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Optimal Switching in Finite Horizon under State Constraints

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Abstract

We study an optimal switching problem with a state constraint: the controller is only allowed to choose strategies that keep the controlled diffusion in a closed domain. We prove that the value function associated to the weak formulation of this problem is the limit of the value function associated to an unconstrained switching problem with penalized coefficients, as the penalization parameter goes to infinity. This convergence allows to set a dynamic programming principle for the constrained switching problem. We then prove that the value function is a constrained viscosity solution to a system of variational inequalities (SVI for short). We finally prove that the value function is the maximal solution to this SVI. All our results are obtained without any regularity assumption on the constraint domain.

Key words: Optimal switching, state constraints, dynamic programming, variational inequalities, energy management.

Mathematics Subject Classification (2010): 60H10, 60H30, 91G80, 93E20.

1 Introduction

Optimal control of multiples switching regimes consists in looking for the value of an optimization problem where the allowed strategies consist in sequences of interventions. It naturally arises in many applied disciplines where it is not realistic to consider that the involved quantities can be continuously controlled. More precisely, the optimal switching problem supposes that the control strategies are sequences $\alpha = (\tau_k, \zeta_k)_{k \geq 1}$ where the sequence $(\tau_k)_{k \geq 1}$ represents the intervention times of the controller and ζ_k corresponds to the level of intervention of the agent at each time τ_k .

Such a class of strategies allows to consider that the action of the controller is not continuous which is more realistic than continuous time controls. Therefore, the modelization with optimal switching problem has attracted a lot of interest during the last decades (see e.g. Brennan and Schwarz [2] for resource extraction, Dixit [6] for production facility

problems, Carmona and Ludkovski [3] for power plant management or Ly Vath, Pham and Villeneuve [11] for dividend decision problem with reversible technology investment).

Another specificity to take into account in the modelization with optimal switching is the limitation of the quantities involved in the control problem. Indeed, in most of management problems the controlled system is subject to a constraint on the possible states that it can take. For example, a solvency condition is usually imposed to the investors of a financial market and the energy producer has to take into account the limited storage capacities. This leads to impose a state constraint on the controlled diffusion X of our switching problem of the form

$$X_s \in \bar{\mathcal{O}} \quad \text{for all } s ,$$

where $\bar{\mathcal{O}}$ is the closure of an open set \mathcal{O} . We therefore need to restrict our control problem to the set $\mathcal{A}_{t,x}^{\bar{\mathcal{O}}}$ of strategies that keep the controlled diffusion starting from (t, x) in the constraint domain $\bar{\mathcal{O}}$. Unfortunately, such a constraint leads to strong difficulties due, in particular, to the complicated structure of the set valued function $(t, x) \mapsto \mathcal{A}_{t,x}^{\bar{\mathcal{O}}}$.

In the continuous time control case, H. M. Soner gives in [12] a first study of the constrained problem in a deterministic framework where he introduces the notion of constrained viscosity solutions. To characterize the value function, his approach relies on a continuity argument under an assumption on the boundary of the constraint domain $\partial\mathcal{O}$. He then extends this result to the case of piecewise deterministic processes in [13]. The continuous time stochastic control case is studied by M. A. Katsoulakis in [9]. His approach also relies on continuity and he imposes some regularity condition on the constraint domain $\bar{\mathcal{O}}$. In our case, such an approach is not possible since the value function may be discontinuous even for a smooth domain $\bar{\mathcal{O}}$ as shown by the counterexample presented in Subsection 5.1.

Let us also mention the recent approach of D. Goreac *et al.* presented in [8]. They formulate the initial problem as a linear problem which concerns the occupation measures induced by the controlled diffusion processes. Under convexity assumptions, the authors characterize (see Theorem 11 in [8]) the value function associated to the weak formulation of the continuous time stochastic control problem under state constraints (the weak formulation means that the controller is allowed to choose the probability space in addition to the control strategy).

In this work, we use an original approach which does not need any regularity or convexity assumption. We take advantage of the simple structure of switching controls. Indeed, they can be seen as random variables taking values in $([0, T] \times \mathcal{I})^{\mathbb{N}}$ where \mathcal{I} is a finite set and $T > 0$ is a given constant. From Tychonov theorem, we get the compactness of this space which allows to prove the tightness hence the convergence in law of a sequence of strategies. Then from Skorokhod representation theorem, we are able to provide a probability space on which we have an almost surely convergence.

We then use this sequential compactness property in the following way. We first introduce a sequence $(v_n)_n$ of unconstrained switching problems with n -penalized terminal and running reward coefficients out of the constraint domain $\bar{\mathcal{O}}$. For each penalized switching problems v_n , we consider an $\frac{1}{n}$ -almost optimal strategy α^n . Using the previous compactness property, we make the sequence $(\alpha^n)_n$ converge to some strategy α^∞ that is proved

to be optimal for the switching problem under constraint. As a byproduct, we get that the sequence of unconstrained penalized switching problems converges to the constrained one. Then using the results on classical optimal switching problems, this convergence allows to set a dynamic programming principle for the constrained switching problem.

We then focus on the PDE characterization of the value function. Using the dynamic programming principle proved before, we show that the value function is a constrained viscosity solution to a system of variational inequalities (SVI for short) defined on the constraint domain $\bar{\mathcal{O}}$. We then investigate the uniqueness of a solution to this SVI. The usual approach to get uniqueness of a viscosity solution consists in proving a comparison theorem for the PDE. Such a result tells that a supersolution is greater than a subsolution. As a consequence of this comparison theorem, the unique solution has to be continuous. Unfortunately, the continuity of the value functions is not true in general as shown by the counterexample given in Subsection 5.1. Therefore, we cannot hope to state such a uniqueness result for the SVI on $\bar{\mathcal{O}}$. Instead, we characterize our value function as the maximal viscosity solution of the SVI under an additional linear growth assumption. This maximality property is also obtained from the convergence on the penalized unconstrained problems to the constrained one.

We end the introduction by the description of the organization of the paper. In Section 2 we expose in detail the weak formulation of the optimal switching problem under state constraints. We then give an application to electricity production management. In Section 3, we provide an approximation of our constrained problem by unconstrained problem with penalized coefficients. We prove the convergence of the penalized problems to the constrained one when the penalization parameter goes to infinity. In Section 4 we state a dynamic programming principle and we prove that the value function is a constrained viscosity solution to a system of variational inequalities. Finally, in section 5 we deal with uniqueness. We first show by a counterexample that we cannot prove uniqueness of a solution to the SVI. Under an additional growth assumption, we characterize the value function as the maximal constrained viscosity solution to the SVI. We end by giving examples where this additional growth condition is satisfied.

2 Problem formulation

2.1 Optimal switching under state constraints

We consider the following weak formulation of optimal switching problem under state constraints.

Controls. Using the terminology of [15] we define a set-up as a quadruple \mathcal{X} given by

$$\mathcal{X} = (\Omega, \mathcal{G}, \mathbb{P}, W, \mathbb{F})$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space endowed with a d -dimensional Brownian motion $W = (W_t)_{t \geq 0}$ and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the filtration satisfying usual conditions generated by W . We fix a terminal time $T > 0$. For $t \in [0, T]$, we denote by $\mathbb{F}^t = (\mathcal{F}_s)_{s \geq t}$ the complete

right-continuous filtration generated by $(W_s - W_t)_{s \geq t}$. We then define the set $\mathcal{A}_t(\mathcal{X})$ of \mathcal{X} -admissible switching controls at time $t \in [0, T]$ as the set of double sequences $\alpha = (\tau_k, \zeta_k)_{k \geq 0}$ where $(\tau_k)_{k \geq 0}$ is a nondecreasing sequence of \mathbb{F}^t -stopping times with $\tau_0 = t$ and $\lim_{k \rightarrow \infty} \tau_k = +\infty$, and ζ_k are $\mathcal{F}_{\tau_k}^t$ -measurable random variables valued in the set \mathcal{I} defined by $\mathcal{I} = \{1, \dots, m\}$. To a strategy $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t(\mathcal{X})$ we associate the process $(\alpha_s)_{s \geq t}$ defined by

$$\alpha_s = \sum_{k \geq 0} \zeta_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(s), \quad s \geq t.$$

Controlled diffusion. We then are given two functions $\mu : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}^{d \times d}$. We make the following assumption.

(H1) There exists a constant L such that

$$|\mu(x, i) - \mu(x', i)| + |\sigma(x, i) - \sigma(x', i)| \leq L|x - x'|,$$

for all $(x, x', i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{I}$.

For $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}_t(\mathcal{X})$ we consider the controlled diffusion $X^{\mathcal{X}, t, x, \alpha}$ defined on \mathcal{X} by the following SDE

$$X_s^{\mathcal{X}, t, x, \alpha} = x + \int_t^s \mu(X_r^{\mathcal{X}, t, x, \alpha}, \alpha_r) dr + \int_t^s \sigma(X_r^{\mathcal{X}, t, x, \alpha}, \alpha_r) dW_r, \quad s \geq t. \quad (2.1)$$

Under **(H1)**, we have existence and uniqueness of an \mathbb{F}^t -adapted solution $X^{\mathcal{X}, t, x, \alpha}$ to (2.1) for any set-up \mathcal{X} , any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$ and any switching control $\alpha \in \mathcal{A}_t(\mathcal{X})$.

We have the following classical estimate (see e.g. Corollary 12, Section 5, Chapter 2 in [10]): for any $q \geq 1$ there exists a constant C_q such that

$$\sup_{\mathcal{X}} \sup_{\alpha \in \mathcal{A}_t(\mathcal{X})} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [t, T]} |X_s^{\mathcal{X}, t, x, \alpha}|^q \right] \leq C_q (1 + |x|^q) \quad (2.2)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Expected Payoff. We are then given a terminal and running reward functions $g : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ and $f : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ and a cost function $c : \mathbb{R}^d \times \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}$ on which we impose the following assumption:

(H2)

(i) There exists a constant L such that

$$|g(x, i) - g(x', i)| + |f(x, i) - f(x', i)| + |c(x, i, j) - c(x', i, j)| \leq L|x - x'|,$$

for all $x, x' \in \mathbb{R}^d$ and $i, j \in \mathcal{I}$.

(ii) There exists a constant $\bar{c} > 0$, such that

$$c(x, i, j) \geq \bar{c},$$

for all $x \in \mathbb{R}^d$ and $i, j \in \mathcal{I}$.

We then define the functional pay-off J by

$$J(\mathcal{X}, t, x, \alpha) = \mathbb{E} \left[g(X_T^{\mathcal{X}, t, x, \alpha}, \alpha_T) + \int_t^T f(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) ds - \sum_{k \geq 1} c(X_{\tau_k}^{\mathcal{X}, t, x, \alpha}, \zeta_{k-1}, \zeta_k) \mathbb{1}_{\tau_k \leq T} \right]$$

for all set-up \mathcal{X} , $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}_t(\mathcal{X})$.

Under **(H1)** and **(H2)** we get from (2.2) that $J(\mathcal{X}, t, x, \alpha)$ is well defined for any set-up \mathcal{X} , any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$ and any control $\alpha \in \mathcal{A}_t(\mathcal{X})$.

State constraint. Let $\bar{\mathcal{O}}$ be a nonempty open subset of \mathbb{R}^d . For $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ we denote by $\mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X})$ the set of strategies $\alpha \in \mathcal{A}_t(\mathcal{X})$ such that $\zeta_0 = i$ and

$$\mathbb{P} \left(X_s^{\mathcal{X}, t, x, \alpha} \in \bar{\mathcal{O}} \text{ for all } s \in [t, T] \right) = 1.$$

Value function. We then define the value function v associated to the switching problem under state constraint by

$$v(t, x, i) = \sup_{\mathcal{X}} \sup_{\alpha \in \mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X})} J(\mathcal{X}, t, x, \alpha) \quad (2.3)$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$, with the convention $v(t, x, i) = -\infty$ if $\mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X}) = \emptyset$ for all set-up \mathcal{X} . Our aim is to give an analytic characterization of the function v .

2.2 A hydroelectric pumped storage model

We present in this Subsection an energy management model involving an optimal switching problem under state constraint.

The following -simplified- hydroelectric pumped storage model is inspired from [3]. Pumped Storage (currently, the dominant type of electricity storage) consists of large reservoir of water held by a hydroelectric dam at a higher elevation. When desired, the dam can be opened which activates the turbines and moves the water to another, lower reservoir. The generated electricity is sold to a power grid. As the water flows, the upper reservoir is depleted. Conversely, in times of low electricity demand, the water can be pumped back into the reservoir with required energy purchased from grid. For a fixed set-up $\mathcal{X} = (\Omega, \mathcal{A}, \mathbb{P}, W, \mathbb{F})$, a strategy α consists in a sequence of \mathbb{F} -stopping times $(\tau_k)_k$ representing the intervention times and a sequence of \mathcal{F}_{τ_k} -measurable random variables $(\zeta_k)_k$ representing the changes of regime. There are three possible regimes (i) $\zeta_k = 1$:

pump, (ii) $\zeta_k = 0$: store and (iii) $\zeta_k = -1$: generate. For a given strategy $\alpha = (\tau_k, \zeta_k)_k$, we denote by L_t^α the controlled water level in the upper reservoir. It satisfies the equation

$$dL_t^\alpha = \zeta_k dt \quad \text{for } t \in [\tau_k, \tau_{k+1}), \quad (2.4)$$

Denote by P the electricity price process and suppose that it is a diffusion defined on \mathcal{X} . Suppose also that the cost of changing the regime from i to j is given by a constant $c(i, j)$. The expected gain for a given strategy α is then given by

$$\begin{aligned} J(\mathcal{X}, \alpha) &= \mathbb{E}^\mathbb{P} \left[\int_0^T -P_t dL_t^\alpha - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k) \right] \\ &= \mathbb{E}^\mathbb{P} \left[\int_0^T f(P_t, L_t^\alpha, \alpha_t) dt - \sum_{\tau_k \leq T} c(\zeta_{k-1}, \zeta_k) \right] \end{aligned}$$

where f is defined by $f(p, \ell, i) = -p \times i$ for all $(p, \ell, i) \in \mathbb{R} \times \mathbb{R} \times \{-1, 0, 1\}$.

Since the reservoir capacity is not infinite, the strategy α has to satisfy the constraint $0 \leq L_t^\alpha \leq \ell_{max}$ for all $t \in [0, T]$. This correspond to the general constraint $X_t^\alpha \in \bar{\mathcal{O}}$ where $X^\alpha = (P, L^\alpha)$ and $\bar{\mathcal{O}} = \mathbb{R} \times [0, \ell_{max}]$. The goal of the energy producer is to maximize $J(\mathcal{X}, \alpha)$ over the strategies α satisfying the constraint on the water level L^α .

3 Unconstrained penalized switching problem

3.1 An unconstrained penalized approximating problem

We now introduce an approximation of our initial constrained problem. This approximation consists in a penalization of the gain coefficients f and g out of the domain $\bar{\mathcal{O}}$ where the controlled underlying diffusion is constrained to stay.

Consider, for $n \geq 1$, the functions $f_n : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ and $g_n : \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ defined by

$$f_n(x, i) = (1 - \Theta_n(x))f(x, i) - n\Theta_n(x), \quad (3.1)$$

$$g_n(x, i) = (1 - \Theta_n(x))g(x, i) - n\Theta_n(x), \quad (3.2)$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$, where the function $\Theta_n : \mathbb{R}^d \rightarrow [0, 1]$ is given by

$$\Theta_n(x) = n \left(d(x, \bar{\mathcal{O}}) \wedge \frac{1}{n} \right) = nd(x, \bar{\mathcal{O}}) \wedge 1, \quad (3.3)$$

with $d(x, \bar{\mathcal{O}}) = \inf_{x' \in \bar{\mathcal{O}}} |x - x'|$ for all $x \in \mathbb{R}^d$.

Given a set-up \mathcal{X} , an initial condition (t, x) and a switching control $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t(\mathcal{X})$, we consider the total penalized profit starting from $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ at horizon T , defined by:

$$J_n(\mathcal{X}, t, x, \alpha) = \mathbb{E} \left[g_n(X_T^{\mathcal{X}, t, x, \alpha}, \alpha_T) + \int_t^T f_n(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) ds - \sum_{k \geq 1} c(X_{\tau_k}^{\mathcal{X}, t, x, \alpha}, \zeta_{k-1}, \zeta_k) \mathbf{1}_{\tau_k \leq T} \right]$$

We can then define the penalized unconstrained value function $v_n : [0, T] \times \mathbb{R}^d \times \mathcal{I} \rightarrow \mathbb{R}$ by

$$v_n(t, x, i) = \sup_{\mathcal{X}} \sup_{\alpha \in \mathcal{A}_{t,i}(\mathcal{X})} J_n(\mathcal{X}, t, x, \alpha), \quad (3.4)$$

for all $n \geq 1$ and all $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$, where $\mathcal{A}_{t,i}(\mathcal{X})$ is the set of strategies $\alpha = (\tau_k, \zeta_k)_{k \geq 0} \in \mathcal{A}_t(\mathcal{X})$ such that $\zeta_0 = i$.

3.2 Convergence of the penalized unconstrained problems

We now state the main result of this section.

Theorem 3.1. *Under (H1) and (H2), the sequence $(v_n)_{n \geq 1}$ is nonincreasing and converges on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ to the function v :*

$$v_n(t, x, i) \downarrow v(t, x, i) \quad \text{as } n \uparrow +\infty, \quad (3.5)$$

for all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. Moreover, there exists a set-up $\mathcal{X}^* = (\Omega^*, \mathcal{G}^*, \mathbb{P}^*, W^*, \mathbb{F}^*)$ such that

$$v_n(t, x, i) = \sup_{\alpha \in \mathcal{A}_{t,i}(\mathcal{X}^*)} J_n(\mathcal{X}^*, t, x, \alpha)$$

for all $n \geq 1$ and all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.

Proof. Fix $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. Since $f_{n+1} \leq f_n$ and $g_{n+1} \leq g_n$ we get

$$J_{n+1}(\mathcal{X}, t, x, \alpha) \leq J_n(\mathcal{X}, t, x, \alpha), \quad n \geq 1,$$

for any set-up \mathcal{X} and any $\alpha \in \mathcal{A}_t(\mathcal{X})$. From this last inequality we deduce that

$$v_{n+1}(t, x, i) \leq v_n(t, x, i), \quad n \geq 1.$$

We now prove that $(v_n)_n$ converges to v . We first notice that

$$J_n(\mathcal{X}, t, x, \alpha) = J(\mathcal{X}, t, x, \alpha),$$

for any $n \geq 1$, any set-up \mathcal{X} , any initial condition $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ and any switching strategy $\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X})$. Therefore, we get $v_n \geq v$ for all $n \geq 1$. Denote by \bar{v} the pointwise limit of $(v_n)_n$:

$$\bar{v}(t, x, i) = \lim_{n \rightarrow \infty} v_n(t, x, i), \quad (t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}.$$

Then we have $\bar{v}(t, x, i) \geq v(t, x, i)$. If $\bar{v}(t, x, i) = -\infty$ then we obviously have $\bar{v}(t, x, i) = v(t, x, i)$.

We now suppose that $\bar{v}(t, x, i) > -\infty$ and prove that $\bar{v}(t, x, i) \leq v(t, x, i)$. We proceed in 8 steps.

Step 1. *Sequence of almost optimal strategies for the unconstrained problems.*

For $n \geq 1$, let $\mathcal{X}^n = (\Omega^n, \mathcal{G}^n, \mathbb{P}^n, W^n, \mathbb{F}^n)$ be a set-up and $\alpha^n = (\tau_k^n, \zeta_k^n)_{k \geq 0} \in \mathcal{A}_{t,i}(\mathcal{X}^n)$ a switching strategy such that

$$J_n(\mathcal{X}^n, t, x, \alpha^n) \geq v_n(t, x, i) - \frac{1}{n}.$$

Step 2. *Tightness of the sequence (W^n, α^n) .*

We now prove that the sequence of $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$ -valued random variables $(W^n, \alpha^n)_{n \geq 1}$ is tight.

We first notice that we can suppose w.l.o.g. that

$$\tau_k^n \in [0, T + 1] \quad \mathbb{P}^n - a.s. \quad (3.6)$$

for all $n \geq 1$ and all $k \geq 0$. Indeed, fix $n \geq 1$ and consider the strategy $\tilde{\alpha}^n = (\tilde{\tau}_k^n, \tilde{\zeta}_k^n)_{k \geq 0} \in \mathcal{A}_{t,i}(\mathcal{X}^n)$ defined by

$$\begin{aligned} \tilde{\tau}_k^n &= \tau_k^n \wedge (T + 1) \\ \tilde{\zeta}_k^n &= \zeta_k^n \mathbb{1}_{\tau_k^n \leq T+1} + i \mathbb{1}_{\tau_k^n > T+1} \end{aligned}$$

Then we have $J_n(\mathcal{X}^n, t, x, \alpha^n) = J_n(\mathcal{X}^n, t, x, \tilde{\alpha}^n)$.

We now define for $\eta > 0$ and $C > 0$ the subset \mathcal{K}_η^C of $C([0, T], \mathbb{R}^d)$ by

$$\mathcal{K}_\eta^C = \left\{ h \in C([0, T], \mathbb{R}^d) : h(0) = 0 \text{ and } \text{mc}_\delta(h) \leq C \frac{\delta \ln\left(\frac{2T}{\delta}\right)}{\eta} \text{ for all } \delta > 0 \right\}$$

where mc_δ denotes the modulus of continuity defined by

$$\text{mc}_\delta(h) = \sup_{\substack{s, t \in [0, T] \\ |s - t| \leq \delta}} |h(s) - h(t)|$$

for any $h \in C([0, T], \mathbb{R}^d)$. From Arzéla-Ascoli Theorem, we know that \mathcal{K}_η^C is a compact subset of $C([0, T], \mathbb{R}^d)$.

We now define the subset \mathbf{K}_η^C of $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$ by

$$\mathbf{K}_\eta^C = \mathcal{K}_\eta^C \times ([0, T + 1] \times \mathcal{I})^{\mathbb{N}}.$$

From Tychonov theorem and since \mathcal{K}_η^C is compact, we get that \mathbf{K}_η^C is a compact subset of $C([0, T], \mathbb{R}^d) \times (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$ endowed with the norm $\|\cdot\|$ defined by

$$\|(h, (t_k, z_k)_{k \geq 0})\| = \sup_{t \in [0, T]} |h(t)| + \sum_{k \geq 0} \frac{(|t_k| + |z_k|) \wedge 1}{2^k}$$

for all $h \in C([0, T], \mathbb{R}^d)$ and $(t_k, z_k)_{k \geq 0} \in (\mathbb{R}_+ \times \mathcal{I})^{\mathbb{N}}$. We then have from (3.6)

$$\mathbb{P}^n\left((W^n, \alpha^n) \in \mathbf{K}_\eta^C\right) = \mathbb{P}^n\left(W^n \in \mathcal{K}_\eta^C\right)$$

for all $\eta > 0$, $C > 0$ and $n \geq 1$. Then using Markov inequality we get

$$\begin{aligned} \mathbb{P}^n\left(W^n \in \mathcal{K}_\eta^C\right) &= 1 - \mathbb{P}^n\left(W^n \notin \mathcal{K}_\eta^C\right) \\ &\geq 1 - \frac{\mathbb{E}^{\mathbb{P}^n}\left[\text{mc}_\delta(W^n)\right]}{C \frac{\delta \ln\left(\frac{2T}{\delta}\right)}{\eta}}. \end{aligned} \quad (3.7)$$

Then, from Theorem 1 in [7], there exists a constant C^* such that

$$\mathbb{E}^{\mathbb{P}^n}\left[\text{mc}_\delta(W^n)\right] \leq C^* \frac{\delta \ln\left(\frac{2T}{\delta}\right)}{\eta}. \quad (3.8)$$

Therefore, we get from (3.7) and (3.8)

$$\mathbb{P}^n \left((W^n, \alpha^n) \in \mathbf{K}_\eta^{C^*} \right) \geq 1 - \eta,$$

for all $\eta \in (0, 1)$, and the sequence $(W^n, \alpha^n)_n$ is tight.

Step 3. *Convergence of the almost optimal strategies.*

We then deduce that, up to a subsequence, the sequence $(W^n, \alpha^n)_n$ converges in law. Since $(C([0, T], \mathbb{R}^d) \times (\mathbb{R} \times \mathcal{I})^{\mathbb{N}}, \|\cdot\|)$ is separable, we deduce from the extension of Skorokhod representation result given by Theorem A.5 that there exists a set-up $\mathcal{X}^* = (\Omega^*, \mathcal{A}^*, \mathbb{P}^*, W^*, \mathbb{F}^*)$ and random variables $\alpha^{*n} = (\tau_k^{*n}, \zeta_k^{*n})_{k \geq 0}$, $n \geq 1$, and $\alpha^* = (\tau_k^*, \zeta_k^*)_{k \geq 0}$ defined on $(\Omega^*, \mathcal{G}^*, \mathbb{P}^*)$ such that

$$\mathbb{P}^* \circ (W^*, \alpha^{*n})^{-1} = \mathbb{P}^n \circ (W^n, \alpha^n)^{-1} \quad (3.9)$$

for all $n \geq 1$ and

$$\left\| (W^*, \alpha^{*n}) - (W^*, \alpha^*) \right\| \xrightarrow[n \rightarrow \infty]{\mathbb{P}^* - a.s.} 0. \quad (3.10)$$

Step 4. *The α^{*n} and α^* are switching strategies of $\mathcal{A}_{t,i}(\mathcal{X}^*)$ and $J_n(\mathcal{X}^*, t, x, \alpha^{*n}) = J_n(\mathcal{X}^n, t, x, \alpha^n)$ for all n .*

We now prove that $\alpha^{*n} \in \mathcal{A}_{t,i}(\mathcal{X}^*)$ for all $n \geq 1$. We first have that τ_k^{*n} is an \mathbb{F}^{*t} -stopping time for all $k \geq 0$ and all $n \geq 1$. Indeed, since τ_k^n is an \mathbb{F}^{nt} -stopping time and \mathbb{F}^{nt} is the complete right-continuous extension of the natural filtration of $(W_s^n - W_t^n)_{s \geq t}$, we can write from Remark 32, Chapter 2 in [5] for any $r \geq 0$ and any $\varepsilon > 0$,

$$\underline{\psi}(W_s^n - W_t^n, s \in [t, t + r + \varepsilon]) \leq \mathbf{1}_{\tau_k^n \leq r} \leq \bar{\psi}(W_s^n - W_t^n, s \in [t, t + r + \varepsilon])$$

and

$$\mathbb{P}^n(\underline{\psi}(W_s^n - W_t^n, s \in [t, t + r + \varepsilon]) \neq \bar{\psi}(W_s^n - W_t^n, s \in [t, t + r + \varepsilon])) = 0$$

where $\underline{\psi}$ and $\bar{\psi}$ are two Borel functions. Since (W^n, τ_k^n) and (W^*, τ_k^{*n}) have the same law we get

$$\mathbb{P}^* \left(\underline{\psi}(W_s^* - W_t^*, s \in [t, t + r + \varepsilon]) \leq \mathbf{1}_{\tau_k^{*n} \leq r} \leq \bar{\psi}(W_s^* - W_t^*, s \in [t, t + r + \varepsilon]) \right) = 1$$

and

$$\mathbb{P}^* \left(\underline{\psi}(W_s^* - W_t^*, s \in [t, t + r + \varepsilon]) \neq \bar{\psi}(W_s^* - W_t^*, s \in [t, t + r + \varepsilon]) \right) = 0.$$

Therefore, τ_k^{*n} is an \mathbb{F}^{*t} -stopping time. By the same argument, we get that the random variable $\zeta_k^{*n} \mathbf{1}_{\tau_k^{*n} \leq r}$ is \mathcal{F}_r^{*t} -measurable for all $r \geq t$, which is equivalent to the $\mathcal{F}_{\tau_k^{*n}}^t$ -measurability of ζ_k^{*n} . Therefore $\alpha^{*n} \in \mathcal{A}_{t,i}(\mathcal{X}^*)$. Since the $\mathbb{P}^* - a.s.$ convergence preserves the measurability, we also have $\alpha^* \in \mathcal{A}_{t,i}(\mathcal{X}^*)$.

Finally, since (W^*, α^{*n}) and (W^n, α^n) have the same law, we deduce from **(H1)** and **(H2)** that

$$J_n(\mathcal{X}^*, t, x, \alpha^{*n}) = J_n(\mathcal{X}^n, t, x, \alpha^n) \geq v_n(t, x, i) - \frac{1}{n}$$

for all $n \geq 1$.

Step 5. *Convergence of $X^{\mathcal{X}^*, t, x, \alpha^{*n}}$ to $X^{\mathcal{X}^*, t, x, \alpha^*}$.*

Since the process $(\alpha_s^{*n})_{t \leq s \leq T}$ converges to $(\alpha_s^*)_{t \leq s \leq T}$, $d\mathbb{P}^* \otimes ds - a.e.$ on $\Omega^* \times [t, T]$, we have from **(H1)** and the dominated convergence theorem

$$\mathbb{E}^{\mathbb{P}^*} \left[\int_t^T |\mu(x', \alpha_s^{*n}) - \mu(x', \alpha_s^*)|^q ds \right] + \mathbb{E}^{\mathbb{P}^*} \left[\int_t^T |\sigma(x', \alpha_s^{*n}) - \sigma(x', \alpha_s^*)|^q ds \right] \xrightarrow{n \rightarrow \infty} 0$$

for all $q \geq 1$ and all $x' \in \mathbb{R}^d$. From Theorem 1, Chapter 2, Section 8 in [10], we get

$$\mathbb{E}^{\mathbb{P}^*} \left[\sup_{s \in [t, T]} \left| X_s^{\mathcal{X}^*, t, x, \alpha^{*n}} - X_s^{\mathcal{X}^*, t, x, \alpha^*} \right|^q \right] \xrightarrow{n \rightarrow \infty} 0 \quad (3.11)$$

for all $q \geq 1$.

Step 6. *The strategy α^* belongs to $\mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$.*

We now prove that $\mathbb{P}^*(X_s^{\mathcal{X}^*, t, x, \alpha^*} \in \bar{\mathcal{O}}) = 1$, for all $s \in [t, T]$. For $\varepsilon > 0$, we define the set \mathcal{O}_ε by

$$\mathcal{O}_\varepsilon = \left\{ x' \in \mathbb{R}^d : d(x', \mathcal{O}) < \varepsilon \right\}.$$

Suppose that there exists some $\varepsilon > 0$ such that

$$\mathbb{E}^{\mathbb{P}^*} \left[\int_t^T \mathbf{1}_{\mathcal{O}_\varepsilon}(X_s^{\mathcal{X}^*, t, x, \alpha^*}) ds \right] > 0.$$

From (3.11), up to a subsequence, we have

$$\sup_{s \in [t, T]} \left| X_s^{\mathcal{X}^*, t, x, \alpha^{*n}} - X_s^{\mathcal{X}^*, t, x, \alpha^*} \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}^* - a.s.} 0.$$

Applying the dominated convergence theorem we can find $\eta > 0$ and $n_\eta \geq 1$ such that

$$\mathbb{E}^{\mathbb{P}^*} \left[\int_t^T \mathbf{1}_{\mathcal{O}_\varepsilon}(X_s^{\mathcal{X}^*, t, x, \alpha^{*n}}) ds \right] > \eta$$

for all $n \geq n_\eta$. From the definition of f_n and g_n there exists a constant C such that

$$J_n(\mathcal{X}^*, t, x, \alpha^{*n}) \leq C \mathbb{E}^{\mathbb{P}^*} \left[\sup_{s \in [t, T]} |X_s^{\mathcal{X}^*, t, x, \alpha^{*n}}| \right] - n \mathbb{E}^{\mathbb{P}^*} \left[\int_t^T \mathbf{1}_{\mathcal{O}_\varepsilon}(X_s^{\mathcal{X}^*, t, x, \alpha^{*n}}) ds \right]$$

for any $n \geq \frac{1}{\varepsilon} \vee n_\eta$. Sending n to infinity we get from (2.2)

$$\bar{v}(t, x, i) = \lim_{n \rightarrow \infty} J_n(\mathcal{X}^*, t, x, \alpha^{*n}) = -\infty$$

which contradicts $\bar{v}(t, x, i) > -\infty$. We therefore obtain

$$\mathbb{E}^{\mathbb{P}^*} \left[\int_t^T \mathbf{1}_{\mathcal{O}_\varepsilon}(X_s^{\mathcal{X}^*, t, x, \alpha^*}) ds \right] = 0$$

for all $\varepsilon > 0$ and therefore $\mathbb{E}^{\mathbb{P}^*} \left[\int_t^T \mathbb{1}_{\{X_s^{\mathcal{X}^*, t, x, \alpha^*} \notin \bar{\mathcal{O}}\}} ds \right] = 0$. Since $X^{\mathcal{X}^*, t, x, \alpha^*}$ is continuous, we get $\alpha^* \in \mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$.

Step 7. *Convergence of the pay-offs.*

We now prove that, up to a subsequence, $\limsup_{n \rightarrow \infty} J_n(\mathcal{X}^*, t, x, \alpha^{*n}) \leq J(\mathcal{X}^*, t, x, \alpha^*)$. We first notice that

$$\limsup_{n \rightarrow \infty} J_n(\mathcal{X}^*, t, x, \alpha^{*n}) \leq \limsup_{n \rightarrow \infty} J_1(\mathcal{X}^*, t, x, \alpha^{*n}).$$

From (3.11) we have, up to a subsequence,

$$\sup_{s \in [t, T]} \left| X_s^{\mathcal{X}^*, t, x, \alpha^{*n}} - X_s^{\mathcal{X}^*, t, x, \alpha^*} \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}^* - a.s.} 0. \quad (3.12)$$

This implies with Step 5 that

$$g_1(X_T^{\mathcal{X}^*, t, x, \alpha^{*n}}, \alpha_T^{*n}) + \int_t^T f_1(X_s^{\mathcal{X}^*, t, x, \alpha^{*n}}, \alpha_s^{*n}) ds \xrightarrow[n \rightarrow \infty]{\mathbb{P}^* - a.s.} g(X_T^{\mathcal{X}^*, t, x, \alpha^*}, \alpha_T^*) + \int_t^T f(X_s^{\mathcal{X}^*, t, x, \alpha^*}, \alpha_s^*) ds.$$

Moreover, since $\bar{v}(t, x, i) > -\infty$ we have from **(H2)** (ii)

$$\sup_{n \geq 1} \#\{k \geq 1 : \tau_k^{*n} \leq T\} < +\infty, \quad \mathbb{P}^* - a.s.$$

This last estimate, (3.10) and (3.12) imply

$$\sum_{k \geq 1} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha^{*n}}, \zeta_{k-1}^{*n}, \zeta_k^{*n}) \mathbb{1}_{\tau_k^{*n} \leq t} \xrightarrow[n \rightarrow \infty]{\mathbb{P}^* - a.s.} \sum_{k \geq 1} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha^*}, \zeta_{k-1}^*, \zeta_k^*) \mathbb{1}_{\tau_k^* \leq t}.$$

We finally conclude by using Fatou's Lemma.

Step 8. *Conclusion.*

We have proved that

$$J(\mathcal{X}^*, t, x, \alpha^{*n}) \geq v_n(t, x, i) - \frac{1}{n}$$

for all $n \geq 1$. We also have seen that α^{*n} converges $\mathbb{P}^* - a.s.$ to $\alpha^* \in \mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ and that

$$\limsup_{n \rightarrow \infty} J(\mathcal{X}^*, t, x, \alpha^{*n}) \leq J(\mathcal{X}^*, t, x, \alpha^*).$$

We then conclude that

$$\begin{aligned} v(t, x, i) &\geq J(\mathcal{X}^*, t, x, \alpha^*) \\ &\geq \limsup_{n \rightarrow \infty} J(\mathcal{X}^*, t, x, \alpha^{*n}) \\ &\geq \limsup_{n \rightarrow \infty} v_n(t, x, i) - \frac{1}{n} = \bar{v}(t, x, i). \end{aligned}$$

□

Remark 3.1. The set-up \mathcal{X}^* and the strategy $\alpha^* \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ constructed in the proof of Theorem 3.1 are optimal for the constrained switching problem. We can therefore rewrite the optimal switching problem (2.3) under the following strong form:

$$v(t, x, i) = \sup_{\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)} J(\mathcal{X}^*, t, x, \alpha) \quad (3.13)$$

for all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.

In general, proving a regularity result on the value function of a constrained optimization problem is very technical (see e.g. [12] or [9]). In our case, Theorem 3.1 gives a semi-regularity for v .

Corollary 3.1. *The function v is upper semicontinuous on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.*

Proof. Using Remark 3.1, we have $v = \lim_{n \rightarrow \infty} v_n^{\mathcal{X}^*}$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ where $v_n^{\mathcal{X}^*}$ is defined by

$$v_n^{\mathcal{X}^*}(t, x, i) = \sup_{\alpha \in \mathcal{A}_{t,i}(\mathcal{X}^*)} J_n(\mathcal{X}^*, t, x, \alpha),$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$. From Assumptions **(H1)** and **(H2)**, the function $v_n^{\mathcal{X}^*}(\cdot, i)$ is continuous on $[0, T] \times \mathbb{R}^d$, for all $(n, i) \in \mathbb{N}^* \times \mathcal{I}$ (see e.g. [16] or [1]). Therefore, the function v is upper semicontinuous on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ as an infimum of continuous functions. \square

4 Dynamic programming and variational inequalities

4.1 The Dynamic programming principle

In this section we state the dynamic programming principle. We first need the following lemmata

Lemma 4.1. *Under **(H2)**, the functions f_n and g_n , satisfy the local Lipschitz property: there exists a constant C_n such that*

$$|g_n(x, i) - g_n(x', i)| + |f_n(x, i) - f_n(x', i)| \leq C_n(1 + |x| + |x'|)|x - x'|,$$

for all $x, x' \in \mathbb{R}^d$, $i \in \mathcal{I}$ and $n \in \mathbb{N}^*$.

Proof. Fix $i \in \mathcal{I}$. From the definition of f_n we have

$$|f_n(x, i) - f_n(x', i)| \leq |f(x, i)(\Theta_n(x) - \Theta_n(x'))| + |\Theta_n(x')(f(x, i) - f(x', i))|,$$

for all $x, x' \in \mathbb{R}^d$ and $i \in \mathcal{I}$. Since f is Lipschitz continuous and Θ_n is bounded we get

$$|f_n(x, i) - f_n(x', i)| \leq C_n(1 + |x| + |x'|)|x - x'|.$$

for all $x, x' \in \mathbb{R}^d$. The proof is the same for g_n . \square

Lemma 4.2. *Under (H1) and (H2), there exists a constant C such that*

$$v_n(t, x, i) \leq C(1 + |x|) \quad (4.1)$$

for all $n \geq 1$ and all $(t, x, i) \in [0, T] \times \mathcal{O} \times \mathcal{I}$.

Proof. Fix $n \geq 1$ and $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. Using the definition of f_n and g_n we have

$$J_n(\mathcal{X}, t, x, \alpha) \leq J_1(\mathcal{X}, t, x, \alpha) \quad (4.2)$$

for any $\alpha \in \mathcal{A}_t(\mathcal{X})$. From (2.2) and (H2) there exists a constant C such that

$$J_1(\mathcal{X}, t, x, \alpha) \leq C(1 + |x|)$$

for any $\alpha \in \mathcal{A}_t(\mathcal{X})$. From (4.2) and the definition of $v(t, x, i)$, we get (4.1). \square

We are now able to state the dynamic programming principle.

Theorem 4.1. *Under (H1) and (H2), the value function v satisfies the following dynamic programming relation:*

$$v(t, x, i) = \sup_{\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)} \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{\mathcal{X}^*, t, x, \alpha}, \alpha_\nu) \right]. \quad (4.3)$$

for any $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$, and any stopping times ν valued in $[t, T]$.

Proof. Fix $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. If $\mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*) = \emptyset$ then the two hand sides of (4.3) are equal to $-\infty$ so the equality holds.

Suppose now that $\mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*) \neq \emptyset$ and let $\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ and ν a stopping time valued in $[t, T]$. From (3.13) and the local Lipschitz properties of f_n and g_n given by Lemma 4.1, we have by Lemma 4.4 in [1]

$$v_n(t, x, i) \geq \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f_n(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) + v_n(\nu, X_\nu^{\mathcal{X}^*, t, x, \alpha}, \alpha_\nu) \right], \quad (4.4)$$

for all $n \geq 1$. Since $\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ we have from the definition of f_n , $f_n(X_s^{t,x,\alpha}, \alpha_s) = f(X_s^{t,x,\alpha}, \alpha_s)$ for almost all $(s, \omega) \in [t, T] \times \Omega^*$. From Theorem 3.1, Lemma 4.2, (2.2) and the monotone convergence theorem, we get by sending n to infinity

$$v(t, x, i) \geq \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{\mathcal{X}^*, t, x, \alpha}, \alpha_\nu) \right].$$

We now prove the reverse inequality. From the definitions of the performance criterion and the value functions, the law of iterated conditional expectations and Markov property of

our model, we get the successive relations

$$\begin{aligned}
& J(\mathcal{X}^*, t, x, \alpha) = \\
& \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f(s, X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) \right. \\
& \left. + \mathbb{E}^{\mathbb{P}^*} \left[g(X_T^{\mathcal{X}^*, t, x, \alpha}) + \int_\nu^T f(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{\nu < \tau_k \leq T} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) \middle| \mathcal{F}_\nu^{*t} \right] \right] = \\
& \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) + J(\mathcal{X}^*, \nu, X_\nu^{\mathcal{X}^*, t, x, \alpha}, \alpha) \right] \leq \\
& \mathbb{E}^{\mathbb{P}^*} \left[\int_t^\nu f(X_s^{\mathcal{X}^*, t, x, \alpha}, \alpha_s) ds + \sum_{t < \tau_k \leq \nu} c(X_{\tau_k}^{\mathcal{X}^*, t, x, \alpha}, \zeta_{k-1}, \zeta_k) + v(\nu, X_\nu^{\mathcal{X}^*, t, x, \alpha}, \alpha_\nu) \right].
\end{aligned}$$

Since ν and α are arbitrary, we obtain the required inequality. \square

4.2 Viscosity properties

We prove in this section that the function v is a solution to a system of variational inequalities. More precisely we consider the following PDE

$$\min \left[-\frac{\partial v}{\partial t} - \mathcal{L}v - f, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}, \quad (4.5)$$

$$\min \left[v - g, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad \{T\} \times \bar{\mathcal{O}} \times \mathcal{I}. \quad (4.6)$$

where \mathcal{L} is the second order local operator defined by

$$\mathcal{L}v(t, x, i) = \left(\mu \cdot Dv + \frac{1}{2} \text{tr}[\sigma \sigma^\top D^2 v] \right)(t, x, i)$$

and \mathcal{H} is the nonlocal operator defined by

$$\mathcal{H}v(t, x, i) = \max_{\substack{j \in \mathcal{I} \\ j \neq i}} [v(t, x, j) - c(x, i, j)]$$

for all $t \in [0, T]$, $x \in \bar{\mathcal{O}}$ and $i \in \mathcal{I}$. As usual, the value functions need not be smooth, and even not known to be continuous a priori. So, we shall work with the notion of (discontinuous) viscosity solutions (see [4]). Generally, for PDEs arising in optimal control problems involving state constraints, we need the notion of constrained viscosity solution introduced by [12] for first-order equations to take into account the boundary conditions induced by the state constraints.

For a locally bounded function u on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$, we define its lower semicontinuous (lsc in short) envelope u_* , and upper semicontinuous (usc in short) envelope u^* by

$$u_*(t, x, i) = \liminf_{\substack{(t', x') \rightarrow (t, x), \\ (t', x') \in [0, T] \times \bar{\mathcal{O}}}} u(t', x', i), \quad u^*(t, x, i) = \limsup_{\substack{(t', x') \rightarrow (t, x), \\ (t', x') \in [0, T] \times \bar{\mathcal{O}}}} u(t', x', i).$$

for all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.

Remark 4.1. From Corollary 3.1 and the definition of the usc envelope we have $v = v^*$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. However, this equality may not be true on $\{T\} \times \bar{\mathcal{O}} \times \mathcal{I}$.

We now give the definition of a constrained viscosity solutions to (4.5) and (4.6).

Definition 4.1 (Constrained viscosity solutions to (4.5)-(4.6)).

(i) A function u , lsc (resp. usc) on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$, is called a viscosity supersolution on $[0, T] \times \mathcal{O} \times \mathcal{I}$ (resp. subsolution on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$) to (4.5)-(4.6) if we have

$$\min \left[-\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(x, i), u(t, x, i) - \mathcal{H}u(t, x, i) \right] \geq (\text{resp. } \leq) 0$$

for any $(t, x, i) \in [0, T] \times \mathcal{O} \times \mathcal{I}$ (resp. $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$), and any $\varphi \in C^{1,2}([0, T] \times \bar{\mathcal{O}}, \mathbb{R})$ such that

$$(\varphi - u)(t, x, i) = \max_{[0, T] \times \bar{\mathcal{O}}} (\varphi - u(\cdot, i)) \quad (\text{resp. } \min_{[0, T] \times \bar{\mathcal{O}}} (\varphi - u(\cdot, i)))$$

and

$$\min \left[u(T, x, i) - g(x, i), u(T, x, i) - \mathcal{H}u(T, x, i) \right] \geq (\text{resp. } \leq) 0$$

for any $x \in \mathcal{O}$ (resp. $x \in \bar{\mathcal{O}}$).

(ii) A locally bounded function on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ is called a constrained viscosity solution to (4.5)-(4.6) if its lsc envelope u_* is a viscosity supersolution to (4.5)-(4.6) on $[0, T] \times \mathcal{O} \times \mathcal{I}$ and its usc envelope u^* is a viscosity subsolution on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ to (4.5)-(4.6).

We can now state the viscosity property of v .

Theorem 4.2. Suppose that the function v is locally bounded. Under **(H1)** and **(H2)**, v is a constrained viscosity solution to (4.5)-(4.6).

Proof of the supersolution property on $[0, T] \times \mathcal{O} \times \mathcal{I}$. First, for any $(t, x, i) \in [0, T] \times \mathcal{O} \times \mathcal{I}$, we see, as a consequence of (4.3) applied to $\nu = t$, and by choosing any admissible control $\alpha \in \mathcal{A}_{t, x, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ with immediate switch j at t , that

$$v(t, x, i) \geq \mathcal{H}v(t, x, i). \quad (4.7)$$

Now, let $(\bar{t}, \bar{x}, i) \in [0, T] \times \mathcal{O} \times \mathcal{I}$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ s.t.

$$\varphi(\bar{t}, \bar{x}) - v_*(\bar{t}, \bar{x}, i) = \max_{[0, T] \times \bar{\mathcal{O}}} (\varphi - v_*(\cdot, i)). \quad (4.8)$$

Since $v \geq \mathcal{H}v$ on $[0, T] \times \mathcal{O} \times \mathcal{I}$, we get from the definition of the operator \mathcal{H} and **(H2)** (i)

$$v_*(\bar{t}, \bar{x}, j) \geq v_*(\bar{t}, \bar{x}, j) + c(\bar{x}, i, j),$$

for all $j \in \mathcal{I}$. Therefore we obtain

$$v_*(\bar{t}, \bar{x}, i) \geq \mathcal{H}v_*(\bar{t}, \bar{x}, i).$$

So it remains to show that

$$-\frac{\partial\varphi}{\partial t}(\bar{t}, \bar{x}, i) - \mathcal{L}\varphi(\bar{t}, \bar{x}, i) - f(\bar{x}, i) \geq 0. \quad (4.9)$$

From the definition of v_* there exists a sequence $(t_m, x_m)_m$ valued in $[0, T) \times \mathcal{O}$ s.t.

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow{m \rightarrow \infty} (\bar{t}, \bar{x}, v_*(\bar{t}, \bar{x}, i)).$$

By continuity of φ we also have $\gamma_m := v(t_m, x_m, i) - \varphi(t_m, x_m) - v_*(\bar{t}, \bar{x}, i) + \varphi(\bar{t}, \bar{x})$ converges to 0 as m goes to infinity. Since $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}$, there exists $\eta > 0$ s.t. for m large enough, $t_m < T$ and

$$((t_m - \frac{\eta}{2}) \wedge 0, t_m + \frac{\eta}{2}) \times B(x_m, \frac{\eta}{2}) \subset ((t - \eta) \wedge 0, t + \eta) \times B(x, \eta) \subset [0, T) \times \mathcal{O}.$$

Let us consider an admissible control α in $\mathcal{A}_{t_m, x_m, i}^{\mathcal{O}}(\mathcal{X}^*)$ with no switch until the first exist time τ_m before T of the associated process $(s, X_s^m) := (s, X_s^{\mathcal{X}^*, t_m, x_m, \alpha})$ from $(t_m - \frac{\eta}{2}, t_m + \frac{\eta}{2}) \times B(x_m, \frac{\eta}{2})$:

$$\tau_m := \inf\{s \geq t_m : (s - t_m) \vee |X_s^m - x_m| \geq \frac{\eta}{2}\} \wedge T.$$

Consider also a strictly positive sequence $(h_m)_m$ s.t. h_m and γ_m/h_m converge to 0 as m goes to infinity. By using the dynamic programming principle (4.3) for $v(t_m, x_m, i)$ and $\nu = \hat{\tau}_m := \tau_m \wedge (t_m + h_m)$, we get

$$\begin{aligned} v(t_m, x_m, i) &= \gamma_m + v_*(\bar{t}, \bar{x}, i) - \varphi(\bar{t}, \bar{x}, i) + \varphi(t_m, x_m, i) \\ &\geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\hat{\tau}_m} f(X_s^m, i) ds + v(\hat{\tau}_m, X_{\hat{\tau}_m}^m, i) \right]. \end{aligned}$$

Using (4.8), we obtain

$$v(t_m, x_m, i) \geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\hat{\tau}_m} f(X_s^m, i) ds + \varphi(\hat{\tau}_m, X_{\hat{\tau}_m}^m) \right].$$

Applying Itô's formula to $\varphi(s, X_s^m)$ between t_m and $\hat{\tau}_m$ and since $\sigma(X_s^m, i)D\varphi(s, X_s^m)$ is bounded for $s \in [t_m, \hat{\tau}_m]$, we obtain

$$\frac{\gamma_m}{h_m} + \mathbb{E}^{\mathbb{P}^*} \left[\frac{1}{h_m} \int_{t_m}^{\hat{\tau}_m} \left(-\frac{\partial\varphi}{\partial t} - \mathcal{L}\varphi - f \right)(s, X_s^m, i) ds \right] \geq 0, \quad (4.10)$$

for all $m \geq 1$. From the continuity of the process X^m , we have

$$\mathbb{P}^* \left(\exists m, \forall m' \geq m : \hat{\tau}_{m'} = t_{m'} + h_{m'} \right) = 1.$$

Hence, by the mean-value theorem, the random variable inside the expectation in (4.10) converges a.s. to $(-\frac{\partial\varphi}{\partial t} - \mathcal{L}\varphi - f)(\bar{t}, \bar{x}, i)$ as m goes to infinity. We conclude by the dominated convergence theorem and get (4.9). \square

Proof of the subsolution property on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. We first recall that $v^* = v$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ from Remark 4.1. Let $(\bar{t}, \bar{x}, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ s.t.

$$\varphi(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}, i) = \min_{[0, T] \times \bar{\mathcal{O}}} (\varphi - v(\cdot, i)). \quad (4.11)$$

If $v(\bar{t}, \bar{x}, i) \leq \mathcal{H}v(\bar{t}, \bar{x}, i)$ then the subsolution property trivially holds. Consider now the case where $v(\bar{t}, \bar{x}, i) > \mathcal{H}v(\bar{t}, \bar{x}, i)$ and argue by contradiction by assuming on the contrary that

$$\eta := -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \mathcal{L}\varphi(\bar{t}, \bar{x}, i) - f(\bar{x}, i) > 0.$$

By continuity of φ and its derivatives, there exists some $\delta > 0$ such that $\bar{t} + \delta < T$ and

$$\left(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f\right)(t, x, i) \geq \frac{\eta}{2}, \quad (4.12)$$

for all $(t, x) \in \mathcal{V} := \left((\bar{t} - \delta, \bar{t} + \delta) \cap [0, T]\right) \times B(\bar{x}, \delta)$. By the dynamic programming principle (4.3), given $m \geq 1$, there exists $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_n \in \mathcal{A}_{\bar{t}, \bar{x}, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ s.t. for any stopping time τ valued in $[\bar{t}, T]$, we have

$$v(\bar{t}, \bar{x}, i) \leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\tau} f(\hat{X}_s^m, i) + \sum_{\bar{t} < \hat{\tau}_n^m \leq \tau} c(\hat{X}_{\hat{\tau}_n^m}^m, \hat{\zeta}_n^m, \hat{\zeta}_n^m) + v(\tau, \hat{X}_\tau^m, i) \right] + \frac{1}{m}.$$

Here \hat{X}^m is the state process, starting from \bar{x} at \bar{t} and controlled by $\hat{\alpha}^m$. By choosing $\tau = \bar{\tau}_m := \hat{\tau}_1^m \wedge \nu^m$ where

$$\nu^m := \inf\{s \geq t_m : (s, \hat{X}_s^m) \notin \mathcal{V}\}$$

is the first exit time of (s, \hat{X}_s^m) from \mathcal{V} , we then get

$$\begin{aligned} v(\bar{t}, \bar{x}, i) &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} \left[v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbf{1}_{\tau_\delta^m < \hat{\tau}_1^m} \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[[v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, \hat{\zeta}_1^m) + c(\hat{X}_{\bar{\tau}^m}^m, i, \hat{\zeta}_1^m)] \mathbf{1}_{\tau_\delta^m \geq \hat{\tau}_1^m} \right] + \frac{1}{m} \\ &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} \left[v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbf{1}_{\tau_\delta^m \leq \hat{\tau}_1^m} \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[\mathcal{H}v(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m, i) \mathbf{1}_{\tau_\delta^m > \hat{\tau}_1^m} \right] + \frac{1}{m}. \end{aligned} \quad (4.13)$$

Now, since $v \geq \mathcal{H}v$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ and $\hat{\alpha}^m \in \mathcal{A}_{\bar{t}, \bar{x}, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$, we obtain from (4.11)

$$\varphi(\bar{t}, \bar{x}, i) \leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds + \varphi(\bar{\tau}^m, \hat{X}_{\bar{\tau}^m}^m) \right] + \frac{1}{m}.$$

Applying Itô's formula to $\varphi(s, \hat{X}_s^m)$ between t_m and $\bar{\tau}^m$ we get:

$$0 \leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\bar{\tau}^m} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi + f \right)(s, \hat{X}_s^m, i) \right] + \frac{1}{m} \leq -\frac{\eta}{2} \mathbb{E}^{\mathbb{P}^*} [\bar{\tau}^m - \bar{t}] + \frac{1}{m}.$$

This implies

$$\lim_{m \rightarrow +\infty} \mathbb{E}^{\mathbb{P}^*}[\bar{\tau}^m] = \bar{t}. \quad (4.14)$$

From the definition of ν^m and (4.14) we have, up to a subsequence,

$$\mathbb{P}^*(\nu^m \geq \hat{\tau}_1^m) \xrightarrow{m \rightarrow \infty} 1. \quad (4.15)$$

On the other hand, we get from (4.13)

$$\begin{aligned} v(\bar{t}, \bar{x}, i) &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{P}^*(\nu^m < \hat{\tau}_1^m) \sup_{(t', x') \in \mathcal{V}} v(t', x') \\ &\quad + \mathbb{P}^*(\nu^m \geq \hat{\tau}_1^m) \sup_{(t', x') \in \mathcal{V}} \mathcal{H}v(t', x', i) + \frac{1}{m}. \end{aligned}$$

This gives

$$v(\bar{t}, \bar{x}, i) \mathbb{P}^*(\nu^m \geq \hat{\tau}_1^m) \leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{\bar{t}}^{\bar{\tau}^m} f(\hat{X}_s^m, i) ds \right] + \mathbb{P}(\nu^m \geq \hat{\tau}_1^m) \sup_{(t', x') \in \mathcal{V}} \mathcal{H}v(t', x', i) + \frac{1}{m}.$$

From (4.14) and (4.15) and since $v = v^*$, we get by sending m to infinity and δ to zero

$$v(\bar{t}, \bar{x}, i) \leq (\mathcal{H}v)^*(\bar{t}, \bar{x}, i) \leq \mathcal{H}v(\bar{t}, \bar{x}, i),$$

which is the required contradiction. \square

Proof of the viscosity supersolution property on $\{T\} \times \mathcal{O} \times \mathcal{I}$. Fix some $(\bar{x}, i) \in \mathcal{O} \times \mathcal{I}$, and consider some sequence $(t_m, x_m)_{m \geq 1}$ valued in $[0, T] \times \mathcal{O}$, such that

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow{m \rightarrow \infty} (T, \bar{x}, v_*(T, \bar{x}, i)).$$

Let $\delta > 0$ s.t. $B(\bar{x}, \delta) \in \mathcal{O}$. By taking a strategy $\alpha^m = (\tau_k^m, \zeta_k^m)_k \in \mathcal{A}_{t_m, x_m, i}^{\bar{\mathcal{O}}}(\mathcal{X}^*)$ with no switch before $\tau_m := \inf\{s \geq t_m, X_s^m \notin B(x_m, \frac{\delta}{2})\} \wedge T$ with $X^m := X^{\mathcal{X}^*, t_m, x_m, \alpha^m}$, we have from (4.3) applied to τ_m and α_m

$$\begin{aligned} v(t_m, x_m, i) &\geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} [v(\tau^m, X_{\tau^m}^m, i) \mathbf{1}_{\tau^m < \tau_1^m}] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[(v(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m) + c(X_{\tau_1^m}^m, i, \zeta_1^m)) \mathbf{1}_{\tau^m \geq \tau_1^m} \right]. \end{aligned}$$

Then, using $v \geq \mathcal{H}v$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$, we get

$$v(t_m, x_m, i) \geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} [v(\tau^m, X_{\tau^m}^m, i)]$$

Since $v(T, \cdot) = g$ we obtain

$$\begin{aligned} v(t_m, x_m, i) &\geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} [v(\tau^m, X_{\tau^m}^m, i) \mathbf{1}_{\tau^m < T}] + \mathbb{E}^{\mathbb{P}^*} [g(X_{\tau^m}^m, i) \mathbf{1}_{\tau^m = T}] \\ &\geq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\tau^m} f(X_s^m, i) ds \right] + \mathbb{P}^*(\tau^m < T) \inf_{\substack{t < T \\ x \in B(x_m, \frac{\delta}{2})}} v(t, x, i) \\ &\quad + \mathbb{P}^*(\tau^m = T) \inf_{x \in B(x_m, \frac{\delta}{2})} g(x). \end{aligned} \quad (4.16)$$

Since $\mathbb{E}^{\mathbb{P}^*}[\sup_{s \in [t_m, T]} |X_s^m - x_m|]$ converges to zero (see e.g. Corollary 12, Section 5, Chapter 2 in [10]), we have, up to a subsequence,

$$\sup_{s \in [t_m, T]} |X_s^m - x_m| \xrightarrow[m \rightarrow \infty]{\mathbb{P}^* \text{-a.s.}} 0.$$

Since $(x_m)_m$ converges to $x \in \mathcal{O}$, we deduce that

$$\mathbb{P}^*(\tau^m = T) \xrightarrow[m \rightarrow \infty]{} 1.$$

Sending m to infinity and δ to 0 in (4.16) we get

$$v_*(T, \bar{x}, i) \geq g(\bar{x}, i). \quad (4.17)$$

On the other hand, we know from (4.7) that $v \geq \mathcal{H}v$ on $[0, T] \times \mathcal{O}$, and thus

$$v(t_m, x_m, i) \geq \mathcal{H}v(t_m, x_m, i) \geq \mathcal{H}v_*(t_m, x_m, i),$$

for all $m \geq 1$. Recalling that $\mathcal{H}v_*$ is lsc, we obtain by sending m to infinity

$$v_*(T, \bar{x}, i) \geq \mathcal{H}v_*(T, \bar{x}, i).$$

Together with (4.17), this proves the required viscosity supersolution property of (4.6). \square

Proof of the viscosity subsolution property on $\{T\} \times \bar{\mathcal{O}} \times \mathcal{I}$. We argue by contradiction by assuming that there exists $(\bar{x}, i) \in \bar{\mathcal{O}} \times \mathcal{I}$ such that

$$\min[v^*(T, \bar{x}, i) - g(\bar{x}, i), \mathcal{H}v^*(T, \bar{x}, i)] := 2\varepsilon > 0. \quad (4.18)$$

One can find a sequence of smooth functions $(\varphi^n)_{n \geq 0}$ on $[0, T] \times \bar{\mathcal{O}}$ such that φ^n converges pointwisely to $v^*(\cdot, i)$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$ as $n \rightarrow \infty$. Moreover, by (4.18) and the upper semicontinuity of v^* , we may assume that the inequality

$$\min[\varphi^n - g(\cdot, i), \varphi^n - \max_{j \in \mathcal{I}}\{v^*(\cdot, j) + c(\cdot, i, j)\}] \geq \varepsilon, \quad (4.19)$$

holds on some bounded neighborhood B^n of (T, \bar{x}) in $[0, T] \times \bar{\mathcal{O}}$, for n large enough. Let $(t_m, x_m)_{m \geq 1}$ be a sequence in $[0, T] \times \bar{\mathcal{O}}$ such that

$$(t_m, x_m, v(t_m, x_m, i)) \xrightarrow[m \rightarrow \infty]{} (T, \bar{x}, v^*(T, \bar{x}, i)).$$

Then there exists $\delta^n > 0$ such that $B_m^n := [t_m, T] \times B(x_m, \delta^n) \subset B^n$ for m large enough, so that (4.19) holds on B_m^n . Since v is locally bounded, there exists some $\eta > 0$ such that $|v^*| \leq \eta$ on B^n . We can then assume that $\varphi^n \geq -2\eta$ on B^n . Let us define the smooth function $\tilde{\varphi}_m^n$ by

$$\tilde{\varphi}_m^n(t, x) := \varphi^n(t, x) + \left(4\eta \frac{|x - x_m|^2}{|\delta^n|^2} + \sqrt{T - t}\right)$$

for $(t, x) \in [0, T] \times \mathcal{O}$ and observe that

$$(v^* - \tilde{\varphi}_m^n)(t, x, i) \leq -\eta, \quad (4.20)$$

for $(t, x) \in [t_m, T] \times \partial B(x_m, \delta^n)$. Since $\frac{\partial \sqrt{T-t}}{\partial t} \rightarrow -\infty$ as $t \rightarrow T$, we have for m large enough

$$-\frac{\partial \tilde{\varphi}_m^n}{\partial t} - \mathcal{L} \tilde{\varphi}_m^n(\cdot, i) \geq 0 \text{ on } B_m^n. \quad (4.21)$$

Let $\alpha^m = (\tau_j^m, \zeta_j^m)_{j \geq 1}$ be a $\frac{1}{m}$ -optimal control for $v(t_m, x_m, i)$ with corresponding state process $X^m = X^{\alpha^m, t_m, x_m}$, and denote by $\theta_n^m = \inf \{s \geq t_m : (s, X_s^m) \notin B_m^n\} \wedge \tau_1^m$. From (4.3), this means that

$$\begin{aligned} v(t_m, x_m, i) - \frac{1}{m} &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\theta_n^m} f(X_s^m, i) ds \right] + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\theta_n^m < \tau_1^m \wedge T} v(\theta_n^m, X_{\theta_n^m}^m, i) \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\theta_n^m = T < \tau_1^m} g(X_{\theta_n^m}^m, i) \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\tau_1^m \leq \theta_n^m} \left(v(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m) + c(X_{\tau_1^m}^m, i, \zeta_1^m) \right) \right]. \end{aligned} \quad (4.22)$$

Now, by applying Itô's Lemma to $\tilde{\varphi}_m^n(s, X_s^m)$ between t_m and θ_n^m we get from (4.19), (4.20) and (4.21),

$$\begin{aligned} \tilde{\varphi}_m^n(t_m, x_m) &\geq \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\theta_n^m < \tau_1^m} \tilde{\varphi}_m^n(\theta_n^m, X_{\theta_n^m}^m) \right] + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\tau_1^m \leq \theta_n^m} \tilde{\varphi}_m^n(\tau_1^m, X_{\tau_1^m}^m) \right] \\ &\geq \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\theta_n^m < \tau_1^m \wedge T} \left(v^*(\theta_n^m, X_{\theta_n^m}^m, i) + \eta \right) \right] + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\theta_n^m = T < \tau_1^m} \left(g(X_{\theta_n^m}^m, i) + \varepsilon \right) \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[\mathbf{1}_{\tau_1^m \leq \theta_n^m} \left(v^*(\tau_1^m, X_{\tau_1^m}^m, \zeta_1^m) + c(X_{\tau_1^m}^m, i, \zeta_1^m) + \varepsilon \right) \right]. \end{aligned}$$

Together with (4.22), this implies

$$\tilde{\varphi}_m^n(t_m, x_m) \geq v(t_m, x_m, i) - \mathbb{E}^{\mathbb{P}^*} \left[\int_{t_m}^{\theta_n^m} f(X_s^m, i) ds \right] - \frac{1}{m} + \varepsilon \wedge \eta.$$

Sending m , and then n to infinity, we get the required contradiction: $v^*(T, \bar{x}, i) \geq v^*(T, \bar{x}, i) + \varepsilon \wedge \eta$. \square

5 Uniqueness result

This section deals with the uniqueness issue for the SVI (4.5)-(4.6). Unfortunately, we cannot provide a comparison result as the counterexample presented below shows. We then give a weaker characterization of v as a maximal solution.

5.1 A counterexample for comparison

In general, the uniqueness of a viscosity solution to some PDE is given by a comparison theorem. Such a result says that for u an usc supersolution and w a l.s.c. subsolution, we have $u \geq w$. Applying this result to $u = v_*$ the lsc envelope of v and $w = v^*$ the u.s.c. envelope of v we get that $v_* = v^*$ and v is continuous. We provide here an example of a switching problem under state constraint where the value function v is discontinuous.

Fix $d = 2$ and consider the case where $\bar{\mathcal{O}}$ is the smooth domain $\mathbb{R} \times \mathbb{R}_+$. Take $\mathcal{I} = \{1, 2\}$ and define the diffusion coefficients μ and σ by

$$\mu(x, 1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mu(x, 2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\sigma(x, 1) = \sigma(x, 2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x \in \mathbb{R}^2$. Define the gain coefficients g and f by

$$g(x, 1) = g(x, 2) = 0 \quad \text{and} \quad f(x, 1) = f(x, 2) = 1,$$

for all $x \in \mathbb{R}^2$, and the cost coefficients $c(\cdot, 1, 2)$ and $c(\cdot, 2, 1)$ by

$$c(x, 1, 2) = c(x, 2, 1) = c > 0,$$

for all $x \in \mathbb{R}^2$. Then we can directly compute the value function and, due to the state constraints, we have

$$v(t, x, 1) = \begin{cases} T - t & \text{if } x_2 \geq T - t, \\ T - t - c & \text{if } x_2 < T - t, \end{cases}$$

for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \bar{\mathcal{O}}$ and all $t \in [0, T]$. In particular the function $v(\cdot, 1)$ is discontinuous at each point $(t, (x_1, T - t))$ for all $t \in [0, T]$ and all $x_1 \in \mathbb{R}$. Hence the function v is discontinuous even on the interior \mathcal{O} .

5.2 Maximality of the value function as a solution to the SVI

The previous example shows that we cannot obtain a comparison theorem for SVI (4.5)-(4.6) to characterize the value function v . We provide in this subsection a weaker characterization of v . To this end, we introduce, for $n \geq 1$, the SVI with penalized coefficients defined on the whole space $[0, T] \times \mathbb{R}^d \times \mathcal{I}$:

$$\min \left[-\frac{\partial v}{\partial t} - \mathcal{L}v - f_n, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad [0, T] \times \mathbb{R}^d \times \mathcal{I}, \quad (5.23)$$

$$\min \left[v - g_n, v - \mathcal{H}v \right] = 0 \quad \text{on} \quad \{T\} \times \mathbb{R}^d \times \mathcal{I}. \quad (5.24)$$

Under assumption **(H1)** and **(H2)**, we can use Lemma 4.1 to apply Proposition 5.1 in [1] and we get from Proposition 4.12 in [1] the following comparison result for this PDE.

Theorem 5.3. *Suppose that **(H1)** and **(H2)** holds. Let u and w be respectively a subsolution and a supersolution to (5.23)-(5.24). Suppose that there exists two constants $C_u > 0$ and $C_w > 0$ and an integer $\gamma \geq 1$ such that*

$$\begin{aligned} u(t, x, i) &\leq C_u(1 + |x|^\gamma) \\ w(t, x, i) &\geq -C_w(1 + |x|^\gamma) \end{aligned}$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$. Then under **(H1)** and **(H2)**, we have $u \leq w$ on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$.

We then need the following additional assumption on the function v .

(H3) There exists a constant $C > 0$ and an integer $\eta \geq 1$ such that

$$v(t, x, i) \geq -C(1 + |x|^\eta) \quad (5.25)$$

for all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.

We give in the next Subsection, some examples where **(H3)** is satisfied. We can now state our result.

Theorem 5.4. *Under **(H1)**, **(H2)** and **(H3)** the function v is the maximal constrained viscosity solution to SVI (4.5)-(4.6) satisfying (5.25): for any function $w : [0, T] \times \bar{\mathcal{O}} \times \mathcal{I} \rightarrow \mathbb{R}$ such that*

- w is a constrained viscosity solution to (4.5)-(4.6),
- there exists a constant C and an integer $\eta \geq 1$ such that

$$w(t, x, i) \geq -C(1 + |x|^\eta) \quad (5.26)$$

for all $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$,

we have $v \geq w$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$.

Proof. Let $w : [0, T] \times \bar{\mathcal{O}} \times \mathcal{I} \rightarrow \mathbb{R}$ be a constrained viscosity solution to (4.5)-(4.6) satisfying (5.26). We proceed in four steps to prove that $w \leq v$.

Step 1. *Extension of the definition of w to $[0, T] \times \mathbb{R}^d \times \mathcal{I}$.*

Define the function \tilde{w} on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$ by

$$\tilde{w}(t, x, i) = \begin{cases} w(t, x, i) & \text{for } (t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}, \\ -Ce^{-\rho t}(1 + |x|^{2\eta}) & \text{for } (t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \bar{\mathcal{O}}) \times \mathcal{I}. \end{cases} \quad (5.27)$$

where ρ and C are two positive constants. From **(H1)**, **(H2)** and (5.26), we can find ρ and C (large enough) such that

$$-\frac{\partial \tilde{w}}{\partial t} - \mathcal{L}\tilde{w} - f_n \leq 0 \quad \text{on } [0, T] \times (\mathbb{R}^d \setminus \bar{\mathcal{O}}) \times \mathcal{I}, \quad (5.28)$$

$$\tilde{w} - g_n \leq 0 \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}, \quad (5.29)$$

and

$$\tilde{w}(t, x, i) \geq -Ce^{-\rho t}(1 + |x|^{2\eta}) \quad \text{for } (t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}. \quad (5.30)$$

Step 2. *Viscosity property of \tilde{w} .*

For C and ρ such that (5.28)-(5.29) and (5.30) hold, we obtain that \tilde{w} is a viscosity subsolution to (5.23)-(5.24). Indeed, let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ and $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ such that

$$(\tilde{w}^* - \varphi)(t, x, i) = \max_{[0, T] \times \mathbb{R}^d \times \mathcal{I}} (\tilde{w}^* - \varphi). \quad (5.31)$$

We first notice from (5.30) that the upper semicontinuous envelope \tilde{w}^* of \tilde{w} is given by

$$\tilde{w}^*(t, x, i) = \begin{cases} w^*(t, x, i) & \text{for } (t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}, \\ -Ce^{-\rho t}(1 + |x|^2) & \text{for } (t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \bar{\mathcal{O}}) \times \mathcal{I}. \end{cases} \quad (5.32)$$

We now prove that \tilde{w} is a subsolution to (5.23)-(5.24). Using (5.28), (5.32) and the viscosity subsolution property of w , we get

$$\tilde{w}^* \leq g \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathcal{I}.$$

For the viscosity property on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$, we distinguish two cases.

- Case 1: $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. From (5.31) and (5.32), we have

$$(\tilde{w}^* - \varphi)(t, x, i) = \max_{[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}} (\tilde{w}^* - \varphi).$$

Since w is a constrained viscosity solution we get

$$\min \left[-\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(t, x, i), \varphi(t, x, i) - \mathcal{H}w^*(t, x, i) \right] \leq 0.$$

- Case 2: $(t, x, i) \in [0, T] \times (\mathbb{R}^d \setminus \bar{\mathcal{O}}) \times \mathcal{I}$. From (5.28), (5.32) we also get

$$\min \left[-\frac{\partial \varphi}{\partial t}(t, x, i) - \mathcal{L}\varphi(t, x, i) - f(t, x, i), \varphi(t, x, i) - \mathcal{H}w^*(t, x, i) \right] \leq 0.$$

Therefore, \tilde{w} is a viscosity subsolution to (5.23)-(5.24).

Step 3. *Growth condition on v_n .*

We prove that for each $n \geq 1$ there exists a constant $C_n > 0$ such that

$$v_n(t, x, i) \geq -C_n(1 + |x|^{2\eta}), \quad (t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}.$$

Fix a set-up \mathcal{X} and $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$, and denote by ${}^0\alpha = ({}^0\tau_k, {}^0\zeta_k)_k$ the trivial strategy of $\mathcal{A}_{t,i}(\mathcal{X})$ i.e. ${}^0\tau_0 = t$, ${}^0\zeta_0 = i$ and ${}^0\tau_k > T$ for $k \geq 1$. Then we have

$$v_n(t, x, i) \geq J(\mathcal{X}, t, x, {}^0\alpha)$$

From the definition of J_n , (2.2) and Lemma 4.1 there exists a constant $\tilde{C}_n > 0$ such that

$$v_n(t, x, i) \geq -\tilde{C}_n(1 + |x|).$$

Since $\eta \geq 1$, this implies

$$v_n(t, x, i) \geq -C_n(1 + |x|^{2\eta}).$$

for some constant C_n .

Step 4. *Comparison on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$.* From Proposition 4.2 in [1], we know that v_n is a viscosity solution to (5.23)-(5.24). Using the results of Steps 2 and 3, we can apply Theorem 5.3 to \tilde{w} and v_n with $\gamma = 2\eta$, and we get

$$\tilde{w} \leq \tilde{w}^*(t, x, i) \leq v_n(t, x, i),$$

for all $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$. Sending n to infinity and using Theorem 3.1, we get $w \leq w^* \leq v$ on $[0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. \square

5.3 Some examples for assumption (H3)

We end this Section by explicit examples where (H3) is satisfied. The first one concerns the case of a regime that stops the controlled diffusion.

Proposition 5.1. *Suppose that for any $x \in \partial\mathcal{O}$ there exists $i_x \in \mathcal{I}$ such that $\mu(x, i_x) = 0$ and $\sigma(x, i_x) = 0$, then assumption (H3) is satisfied.*

Proof. Fix a set-up \mathcal{X} and an initial condition $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$. Let $X^{t,x}$ be the diffusion defined on \mathcal{X} by

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i) dr + \int_t^s \sigma(X_r^{t,x}, i) dW_r, \quad s \geq t.$$

Consider the strategy $\alpha : (\tau_k, \zeta_k)_k$ defined by $(\tau_0, \zeta_0) = (t, i)$,

$$\begin{aligned} \tau_1 &= \inf \{s \geq 0 : X_s \in \partial\mathcal{O}\} \\ \zeta_1 &= i_{X_{\tau_1}} \end{aligned}$$

and $\tau_k > T$ and $\zeta_k = \zeta_1$ for $k \geq 2$. We then have $\mu(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) = 0$ and $\sigma(X_s^{\mathcal{X}, t, x, \alpha}, \alpha_s) = 0$ for $s \in [\tau_1, T]$. Therefore, we get $\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X})$ and

$$v(t, x, i) \geq J(\mathcal{X}, t, x, \alpha).$$

From (2.2) and (H2) there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|).$$

By combining this inequality with Lemma 4.2, we get (H3). \square

We now consider the case where for any initial condition, we can find a regime that keeps the diffusion in $\bar{\mathcal{O}}$.

Proposition 5.2. *Fix a set-up \mathcal{X} . Suppose that for each $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, there exists $i_{t,x} \in \mathcal{I}$ such that the process $X^{t,x}$ defined on \mathcal{X} by*

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}, i_{t,x}) dr + \int_t^s \sigma(X_r^{t,x}, i_{t,x}) dW_r, \quad s \geq t,$$

satisfies

$$\mathbb{P}(X_s^{t,x} \in \bar{\mathcal{O}} \forall s \in [t, T]) = 1. \quad (5.33)$$

Then assumption (H3) is satisfied.

Proof. Fix $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times [0, T]$. Consider the strategy $\alpha : (\tau_k, \zeta_k)_k$ defined by $(\tau_0, \zeta_0) = (t, i)$, $(\tau_1, \zeta_1) = (t, i_{t,x})$ and $\tau_k > T$ and $\zeta_k = \zeta_1$ for $k \geq 2$. From (5.33) we get $\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X})$. We then have

$$v(t, x, i) \geq J(\mathcal{X}, t, x, \alpha).$$

From (2.2) and **(H2)** there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|).$$

By combining this inequality with Lemma 4.2, we get **(H3)**. \square

We end this subsection by using a viability result in the case of a convex constraint. For $x \in \partial\mathcal{O}$, we define the second order normal cone to $\bar{\mathcal{O}}$ at x by

$$\mathcal{N}_{\bar{\mathcal{O}}}(x) = \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}^d : p \cdot (y - x) + \frac{1}{2} A(y - x) \cdot (y - x) \leq o(|y - x|^2) \right. \\ \left. \text{as } y \rightarrow x \text{ and } y \in K \right\},$$

where \mathbb{S}^d is the set of $d \times d$ symmetric matrices.

Proposition 5.3. *Suppose that $\bar{\mathcal{O}}$ is convex and there exists $i^* \in \mathcal{I}$ such that*

$$p \cdot \mu(x, i^*) + \frac{1}{2} \text{tr}[\sigma(x, i^*) \sigma(x, i^*)^\top A] \leq 0$$

for all $x \in \partial\mathcal{O}$ and all $(p, A) \in \mathcal{N}_{\bar{\mathcal{O}}}^2(x)$. Then assumption **(H3)** is satisfied.

Proof. From Proposition 8 and Remark 9 in [8] we get that for any set-up \mathcal{X} and any initial condition $(t, x, i) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{I}$, the control $\alpha = (\tau_k, \zeta_k)_k$ define by

$$\begin{aligned} (\tau_0, \zeta_0) &= (t, i) \\ (\tau_1, \zeta_1) &= (t, i^*) \end{aligned}$$

and $\tau_k > T$ and $\zeta_k = \zeta_1$ for $k \geq 2$, satisfies $\alpha \in \mathcal{A}_{t,x,i}^{\bar{\mathcal{O}}}(\mathcal{X})$. We then have

$$v(t, x, i) \geq J(\mathcal{X}, t, x, \alpha).$$

From (2.2) and **(H2)** there exists a constant $C > 0$ such that

$$v(t, x, i) \geq -C(1 + |x|).$$

By combining this inequality with Lemma 4.2, we get **(H3)**. \square

A Skorokhod representation theorem a with constant marginal law

In this section we extend the Skorokhod representation theorem (see [14]) in the case where a marginal law is supposed to be constant. More precisely, we fix two complete separable metric spaces (\mathcal{S}_1, d_1) and (\mathcal{S}_2, d_2) . We denote by (\mathcal{S}, d) the complete separable metric space defined by

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_1 \times \mathcal{S}_2, \\ d((x_1, x_2), (y_1, y_2)) &= d_1(x_1, y_1) + d_2(x_2, y_2), \quad (x_1, x_2), (y_1, y_2) \in \mathcal{S}. \end{aligned}$$

Theorem A.5. Let μ_0 and μ_n , $n \geq 1$ be probability measures on (\mathcal{S}, d) such that μ_n converges weakly to μ_0 . Suppose that the marginal laws $\mu_n(\cdot \times \mathcal{S}_2)$ do not depend on n . Then there exists a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and random variables X, Y and Y_n , $n \geq 1$ defined on $(\Omega, \mathcal{G}, \mathbb{P})$ such that

- X is valued in \mathcal{S}_1 , Y and Y_n , $n \geq 1$ are valued in \mathcal{S}_2 ,
- the law of (X, Y) is μ ,
- the law of (X, Y_n) is μ_n for all $n \geq 1$,
- Y_n converges \mathbb{P} -a.s. to Y .

Proof. Let $(x_n^1)_{n \geq 1}$ and $(x_n^2)_{n \geq 1}$ be two sequences that are respectively dense in (\mathcal{S}_1, d_1) and (\mathcal{S}_2, d_2) . For any $k \geq 1$, we can choose $r_k \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$ such that

$$\mu_0\left(\partial(B(x_i^1, r_k) \times B(x_i^2, r_k))\right) = 0.$$

Such an r_k exists since μ_0 is finite and hence the set of radiuses $r \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$ for which $\mu_0\left(\partial(B(x_i^1, r) \times B(x_i^2, r))\right) > 0$ is at most a countable.

For $\ell = 1, 2$, we define the subsets $D_i^{\ell, k}$ of \mathcal{S}_ℓ by

$$D_i^{\ell, k} = B(x_i^\ell, r_k) \setminus \left(\bigcup_{j=1}^{i-1} B(x_j^\ell, r_k) \right)$$

for $k \geq 1$ and $i \geq 1$. Obviously $(D_i^{\ell, k})_{i \geq 1}$ is a partition of \mathcal{S}_ℓ for $\ell = 1, 2$. We also define the sets

$$S_{i_1, \dots, i_k}^\ell = D_{i_1}^{\ell, 1} \cap \dots \cap D_{i_k}^{\ell, k}$$

for $k \geq 1$ and $i_1, \dots, i_k \geq 1$. They satisfy the following properties:

- $S_{i_1, \dots, i_k}^\ell \cap S_{i'_1, \dots, i'_k}^\ell = \emptyset$ if there exists some j such that $i_j \neq i'_j$,
- $\bigcup_{i_k \geq 1} S_{i_1, \dots, i_k}^\ell = S_{i_1, \dots, i_{k-1}}^\ell$ for $k \geq 2$ and $\bigcup_{i_1 \geq 1} S_{i_1}^\ell = \mathcal{S}_\ell$,
- The diameter of S_{i_1, \dots, i_k}^ℓ is smaller than 2^{-k+1} :

$$\sup \left\{ d_\ell(x, y) : x, y \in S_{i_1, \dots, i_k}^\ell \right\} \leq 2^{-k+1}$$

for all $k \geq 1$,

- $\mu_0\left(\partial(S_{i_1, \dots, i_k}^1 \times S_{i_1, \dots, i_k}^2)\right) = 0$.

For $k \geq 1$ and $i_1, \dots, i_k \geq 1$, we define the subintervals I_{i_1, \dots, i_k} of $[0, 1]$ as follows.

- $I_{i_1, \dots, i_k} \cap I_{i_1, \dots, i'_k} = \emptyset$ if $i_k \neq i'_k$.
- $\bigcup_{i_k \geq 1} I_{i_1, \dots, i_k} = I_{i_1, \dots, i_{k-1}}$.

- I_{i_1, \dots, i_k} lies in the left of I_{i_1, \dots, i'_k} for $i_k < i'_k$.
- The length of I_{i_1, \dots, i_k} is $\mu_n(S_{i_1, \dots, i_k}^1 \times \mathcal{S}_2)$ (which does not depend on n by assumption).

We also define, for $n \geq 0$, $k, m \geq 1$, $i_1, \dots, i_k \geq 1$ and $j_1, \dots, j_m \geq 1$, the subintervals ${}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$ of $[0, 1]$ as follows.

- ${}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k} \cap {}^n J_{j_1, \dots, j'_m}^{i_1, \dots, i_k} = \emptyset$ if $j_m \neq j'_m$.
- $\bigcup_{j_m \geq 1} {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k} = {}^n J_{j_1, \dots, j_{m-1}}^{i_1, \dots, i_k}$.
- ${}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$ lies in the left of ${}^n J_{j_1, \dots, j'_m}^{i_1, \dots, i_k}$ for $j_m < j'_m$.
- The length of ${}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$ is $\frac{\mu_n(S_{i_1, \dots, i_k}^1 \times S_{j_1, \dots, j_m}^2)}{\mu_n(S_{i_1, \dots, i_k}^1 \times \mathcal{S}_2)}$ if $\mu_n(S_{i_1, \dots, i_k}^1 \times \mathcal{S}_2) \neq 0$ and $\nu(S_{j_1, \dots, j_m}^2)$ else, where ν is some fixed probability measure on \mathcal{S}_2 .

We now fix a point $x_{i_1, \dots, i_k}^\ell \in S_{i_1, \dots, i_k}^\ell$ for $\ell = 1, 2$, $k \geq 1$ and each $i_1, \dots, i_k \geq 1$. We consider the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ where $\Omega = [0, 1] \times [0, 1]$, $\mathcal{G} = \mathcal{B}([0, 1] \times [0, 1])$ and \mathbb{P} the Lebesgue measure on $[0, 1] \times [0, 1]$. We then define the couple of random variable $(X^m, Y_n^{k, m})$ from $[0, 1] \times [0, 1]$ to $\mathcal{S}_1 \times \mathcal{S}_2$ by

$$\begin{aligned} X^m(t_1, t_2) &= x_{i_1, \dots, i_k}^1 && \text{for } (t_1, t_2) \in I_{i_1, \dots, i_k} \times [0, 1], \\ Y_n^{k, m} &= x_{i_1, \dots, i_m}^2 && \text{for } (t_1, t_2) \in I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}, \end{aligned}$$

for all $k, m \geq 1$, $i_1, \dots, i_k \geq 1$ and $j_1, \dots, j_m \geq 1$. Since $\bigcup_{i_1, \dots, i_k \geq 1} I_{i_1, \dots, i_k} = [0, 1]$, we can rewrite $(X^m, Y_n^{k, m})$ in the following way

$$(X^m, Y_n^{k, m})(t_1, t_2) = (x_{i_1, \dots, i_k}^1, x_{j_1, \dots, j_m}^2) \text{ for } (t_1, t_2) \in I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$$

for all $k, m \geq 1$, $i_1, \dots, i_k \geq 1$ and $j_1, \dots, j_m \geq 1$.

We then have

$$d\left((X^k, Y_n^{k, m})(t_1, t_2), (X^{k+p}, Y_n^{k+p, m+q})(t_1, t_2)\right) \leq 2^{-k+1} + 2^{-m+1} \quad (\text{A.34})$$

for all $(t_1, t_2) \in [0, 1] \times [0, 1]$ and all $p, q, n \geq 1$. Indeed, we can write

$$\begin{aligned} (X^{k+p}, Y_n^{k+p, m+q})(t_1, t_2) &= (x_{i_1, \dots, i_k}^1, x_{j_1, \dots, j_m}^2) \\ &\in S_{i_1, \dots, i_{k+p}}^1 \times S_{j_1, \dots, j_{m+q}}^2 \subset S_{i_1, \dots, i_k}^1 \times S_{j_1, \dots, j_m}^2 \end{aligned}$$

for all $i_1, \dots, i_{k+p} \geq 1$, $j_1, \dots, j_{m+q} \geq 1$ and $(t_1, t_2) \in I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$. Since the diameter of $S_{i_1, \dots, i_k}^1 \times S_{j_1, \dots, j_m}^2$ is smaller than $2^{-k+1} + 2^{-m+1}$, we get (A.34).

Since (\mathcal{S}, d) is complete, there exist random variables (X, Y_n) such that

$$d\left((X^k, Y_n^{k, m})(t_1, t_2), (X, Y_n)(t_1, t_2)\right) \longrightarrow 0 \quad \text{as } k, m \rightarrow \infty$$

for all $(t_1, t_2) \in [0, 1] \times [0, 1]$.

Moreover we notice, from the convergence of μ_n to μ_0 , that the area of $I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$ converges to the area of $I_{i_1, \dots, i_k} \times {}^0 J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$ as n goes to infinity. Therefore, for $(t_1, t_2) \in \text{Int}(I_{i_1, \dots, i_k} \times {}^0 J_{j_1, \dots, j_m}^{i_1, \dots, i_k})$ there exists n^* such that

$$(t_1, t_2) \in I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$$

for any $n \geq n^*$. In particular we have

$$\begin{aligned} d\left((X, Y_0)(t_1, t_2), (X, Y_n)(t_1, t_2)\right) &\leq d\left((X, Y_0)(t_1, t_2), (X^m, Y_0^m)(t_1, t_2)\right) \\ &\quad + d\left((X^m, Y_0^m)(t_1, t_2), (X^m, Y_n^m)(t_1, t_2)\right) \\ &\quad + d\left((X^m, Y_n^m)(t_1, t_2), (X, Y_n)(t_1, t_2)\right) \\ &\leq 2^{-k+1} + d\left((X^m, Y_n^m)(t_1, t_2), (X, Y_n)(t_1, t_2)\right) + 2^{-m+1} \\ &\leq 2^{-k+1} + 2^{-m+1} \end{aligned}$$

for all $n \geq n^*$.

Hence, except for a \mathbb{P} -negligible set (which consists at least in the boundaries of the sets $I_{i_1, \dots, i_k} \times {}^n J_{j_1, \dots, j_m}^{i_1, \dots, i_k}$) we get that

$$\lim_{n \rightarrow \infty} Y_n = Y_0, \quad \mathbb{P} - a.s. \quad \text{on} \quad \Omega.$$

To complete the proof, we now show that the law of (X, Y_n) is μ_n for each $n \geq 0$. By construction, the law of (X, Y_n) coincides with μ_n on

$$\mathfrak{P} := \left\{ S_{i_1, \dots, i_k}^1 \times S_{i_1, \dots, i_m}^2, \quad m, k \geq 1 \right\}$$

We then notice that \mathfrak{P} contains the empty set and is stable by intersection. From the monotone class Theorem (see e.g. Chapter I Theorem 19 in [5]) we get that the law of (X, Y_n) coincides with μ_n on $\sigma(\mathfrak{P})$.

We now prove that $\sigma(\mathfrak{P}) = \mathcal{B}(\mathcal{S})$. We first notice that $\mathfrak{P} \subset \mathcal{B}(\mathcal{S})$. Therefore we get $\sigma(\mathfrak{P}) \subset \mathcal{B}(\mathcal{S})$. We now consider the reverse inclusion. Let \mathcal{O}_1 and \mathcal{O}_2 be two nonempty open subsets of \mathcal{S}_1 and \mathcal{S}_2 . For $(x, y) \in \mathcal{O}_1 \times \mathcal{O}_2$, we can find $k, m \geq 1$ such that $B(x, 2^{-k}) \times B(y, 2^{-m}) \subset \mathcal{O}_1 \times \mathcal{O}_2$. Since $\{S_{i_1, \dots, i_{k+1}}^1 \times S_{j_1, \dots, j_{m+1}}^2\}$ is a partition of $\mathcal{S}_1 \times \mathcal{S}_2$ with diameters smaller than $2^{-k} + 2^{-m}$, we can find integers $i_1, \dots, i_k \geq 1$ and $j_1, \dots, j_m \geq 1$ such that

$$(x, y) \in S_{i_1, \dots, i_k}^1 \times S_{j_1, \dots, j_m}^2 \subset \mathcal{O}_1 \times \mathcal{O}_2.$$

We therefore get

$$\mathcal{O}_1 \times \mathcal{O}_2 = \bigcup \left\{ A \in \mathfrak{P} \mid A \subset \mathcal{O}_1 \times \mathcal{O}_2 \right\}.$$

Since \mathfrak{P} is a countable set, we get $\mathcal{O}_1 \times \mathcal{O}_2 \subset \sigma(\mathfrak{P})$. Finally, we obtain $\mathcal{B}(\mathcal{S}_1) \otimes \mathcal{B}(\mathcal{S}_2) \subset \sigma(\mathfrak{P})$. From the separability of \mathcal{S}_1 and \mathcal{S}_2 we get

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}(\mathcal{S}_1) \otimes \mathcal{B}(\mathcal{S}_2) \subset \sigma(\mathfrak{P}).$$

□

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