of the controller and  $\zeta_k$  corresponds to the level of intervention of the agent at each time  $\tau_k$ .

Such a class of strategies allows to consider discrete actions for the controller which can be more relevant than continuous time controls. Therefore, the modelization with optimal switching problems has attracted a lot of interest during the last decades (see e.g. Brennan

and Schwarz [2] for resource extraction, Dixit [8] for production facility problems, Carmona and Ludkovski [4] for power plant management or Ly Vath, Pham and Villeneuve [14] for dividend decision problem with reversible technology investment).

Another specificity to take into account in the modelization with optimal switching is the limitation of the quantities involved in the control problem. Indeed, in most of management problems the controlled system is subject to a constraint on the possible states that it can take. For example, a solvency condition is usually imposed to the investors of a financial market and the energy producer has to take into account the limited storage capacities. This leads to impose a state constraint on the controlled diffusion  $X$  of the form

$$X_s \in \mathcal{D} \quad \text{for all } s,$$

where  $\mathcal{D}$  is a closed set. We therefore need to restrict our control problem to the set  $\mathcal{A}_{t,x}^{\mathcal{D}}$  of strategies that keep the controlled diffusion starting from  $(t, x)$  in the constraint domain  $\mathcal{D}$ . Unfortunately, such a constraint leads to strong difficulties due, in particular, to the complicated structure of the set valued function  $(t, x) \mapsto \mathcal{A}_{t,x}^{\mathcal{D}}$ . To the best of our knowledge, no rigorous study of the optimal switching problem in the constrained case has been done before and our aim is to fill this gap.

In the continuous time control case, H. M. Soner gives in [15] a first study of the constrained problem in a deterministic framework where he introduces the notion of constrained viscosity solutions. To characterize the value function, his approach relies on a continuity argument under an assumption on the boundary of the constraint domain  $\partial\mathcal{D}$ . He then extends this result to the case of piecewise deterministic processes in [16]. The continuous time stochastic control case is studied by M. A. Katsoulakis in [12]. His approach is also based on continuity and he imposes some regularity condition on the constraint domain  $\mathcal{D}$ . In our case, such an approach is not possible since the value function may be discontinuous even for a smooth domain  $\mathcal{D}$  as shown by the counterexample presented in Subsection 5.1.

Let us also mention the recent approach of D. Goreac *et al.* presented in [10]. They formulate the initial problem as a linear problem which concerns the occupation measures induced by the controlled diffusion processes. Under convexity assumptions, the authors characterize (see Theorem 11 in [10]) the value function associated to the weak formulation of the continuous time stochastic control problem under state constraints (the weak formulation means that the controller is allowed to choose the probability space in addition to the control strategy). Unfortunately, such an approach cannot be applied to the optimal switching under state constraints since the the set of values ??taken by the controls is not convex.

In this work, we present an original approach which allows to deal with the lack of regularity of the associated value function. Moreover, our method does not need any regularity or convexity assumption. In particular, we only need to assume that the constraint domain  $\mathcal{D}$  is closed.

To be more precise, our approach relies on the simple structure of switching controls. Indeed, they can be seen as random variables taking values in  $([0, T] \times \mathcal{I})^{\mathbb{N}}$  where  $\mathcal{I}$  is a finite set and  $T > 0$  is a given constant. From Tychonov theorem, we get the compactness of this space which allows to prove the tightness of a sequence  $(\alpha^n)_n$  of switching strategies and

hence the convergence in law up to a subsequence. Then applying Skorokhod representation theorem, we are able to provide a probability space and a sequence  $(\tilde{\alpha}^n)_n$  that converges almost surely to some  $\tilde{\alpha}$  and such that  $\tilde{\alpha}^n$  is equal in law to  $\alpha^n$  for all  $n$ .

We use this sequential compactness property in the following way. We first introduce a sequence  $(v_n)_n$  of unconstrained switching problems with  $n$ -penalized terminal and running reward coefficients out of the constraint domain  $\mathcal{D}$ . For each penalized switching problems  $v_n$ , we take  $\alpha^n$  as a  $\frac{1}{n}$ -almost optimal strategy for  $v_n$ . Then we construct a switching strategy  $\alpha^*$  which is equal in law to  $\tilde{\alpha}$ . The strong convergence of  $\tilde{\alpha}^n$  to  $\tilde{\alpha}$  allows to prove that  $\alpha^*$  is optimal for the switching problem under constraint. As a byproduct, we get the convergence of the unconstrained penalized switching problems to the constrained one. Using existing results on classical optimal switching problems, this convergence allows to set a dynamic programming principle for the constrained switching problem.

We then focus on the PDE characterization of the value function. Using the dynamic programming principle proved before, we show that the value function is a constrained viscosity solution to a system of variational inequalities (SVI for short) defined on the constraint domain  $\mathcal{D}$ . We then investigate the uniqueness of a solution to this SVI. The usual approach to get uniqueness of a viscosity solution consists in proving a comparison theorem for the PDE. Such a result tells that a supersolution is greater than a subsolution. As a consequence of this comparison theorem, the unique solution has to be continuous. Unfortunately, the continuity of the value functions is not true in general as shown by the counterexample given in Subsection 5.1. Therefore, we cannot hope to state such a uniqueness result for the SVI on  $\mathcal{D}$ . Instead, we characterize our value function as the maximal viscosity solution of the SVI under an additional growth assumption. This maximality property is also obtained from the convergence on the penalized unconstrained problems to the constrained one.

We end the introduction by the description of the organization of the paper. In Section 2 we expose in detail the formulation of the optimal switching problem under state constraints. We then give an application to electricity production management. In Section 3, we provide an approximation of our constrained problem by unconstrained problems with penalized coefficients. We prove the convergence of the penalized problems to the constrained one as the penalization parameter goes to infinity. In Section 4 we state a dynamic programming principle and we prove that the value function is a constrained viscosity solution to a SVI. Finally, in section 5 we focus on uniqueness. We first show by a counterexample that we cannot prove uniqueness of a solution to the SVI. Under an additional growth assumption, we characterize the value function as the maximal constrained viscosity solution to the SVI. We end by giving examples where this additional growth condition is satisfied.

## 2 Problem formulation

### 2.1 Optimal switching under state constraints

We consider the following formulation of optimal switching problem under state constraints.

We fix a complete probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  which is endowed with a Brownian motion

$W = (W_t)_{t \geq 0}$  valued in  $\mathbb{R}^d$ . We denote by  $\mathbb{F}$  the complete and right continuous filtration generated by  $W$  *i.e.* the smallest complete and right continuous filtration such that each  $Z_n$  is  $\mathcal{F}_0$ -measurable and  $\sigma(W_s, s \in [0, t]) \subset \mathcal{F}_t$  for all  $t \geq 0$ .

**Controls.** We then define the set  $\mathcal{A}_t$  of admissible switching controls at time  $t \in [0, T]$  as the set of double sequences  $\alpha = (\tau_k, \zeta_k)_{k \geq 0}$  where

- $(\tau_k)_{k \geq 0}$  is a nondecreasing sequence of  $\mathbb{F}$ -stopping times with  $\tau_0 = t$  and  $\lim_{k \rightarrow \infty} \tau_k = +\infty$ ,
- $\zeta_k$  is an  $\mathcal{F}_{\tau_k}$ -measurable random variables valued in the set  $\mathcal{I}$  defined by  $\mathcal{I} = \{1, \dots, m\}$ .

With a strategy  $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t$  we associate the process  $(\alpha_s)_{s \geq t}$  defined by

$$\alpha_s = \sum_{k \geq 0} \zeta_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(s), \quad s \geq t.$$

**Controlled diffusion.** We are given two functions  $\mu : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^{d \times d}$ . We make the following assumption.

**(H1)** There exists a constant  $L$  such that

$$|\mu(x, i) - \mu(x', i)| + |\sigma(x, i) - \sigma(x', i)| \leq L|x - x'|,$$

for all  $(x, x', i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{I}$ .

For  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $\alpha \in \mathcal{A}_t$  we consider the controlled diffusion  $X^{t,x,\alpha}$  defined by the following SDE

$$X_s^{t,x,\alpha} = x + \int_t^s \mu(X_r^{t,x,\alpha}, \alpha_r) dr + \int_t^s \sigma(X_r^{t,x,\alpha}, \alpha_r) dW_r, \quad s \geq t. \quad (2.1)$$

Under **(H1)**, we have existence and uniqueness of an  $\mathbb{F}$ -adapted solution  $X^{t,x,\alpha}$  to (2.1) for any initial condition  $(t, x) \in [0, T] \times \mathbb{R}^d$  and any switching control  $\alpha \in \mathcal{A}_t$ .

We also have the following classical estimate (see e.g. Corollary 12, Section 5, Chapter 2 in [13]): for any  $q \geq 1$  there exists a constant  $C_q$  such that

$$\sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[ \sup_{s \in [t, T]} |X_s^{t,x,\alpha}|^q \right] \leq C_q (1 + |x|^q) \quad (2.2)$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ .

**Expected Payoff.** We consider a terminal and running reward functions  $g : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$  and  $f : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$  and a cost function  $c : \mathbb{R}^d \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}$  on which we impose the following assumption.

**(H2)**

(i) There exists a constant  $L$  such that

$$|g(x, i) - g(x', i)| + |f(x, i) - f(x', i)| + |c(x, i, j) - c(x', i, j)| \leq L|x - x'|,$$

for all  $x, x' \in \mathbb{R}^d$  and  $i, j \in \mathcal{I}$ .

(ii) There exists a constant  $\bar{c} > 0$ , such that

$$c(x, i, j) \geq \bar{c},$$

for all  $x \in \mathbb{R}^d$  and  $i, j \in \mathcal{I}$ .

We then define the functional pay-off  $J$  by

$$J(t, x, \alpha) = \mathbb{E} \left[ g(X_T^{t,x,\alpha}, \alpha_T) + \int_t^T f(X_s^{t,x,\alpha}, \alpha_s) ds - \sum_{k \geq 1} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) \mathbb{1}_{\tau_k < T} \right]$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $\alpha \in \mathcal{A}_t$ .

Under **(H1)** and **(H2)** we get from (2.2) that  $J(t, x, \alpha)$  is well defined for any initial condition  $(t, x) \in [0, T] \times \mathbb{R}^d$  and any control  $\alpha \in \mathcal{A}_t$ .

**State constraint.** Let  $\mathcal{D}$  be a nonempty closed subset of  $\mathbb{R}^d$ . For  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$  we denote by  $\mathcal{A}_{t,x,i}^{\mathcal{D}}$  the set of strategies  $\alpha \in \mathcal{A}_t$  such that  $\zeta_0 = i$  and

$$\mathbb{P} \left( X_s^{t,x,\alpha} \in \mathcal{D} \text{ for all } s \in [t, T] \right) = 1.$$

**Value function.** We then define the value function  $v$  associated with the switching problem under state constraints by

$$v(t, x, i) = \sup_{\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}} J(t, x, \alpha) \tag{2.3}$$

for all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ , with the convention  $v(t, x, i) = -\infty$  if  $\mathcal{A}_{t,x,i}^{\mathcal{D}} = \emptyset$ . Our aim is to give an analytic characterization of the function  $v$ .

## 2.2 An example of application: a hydroelectric pumped storage model

We present in this subsection an energy management model involving an optimal switching problem under state constraint.

The following simplified hydroelectric pumped storage model is inspired by [4]. Pumped Storage (currently, the dominant type of electricity storage) consists of large reservoir of water held by a hydroelectric dam at a higher elevation. When desired, the dam can be opened which activates the turbines and moves the water to another, lower reservoir. The generated electricity is sold to a power grid. As the water flows, the upper reservoir is depleted. Conversely, in times of low electricity demand, the water can be pumped back into the reservoir with required energy purchased from grid. A strategy  $\alpha$  consists in a sequence of  $\mathbb{F}$ -stopping times  $(\tau_k)_k$  representing the intervention times and a sequence of  $\mathcal{F}_{\tau_k}$ -measurable random variables  $(\zeta_k)_k$  representing the changes of regime. There are three possible regimes.

- (i)  $\zeta_k = 1$  : pump, in this case we set  $\mu_1(x, 1) = 1$  and  $\sigma_1(x, 1) = 0$ .
- (ii)  $\zeta_k = 2$ : store, in this case we set  $\mu_1(x, 1) = 0$  and  $\sigma_1(x, 1) = 0$ .
- (iii)  $\zeta_k = 3$ : generate, in this case we set  $\mu_1(x, 1) = -1$  and  $\sigma_1(x, 1) = 0$ .

For a given strategy  $\alpha = (\tau_k, \zeta_k)_k$ , we denote by  $L_t^\alpha$  the controlled water level in the upper reservoir. It satisfies the equation

$$L_t^\alpha = L_0 + \int_0^t \mu_1(L_s^\alpha, \alpha_s) ds + \int_0^t \sigma_1(L_s^\alpha, \alpha_s) dW_s, \quad t \geq 0.$$

Denote by  $P$  the electricity price process and suppose that it is a diffusion defined on  $(\Omega, \mathcal{G}, \mathbb{P})$  by

$$P_t = P_0 + \int_0^t \mu_2(P_s) ds + \int_0^t \sigma_2(P_s) dW_s, \quad t \geq 0.$$

If we denote by  $X^\alpha$  the controlled process defined by  $X^\alpha = \begin{pmatrix} L^\alpha \\ P \end{pmatrix}$  then it satisfies the SDE

$$X_t^\alpha = X_0 + \int_0^t \mu(X_s^\alpha, \alpha_s) ds + \int_0^t \sigma(X_s^\alpha, \alpha_s) dW_s$$

with  $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$  and  $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ . Suppose also that the cost of changing the regime from  $i$  to  $j$  is given by a constant  $c(i, j)$ . The expected gain for a given strategy  $\alpha$  is then given by

$$\begin{aligned} J(0, X_0, \alpha) &= \mathbb{E} \left[ \int_0^T -P_t dL_t^\alpha - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k) \right] \\ &= \mathbb{E} \left[ \int_0^T f(X_t^\alpha, \alpha_t) dt - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k) \right] \end{aligned}$$

where  $f$  is defined by  $f(p, \ell, i) = -p \times \mu_1(\ell, i)$  for all  $(p, \ell, i) \in \mathbb{R} \times \mathbb{R} \times \{0, 1, 2\}$ .

Since the reservoir capacity is not infinite, the strategy  $\alpha$  has to satisfy the constraint  $0 \leq L_t^\alpha \leq \ell_{max}$  for all  $t \in [0, T]$ . This corresponds to the general constraint  $X_t^\alpha \in \mathcal{D}$  where  $\mathcal{D} = \mathbb{R} \times [0, \ell_{max}]$ . The goal of the energy producer is to maximize  $J(0, X_0, \alpha)$  over the strategies  $\alpha$  satisfying the constraint on the water level  $L^\alpha$ .

### 3 Unconstrained penalized switching problem

#### 3.1 An unconstrained penalized approximating problem

We now introduce an approximation of our initial constrained problem. This approximation consists in a penalization of the gain coefficients  $f$  and  $g$  out of the domain  $\mathcal{D}$  where the controlled underlying diffusion is constrained to stay.

Consider, for  $n \geq 1$ , the functions  $f_n : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$  and  $g_n : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$  defined by

$$f_n(x, i) = f(x, i) - n\Theta_n(x), \quad (3.1)$$

$$g_n(x, i) = g(x, i) - n\Theta_n(x), \quad (3.2)$$

for all  $(x, i) \in \mathbb{R}^d \times \mathcal{I}$ , where the function  $\Theta_n : \mathbb{R}^d \rightarrow [0, 1]$  is given by

$$\Theta_n(x) = n \left( d(x, \mathcal{D}) \wedge \frac{1}{n} \right) = nd(x, \mathcal{D}) \wedge 1, \quad (3.3)$$

with  $d(x, \mathcal{D}) = \inf_{x' \in \mathcal{D}} |x - x'|$  for all  $x \in \mathbb{R}^d$ .

Given an initial condition  $(t, x)$  and a switching control  $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t$ , we consider the total penalized profit starting from  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$  at horizon  $T$ , defined by:

$$J_n(t, x, \alpha) = \mathbb{E} \left[ g_n(X_T^{t,x,\alpha}, \alpha_T) + \int_t^T f_n(X_s^{t,x,\alpha}, \alpha_s) ds - \sum_{k \geq 1} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) \mathbb{1}_{\tau_k < T} \right]$$

We can then define the penalized unconstrained value function  $v_n : [0, T] \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$  by

$$v_n(t, x, i) = \sup_{\alpha \in \mathcal{A}_{t,i}} J_n(t, x, \alpha), \quad (3.4)$$

for all  $n \geq 1$  and all  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ , where  $\mathcal{A}_{t,i}$  is the set of strategies  $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t$  such that  $\zeta_0 = i$ .

## 3.2 Convergence of the penalized unconstrained problems

We now state the main result of this section.

**Theorem 3.1.** *Under (H1) and (H2), the sequence  $(v_n)_{n \geq 1}$  is nonincreasing and converges on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  to the function  $v$ :*

$$v_n(t, x, i) \downarrow v(t, x, i) \quad \text{as } n \uparrow +\infty, \quad (3.5)$$

for all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . Moreover, for any  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ , there exists a strategy  $\alpha^* \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  such that

$$v(t, x, i) = J(t, x, \alpha^*).$$

**Proof.** Fix  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . Since  $f_{n+1} \leq f_n$  and  $g_{n+1} \leq g_n$  we get

$$J_{n+1}(t, x, \alpha) \leq J_n(t, x, \alpha),$$

for all  $n \geq 1$  and any  $\alpha \in \mathcal{A}_t$ . From this last inequality we deduce that

$$v_{n+1}(t, x, i) \leq v_n(t, x, i), \quad n \geq 1.$$

We now prove that  $(v_n)_n$  converges to  $v$ . We first notice that

$$J_n(t, x, \alpha) = J(t, x, \alpha),$$



for any  $n \geq 1$ , any initial condition  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$  and any switching strategy  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ . Therefore, we get  $v_n \geq v$  for all  $n \geq 1$ . Denote by  $\bar{v}$  the pointwise limit of  $(v_n)_n$ :

$$\bar{v}(t, x, i) = \lim_{n \rightarrow \infty} v_n(t, x, i), \quad (t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}.$$

Then we have  $\bar{v}(t, x, i) \geq v(t, x, i)$ . If  $\bar{v}(t, x, i) = -\infty$  we obviously get  $\bar{v}(t, x, i) = v(t, x, i)$ .

We now suppose that  $\bar{v}(t, x, i) > -\infty$  and prove that  $\bar{v}(t, x, i) \leq v(t, x, i)$ . We proceed in 3 steps.

**Step 1.** *Convergence of a sequence of almost optimal strategies for the unconstrained problems.*

**Step 1.1.** *Bounded sequence of almost optimal strategies.*

For  $n \geq 1$ , let  $\alpha^n = (\tau_k^n, \zeta_k^n)_{k \geq 0} \in \mathcal{A}_{t,i}$  a switching strategy such that

$$J_n(t, x, \alpha^n) \geq v_n(t, x, i) - \frac{1}{n}.$$

We can suppose without loss of generality that

$$\tau_k^n \in [0, T + 1] \quad \mathbb{P} - a.s. \quad (3.6)$$

for all  $n \geq 1$  and all  $k \geq 0$ . Indeed, fix  $n \geq 1$  and consider the strategy  $\hat{\alpha}^n = (\hat{\tau}_k^n, \hat{\zeta}_k^n)_{k \geq 0} \in \mathcal{A}_{t,i}$  defined by

$$\begin{aligned} \hat{\tau}_k^n &= \tau_k^n \wedge (T + 1) \\ \hat{\zeta}_k^n &= \zeta_k^n \mathbf{1}_{\tau_k^n \leq T+1} + i \mathbf{1}_{\tau_k^n > T+1} \end{aligned}$$

Then we have  $J_n(t, x, \alpha^n) = J_n(t, x, \hat{\alpha}^n)$  and we can replace  $\alpha^n$  by  $\hat{\alpha}^n$  which satisfies (3.6).

**Substep 1.2.** *Tightness and convergence of  $(W, \alpha^n)_n$ .*

We now prove that the sequence of  $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$ -valued random variables  $(W, \alpha^n)_{n \geq 1}$  is tight. Fix a sequence  $(\delta_\ell)_\ell$  of positive numbers such that

$$\delta_\ell \xrightarrow{\ell \rightarrow \infty} 0 \quad \text{and} \quad 2^\ell \delta_\ell \ln\left(\frac{2T}{\delta_\ell}\right) \xrightarrow{\ell \rightarrow \infty} 0. \quad (3.7)$$

We define for  $\eta > 0$  and  $C > 0$  the subset  $\mathcal{K}_\eta^C$  of  $C([0, T], \mathbb{R}^d)$  by

$$\mathcal{K}_\eta^C = \bigcap_{\ell \geq 1} \mathcal{K}_{\eta, \ell}^C$$

where

$$\mathcal{K}_{\eta, \ell}^C = \left\{ h \in C([0, T], \mathbb{R}^d) : h(0) = 0 \text{ and } \text{mc}_{\delta_\ell}(h) \leq C \frac{2^\ell \delta_\ell \ln\left(\frac{2T}{\delta_\ell}\right)}{\eta} \right\}$$

and  $\text{mc}$  denotes the modulus of continuity defined by

$$\text{mc}_\delta(h) = \sup_{\substack{s, t \in [0, T] \\ |s - t| \leq \delta}} |h(s) - h(t)|$$

for any  $h \in C([0, T], \mathbb{R}^d)$  and any  $\delta > 0$ . Using Arzela-Ascoli Theorem, we get from (3.7) that  $\mathcal{K}_\eta^C$  is a compact subset of  $C([0, T], \mathbb{R}^d)$ . We now define the subset  $\mathbf{K}_\eta^C$  of  $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$  by

$$\mathbf{K}_\eta^C = \mathcal{K}_\eta^C \times ([0, T + 1] \times \mathcal{I})^{\mathbb{N}}.$$

From Tychonov theorem and since  $\mathcal{K}_\eta^C$  is compact, we get that  $\mathbf{K}_\eta^C$  is a compact subset of  $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$  endowed with the norm  $\|\cdot\|$  defined by

$$\|(h, (t_k, z_k)_{k \geq 0})\| = \sup_{t \in [0, T]} |h(t)| + \sum_{k \geq 0} \frac{(|t_k| + |z_k|) \wedge 1}{2^k}$$

for all  $h \in C([0, T], \mathbb{R}^d)$  and  $(t_k, z_k)_{k \geq 0} \in (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$ . We then have from (3.6)

$$\mathbb{P}\left((W, \alpha^n) \in \mathbf{K}_\eta^C\right) = \mathbb{P}\left(W \in \mathcal{K}_\eta^C\right)$$

for all  $\eta > 0$ ,  $C > 0$  and  $n \geq 1$ . Using Markov inequality we get

$$\begin{aligned} \mathbb{P}\left(W \in \mathcal{K}_\eta^C\right) &= 1 - \mathbb{P}\left(W \notin \mathcal{K}_\eta^C\right) \\ &\geq 1 - \sum_{\ell \geq 1} \mathbb{P}\left(W \notin \mathcal{K}_{\eta, \ell}^C\right) \\ &\geq 1 - \sum_{\ell \geq 1} \frac{\mathbb{E}\left[\text{mc}_{\delta_\ell}(W)\right]}{C \frac{2^\ell \delta_\ell \ln\left(\frac{2T}{\delta_\ell}\right)}{\eta}}. \end{aligned} \quad (3.8)$$

From Theorem 1 in [9], there exists a constant  $C^*$  such that

$$\mathbb{E}\left[\text{mc}_\delta(W)\right] \leq C^* \frac{\delta \ln\left(\frac{2T}{\delta}\right)}{\eta}. \quad (3.9)$$

for all  $\delta > 0$ . Therefore, we get from (3.8) and (3.9)

$$\mathbb{P}\left((W, \alpha^n) \in \mathbf{K}_\eta^{C^*}\right) \geq 1 - \eta,$$

for all  $\eta \in (0, 1)$ , and the sequence  $(W, \alpha^n)_n$  is tight.

We deduce from Prokhorov theorem that, up to a subsequence,

$$\mathbb{P} \circ (W, \alpha^n)^{-1} \xrightarrow{n \rightarrow \infty} \mathcal{L}.$$

with  $\mathcal{L}$  a probability measure on  $(C([0, T], \mathbb{R}^d) \times (\mathbb{R} \times \mathcal{I})^{\mathbb{N}}, \|\cdot\|)$ .

**Step 2. Change of probability space.**

Since  $(C([0, T], \mathbb{R}^d) \times (\mathbb{R} \times \mathcal{I})^{\mathbb{N}}, \|\cdot\|)$  is separable, we get from the Skorokhod representation theorem that there exists a probability space  $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}})$  on which are defined Brownian motions  $\tilde{W}^n$ ,  $n \geq 1$ , and  $\tilde{W}$ , and random variables  $\tilde{\alpha}^n = (\tilde{\tau}_k^n, \tilde{\zeta}_k^n)_{k \geq 0}$ ,  $n \geq 1$ , and  $\tilde{\alpha} = (\tilde{\tau}_k, \tilde{\zeta}_k)_{k \geq 0}$  such that

$$\tilde{\mathbb{P}} \circ (\tilde{W}^n, \tilde{\alpha}^n)^{-1} = \mathbb{P} \circ (W, \alpha^n)^{-1} \quad (3.10)$$

for all  $n \geq 1$  and

$$\left\| (\tilde{W}^n, \tilde{\alpha}^n) - (\tilde{W}, \tilde{\alpha}) \right\| \xrightarrow[n \rightarrow \infty]{\tilde{\mathbb{P}}\text{-a.s.}} 0. \quad (3.11)$$

In particular we get

$$\mathcal{L} = \tilde{\mathbb{P}} \circ (\tilde{W}, \tilde{\alpha})^{-1}.$$

**Substep 2.1** *Measurability properties for  $\tilde{\alpha}^n$  and  $\tilde{\alpha}$ .*

We now prove that each  $\tilde{\tau}_k$  is an  $\tilde{\mathbb{F}}$ -stopping time and  $\zeta_k$  is  $\tilde{\mathcal{F}}_{\tilde{\tau}_k}$ -measurable where  $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$  is the complete right-continuous filtration generated by  $\tilde{W}$ .

For  $n \geq 1$ , denote by  $\tilde{\mathbb{F}}^n = (\tilde{\mathcal{F}}_t^n)_{t \geq 0}$  the complete right-continuous filtration generated by  $\tilde{W}^n$ . Using Proposition A.5, we get from (3.10) that  $\tilde{\tau}_k^n$  is an  $\tilde{\mathbb{F}}^n$ -stopping time and that  $\tilde{\zeta}_k^n$  is  $\tilde{\mathcal{F}}_{\tilde{\tau}_k^n}^n$ -measurable for all  $n \geq 1$  and  $k \geq 0$ . Then using Proposition A.6, we get from (3.11) that  $\tilde{\tau}_k$  is an  $\tilde{\mathbb{F}}$ -stopping time and that  $\tilde{\zeta}_k$  is  $\tilde{\mathcal{F}}_{\tilde{\tau}_k}$ -measurable for all  $k \geq 0$ .

**Substep 2.3.** *Equality of the penalized gains and convergence of the associated controlled diffusions.*

From the previous substep, we can define the diffusions  $\tilde{X}^{t,x,\tilde{\alpha}^n}$  and  $\tilde{X}^{t,x,\tilde{\alpha}}$  on  $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}})$  by

$$\tilde{X}_s^{t,x,\tilde{\alpha}^n} = x + \int_t^s b(\tilde{X}_r^{t,x,\tilde{\alpha}^n}, \tilde{\alpha}_r^n) dr + \int_t^s \sigma(\tilde{X}_r^{t,x,\tilde{\alpha}^n}, \tilde{\alpha}_r^n) d\tilde{W}_r^n, \quad s \geq t,$$

and

$$\tilde{X}_s^{t,x,\tilde{\alpha}} = x + \int_t^s b(\tilde{X}_r^{t,x,\tilde{\alpha}}, \tilde{\alpha}_r) dr + \int_t^s \sigma(\tilde{X}_r^{t,x,\tilde{\alpha}}, \tilde{\alpha}_r) d\tilde{W}_r, \quad s \geq t,$$

and the associated gains  $J_n(t, x, \tilde{\alpha}^n)$  and  $J(t, x, \tilde{\alpha})$  by

$$\tilde{J}_n(t, x, \tilde{\alpha}^n) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[ g_n(\tilde{X}_T^{t,x,\tilde{\alpha}^n}, \tilde{\alpha}_T^n) + \int_t^T f_n(\tilde{X}_s^{t,x,\tilde{\alpha}^n}, \tilde{\alpha}_s^n) ds - \sum_{k \geq 1} c(\tilde{X}_{\tilde{\tau}_k^n}^{t,x,\tilde{\alpha}^n}, \tilde{\zeta}_{k-1}^n, \tilde{\zeta}_k^n) \mathbf{1}_{\tilde{\tau}_k^n < T} \right]$$

and

$$\tilde{J}(t, x, \tilde{\alpha}) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[ g(\tilde{X}_T^{t,x,\tilde{\alpha}}, \tilde{\alpha}_T) + \int_t^T f(\tilde{X}_s^{t,x,\tilde{\alpha}}, \tilde{\alpha}_s) ds - \sum_{k \geq 1} c(\tilde{X}_{\tilde{\tau}_k}^{t,x,\tilde{\alpha}}, \tilde{\zeta}_{k-1}, \tilde{\zeta}_k) \mathbf{1}_{\tilde{\tau}_k < T} \right].$$

Since  $(W, \alpha^n)$  and  $(\tilde{W}^n, \tilde{\alpha}^n)$  have the same law, we deduce from **(H1)** and **(H2)** that

$$J_n(t, x, \alpha^n) = \tilde{J}_n(t, x, \tilde{\alpha}^n) \geq v_n(t, x, i) - \frac{1}{n}, \quad n \geq 1. \quad (3.12)$$

We now prove that, up to a subsequence,

$$\limsup_{n \rightarrow \infty} \tilde{J}_n(t, x, \tilde{\alpha}^n) \leq \tilde{J}(t, x, \tilde{\alpha}). \quad (3.13)$$

We first notice that  $\limsup_{n \rightarrow \infty} \tilde{J}_n(t, x, \tilde{\alpha}^n) \leq \limsup_{n \rightarrow \infty} \tilde{J}(t, x, \tilde{\alpha}^n)$ . From Proposition A.7 and (3.11) we have

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \sup_{s \in [t, T]} |\tilde{X}_s^{t, x, \tilde{\alpha}} - \tilde{X}_s^{t, x, \tilde{\alpha}^n}|^2 \right] \xrightarrow{n \rightarrow \infty} 0. \quad (3.14)$$

We therefore get, up to a subsequence,

$$\sup_{s \in [t, T]} |\tilde{X}_s^{t, x, \tilde{\alpha}^n} - \tilde{X}_s^{t, x, \tilde{\alpha}}| \xrightarrow[n \rightarrow \infty]{\tilde{\mathbb{P}}-a.s.} 0. \quad (3.15)$$

This implies with **(H2)** (i) and (3.11)

$$g(\tilde{X}_T^{t, x, \tilde{\alpha}^n}, \tilde{\alpha}_T^n) + \int_t^T f(\tilde{X}_s^{t, x, \tilde{\alpha}^n}, \tilde{\alpha}_s^n) ds \xrightarrow[n \rightarrow \infty]{\tilde{\mathbb{P}}-a.s.} g(\tilde{X}_T^{t, x, \tilde{\alpha}}, \tilde{\alpha}_T) + \int_t^T f(\tilde{X}_s^{t, x, \tilde{\alpha}}, \tilde{\alpha}_s) ds.$$

Moreover, since  $\bar{v}(t, x, i) > -\infty$  we have from **(H2)** (ii)

$$\sup_{n \geq 1} \#\{k \geq 1 : \tilde{\tau}_k^n \leq T\} < +\infty, \quad \tilde{\mathbb{P}}-a.s.$$

This last estimate, (3.11) and (3.15) imply

$$\liminf_{n \rightarrow \infty} \sum_{k \geq 1} c(\tilde{X}_{\tilde{\tau}_k}^{t, x, \tilde{\alpha}^n}, \tilde{\zeta}_{k-1}^n, \tilde{\zeta}_k^n) \mathbf{1}_{\tilde{\tau}_k^n < T} \geq \sum_{k \geq 1} c(\tilde{X}_{\tilde{\tau}_k}^{t, x, \tilde{\alpha}}, \tilde{\zeta}_{k-1}, \tilde{\zeta}_k) \mathbf{1}_{\tilde{\tau}_k < T}, \quad \tilde{\mathbb{P}}-a.s.$$

We finally conclude by using Fatou's Lemma.

**Substep 2.4** *The process  $\tilde{X}_s^{t, x, \tilde{\alpha}}$  satisfies the constraint  $\tilde{X}_s^{t, x, \tilde{\alpha}} \in \mathcal{D}$  for all  $s \in [t, T]$ .*

For  $\varepsilon > 0$ , we define the set  $\mathcal{D}_\varepsilon$  by

$$\mathcal{D}_\varepsilon = \left\{ x' \in \mathbb{R}^d : d(x', \mathcal{D}) < \varepsilon \right\}.$$

Suppose that there exists some  $\varepsilon > 0$  such that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_t^T \mathbf{1}_{\mathcal{D}_\varepsilon^c}(\tilde{X}_s^{t, x, \tilde{\alpha}}) ds \right] > 0.$$

From (3.15) and the dominated convergence theorem we can find  $\eta > 0$  and  $n_\eta \geq 1$  such that, up to a subsequence,

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_t^T \mathbf{1}_{\mathcal{D}_\varepsilon^c}(\tilde{X}_s^{t, x, \tilde{\alpha}^n}) ds \right] \geq \eta$$

for all  $n \geq n_\eta$ . From the definition of  $f_n$  and  $g_n$  and the previous inequality, there exists a constant  $C$  such that

$$\tilde{J}(t, x, \tilde{\alpha}^n) \leq C \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \sup_{s \in [t, T]} |\tilde{X}_s^{t, x, \tilde{\alpha}^n}| \right] - n\eta$$

for any  $n \geq \frac{1}{\varepsilon} \vee n_\eta$ . Sending  $n$  to infinity we get from (3.12) and (2.2) applied on  $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}})$

$$\bar{v}(t, x, i) = \lim_{n \rightarrow \infty} \tilde{J}_n(t, x, \tilde{\alpha}^n) = -\infty$$

which contradicts  $\bar{v}(t, x, i) > -\infty$ . We therefore obtain

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T \mathbb{1}_{\mathcal{D}_\varepsilon}(\tilde{X}_s^{t,x,\tilde{\alpha}}) ds\right] = 0$$

for all  $\varepsilon > 0$  and  $\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T \mathbb{1}_{\{\tilde{X}_s^{t,x,\tilde{\alpha}} \notin \mathcal{D}\}} ds\right] = 0$ . Since  $\tilde{X}^{t,x,\tilde{\alpha}}$  is continuous, we get

$$\tilde{\mathbb{P}}\left(\tilde{X}^{t,x,\tilde{\alpha}} \in \mathcal{D}, \forall s \in [t, T]\right) = 1.$$

**Step 3.** *Back to  $(\Omega, \mathcal{G}, \mathbb{P})$  and conclusion.*

We construct  $\alpha^* \in \mathcal{A}_{t,i}$  such that  $(W, \alpha^*)$  has the same law as  $(\tilde{W}, \tilde{\alpha})$ . Using Proposition A.4 we can find Borel functions  $\psi_k$  and  $\phi_k$ ,  $k \geq 1$  such that

$$\tilde{\tau}_k = \psi_k((\tilde{W}_s)_{s \in [0, T]}) \quad \text{and} \quad \tilde{\zeta}_k = \phi_k((\tilde{W}_s)_{s \in [0, T+1]}) \quad \mathbb{P} - a.s.$$

for all  $k \geq 0$ . Define the strategy  $\alpha^* = (\tau_k^*, \zeta_k^*)_{k \geq 0}$  by

$$\tau_k^* = \psi_k((W_s)_{s \in [0, T]}) \quad \text{and} \quad \zeta_k^* = \phi_k((W_s)_{s \in [0, T+1]})$$

for all  $k \geq 0$ . Obviously  $(W, \alpha^*)$  has the same law as  $(\tilde{W}, \tilde{\alpha})$ . Moreover, from Proposition A.5, each  $\tau_k^*$  is an  $\mathbb{F}$ -stopping time and each  $\zeta_k^*$  is  $\mathcal{F}_{\tau_k^*}$ -measurable. We deduce that  $\alpha^* \in \mathcal{A}_{t,i}$ . Using Substep 2.4 we also get  $\alpha^* \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ . From (3.12) and (3.13) we get, up to a subsequence,

$$\tilde{J}(t, x, \tilde{\alpha}) \geq \limsup_{n \rightarrow \infty} \tilde{J}_n(t, x, \tilde{\alpha}^n) = \limsup_{n \rightarrow \infty} J_n(t, x, \alpha^n) \geq \bar{v}(t, x, i).$$

Since  $(W, \alpha^*)$  and  $(\tilde{W}, \tilde{\alpha})$  have the same law and  $\alpha^* \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  we get

$$v(t, x, i) \geq J(t, x, \alpha^*) = \tilde{J}(t, x, \tilde{\alpha}) \geq \bar{v}(t, x, i).$$

□

In general, proving a regularity result on the value function of a constrained optimization problem is very technical (see e.g. [15] or [12]). In our case, Theorem 3.1 gives a semi-regularity for  $v$ .

**Corollary 3.1.** *The function  $v$  is upper semicontinuous on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ .*

**Proof.** Under **(H1)** and **(H2)** the value function  $v_n(\cdot, i)$  associated to the penalized optimal switching problem is continuous (see e.g. [17] or [1]). From Theorem 3.1, the function  $v$  is upper semicontinuous on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  as an infimum of continuous functions. □

## 4 Dynamic programming and variational inequalities

### 4.1 The Dynamic programming principle

In this section we state the dynamic programming principle. We first need the following lemmata

**Lemma 4.1.** Under **(H2)**, the functions  $f_n$  and  $g_n$  are Lipschitz continuous: for any  $n \geq 1$  there exists a constant  $C_n$  such that

$$|g_n(x, i) - g_n(x', i)| + |f_n(x, i) - f_n(x', i)| \leq C_n |x - x'|,$$

for all  $x, x' \in \mathbb{R}^d$ ,  $i \in \mathcal{I}$ .

**Proof.** Fix  $n \geq 1$  and  $i \in \mathcal{I}$ . From the definition of  $f_n$  we have

$$|f_n(x, i) - f_n(x', i)| \leq n |\Theta_n(x) - \Theta_n(x')| + |f(x, i) - f(x', i)|,$$

for all  $x, x' \in \mathbb{R}^d$  and  $i \in \mathcal{I}$ . Since  $f$  and  $d(\cdot, \mathcal{D})$  are Lipschitz continuous, we get from the definition of  $\Theta_n$  the existence of a constant  $C_n$  such that

$$|f_n(x, i) - f_n(x', i)| \leq C_n |x - x'|,$$

for all  $x, x' \in \mathbb{R}^d$ . The proof is the same for  $g_n$ .  $\square$

**Lemma 4.2.** Under **(H1)** and **(H2)**, there exists a constant  $C$  such that

$$v_n(t, x, i) \leq C(1 + |x|) \quad (4.1)$$

for all  $n \geq 1$  and all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ .

**Proof.** Fix  $n \geq 1$  and  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . Using the definition of  $f_n$  and  $g_n$  we have

$$J_n(t, x, \alpha) \leq J_1(t, x, \alpha) \quad (4.2)$$

for any  $\alpha \in \mathcal{A}_{t,i}$ . From (2.2) and **(H2)** there exists a constant  $C$  such that

$$J_1(t, x, \alpha) \leq C(1 + |x|)$$

for any  $\alpha \in \mathcal{A}_{t,i}$ . From (4.2) and the definition of  $v(t, x, i)$ , we get (4.1).  $\square$

We are now able to state the dynamic programming principle.

**Theorem 4.1.** Under **(H1)** and **(H2)**, the value function  $v$  satisfies the following dynamic programming equality:

$$v(t, x, i) = \sup_{\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}_{t,x,i}^{\mathcal{D}}} \mathbb{E} \left[ \int_t^\nu f(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu) \right]. \quad (4.3)$$

for any  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ , and any stopping times  $\nu$  valued in  $[t, T]$ .

**Proof.** We first notice that the l.h.s. of (4.3) is well defined. Indeed, for a given stopping time  $\nu$  valued in  $[t, T]$  and a strategy  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ , we get from the regularity of  $v$  given by Corollary 3.1 that the random quantity  $v(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu)$  is measurable. Moreover, from Lemma 4.2, (2.2) and the inequality  $v \leq v_n$ , we get that its expectation is well defined.

Fix  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . If  $\mathcal{A}_{t,x,i}^{\mathcal{D}} = \emptyset$  then the two hand sides of (4.3) are equal to  $-\infty$  so the equality holds.

Suppose now that  $\mathcal{A}_{t,x,i}^{\mathcal{D}} \neq \emptyset$  and let  $\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  and  $\nu$  a stopping time valued in  $[t, T]$ . From Lipschitz properties of  $f_n$  and  $g_n$  given by Lemma 4.1, we have by Lemma 4.4 in [1]

$$v_n(t, x, i) \geq \mathbb{E} \left[ \int_t^\nu f_n(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) + v_n(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu) \right],$$

for all  $n \geq 1$ . Since  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  we have from the definition of  $f_n$ ,

$$f_n(X_s^{t,x,\alpha}, \alpha_s) = f(X_s^{t,x,\alpha}, \alpha_s)$$

for  $d\mathbb{P} \otimes ds$ -almost all  $(s, \omega) \in [t, T] \times \Omega$ . From Theorem 3.1, Lemma 4.2, (2.2) and the monotone convergence theorem, we get by sending  $n$  to infinity

$$v(t, x, i) \geq \mathbb{E} \left[ \int_t^\nu f(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu) \right].$$

We now prove the reverse inequality. From the definitions of the performance criterion and the value functions, the law of iterated conditional expectations and Markov property of our model, we get the successive relations

$$\begin{aligned} J(t, x, \alpha) &= \\ &\mathbb{E} \left[ \int_t^\nu f(s, X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) \right. \\ &\left. + \mathbb{E} \left[ g(X_T^{t,x,\alpha}) + \int_\nu^T f(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{\nu \leq \tau_k < T} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) \middle| \mathcal{F}_\nu \right] \right] = \\ &\mathbb{E} \left[ \int_t^\nu f(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) + J(\nu, X_\nu^{X^*, t, x, \alpha}, \alpha) \right] \leq \\ &\mathbb{E} \left[ \int_t^\nu f(X_s^{t,x,\alpha}, \alpha_s) ds + \sum_{t \leq \tau_k < \nu} c(X_{\tau_k}^{t,x,\alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{t,x,\alpha}, \alpha_\nu) \right]. \end{aligned}$$

Since  $\nu$  and  $\alpha$  are arbitrary, we obtain the required inequality.  $\square$

## 4.2 Viscosity properties

We prove in this section that the function  $v$  is a solution to a system of variational inequalities. More precisely we consider the following PDE

$$\min \left[ -\frac{\partial v}{\partial t} - \mathcal{L}v - f, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad [0, T] \times \mathcal{D} \times \mathcal{I}, \quad (4.4)$$

$$\min \left[ v - g, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad \{T\} \times \mathcal{D} \times \mathcal{I}. \quad (4.5)$$

where  $\mathcal{L}$  is the second order local operator defined by

$$\mathcal{L}v(t, x, i) = \left( \mu^\top Dv + \frac{1}{2} \text{tr}[\sigma \sigma^\top D^2 v] \right)(t, x, i)$$

and  $\mathcal{H}$  is the nonlocal operator defined by

$$\mathcal{H}v(t, x, i) = \max_{\substack{j \in \mathcal{I} \\ j \neq i}} [v(t, x, j) - c(x, i, j)]$$

for all  $t \in [0, T]$ ,  $x \in \mathcal{D}$  and  $i \in \mathcal{I}$ . As usual, the value functions need not be smooth, and even not known to be continuous a priori. So, we shall work with the notion of (discontinuous) viscosity solutions (see [6]). Generally, for PDEs arising in optimal control problems involving state constraints, we need the notion of constrained viscosity solution introduced by [15] for first order equations to take into account the boundary conditions induced by the state constraints.

For a locally bounded function  $u$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ , we define its lower semicontinuous (lsc in short) envelope  $u_*$ , and upper semicontinuous (usc in short) envelope  $u^*$  by

$$u_*(t, x, i) = \liminf_{\substack{(t', x') \rightarrow (t, x), \\ (t', x') \in [0, T] \times \mathcal{D}}} u(t', x', i), \quad u^*(t, x, i) = \limsup_{\substack{(t', x') \rightarrow (t, x), \\ (t', x') \in [0, T] \times \mathcal{D}}} u(t', x', i).$$

for all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ .

**Remark 4.1.** From Corollary 3.1 and the definition of the usc envelope we have  $v = v^*$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ . However, this equality may not to be true on  $\{T\} \times \mathcal{D} \times \mathcal{I}$ .

We now give the definition of a constrained viscosity solutions to (4.4) and (4.5).

**Definition 4.1** (Constrained viscosity solutions to (4.4)-(4.5)).

(i) A function  $u$ , lsc (resp. usc) on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ , is called a viscosity supersolution on  $[0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$  (resp. subsolution on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ ) to (4.4)-(4.5) if we have

$$\min \left[ -\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(x, i), u(t, x, i) - \mathcal{H}u(t, x, i) \right] \geq (\text{resp. } \leq) 0$$

for any  $(t, x, i) \in [0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$  (resp.  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ ), and any  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$  such that

$$\varphi(t, x) - u(t, x, i) = \max_{[0, T] \times \mathcal{D}} (\varphi - u(\cdot, i)) \quad (\text{resp. } \min_{[0, T] \times \mathcal{D}} (\varphi - u(\cdot, i)))$$

and

$$\min \left[ u(T, x, i) - g(x, i), u(T, x, i) - \mathcal{H}u(T, x, i) \right] \geq (\text{resp. } \leq) 0$$

for any  $x \in \text{Int}(\mathcal{D})$  (resp.  $x \in \mathcal{D}$ ).

(ii) A locally bounded function  $u$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  is called a constrained viscosity solution to (4.4)-(4.5) if its lsc envelope  $u_*$  is a viscosity supersolution to (4.4)-(4.5) on  $[0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$  and its usc envelope  $u^*$  is a viscosity subsolution on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  to (4.4)-(4.5).

We can now state the viscosity property of  $v$ .



**Theorem 4.2.** *Suppose that the function  $v$  is locally bounded. Under **(H1)** and **(H2)**,  $v$  is a constrained viscosity solution to (4.4)-(4.5).*

**Proof of the supersolution property on  $[0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$ .** First, for any  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ , we see, as a consequence of (4.3) applied to  $\nu = t$ , and by choosing any admissible control  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  with immediate switch  $j$  at  $t$ , that

$$v(t, x, i) \geq \mathcal{H}v(t, x, i). \quad (4.6)$$

Now, let  $(\bar{t}, \bar{x}, i) \in [0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$  and  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$  s.t.

$$\varphi(\bar{t}, \bar{x}) - v_*(\bar{t}, \bar{x}, i) = \max_{[0, T] \times \mathcal{D}} (\varphi - v_*(\cdot, i)). \quad (4.7)$$

Since  $v \geq \mathcal{H}v$  on  $[0, T] \times \text{Int}(\mathcal{D}) \times \mathcal{I}$ , we get from the definition of the operator  $\mathcal{H}$  and **(H2)** (i)

$$v_*(\bar{t}, \bar{x}, j) \geq v_*(\bar{t}, \bar{x}, j) + c(\bar{x}, i, j),$$

for all  $j \in \mathcal{I}$ . Therefore we obtain

$$v_*(\bar{t}, \bar{x}, i) \geq \mathcal{H}v_*(\bar{t}, \bar{x}, i).$$

So it remains to show that

$$-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}, i) - \mathcal{L}\varphi(\bar{t}, \bar{x}, i) - f(\bar{x}, i) \geq 0. \quad (4.8)$$

From the definition of  $v_*$  there exists a sequence  $(t_m, x_m)_m$  valued in  $[0, T] \times \text{Int}(\mathcal{D})$  s.t.

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow{m \rightarrow \infty} (\bar{t}, \bar{x}, v_*(\bar{t}, \bar{x}, i)).$$

By continuity of  $\varphi$ ,  $\gamma_m := v(t_m, x_m, i) - \varphi(t_m, x_m) - v_*(\bar{t}, \bar{x}, i) + \varphi(\bar{t}, \bar{x})$  converges to 0 as  $m$  goes to infinity. Since  $(\bar{t}, \bar{x}) \in [0, T] \times \text{Int}(\mathcal{D})$ , there exists  $\eta > 0$  s.t. for  $m$  large enough,  $t_m < T$  and

$$((t_m - \frac{\eta}{2}) \wedge 0, t_m + \frac{\eta}{2}) \times B(x_m, \frac{\eta}{2}) \subset ((t - \eta) \wedge 0, t + \eta) \times B(x, \eta) \subset [0, T] \times \text{Int}(\mathcal{D}).$$

Let us consider an admissible control  $\alpha^m$  in  $\mathcal{A}_{t_m, x_m, i}^{\mathcal{D}}$  with no switch until the first exit time  $\tau_m$  before  $T$  of the associated process  $(s, X_s^m) := (s, X_s^{t_m, x_m, \alpha^m})$  from  $(t_m - \frac{\eta}{2}, t_m + \frac{\eta}{2}) \times B(x_m, \frac{\eta}{2})$ :

$$\tau_m := \inf\{s \geq t_m : (s - t_m) \vee |X_s^m - x_m| \geq \frac{\eta}{2}\} \wedge T.$$

Consider also a strictly positive sequence  $(h_m)_m$  s.t.  $h_m$  and  $\gamma_m/h_m$  converge to 0 as  $m$  goes to infinity. By using the dynamic programming principle (4.3) for  $v(t_m, x_m, i)$  and  $\nu = \hat{\tau}_m := \tau_m \wedge (t_m + h_m)$ , we get

$$\begin{aligned} v(t_m, x_m, i) &= \gamma_m + v_*(\bar{t}, \bar{x}, i) - \varphi(\bar{t}, \bar{x}, i) + \varphi(t_m, x_m, i) \\ &\geq \mathbb{E}\left[\int_{t_m}^{\hat{\tau}_m} f(X_s^m, i) ds + v(\hat{\tau}_m, X_{\hat{\tau}_m}^m, i)\right]. \end{aligned}$$

Using (4.7), we obtain

$$v(t_m, x_m, i) \geq \mathbb{E} \left[ \int_{t_m}^{\hat{\tau}_m} f(X_s^m, i) ds + \varphi(\hat{\tau}_m, X_{\hat{\tau}_m}^m) \right].$$

Applying Itô's formula to  $\varphi(s, X_s^m)$  between  $t_m$  and  $\hat{\tau}_m$  and since  $\sigma(X_s^m, i)D\varphi(s, X_s^m)$  is bounded for  $s \in [t_m, \hat{\tau}_m]$ , we obtain

$$\frac{\gamma_m}{h_m} + \mathbb{E} \left[ \frac{1}{h_m} \int_{t_m}^{\hat{\tau}_m} \left( -\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f \right)(s, X_s^m, i) ds \right] \geq 0, \quad (4.9)$$

for all  $m \geq 1$ . From the continuity of the process  $X^m$ , we have

$$\mathbb{P} \left( \exists m, \forall m' \geq m : \hat{\tau}_{m'} = t_{m'} + h_{m'} \right) = 1.$$

Hence, by the mean-value theorem, the random variable inside the expectation in (4.9) converges a.s. to  $(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f)(\bar{t}, \bar{x}, i)$  as  $m$  goes to infinity. We conclude by the dominated convergence theorem and get (4.8).  $\square$

**Proof of the subsolution property on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ .** We first recall that  $v^* = v$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  from Corollary 3.1. Let  $(\bar{t}, \bar{x}, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$  and  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$  s.t.

$$\varphi(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}, i) = \min_{[0, T] \times \mathcal{D}} (\varphi - v(\cdot, i)). \quad (4.10)$$

If  $v(\bar{t}, \bar{x}, i) \leq \mathcal{H}v(\bar{t}, \bar{x}, i)$  then the subsolution property trivially holds. Consider now the case  $v(\bar{t}, \bar{x}, i) > \mathcal{H}v(\bar{t}, \bar{x}, i)$  and argue by contradiction by assuming on the contrary that

$$\eta := -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \mathcal{L}\varphi(\bar{t}, \bar{x}, i) - f(\bar{x}, i) > 0.$$

By continuity of  $\varphi$  and its derivatives, there exists some  $\delta > 0$  such that  $\bar{t} + \delta < T$  and

$$\left( -\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f \right)(t, x, i) \geq \frac{\eta}{2}, \quad (4.11)$$

for all  $(t, x) \in \mathcal{V} := \left( (\bar{t} - \delta, \bar{t} + \delta) \cap [0, T] \right) \times B(\bar{x}, \delta)$ . By the dynamic programming principle (4.3), given  $m \geq 1$ , there exists  $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_n \in \mathcal{A}_{\bar{t}, \bar{x}, i}^{\mathcal{D}}$  s.t. for any stopping time  $\tau$  valued in  $[\bar{t}, T]$ , we have

$$v(\bar{t}, \bar{x}, i) \leq \mathbb{E} \left[ \int_{\bar{t}}^{\tau} f(\hat{X}_s^m, i) + \sum_{\bar{t} \leq \hat{\tau}_n^m < \tau} c(\hat{X}_{\hat{\tau}_n^m}^m, \hat{\zeta}_n^m, \hat{\zeta}_n^m) + v(\tau, \hat{X}_\tau^m, i) \right] + \frac{1}{m}$$

where  $\hat{X}^m := X^{\bar{t}, \bar{x}, \hat{\alpha}^m}$ . By choosing  $\tau = \bar{\tau}_m := \hat{\tau}_1^m \wedge \nu^m$  where

$$\nu^m := \inf \{ s \geq \bar{t} : (s, \hat{X}_s^m) \notin \mathcal{V} \}$$

is the first exit time of  $(s, \hat{X}_s^m)$  from  $\mathcal{V}$ , we then get

$$\begin{aligned}
v(\bar{t}, \bar{x}, i) &\leq \mathbb{E} \left[ \int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{E} \left[ v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbb{1}_{\nu^m \leq \hat{\tau}_1^m} \right] \\
&\quad + \mathbb{E} \left[ [v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, \hat{\zeta}_1^m) + c(\hat{X}_{\bar{\tau}^m}^m, i, \hat{\zeta}_1^m)] \mathbb{1}_{\nu^m > \hat{\tau}_1^m} \right] + \frac{1}{m} \\
&\leq \mathbb{E} \left[ \int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{E} \left[ v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbb{1}_{\nu^m \leq \hat{\tau}_1^m} \right] \\
&\quad + \mathbb{E} \left[ \mathcal{H}v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbb{1}_{\nu^m > \hat{\tau}_1^m} \right] + \frac{1}{m}. \tag{4.12}
\end{aligned}$$

Now, since  $v \geq \mathcal{H}v$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  and  $\hat{\alpha}^m \in \mathcal{A}_{\bar{t}, \bar{x}, i}^{\mathcal{D}}$ , we obtain from (4.10)

$$\varphi(\bar{t}, \bar{x}, i) \leq \mathbb{E} \left[ \int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds + \varphi(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m) \right] + \frac{1}{m}.$$

Applying Itô's formula to  $\varphi(s, \hat{X}_s^m)$  between  $t_m$  and  $\bar{\tau}^m$  we get:

$$0 \leq \mathbb{E} \left[ \int_{t_m}^{\bar{\tau}^m} \left( \frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi + f \right) (s, \hat{X}_s^m, i) ds \right] + \frac{1}{m} \leq -\frac{\eta}{2} \mathbb{E} [\bar{\tau}^m - \bar{t}] + \frac{1}{m}.$$

This implies

$$\lim_{m \rightarrow +\infty} \mathbb{E} [\bar{\tau}^m] = \bar{t}. \tag{4.13}$$

From the definition of  $\nu^m$  and (4.13) we have, up to a subsequence,

$$\mathbb{P}(\nu^m > \hat{\tau}_1^m) \xrightarrow{m \rightarrow \infty} 1. \tag{4.14}$$

On the other hand, we get from (4.12)

$$\begin{aligned}
v(\bar{t}, \bar{x}, i) &\leq \mathbb{E} \left[ \int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{P}(\nu^m \leq \hat{\tau}_1^m) \sup_{(t', x') \in \text{Adh}(\mathcal{V})} v(t', x', i) \\
&\quad + \mathbb{P}(\nu^m > \hat{\tau}_1^m) \sup_{(t', x') \in \text{Adh}(\mathcal{V})} \mathcal{H}v(t', x', i) + \frac{1}{m}.
\end{aligned}$$

From Lemma 4.2, (4.13) and (4.14) we get by sending  $m$  to  $\infty$

$$v(\bar{t}, \bar{x}, i) \leq \sup_{(t', x') \in \text{Adh}(\mathcal{V})} \mathcal{H}v(t', x', i).$$

Since  $v = v^*$ , we get by sending  $m$  to infinity and  $\delta$  to zero

$$v(\bar{t}, \bar{x}, i) \leq (\mathcal{H}v)^*(\bar{t}, \bar{x}, i) \leq \mathcal{H}v(\bar{t}, \bar{x}, i),$$

which is the required contradiction.  $\square$

**Proof of the viscosity supersolution property on  $\{T\} \times \text{Int}(\mathcal{D}) \times \mathcal{I}$ .** Fix some  $(\bar{x}, i) \in \text{Int}(\mathcal{D}) \times \mathcal{I}$ , and consider a sequence  $(t_m, x_m)_{m \geq 1}$  valued in  $[0, T] \times \text{Int}(\mathcal{D})$ , such that

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow{m \rightarrow \infty} (T, \bar{x}, v_*(T, \bar{x}, i)).$$

Let  $\delta > 0$  s.t.  $B(\bar{x}, \delta) \in \text{Int}(\mathcal{D})$ . We first can suppose w.l.o.g. that

$$B(x_m, \frac{\delta}{2}) \subset B(\bar{x}, \delta) \quad (4.15)$$

for all  $m \geq 1$ . By taking a strategy  $\alpha^m = (\tau_k^m, \zeta_k^m)_k \in \mathcal{A}_{t_m, x_m, i}^{\mathcal{D}}$  with no switch before  $\tau_m := \inf\{s \geq t_m, X_s^m \notin B(x_m, \frac{\delta}{2})\} \wedge T$  with  $X^m := X^{t_m, x_m, \alpha^m}$ , we have from (4.3) applied to  $\tau_m$  and  $\alpha_m$

$$v(t_m, x_m, i) \geq \mathbb{E} \left[ \int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{E} [v(\tau^m, X_{\tau^m}^m, i)]$$

Since  $v(T, \cdot) = g$  we obtain from (4.15)

$$\begin{aligned} v(t_m, x_m, i) &\geq \mathbb{E} \left[ \int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{E} [v(\tau^m, X_{\tau^m}^m, i) \mathbf{1}_{\tau^m < T}] + \mathbb{E} [g(X_{\tau^m}^m, i) \mathbf{1}_{\tau^m = T}] \\ &\geq \mathbb{E} \left[ \int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{P}(\tau^m < T) \inf_{\substack{t < T \\ x \in \text{Adh}(B(\bar{x}, \delta))}} v(t, x, i) \\ &\quad + \mathbb{P}(\tau^m = T) \inf_{x \in \text{Adh}(B(\bar{x}, \delta))} g(x). \end{aligned} \quad (4.16)$$

Since  $\mathbb{E}[\sup_{s \in [t_m, T]} |X_s^m - x_m|]$  converges to zero (see e.g. Corollary 12, Section 5, Chapter 2 in [13]), we have, up to a subsequence,

$$\sup_{s \in [t_m, T]} |X_s^m - x_m| \xrightarrow[m \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0.$$

From the convergence of  $(x_m)_m$  to  $x \in \text{Int}(\mathcal{D})$ , we deduce that

$$\mathbb{P}(\tau^m = T) \xrightarrow[m \rightarrow \infty]{} 1.$$

Sending  $m$  to infinity and  $\delta$  to 0 in (4.16) we get

$$v_*(T, \bar{x}, i) \geq g(\bar{x}, i). \quad (4.17)$$

On the other hand, we know from (4.6) that  $v \geq \mathcal{H}v$  on  $[0, T] \times \text{Int}(\mathcal{D})$ , and thus

$$v(t_m, x_m, i) \geq \mathcal{H}v(t_m, x_m, i) \geq \mathcal{H}v_*(t_m, x_m, i),$$

for all  $m \geq 1$ . Recalling that  $\mathcal{H}v_*$  is lsc, we obtain by sending  $m$  to infinity

$$v_*(T, \bar{x}, i) \geq \mathcal{H}v_*(T, \bar{x}, i).$$

Together with (4.17), this proves the required viscosity supersolution property of (4.5).  $\square$

**Proof of the viscosity subsolution property on  $\{T\} \times \mathcal{D} \times \mathcal{I}$ .** We argue by contradiction by assuming that there exists  $(\bar{x}, i) \in \mathcal{D} \times \mathcal{I}$  such that

$$\min [v^*(T, \bar{x}, i) - g(\bar{x}, i), \mathcal{H}v^*(T, \bar{x}, i)] := 2\varepsilon > 0. \quad (4.18)$$

One can find a sequence of smooth functions  $(\varphi^n)_{n \geq 0}$  on  $[0, T] \times \mathbb{R}^d$  such that  $\varphi^n$  converges pointwisely to  $v^*(\cdot, i)$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$  as  $n \rightarrow \infty$ . Moreover, by (4.18) and the upper semicontinuity of  $v^*$ , we may assume that the inequality

$$\min [\varphi^n - g(\cdot, i), \varphi^n - \max_{j \in \mathcal{I}} \{v^*(\cdot, j) + c(\cdot, i, j)\}] \geq \varepsilon, \quad (4.19)$$

holds on some bounded neighborhood  $B^n$  of  $(T, \bar{x})$  in  $[0, T] \times \mathcal{D}$ , for  $n$  large enough. Let  $(t_m, x_m)_{m \geq 1}$  be a sequence in  $[0, T] \times \mathcal{D}$  such that

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow{m \rightarrow \infty} (T, \bar{x}, v^*(T, \bar{x}, i)).$$

Then there exists  $\delta^n > 0$  such that  $B_m^n := [t_m, T] \times B(x_m, \delta^n) \subset B^n$  for  $m$  large enough, so that (4.19) holds on  $B_m^n$ . Since  $v$  is locally bounded, there exists some  $\eta > 0$  such that  $|v^*| \leq \eta$  on  $B^n$ . We can then assume that  $\varphi^n \geq -2\eta$  on  $B^n$ . Let us define the smooth function  $\tilde{\varphi}_m^n$  by

$$\tilde{\varphi}_m^n(t, x) := \varphi^n(t, x) + \left(4\eta \frac{|x - x_m|^2}{|\delta^n|^2} + \sqrt{T - t}\right)$$

for  $(t, x) \in [0, T] \times \text{Int}(\mathcal{D})$  and observe that

$$(v^* - \tilde{\varphi}_m^n)(t, x, i) \leq -\eta, \quad (4.20)$$

for  $(t, x) \in [t_m, T] \times \partial B(x_m, \delta^n)$ . Since  $\frac{\partial \sqrt{T-t}}{\partial t} \rightarrow -\infty$  as  $t \rightarrow T$ , we have for  $m$  large enough

$$-\frac{\partial \tilde{\varphi}_m^n}{\partial t} - \mathcal{L} \tilde{\varphi}_m^n(\cdot, i) \geq 0 \text{ on } B_m^n. \quad (4.21)$$

Let  $\alpha^m = (\tau_j^m, \zeta_j^m)_j$  be a  $\frac{1}{m}$ -optimal control for  $v(t_m, x_m, i)$  with corresponding state process  $X^m = X^{t_m, x_m, \alpha^m}$ , and denote by  $\theta_n^m = \inf \{s \geq t_m : (s, X_s^m) \notin B_m^n\} \wedge \tau_1^m \wedge T$ . From (4.3) we have

$$\begin{aligned} v(t_m, x_m, i) - \frac{1}{m} &\leq \mathbb{E} \left[ \int_{t_m}^{\theta_n^m} f(X_s^m, i) ds \right] + \mathbb{E} \left[ \mathbf{1}_{\theta_n^m < \tau_1^m \wedge T} v(\theta_n^m, X_{\theta_n^m}^m, i) \right] \\ &\quad + \mathbb{E} \left[ \mathbf{1}_{\theta_n^m = T \leq \tau_1^m} g(X_{\theta_n^m}^m, i) \right] \\ &\quad + \mathbb{E} \left[ \mathbf{1}_{\tau_1^m = \theta_n^m < T} \left( v(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m) + c(X_{\tau_1^m}^m, i, \zeta_1^m) \right) \right]. \end{aligned} \quad (4.22)$$

Now, by applying Itô's Lemma to  $\tilde{\varphi}_n^m(s, X_s^m)$  between  $t_m$  and  $\theta_n^m$  we get from (4.19), (4.20) and (4.21)

$$\begin{aligned} \tilde{\varphi}_m^n(t_m, x_m) &\geq \mathbb{E} \left[ \mathbf{1}_{\theta_n^m < \tau_1^m} \tilde{\varphi}_m^n(\theta_n^m, X_{\theta_n^m}^m) \right] + \mathbb{E} \left[ \mathbf{1}_{\tau_1^m \leq \theta_n^m} \tilde{\varphi}_m^n(\tau_1^m, X_{\tau_1^m}^m) \right] \\ &\geq \mathbb{E} \left[ \mathbf{1}_{\theta_n^m < \tau_1^m \wedge T} \left( v^*(\theta_n^m, X_{\theta_n^m}^m, i) + \eta \right) \right] + \mathbb{E} \left[ \mathbf{1}_{\theta_n^m = T \leq \tau_1^m} \left( g(X_{\theta_n^m}^m, i) + \varepsilon \right) \right] \\ &\quad + \mathbb{E} \left[ \mathbf{1}_{\tau_1^m \leq \theta_n^m < T} \left( v^*(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m) + c(X_{\tau_1^m}^m, i, \zeta_1^m) + \varepsilon \right) \right]. \end{aligned}$$

Together with (4.22), this implies

$$\tilde{\varphi}_m^n(t_m, x_m) \geq v(t_m, x_m, i) - \mathbb{E} \left[ \int_{t_m}^{\theta_n^m} f(X_s^m, i) ds \right] - \frac{1}{m} + \varepsilon \wedge \eta.$$

Sending  $m$ , and then  $n$  to infinity, we get the required contradiction:  $v^*(T, \bar{x}, i) \geq v^*(T, \bar{x}, i) + \varepsilon \wedge \eta$ .  $\square$

## 5 Uniqueness result

This section deals with the uniqueness issue for the SVI (4.4)-(4.5). Unfortunately, we cannot provide a comparison result as the counterexample presented below shows. We then give a weaker characterization of  $v$  as a maximal solution.

### 5.1 A counterexample for comparison

In general, the uniqueness of a viscosity solution to some PDE is given by a comparison theorem. Such a result says that for  $u$  an *usc* supersolution and  $w$  a *lsc* subsolution, we have  $u \geq w$ . Applying this result to  $u = v_*$  the *lsc* envelope of  $v$  and  $w = v^*$  the *usc* envelope of  $v$  we get that  $v_* = v^*$  and  $v$  is continuous. We provide here an example of a switching problem under state constraints where the value function  $v$  is discontinuous.

Fix  $d = 2$  and consider the case where  $\mathcal{D}$  is the smooth domain  $\mathbb{R} \times \mathbb{R}_+$ . Take  $\mathcal{I} = \{1, 2\}$  and define the diffusion coefficients  $\mu$  and  $\sigma$  by

$$\mu(x, 1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mu(x, 2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\sigma(x, 1) = \sigma(x, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all  $x \in \mathbb{R}^2$ . Define the gain coefficients  $g$  and  $f$  by

$$g(x, 1) = g(x, 2) = 0 \quad \text{and} \quad f(x, 1) = f(x, 2) = 1,$$

for all  $x \in \mathbb{R}^2$ , and the cost coefficients  $c(\cdot, 1, 2)$  and  $c(\cdot, 2, 1)$  by

$$c(x, 1, 2) = c(x, 2, 1) = c > 0,$$

for all  $x \in \mathbb{R}^2$ . Then we can directly compute the value function and, due to the state constraints, we have

$$v(t, x, 1) = \begin{cases} T - t & \text{if } x_2 \geq T - t, \\ T - t - c & \text{if } x_2 < T - t, \end{cases}$$

for all  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}$  and all  $t \in [0, T]$ . In particular the function  $v(\cdot, 1)$  is discontinuous at each point  $(t, (x_1, T - t))$  for all  $t \in [0, T]$  and all  $x_1 \in \mathbb{R}$ . Hence the function  $v$  is discontinuous even on the interior  $\text{Int}(\mathcal{D})$  of the constraint domain.

### 5.2 Maximality of the value function as a solution to the SVI

The previous example shows that we cannot obtain a comparison theorem for SVI (4.4)-(4.5) to characterize the value function  $v$ . We provide in this subsection a weaker characterization

of  $v$ . To this end, we introduce, for  $n \geq 1$ , the SVI with penalized coefficients defined on the whole space  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ :

$$\min \left[ -\frac{\partial v}{\partial t} - \mathcal{L}v - f_n, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad [0, T] \times \mathbb{R}^d \times \mathcal{I}, \quad (5.23)$$

$$\min \left[ v - g_n, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad \{T\} \times \mathbb{R}^d \times \mathcal{I}. \quad (5.24)$$

Under assumption **(H1)** and **(H2)**, we can use Lemma 4.1 to apply Proposition 5.1 in [1] and we get from Proposition 4.12 in [1] the following comparison result for this PDE.

**Theorem 5.3.** *Suppose that **(H1)** and **(H2)** hold. Let  $u$  and  $w$  be respectively a subsolution and a supersolution to (5.23)-(5.24). Suppose that there exists two constants  $C_u > 0$  and  $C_w > 0$  and an integer  $\gamma \geq 1$  such that*

$$\begin{aligned} u(t, x, i) &\leq C_u(1 + |x|^\gamma) \\ w(t, x, i) &\geq -C_w(1 + |x|^\gamma) \end{aligned}$$

for all  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ . Then we have  $u \leq w$  on  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ .

We now introduce the following additional assumption on the function  $v$ .

**(H3)** There exists a constant  $C > 0$  and an integer  $\eta \geq 1$  such that

$$v(t, x, i) \geq -C(1 + |x|^\eta) \quad (5.25)$$

for all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ .

We give in the next subsection, some examples where **(H3)** is satisfied. We can state our maximality result as follows.

**Theorem 5.4.** *Under **(H1)**, **(H2)** and **(H3)** the function  $v$  is the maximal constrained viscosity solution to SVI (4.4)-(4.5) satisfying (5.25): for any function  $w : [0, T] \times \mathcal{D} \times \mathcal{I} \rightarrow \mathbb{R}$  such that*

- $w$  is a constrained viscosity solution to (4.4)-(4.5),
- there exists a constant  $C$  and an integer  $\eta \geq 1$  such that

$$w(t, x, i) \geq -C(1 + |x|^\eta) \quad (5.26)$$

for all  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ ,

we have  $v \geq w$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ .

**Proof.** Let  $w : [0, T] \times \mathcal{D} \times \mathcal{I} \rightarrow \mathbb{R}$  be a constrained viscosity solution to (4.4)-(4.5) satisfying (5.26). We proceed in four steps to prove that  $w \leq v$ .

**Step 1.** *Extension of the definition of  $w$  to  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ .*

Define the function  $\tilde{w}$  on  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$  by

$$\tilde{w}(t, x, i) = \begin{cases} w(t, x, i) & \text{for } (t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}, \\ -Ce^{-\rho t}(1 + |x|^{2\eta}) & \text{for } (t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}. \end{cases} \quad (5.27)$$

where  $\rho$  and  $C$  are two positive constants. From **(H1)**, **(H2)** and (5.26), we can find  $\rho$  and  $C$  (large enough) such that

$$-\frac{\partial \tilde{w}}{\partial t} - \mathcal{L}\tilde{w} - f_n \leq 0 \quad \text{on } [0, T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}, \quad (5.28)$$

$$\tilde{w} - g_n \leq 0 \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}, \quad (5.29)$$

and

$$\tilde{w}(t, x, i) \geq -Ce^{-\rho t}(1 + |x|^{2n}) \quad \text{for } (t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}. \quad (5.30)$$

**Step 2.** *Viscosity property of  $\tilde{w}$ .*

For  $C$  and  $\rho$  such that (5.28)-(5.29) and (5.30) hold, we obtain that  $\tilde{w}$  is a viscosity subsolution to (5.23)-(5.24). Indeed, let  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$  and  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$  such that

$$(\tilde{w}^* - \varphi)(t, x, i) = \max_{[0, T] \times \mathbb{R}^d \times \mathcal{I}} (\tilde{w}^* - \varphi). \quad (5.31)$$

We first notice from (5.30) that the upper semicontinuous envelope  $\tilde{w}^*$  of  $\tilde{w}$  is given by

$$\tilde{w}^*(t, x, i) = \begin{cases} w^*(t, x, i) & \text{for } (t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}, \\ -Ce^{-\rho t}(1 + |x|^2) & \text{for } (t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}. \end{cases} \quad (5.32)$$

We now prove that  $\tilde{w}$  is a subsolution to (5.23)-(5.24). Using (5.28), (5.32) and the viscosity subsolution property of  $w$ , we get

$$\tilde{w}^* \leq g \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}.$$

For the viscosity property on  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ , we distinguish two cases.

- Case 1:  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . From (5.31) and (5.32), we have

$$(\tilde{w}^* - \varphi)(t, x, i) = \max_{[0, T] \times \mathcal{D} \times \mathcal{I}} (\tilde{w}^* - \varphi).$$

Since  $w$  is a constrained viscosity solution we get

$$\min \left[ -\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(t, x, i), \varphi(t, x, i) - \mathcal{H}w^*(t, x, i) \right] \leq 0.$$

- Case 2:  $(t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \mathcal{D}) \times \mathcal{I}$ . From (5.28), (5.32) we also get

$$\min \left[ -\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(t, x, i), \varphi(t, x, i) - \mathcal{H}w^*(t, x, i) \right] \leq 0.$$

Therefore,  $\tilde{w}$  is a viscosity subsolution to (5.23)-(5.24).

**Step 3.** *Growth condition on  $v_n$ .*

We prove that for each  $n \geq 1$  there exists a constant  $C_n > 0$  such that

$$v_n(t, x, i) \geq -C_n(1 + |x|^{2n}), \quad (t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}.$$



Fix  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ , and denote by  ${}^0\alpha = ({}^0\tau_k, {}^0\zeta_k)_k$  the trivial strategy of  $\mathcal{A}_{t,i}$  i.e.  ${}^0\tau_0 = t$ ,  ${}^0\zeta_0 = i$  and  ${}^0\tau_k > T$  for  $k \geq 1$ . Then we have

$$v_n(t, x, i) \geq J_n(\mathcal{X}, t, x, {}^0\alpha)$$

From the definition of  $J_n$ , (2.2) and Lemma 4.1 there exists a constant  $\tilde{C}_n > 0$  such that

$$v_n(t, x, i) \geq -\tilde{C}_n(1 + |x|).$$

Since  $\eta \geq 1$ , this implies

$$v_n(t, x, i) \geq -C_n(1 + |x|^{2\eta}).$$

for some constant  $C_n$ .

**Step 4.** *Comparison on  $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ .* From Proposition 4.2 in [1], we know that  $v_n$  is a viscosity solution to (5.23)-(5.24). Using the results of Steps 2 and 3, we can apply Theorem 5.3 to  $\tilde{w}$  and  $v_n$  with  $\gamma = 2\eta$ , and we get

$$\tilde{w}(t, x, i) \leq \tilde{w}^*(t, x, i) \leq v_n(t, x, i),$$

for all  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ . Sending  $n$  to infinity and using Theorem 3.1, we get  $w \leq w^* \leq v$  on  $[0, T] \times \mathcal{D} \times \mathcal{I}$ .  $\square$

**Remark 5.2.** We notice that the counterexample given in the previous section also satisfies this assumption. In particular this gives an example where the classical uniqueness does not hold and where our maximality result is valid.

### 5.3 Some examples for assumption (H3)

We end this Section by explicit examples where (H3) is satisfied. The first one concerns the case of a regime that stops the controlled diffusion.

**Proposition 5.1.** *Suppose that for any  $x \in \partial\mathcal{D}$  there exists  $i_x \in \mathcal{I}$  such that  $\mu(x, i_x) = 0$  and  $\sigma(x, i_x) = 0$ , then assumption (H3) is satisfied.*

**Proof.** Fix an initial condition  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . Let  $X^{t,x}$  be the diffusion defined by

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i) dr + \int_t^s \sigma(X_r^{t,x}, i) dW_r, \quad s \geq t.$$

Consider the strategy  $\alpha : (\tau_k, \zeta_k)_k$  defined by  $(\tau_0, \zeta_0) = (t, i)$ ,

$$\begin{aligned} \tau_1 &= \inf \{s \geq 0 : X_s \in \partial\mathcal{D}\} \\ \zeta_1 &= i_{X_{\tau_1}} \end{aligned}$$

and  $\tau_k > T$  and  $\zeta_k = \zeta_1$  for  $k \geq 2$ . We then have  $\mu(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) = 0$  and  $\sigma(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) = 0$  for  $s \in [\tau_1, T]$ . Therefore, we get  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$  and

$$v(t, x, i) \geq J(\mathcal{X}, t, x, \alpha).$$

From (2.2) and **(H2)** there exists a constant  $C > 0$  such that

$$v(t, x, i) \geq -C(1 + |x|) .$$

By combining this inequality with Lemma 4.2, we get **(H3)**.  $\square$

We now consider the case where for any initial condition, we can find a regime that keeps the diffusion in  $\mathcal{D}$ .

**Proposition 5.2.** *Suppose that for each  $(t, x) \in [0, T] \times \mathcal{D}$ , there exists  $i_{t,x} \in \mathcal{I}$  such that the process  $X^{t,x}$  defined by*

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i_{t,x}) dr + \int_t^s \sigma(X_r^{t,x}, i_{t,x}) dW_r , \quad s \geq t ,$$

*satisfies*

$$\mathbb{P}(X_s^{t,x} \in \mathcal{D}, \forall s \in [t, T]) = 1 . \quad (5.33)$$

*Then assumption **(H3)** is satisfied.*

**Proof.** Fix  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ . Consider the strategy  $\alpha = (\tau_k, \zeta_k)_k$  defined by  $(\tau_0, \zeta_0) = (t, i)$ ,  $(\tau_1, \zeta_1) = (t, i_{t,x})$  and  $\tau_k > T$  for  $k \geq 2$ . From (5.33) we get  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ . We then have

$$v(t, x, i) \geq J(t, x, \alpha) .$$

From (2.2) and **(H2)** there exists a constant  $C > 0$  such that

$$v(t, x, i) \geq -C(1 + |x|) .$$

By combining this inequality with Lemma 4.2, we get **(H3)**.  $\square$

We end this subsection by using a viability result in the case of a convex constraint. For  $x \in \partial\mathcal{D}$ , we define the second order normal cone to  $\mathcal{D}$  at  $x$  by

$$\mathcal{N}_{\mathcal{D}}(x) = \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}^d : p^\top(y - x) + \frac{1}{2}(y - x)^\top A(y - x) \leq o(|y - x|^2) \right. \\ \left. \text{as } y \rightarrow x \text{ and } y \in K \right\} ,$$

where  $\mathbb{S}^d$  is the set of  $d \times d$  symmetric matrices.

**Proposition 5.3.** *Suppose that  $\mathcal{D}$  is convex and there exists  $i^* \in \mathcal{I}$  such that*

$$p^\top \mu(x, i^*) + \frac{1}{2} \text{tr}[\sigma(x, i^*) \sigma(x, i^*)^\top A] \leq 0$$

*for all  $x \in \partial\mathcal{D}$  and all  $(p, A) \in \mathcal{N}_{\mathcal{D}}^2(x)$ . Then assumption **(H3)** is satisfied.*

**Proof.** From Proposition 8 and Remark 9 in [10] we get that for any initial condition  $(t, x, i) \in [0, T] \times \mathcal{D} \times \mathcal{I}$ , the control  $\alpha = (\tau_k, \zeta_k)_k$  define by

$$\begin{aligned}(\tau_0, \zeta_0) &= (t, i) \\ (\tau_1, \zeta_1) &= (t, i^*)\end{aligned}$$

and  $\tau_k > T$  for  $k \geq 2$ , satisfies  $\alpha \in \mathcal{A}_{t,x,i}^{\mathcal{D}}$ . We then have

$$v(t, x, i) \geq J(t, x, \alpha).$$

From (2.2) and **(H2)** there exists a constant  $C > 0$  such that

$$v(t, x, i) \geq -C(1 + |x|).$$

By combining this inequality with Lemma 4.2, we get **(H3)**. □

## A Additional results on convergence and measurability

We first present two results about stopping times and measurability.

**Proposition A.4.** *Let  $(\Omega, \mathcal{G}, \mathbb{P})$  be a complete probability space endowed with a Brownian motion  $B$ . Let  $\mathbb{H} = (\mathcal{H})_{t \geq 0}$  be the complete right-continuous filtration generated by  $B$ ,  $\tau$  an  $\mathbb{H}$ -stopping time and  $\zeta$  an  $\mathcal{H}_\tau$ -measurable random variable. Suppose that there exists a constant  $M$  such that  $\mathbb{P}(\tau \leq M) = 1$ . Then there exist two Borel function  $\psi$  and  $\phi$  such that*

$$\tau = \psi((B_s)_{s \in [0, M]}) \quad \text{and} \quad \zeta = \phi((B_s)_{s \in [0, M+1]}) \quad \mathbb{P} - a.s.$$

**Proof.** Since  $\tau \leq M$   $\mathbb{P}$ -a.s. we can write

$$\tau = \int_0^M \mathbf{1}_{\tau > s} ds = \lim_{n \rightarrow \infty} \frac{M}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\tau > \frac{k}{n}}, \quad \mathbb{P} - a.s. \quad (\text{A.34})$$

Since  $\tau$  is a  $\mathbb{H}$ -stopping time and  $\mathbb{H}$  is the complete right-continuous extension of the natural filtration of  $B$ , we can write from Remark 32, Chapter 2 in [7]

$$\underline{\psi}_n^k((B_s)_{s \in [0, M]}) \leq \mathbf{1}_{\tau > \frac{k}{n}} \leq \bar{\psi}_n^k((B_s)_{s \in [0, M]}) \quad (\text{A.35})$$

and

$$\mathbb{P}\left(\underline{\psi}_n^k((B_s)_{s \in [0, M]}) \neq \bar{\psi}_n^k((B_s)_{s \in [0, M]})\right) = 0 \quad (\text{A.36})$$

where  $\underline{\psi}_n^k$  and  $\bar{\psi}_n^k$  are two Borel functions for any  $n \geq 1$  and any  $k \in \{0, \dots, n-1\}$ . Define the Borel functions  $\bar{\psi}_n$  and  $\underline{\psi}_n$  by

$$\bar{\psi}_n = \frac{M}{n} \sum_{k=0}^{n-1} \bar{\psi}_n^k \quad \text{and} \quad \underline{\psi}_n = \frac{M}{n} \sum_{k=0}^{n-1} \underline{\psi}_n^k$$

We then get from (A.34), (A.35) and (A.36)

$$\limsup_{n \rightarrow \infty} \underline{\psi}_n((B_s)_{s \in [0, M]}) \leq \tau \leq \limsup_{n \rightarrow \infty} \bar{\psi}_n((B_s)_{s \in [0, M]}), \quad \mathbb{P} - a.s.$$

and

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} \underline{\psi}_n((B_s)_{s \in [0, M]}) \neq \limsup_{n \rightarrow \infty} \bar{\psi}_n^k((B_s)_{s \in [0, M]})\right) = 0$$

Taking  $\psi = \limsup_{n \rightarrow \infty} \bar{\psi}_n$  we get  $\tau = \psi((B_s)_{s \in [0, M]})$   $\mathbb{P}$ -a.s.

We now turn to  $\zeta$ . Since  $\zeta$  is  $\mathcal{G}_\tau$ -measurable,  $\zeta \mathbf{1}_{\tau \leq t}$  is  $\mathcal{G}_t$ -measurable for all  $t \geq 0$ . Using  $\tau \leq M$   $\mathbb{P}$ -a.s. we get  $\zeta$  is  $\mathcal{G}_M$ -measurable. Using Remark 32, Chapter 2 in [7] as previously done, we get a Borel function  $\phi$  such that

$$\zeta = \phi((B_s)_{s \in [0, M+1]}) \quad \mathbb{P} - a.s.$$

□

**Proposition A.5.** *Let  $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$ ,  $i = 1, 2$  be two complete probability spaces. Suppose that each  $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$  is endowed with a Brownian motion  $W^i$  and denote by  $\mathbb{F}^i = (\mathcal{F}_t^i)_t$  the filtration satisfying usual conditions generated by  $W^i$ .*

*Fix  $(\tau^i, \zeta^i)$  a couple of random variables defined on  $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i)$  for  $i = 1, 2$  and suppose that*

- $\tau^1$  is an  $\mathbb{F}^1$ -stopping time,
- $\zeta^1$  is  $\mathcal{F}_{\tau^1}^1$ -measurable
- $(W^2, \tau^2, \zeta^2)$  has the same law as  $(W^1, \tau^1, \zeta^1)$ .

*Then  $\tau^2$  is an  $\mathbb{F}^2$  stopping time and  $\zeta^2$  is  $\mathcal{F}_{\tau^2}^2$ -measurable.*

**Proof.** Since  $\tau^1$  is an  $\mathbb{F}^1$ -stopping time and  $\mathbb{F}^1$  is the complete right-continuous extension of the natural filtration of  $(W_s^1)_{s \geq 0}$  and  $(Z_n^1)_{n \geq 0}$ , we can write from Remark 32, Chapter 2 in [7] for any  $r \geq 0$  and any  $\varepsilon > 0$ ,

$$\underline{\psi}((W_s^1)_{s \in [0, r+\varepsilon]}) \leq \mathbf{1}_{\tau^1 \leq r} \leq \bar{\psi}((W_s^1)_{s \in [0, r+\varepsilon]})$$

and

$$\mathbb{P}^1(\underline{\psi}((W_s^1)_{s \in [0, r+\varepsilon]}) \neq \bar{\psi}((W_s^1)_{s \in [0, r+\varepsilon]})) = 0$$

where  $\underline{\psi}$  and  $\bar{\psi}$  are two Borel functions. Since  $(W^1, \tau^1)$  and  $(W^2, \tau^2)$  have the same law we get

$$\mathbb{P}^2(\underline{\psi}((W_s^2)_{s \in [0, r+\varepsilon]}) \leq \mathbf{1}_{\tau^2 \leq r} \leq \bar{\psi}((W_s^2)_{s \in [0, r+\varepsilon]})) = 1$$

and

$$\mathbb{P}^2(\underline{\psi}((W_s^2)_{s \in [0, r+\varepsilon]}) \neq \bar{\psi}((W_s^2)_{s \in [0, r+\varepsilon]})) = 0.$$

Since  $\mathbb{F}^2$  is complete this implies that  $\mathbb{1}_{\tau^2 \leq r}$  is  $\mathcal{F}_{r+\varepsilon}^2$ -measurable. Using the right-continuity of  $\mathbb{F}^2$ , we deduce that  $\mathbb{1}_{\tau^2 \leq r}$  is  $\mathcal{F}_r^2$ -measurable and  $\tau^2$  is an  $\mathbb{F}^2$ -stopping time.

By the same argument, we get that the random variable  $\zeta^2 \mathbb{1}_{\tau^2 \leq r}$  is  $\mathcal{F}_r^2$ -measurable for all  $r \geq 0$ , which is equivalent to the  $\mathcal{F}_{\tau^2}^2$ -measurability of  $\zeta^2$ .  $\square$

We now provide two results on measurability and convergence for a sequence of processes defined on the same space but with different filtrations.

We fix in the sequel a complete probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  on which is defined a sequence of Brownian motions  $(B^n)_{n \geq 0}$ . For  $n \geq 0$ , we denote by  $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$  the complete right-continuous filtration generated by  $B^n$ .

**Proposition A.6.** *For  $n \geq 1$ , let  $\tau^n$  be an  $\mathbb{F}^n$ -stopping time and  $\zeta^n$  be an  $\mathcal{F}_{\tau^n}^n$ -measurable random variable. We suppose that*

(i)  $B^n$  converges to  $B^0$ :

$$\sup_{t \in [0, T]} |B_t^n - B_t^0| \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0,$$

(ii) the sequences  $(\tau^n)_{n \geq 1}$  and  $(\zeta^n)_{n \geq 1}$  are uniformly bounded,

(iii) there exist random variables  $\tau^0$  and  $\zeta^0$  such that

$$(\tau^n, \zeta^n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} (\tau^0, \zeta^0).$$

Then,  $\tau^0$  is an  $\mathbb{F}^0$ -stopping time and  $\zeta$  is  $\mathcal{F}_{\tau^0}$ -measurable.

**Proof.** We first prove that  $\tau_k^0$  is an  $\mathbb{F}^0$ -stopping time. Fix  $t > 0$  and define for  $p \geq 1$ , the bounded and continuous functions  $\Phi_p$  by

$$\Phi_p(x) = \mathbb{1}_{x \leq t - \frac{1}{p}} + p \mathbb{1}_{t - \frac{1}{p} < x \leq t} (t - x), \quad x \in \mathbb{R}_+$$

From Theorem 3.1 of [3] and we get

$$\mathbb{E}[\Phi_p(\tau^n) | \mathcal{F}_t^n] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbb{E}[\Phi_p(\tau_k^0) | \mathcal{F}_t^0].$$

Since  $\tau^n$  is an  $\mathbb{F}^n$ -stopping time we have  $\mathbb{E}[\Phi_p(\tau^n) | \mathcal{F}_t^n] = \Phi_p(\tau^n)$ . Indeed, we can write  $\Phi_p = \lim_{k \rightarrow \infty} \Phi_p^k$  where  $\Phi_p^k$  is defined by

$$\Phi_p^k(x) = \mathbb{1}_{x \leq t - \frac{1}{p}} + \sum_{j=1}^k \frac{j}{kp} \mathbb{1}_{t - \frac{j}{kp} < x \leq t - \frac{j-1}{kp}}, \quad x \in \mathbb{R}_+.$$

Then since  $\tau^n$  is an  $\mathbb{F}^n$  stopping time, the random variable  $\Phi_p^k(\tau^n)$  is  $\mathcal{F}_t^n$ -measurable. Sending  $k$  to infinity, we get that  $\Phi_p(\tau^n)$  is  $\mathcal{F}_t^n$ -measurable.

Since  $\Phi_p$  is continuous we get from (iii)

$$\Phi_p(\tau_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \Phi_p(\tau^0).$$

Therefore  $\Phi_p(\tau^0) = \mathbb{E}[\Phi_p(\tau^0)|\mathcal{F}_t^0]$ . Sending  $p$  to infinity we get  $\mathbb{1}_{\tau^0 \leq t} = \mathbb{E}[\mathbb{1}_{\tau^0 \leq t}|\mathcal{F}_t^0]$  and  $\tau^0$  is a  $\mathbb{F}^0$ -stopping time.

To prove that  $\zeta^0$  is  $\mathcal{F}_{\tau^0}^0$ -measurable, we proceed in the same way and consider  $\zeta^n \Phi_p(\tau^n)$  instead of  $\Phi_p(\tau^n)$  for  $n \geq 0$ .  $\square$

We now turn to stability diffusions. For  $n \geq 0$ , we fix random functions  $b_n : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $a_n : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ . We suppose that

**(HA)**

- (i) For each  $n \geq 0$ ,  $b_n$  and  $a_n$  are  $\mathbb{F}^n$ -progressive  $\otimes \mathcal{B}(\mathbb{R}^d)$ -measurable,
- (ii) there exists  $\delta > 0$  such that

$$\mathbb{E} \left[ \int_0^T \left( |b^n(t, 0)|^{2+\delta} + |a^n(t, 0)|^{2+\delta} \right) dt \right] < +\infty, \quad n \geq 0,$$

- (iii) there exists a constant  $L$  such that

$$|b^n(t, x) - b^n(t, x')| + |a^n(t, x) - a^n(t, x')| \leq L|x - x'|, \quad x, x' \in \mathbb{R}^d, \quad n \geq 0.$$

Then, for a given deterministic initial condition  $X_0$ , we can define for each  $n \geq 0$ , the solution  $X^n$  to the SDE

$$X_t^n = X_0 + \int_0^t b^n(s, X_s^n) ds + \int_0^t a^n(s, X_s^n) dB_s^n \quad t \geq 0.$$

**Proposition A.7.** *Suppose that*

$$\sup_{t \in [0, T]} |B_t^n - B_t^0| \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0, \quad (\text{A.37})$$

and

$$\mathbb{E} \left[ \int_0^T |a^n(s, x) - a^0(s, x)|^2 ds \right] + \mathbb{E} \left[ \int_0^T |b^n(s, x) - b^0(s, x)|^2 ds \right] \xrightarrow[n \rightarrow +\infty]{} 0, \quad (\text{A.38})$$

for all  $x \in \mathbb{R}^d$ . Then, under **(HA)**, we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^n - X_t^0|^2 \right] \xrightarrow[n \rightarrow \infty]{} 0. \quad (\text{A.39})$$

To prove this result we cannot use classical estimates on diffusions processes since the driving Brownian motion evolves with  $n$ . In particular the stochastic integrals  $\int a^n dB^0$  are not defined. We therefore need to use approximations by step processes as done in the construction of the Itô integral.

**Proof.** We proceed in two steps.

**Step 1.** We first consider the case where the  $b^n$  and  $a^n$  do not depend on the variable  $x$ . For  $p \geq 1$ , Let  $H^p$  be an  $\mathbb{F}$ -adapted piecewise constant process of the form

$$H_t^p = \sum_{k=0}^{N_p} \tilde{H}_k^p \mathbb{1}_{[t_k^p, t_{k+1}^p)}(t), \quad t \in [0, T]$$

where  $\tilde{H}_k^p \in \mathbf{L}^2(\Omega, \mathcal{F}_{t_k^p}, \mathbb{P})$  for  $0 \leq k \leq N_p$ , such that

$$\mathbb{E} \left[ \int_0^T |H_s^p - a_s|^2 ds \right] \leq \frac{1}{p}. \quad (\text{A.40})$$

We then have

$$\mathbb{E} \left[ \left| \int_0^T a^n dB^n - \int_0^T a dB^0 \right|^2 \right] \leq 2 \left( E \left[ \left| \int_0^T a^n dB^n - \int_0^T H^p dB^0 \right|^2 \right] + \frac{1}{p} \right). \quad (\text{A.41})$$

We then define the process  $H^{p,n}$  by

$$H_t^{p,n} = \sum_{k=0}^{N_p} \mathbb{E} \left[ \tilde{H}_k^p | \mathcal{F}_{t_k^p}^n \right] \mathbb{1}_{[t_k^p, t_{k+1}^p)}(t), \quad t \in [0, T]$$

$$\begin{aligned} \mathbb{E} \left[ \left| \int_0^T a^n dB^n - \int_0^T H^{p,n} dB^0 \right|^2 \right] &\leq 2 \left( E \left[ \left| \int_0^T a^n dB^n - \int_0^T H^p dB^0 \right|^2 \right] \right. \\ &\quad \left. + E \left[ \left| \int_0^T H^{p,n} dB^n - \int_0^T H^p dB^0 \right|^2 \right] \right) \end{aligned} \quad (\text{A.42})$$

From (A.37), we can apply Proposition 2 in [5] and we get

$$\mathbb{E} \left[ \tilde{H}_k^p | \mathcal{F}_{t_k^p}^n \right] \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \tilde{H}_k^p, \quad 0 \leq k \leq N_p. \quad (\text{A.43})$$

In particular we get from (A.37) and (A.43)

$$\mathbb{E} \left[ \left| \int_0^T H^{p,n} dB^n - \int_0^T H^p dB^0 \right|^2 \right] \xrightarrow[n \rightarrow +\infty]{} 0. \quad (\text{A.44})$$

Moreover, from Itô Isometry and (A.40) we have

$$\begin{aligned} E \left[ \left| \int_0^T a^n dB^n - \int_0^T H^{p,n} dB^n \right|^2 \right] &= E \left[ \int_0^T |a_s^n - H_s^{p,n}|^2 ds \right] \\ &\leq 3 \left( E \left[ \int_0^T |a_s^n - a_s^0|^2 ds \right] + \frac{1}{p} \right) \\ &\quad + E \left[ \int_0^T |H_s^p - H_s^{p,n}|^2 ds \right]. \end{aligned} \quad (\text{A.45})$$

Then using (A.43), we also get

$$\mathbb{E} \left[ \int_0^T |H_s^p - H_s^{p,n}|^2 ds \right] \xrightarrow[n \rightarrow +\infty]{} 0. \quad (\text{A.46})$$

Therefore, we get from (A.38), (A.45) and (A.46)

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[ \left| \int_0^T a^n dB^n - \int_0^T H^{p,n} dB^n \right|^2 \right] \leq \frac{1}{p}.$$

From this last inequality, (A.41), (A.42) and (A.44) we get

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[ \left| \int_0^T a^n dB^n - \int_0^T a^0 dB^0 \right|^2 \right] \leq \frac{4}{p}, \quad p \geq 1.$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left| \int_0^T a^n dB^n - \int_0^T a^0 dB^0 \right|^2 \right] = 0 .$$

From Theorem 3.1 in [3], we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t a^n dB^n - \int_0^t a^0 dB^0 \right|^2 \right] = 0$$

for all  $\delta \in [0, 1)$ . From this last equality and (A.38), we get (A.39).

**Step 2.** We now consider the general case. For  $n \geq 0$ , we denote by  $(X^{n,p})_{p \geq 0}$  the sequence of processes defined by

$$X_t^{n,0} = X_0, \quad t \in [0, T],$$

and

$$X_t^{n,p+1} = X_0 + \int_0^t b^n(s, X_s^{n,p}) ds + \int_0^t a^n(s, X_s^{n,p}) dB_s^n, \quad t \in [0, T],$$

for  $p \geq 0$ . By induction we get from Step 1 that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^{n,p} - X_t^{0,p}|^2 \right] \xrightarrow{n \rightarrow \infty} 0 \quad (\text{A.47})$$

for all  $p \geq 0$ . Using classical argument on diffusion processes, we have (see e.g. the proof of Theorem 2.9 of Chapter 5 in [11])

$$\sup_{n \geq 0} \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^{n,p} - X_t^n|^2 \right] \leq \psi(p)$$

where  $\psi(p) \rightarrow 0$  as  $p \rightarrow +\infty$ . We then get

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^n - X_t^0|^2 \right] \leq 2\psi(p) + \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^{n,p} - X_t^{0,p}|^2 \right] \leq 2\psi(p) .$$

Sending  $p$  to  $\infty$ , we get the result.  $\square$

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