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Yann Demichel

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RENORMALIZATION OF THE HUTCHINSON OPERATOR

YANN DEMICHEL

ABSTRACT. One of the easiest and common ways of generating fractal sets in \mathbb{R}^D is as attractors of iterated function systems. The classic theory requires that the functions of such systems are contractive. In this paper, we relax this hypothesis considering a new operator H_ρ obtained by renormalizing the usual Hutchinson operator H . Namely, the H_ρ -orbit of a given compact set K_0 is built from the sequence $(H^n(K_0))_n$ with each set being *a priori* rescaled by its distance from 0. We state several results for the convergence of these orbits and give a geometrical description of the corresponding attractors. We link these new sequences to the classic ones, in particular for the linear case, which provides another point of view about the classical theory. We illustrate our results with several various examples. Finally, we discuss some possible generalizations.

1. INTRODUCTION AND NOTATION

The theory and the use of fractal objects, introduced and developed by Mandelbrot (see e.g. [33]), still play an important role today in scientific areas as varied as physics, medicine or finance (see e.g. [24] and references therein). To exhibit theoretical models or to answer practical problems requires to produce various fractal sets. There is a long history of generating fractal sets using Iterated Function Systems. After the fundamental and theoretical works by Hutchinson [28] and Dekking [15], this method was popularized by Barnsley in the 80s (see [2, 1]). Since these years very numerous developments and extensions were made (see e.g. [18, 36, 29, 4] and [6] for an overview of recent advances) making even more enormous the literature related to these topics. The simplicity and the efficiency of this approach have contributed to its success in a lot of domains, for example in image theory (see [3, 25]).

Let us make the mathematical context precise and give the main notation used throughout the paper. Let (M, d) be a metric space. For every map $f : M \rightarrow M$, we define the f -orbit of a point $x_0 \in M$ as the sequence $(x_n)_n$ given by

$$\forall n \geq 0, \quad x_n = f^n(x_0) = (f \circ \dots \circ f)(x_0)$$

where f^n is the n th iterate of f with the convention that f^0 is the identity function Id . In particular, one has $x_{n+1} = f(x_n)$ for all $n \geq 0$, hence, if f is continuous and if $(x_n)_n$ converges to $z \in M$, z is an invariant point for f i.e. $f(z) = z$.

We denote by \mathcal{K}_M the set of all non-empty compact subsets of M . One obtains a metric space endowing it with the Hausdorff metric d_H defined by

$$\forall K, K' \in \mathcal{K}_M, \quad d_H(K, K') = \inf \{ \varepsilon > 0 \mid K \subset K'(\varepsilon) \text{ and } K' \subset K(\varepsilon) \}$$

where $K(\varepsilon) = \{x \in M \mid d(x, K) \leq \varepsilon\}$ and $d(x, K) = \inf\{d(x, y) : y \in K\}$ is the distance between x and K . In this context, a map $F : \mathcal{K}_M \rightarrow \mathcal{K}_M$ will be called an operator on M .

For every map $f : M \rightarrow M$ and every $K \subset M$ one defines the set $f(K) = \{f(x) : x \in K\}$. In the sequel, we will only deal with maps f satisfying $f(K) \in \mathcal{K}_M$ when $K \in \mathcal{K}_M$.

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Let f_1, \dots, f_p be $p \geq 1$ such maps. Then one can define an operator H on M by

$$\forall K \in \mathcal{K}_M, \quad H(K) = \bigcup_{i=1}^p f_i(K). \quad (1)$$

One obtains the so-called *Hutchinson Operator* associated with the family $\{f_1, \dots, f_p\}$ (see e.g. [28, 1, 24]). We will refer to this situation saying that $\{f_1, \dots, f_p\}$ is an *Iterated Function System* (IFS in short). The study of the dynamics of the operator H consists in answering the following questions: Does the orbit $(H^n(K_0))_n$ converge for all compact sets K_0 ? Does its limit depend on K_0 ? What are the geometrical properties of the limit sets?

The ‘classic theory’ of IFS is based on the Contraction Mapping Principle (see e.g. [28, 1, 24] again). A map $f : M \rightarrow M$ is a contraction (or is contractive) with ratio λ if $\lambda \in [0, 1)$ and satisfy

$$\forall x, y \in M, \quad d(f(x), f(y)) \leq \lambda d(x, y).$$

Let us assume moreover that (M, d) is a complete metric space. Then, any contraction is continuous, has a unique invariant point $z \in M$ and, for all $x_0 \in M$, the f -orbit of x_0 converges to z with the basic estimate

$$\forall n \geq 0, \quad d(f^n(x_0), z) \leq \lambda^n d(x_0, z).$$

If f_1, \dots, f_p are all contractive with ratios $\lambda_1, \dots, \lambda_p$ respectively, then the associated Hutchinson operator H is also contractive with ratio $\lambda = \max_{1 \leq i \leq p} \{\lambda_i\}$. Since (\mathcal{K}_M, d_H) inherits the completeness of (M, d) , the operator H has a unique invariant point $L \in \mathcal{K}_M$ which is called the *attractor* of H (or of the IFS $\{f_1, \dots, f_p\}$), and for all $K_0 \in \mathcal{K}_M$ the sequence $(H^n(K_0))_n$ converges to L . Such IFS’s made up with contractive functions are said to be *hyperbolic*. One of the interests is that the sets L obtained in this way are generally fractal sets.

In the sequel, the space M will be essentially \mathbb{R}^D ($D \geq 1$) endowed with the metric induced by the Euclidean norm $\|\cdot\|$. Writing simply \mathcal{K} for \mathcal{K}_M , a subset $K \subset \mathbb{R}^D$ belongs to \mathcal{K} if and only if it is closed and bounded. In this context, one often considers affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$ where A_i is a $D \times D$ matrix and $B_i \in \mathbb{R}^D$ is a vector. Such a map f_i is contractive with $\lambda_i = \|A_i\|$ where $\|A_i\|$ is the norm of A_i given by

$$\|A_i\| = \sup \{ \|A_i x\| : x \in \mathbb{R}^D \text{ with } \|x\| = 1 \} = \inf \{ r > 0 \mid \forall x \in \mathbb{R}^D, \|A_i x\| \leq r \|x\| \}.$$

In particular, classic IFS’s consist of affine transformations involving rotations, symmetries, scalings by a constant ratio, and translations. When the IFS $\{f_1, \dots, f_p\}$ is made up with such contractive affine maps, the corresponding attractor L is said to be a self-affine set. It then enjoys a remarkable geometrical structure since it is equal to a finite union of ‘copies’ of itself. One obtains a nice particular class of such IFS’s when the f_i ’s are of the form $f_i(x) = \alpha_i x + B_i$ where $\alpha_i \geq 0$ is a constant. Indeed, contrarily to general affine maps, f_i contract the distances with the same ratio α_i in all directions. This allows us to give a precise description of L . For example, if the sets $f_i(K_0)$ are mutually disjoint then L is a Cantor set whose fractal dimension is the solution of a very simple equation (see [24, 39]). Cantor sets are fundamental and naturally come when one studies IFS’s (see [20]). A classic family of Cantor sets in \mathbb{R} is the family $\{\Gamma_a : 0 < a < \frac{1}{2}\}$ where Γ_a is the attractor of the IFS $\{f_1, f_2\}$ where $f_1(x) = ax$ and $f_2(x) = ax + (1-a)$ for all $x \in \mathbb{R}$. For example, $\Gamma_{\frac{1}{3}}$ is the usual triadic Cantor set (see [28, 22, 24]). When $\frac{1}{2} \leq a < 1$, the attractor of the previous IFS becomes the whole interval $[0, 1]$. These examples will be extensively used in the sequel.

However, the classic theory suffers from two main drawbacks. The first one is the ‘escape to ∞ ’ problem. If one of the maps f_i is a dilatation, for example in the affine case when $\|A_i\| > 1$, then all the previous results become false and typical orbits fail to converge. Basically, the orbits of some points $x_0 \in K_0$ may then satisfy $\|f_i^n(x_0)\| \rightarrow \infty$, preventing the sequence $(H^n(K_0))_n$ from converging in the space (\mathcal{K}, d_H) . The second one is the ‘vanishing in 0’ problem. When all the f_i ’s are contractive linear maps, then the attractor of the IFS $\{f_1, \dots, f_p\}$ is always $\{0\}$ so does not depend on the fine structure of the A_i ’s but only on their norms. However, in these two degenerate situations, looking at the H -orbit of K_0 we can observe an intriguing geometric structure of the sets $H^n(K_0)$. For example, let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} a & a \\ 0 & a \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a & 0 \\ -a & a \end{bmatrix}$$

with $a \geq 0$ and focus on the H -orbit of the unit ball $B(0, 1)$. For $a = 2$, we have $\|A_1\| = \|A_2\| > 1$ so that the IFS is not hyperbolic, the sequence $(H^n(B(0, 1)))_n$ is not bounded and thus does not converge. For $a = \frac{1}{4}$, we have $\|A_1\| = \|A_2\| < 1$ so that the IFS is now hyperbolic and the sequence $(H^n(B(0, 1)))_n$ converges to $\{0\}$. In each case, one can observe that the sets $H^n(B(0, 1))$ tend to look like a ‘sea urchin’-shaped set whose diameter goes to ∞ in the first case and to 0 in the second one (see Figure 1).

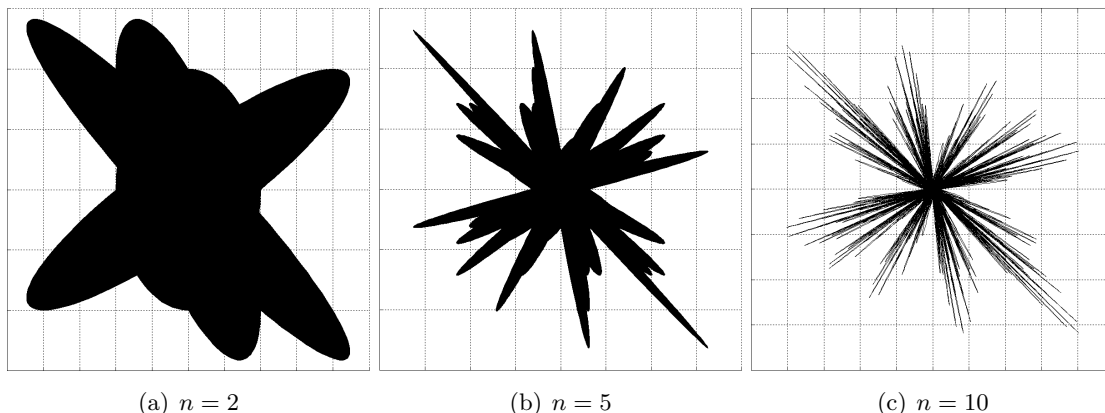


FIGURE 1. Three sets of the H -orbit of $B(0, 1)$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given above. Since the maps are linear, changing the parameter a gives the same sets for each n up to a scaling factor. One has $\text{diam}(H^n(B(0, 1))) \rightarrow \infty$ when $a = 2$ whereas $\text{diam}(H^n(B(0, 1))) \rightarrow 0$ when $a = \frac{1}{4}$. In each case, an adequate renormalization reveals an asymptotic structure.

In this paper, we discuss the possibility to consider IFS avoiding these two drawbacks. Several approaches have already been proposed to solve the first problem, in different contexts: with conformal maps (see [37]), with a random point of view in connection with Markov chains (see [18, 43]), with a weaker notion of contractivity (see [35, 21]), generalizing the definition of attractors (see [5]) or in the setting of multivalued functions (see [32]). At the contrary, the second problem seems not to have been taken into account.

Our strategy to eliminate both these degenerate behaviors consists in modifying the original Hutchinson operator H by dividing $H^n(K_0)$ at each step by its size. The idea of rescale

a sequence of sets $(K_n)_n$ to get its convergence to a non degenerate compact limit is not new and is particularly used in stochastic modeling (see e.g. [40, 13] for famous examples of random growth models and more recently [12, 34, 31] in the context of random graphs and planar maps). Probabilists usually consider the *a posteriori* rescaled sequence of sets $\frac{1}{d_n}K_n$ where d_n estimates the size of K_n , often its diameter $\text{diam}(K_n)$.

Here we choose to proceed slightly differently. In order to keep dealing with the orbit of some operator, we will do an *a priori* renormalization, measuring the size of a compact set with its exact distance from 0. Precisely, we consider the radius function ρ defined on \mathcal{K} by

$$\forall K \in \mathcal{K}, \quad \rho(K) = \sup\{\|x\| : x \in K\} \quad (2)$$

and we denote by H_ρ the operator on \mathbb{R}^D defined by

$$\forall K \in \mathcal{K}, \quad H_\rho(K) = \frac{1}{\rho(\bigcup_{i=1}^p f_i(K))} \bigcup_{i=1}^p f_i(K). \quad (3)$$

The H_ρ -orbit of a set $K_0 \in \mathcal{K}$ then satisfies $\rho(H_\rho^n(K_0)) = 1$ so that it remains bounded and cannot vanish in 0. The operator H_ρ freezes the geometrical structure of $H^n(K_0)$ at each step n of the construction. Typically, H_ρ is not contractive anymore and the classic theory may not be applied. Our objective is then to find new ways to state convergence of the sequence $(H_\rho^n(K_0))_n$ and describe the limit sets. Actually, we will prove that there are strong links between the sequence $(H_\rho^n(K_0))_n$ and some classical H -orbits. However, the H_ρ -orbits may not converge for all starting set K_0 and its possible attractor may be no longer unique, depending on K_0 .

The paper is then organized as follows. In Section 2 we state general results of convergence about rescaling H -orbits of some set K_0 (Lemma 2.1 and Theorem 2.2). We then derive precisely results for asymptotics of sequences $(H_\rho^n(K_0))_n$ and completely solve the case when the f_i 's are homotheties (Theorem 2.8). Section 3 is devoted to IFS's made up with linear maps. Roughly speaking, to renormalize H in this context is nothing but to find classic IFS's that converge to a non-null attractor (Lemma 3.1). We give various examples of such situations and also provide a description of the corresponding attractors, especially in dimension 2. Finally, this suggests to study more general renormalizations, which are discussed in the last section.

2. GENERAL RESULTS

2.1. General renormalization.

We begin with stating two general results which will be extensively used in the sequel. Let $(d_n)_n$ be a sequence of positive real numbers. For $K_0 \in \mathcal{K}$ we consider the sequence of compact sets $(K_n)_n$ defined by

$$\forall n \geq 0, \quad K_{n+1} = \frac{1}{d_n} \bigcup_{i=1}^p f_i(K_n). \quad (4)$$

When $(d_n)_n$ is a constant sequence, say $d_n = d > 0$, $(K_n)_n$ is nothing but $(H_d^n(K_0))_n$ where $H_d = \frac{1}{d}H$ is the Hutchinson operator associated with the IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_p\}$. If $d > \lambda$, $(K_n)_n$ converges to its attractor, L_d . If $d \leq \lambda$, the asymptotic behavior of $(K_n)_n$ is more delicate to study. A such typical situation is presented in the following lemma. We will denote by $\text{cl}(K)$ the closure of a non-empty set K and by $B(0, R)$ the closed ball with center 0 and radius R .

Lemma 2.1. *Let $p \geq 2$ maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$, $d > 0$ and $K_0 \in \mathcal{K}$ such that $H_d(K_0) \supset K_0$. Then the sequence $(H_d^n(K_0))_n$ is increasing thus converges if and only if it is bounded. In this case, its limit is the set*

$$L = \text{cl} \left(\bigcup_{n \geq 0} H_d^n(K_0) \right) \quad (5)$$

and $L \supset K_0$. This situation happens for example when $f_p = d\text{Id}$. In particular, if $f_p = d\text{Id}$ and $\lambda_i < d$ for all $i \in \{1, \dots, p-1\}$, then $(H_d^n(K_0))_n$ converges to L . Moreover, denoting by L' the attractor of the hyperbolic IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_{p-1}\}$, we have $L = L'$ as soon as $K_0 \subset L'$.

Proof. By induction, it is clear that $H_d(K_0) \supset K_0$ implies $K_{n+1} \supset K_n$, hence the first part of the result (see e.g. [22] for details on increasing sequences in \mathcal{K}). In particular, if $f_p = d\text{Id}$ then

$$H_d(K_0) = \bigcup_{i=1}^p \frac{1}{d}f_i(K_0) = K_0 \cup \left(\bigcup_{i=1}^{p-1} \frac{1}{d}f_i(K_0) \right) \supset K_0.$$

In this situation, if $\lambda_i < d$ for all $i \in \{1, \dots, p-1\}$, then these f_i 's are contractions. Thus, if $R > 0$ is large enough to ensure $K_0 \subset B(0, R)$ and

$$\forall i \in \{1, \dots, p-1\} \quad \frac{\|f_i(0)\|}{d - \lambda_i} \leq R,$$

we obtain $\frac{1}{d}f_i(B(0, R)) \subset B(0, R)$ for all $i \in \{1, \dots, p-1\}$. It follows that $K_n \subset B(0, R)$ for all $n \geq 0$ and the convergence of $(K_n)_n$. Finally, observe that, for all $K \in \mathcal{K}$, $H(K) = H'(K) \cup K$ where H' is the Hutchinson operator associated with the hyperbolic IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_{p-1}\}$. Assume that $K_0 \subset L'$. Then, $H(K_0) \subset H(L') = H'(L') \cup L' = L' \cup L' = L'$. By induction it follows that $K_n \subset L'$ for all $n \geq 0$. Taking the limit we get $L \subset L'$. Next, we have $L = H(L) = H'(L) \cup L \supset H'(L)$. By induction it follows that $L \supset (H')^n(L)$ for all $n \geq 0$. Taking the limit we get $L \supset L'$, and finally $L = L'$. \square

Such sequences are related to so-called inhomogeneous IFS's (see e.g. [26]). Assume that $f_p = d\text{Id}$ and $\lambda_i < d$ for all $i \in \{1, \dots, p-1\}$. Let $K_0 \in \mathcal{K}$. Then $K_n = H_0^n(K_0)$ where H_0 is the Hutchinson operator associated with the hyperbolic IFS $\{f_0, \frac{1}{d}f_1, \dots, \frac{1}{d}f_{p-1}\}$ where $f_0 : K \in \mathcal{K} \mapsto K_0$.

When $(d_n)_n$ is no longer a constant sequence, but converges to a positive number $d > \lambda$, we can still prove the convergence of $(K_n)_n$ to L_d .

Theorem 2.2. *Let $p \geq 1$ maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ and $(d_n)_n$ be a sequence of positive real numbers converging to a real number $d > \lambda$. Then, for all $K_0 \in \mathcal{K}$, the sequence $(K_n)_n$ defined by (4) converges to the attractor L_d of the hyperbolic IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_p\}$.*

Proof. Let us write $K'_n = H_d^n(K_0)$. We have to prove that $\varepsilon_n = d_H(K_n, K'_n)$ converges to 0. One has

$$\begin{aligned} d_H(K_{n+1}, K'_{n+1}) &\leq d_H \left(\frac{1}{d_n}H(K_n), \frac{1}{d_n}H(K'_n) \right) + d_H \left(\frac{1}{d_n}H(K'_n), \frac{1}{d}H(K'_n) \right) \\ &\leq \frac{\lambda}{d_n} d_H(K_n, K'_n) + \left| \frac{1}{d_n} - \frac{1}{d} \right| \rho(H(K'_n)). \end{aligned}$$

Since the sequence $(K'_n)_n$ converges, there exists $B \in \mathcal{K}$ such that $K'_n \subset B$ for all $n \geq 0$. Let us then fix $\eta > 0$ and $N \geq 0$ such that $0 < \lambda < d - \eta \leq d_n \leq d + \eta$ for all $n \geq N$. We obtain

$0 \leq \varepsilon_{n+1} \leq \mu \varepsilon_n + m_n$ where $\mu = \frac{\lambda}{d-\eta}$ and $m_n = \left| \frac{1}{d_n} - \frac{1}{d} \right| \rho(H(B))$. It follows that

$$\forall n > N, \quad 0 \leq \varepsilon_n \leq \mu^{n-N} \varepsilon_N + \sum_{k=0}^{n-N-1} \mu^k m_{n-1-k}.$$

Since $\mu \in [0, 1)$ and $m_n \rightarrow 0$ we obtain $\varepsilon_n \rightarrow 0$. \square

2.2. Renormalization with the radius function.

Let us recall that the radius function ρ is defined on \mathcal{K} by $\rho(K) = \max\{\|x\| : x \in K\}$. It satisfies the three following basic properties:

- *continuity* : ρ is continuous with respect to d_H ,
- *monotonicity* : If $K \subset K'$ then $\rho(K) \leq \rho(K')$,
- *homogeneity* : For all $\alpha \in \mathbb{R}$, $\rho(\alpha K) = |\alpha| \rho(K)$.

Moreover, notice that $\rho(K) = 0$ if and only if $K = \{0\}$.

Actually ρ is a nice function because it enjoys an additional stability property:

$$\forall K, K' \in \mathcal{K}, \quad \rho(K \cup K') = \max\{\rho(K), \rho(K')\}. \quad (6)$$

For $K_0 \in \mathcal{K}$, we consider its H_ρ -orbit $(H_\rho^n(K_0))_n$ where H_ρ is the operator defined by (3). We have

$$\forall n \geq 0, \quad H_\rho^{n+1}(K_0) = \frac{1}{d_n} \bigcup_{i=1}^p f_i(H_\rho^n(K_0)) \quad (7)$$

with

$$d_n = \rho\left(\bigcup_{i=1}^p f_i(H_\rho^n(K_0))\right) = \max_{1 \leq i \leq p} \rho(f_i(H_\rho^n(K_0))). \quad (8)$$

In the sequel we will write $K_n = H_\rho^n(K_0)$ in most of the cases and assume that $d_n > 0$, i.e. that $K_n \neq \{0\}$. Notice that for all $n \geq 1$ we have $\rho(K_n) = 1$ with at least one $x_n \in K_n$ such that $\|x_n\| = 1$.

From now on we are interested in the convergence of $(K_n)_n$. A first idea would be to proceed as for the classic theory proving that H_ρ is contractive. Unfortunately, this is rather complicated. Actually the operator H_ρ may be no longer contractive even if the f_i 's are contractions, as it is shown by the following basic example. Assume that $D = 1$ and let f_1, f_2 be the maps defined by $f_1(x) = \frac{1}{3}x + \frac{1}{3}$ and $f_2(x) = \frac{3}{4}x$. Then, H_ρ has (at least) the two invariant points $[0, 1]$ and $[-1, 1]$. Since $(K_n)_n$ may be viewed as in (4), our strategy will consist in linking the convergence of $(K_n)_n$ to the asymptotic behaviour of $(d_n)_n$.

2.3. Convergence of $(d_n)_n$.

In view of Theorem 2.2, we ask the following questions: Does the sequence $(d_n)_n$ always converge? Does its limit be always greater than λ ? A natural idea would be to determine the expression of $(d_n)_n$ from (8) or, at least, a recursive formula to study its convergence. Unfortunately, equality (8) is not tractable even for very simple maps f_i 's and it seems rather difficult to answer the previous questions in a general way. We will focus here on the following problem: if $(d_n)_n$ converges to d , what are the possible values for d ?

Firstly we can give simple upper bounds for d_n and d .

Lemma 2.3. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$ and $(d_n)_n$ the sequence defined in (8). Then,*

$$\forall n \geq 1, \quad d_n \leq \max_{1 \leq i \leq p} \{\rho(f_i(B(0,1)))\} \leq \max_{1 \leq i \leq p} \{\|A_i\| + \|B_i\|\}. \quad (9)$$

In particular, if $(d_n)_n$ converges to d , then d also satisfies (9).

Proof. Since $K_n \subset B(0,1)$ for all $n \geq 1$, using (8) and the monotonicity of ρ we get

$$\forall n \geq 1, \quad d_n = \max_{1 \leq i \leq p} \{\rho(f_i(K_n))\} \leq \max_{1 \leq i \leq p} \{\rho(f_i(B(0,1)))\}.$$

Moreover, one easily proves the following subadditivity property: if $f : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is a map of the form $f(x) = Ax + B$ then

$$\forall K \in \mathcal{K}, \quad \rho(f(K)) = \sup_{x \in K} \{\|Ax + B\|\} \leq \|A\|\rho(K) + \|B\|. \quad (10)$$

Since $\rho(B(0,1)) = 1$, it yields the final bound in (9). \square

Notice that the first inequality in (9) holds even if the f_i 's are not affine maps.

Since ρ is continuous, if $(K_n)_n$ converges to a set $K \in \mathcal{K}$ then $\rho(K) = 1$ and $(d_n)_n$ converges to $d = \rho(\bigcup_{i=1}^p f_i(K)) = \max_{1 \leq i \leq p} \rho(f_i(K))$. When $d > \lambda$, this particular value may be found solving an equation, at least for 'good' maps f_i 's.

Proposition 2.4. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$. If $(d_n)_n$ converges to d , then*

$$\forall i \in \{1, \dots, p\}, \quad 0 \leq \|B_i\| \leq \|d \text{Id} - A_i\|. \quad (11)$$

Moreover, if $i \in \{1, \dots, p\}$ is such that $d \notin \text{Spec}(A_i)$, then

$$0 \leq \|(d \text{Id} - A_i)^{-1} B_i\| \leq 1. \quad (12)$$

In particular, if $d > \lambda$ then

$$\max_{1 \leq i \leq p} \{\|(d \text{Id} - A_i)^{-1} B_i\|\} \leq 1. \quad (13)$$

Proof. Let $i \in \{1, \dots, p\}$ and consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 \in K_1$ and $x_{n+1} = \frac{1}{d_n}(A_i x_n + B_i)$. One has $x_n \in K_n$ and $B_i = d_n x_{n+1} - A_i x_n$. By induction one obtains

$$\forall n \geq 1, \quad n B_i = (d_n x_{n+1} - A_i x_1) + \sum_{k=1}^{n-1} (d_k \text{Id} - A_i) x_{k+1}. \quad (14)$$

Therefore, for all $n > 1$,

$$\begin{aligned} \|B_i\| &\leq \frac{1}{n} \|d_n x_{n+1} - A_i x_1\| + \frac{1}{n} \sum_{k=1}^{n-1} \|(d_k \text{Id} - A_i) x_{k+1}\| \\ &\leq \frac{2}{n} (d_n + \|A_i\|) + \left(\frac{1}{n-1} \sum_{k=1}^{n-1} \|d_k \text{Id} - A_i\| \right). \end{aligned}$$

The first term in the sum above goes to 0 when $n \rightarrow \infty$ and Cesàro's Lemma implies that the term in brackets goes to $\|d \text{Id} - A_i\|$. That gives (11).

Now assume that i is such that $d \notin \text{Spec}(A_i)$. Then the matrix $M_i = d\text{Id} - A_i$ is invertible and (14) yields

$$\forall n \geq 1, \quad n(M_i^{-1}B_i) = M_i^{-1}(d_n x_{n+1} - A_i x_1) + \sum_{k=1}^{n-1} M_i^{-1}(d_k \text{Id} - A_i)x_{k+1}.$$

Thus we obtain in a similar way

$$\|M_i^{-1}B_i\| \leq \frac{2}{n} \|M_i^{-1}\| (d_n + \|A_i\|) + \left(\frac{1}{n-1} \sum_{k=1}^{n-1} \|M_i^{-1}(d_k \text{Id} - A_i)\| \right).$$

We conclude as above using that $\|M_i^{-1}(d_k \text{Id} - A_i)\| \rightarrow \|\text{Id}\| = 1$ as $k \rightarrow \infty$.

Finally, if $d > \lambda$ then $d \notin \bigcap_{i=1}^p \text{Spec}(A_i)$, which concludes the proof. \square

Contrarily to Lemma 2.3, this proposition means that d has to be large enough. We will now show that (13) is an equality when the A_i 's are homotheties. Actually, we will prove again (13) but with a very different approach. We need the following result, which has its own interest. It provides the convex hull of the attractor of an IFS made up with a particular class of similarities (see [19, 16] for other considerations on this topic). We will denote by $\text{ch}(K)$ the convex hull of a non-empty set K .

Lemma 2.5. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = \alpha_i x + B_i$ with $0 \leq \alpha_i < 1$. Let L be the attractor of the IFS defined by $\{f_1, \dots, f_p\}$. Then, the convex hull of L is the polytope $\{z_1, \dots, z_p\}$ where z_i is the unique invariant point of f_i .*

Proof. First the f_i 's are contractions and $z_i = (1 - \alpha_i)^{-1}B_i \in L$ (see [28]). It follows that $\text{ch}(\{z_1, \dots, z_p\}) \subset \text{ch}(L)$. To prove the reverse inclusion we have to state that $L \subset \text{ch}(\{z_1, \dots, z_p\})$. It is enough to prove that $H(\text{ch}(\{z_1, \dots, z_p\})) \subset \text{ch}(\{z_1, \dots, z_p\})$, i.e. that $f_i(\text{ch}(\{z_1, \dots, z_p\})) \subset \text{ch}(\{z_1, \dots, z_p\})$ for all $i \in \{1, \dots, p\}$. Let $z = \sum_{j=1}^p t_j z_j$, $t_j \geq 0$ and $\sum_{j=1}^p t_j = 1$, a point in $\text{ch}(\{z_1, \dots, z_p\})$. We have

$$\forall i \in \{1, \dots, p\}, \quad f_i(z) = \sum_{j=1}^p (t_j \alpha_i) z_j + (1 - \alpha_i) z_i,$$

thus $f_i(z) \in \text{ch}(\{z_1, \dots, z_p\})$. \square

Notice that this result is false in general, even if the A_i 's are as simple as general diagonal matrices.

Proposition 2.6. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$. If $(d_n)_n$ converges to d with $d > \lambda$ then d satisfies the inequality*

$$\rho(\{(d\text{Id} - A_1)^{-1}B_1, \dots, (d\text{Id} - A_p)^{-1}B_p\}) \leq 1. \quad (15)$$

Moreover, if $A_i = \alpha_i \text{Id}$ with $\alpha_i \geq 0$ for all $i \in \{1, \dots, p\}$, then (15) is an equality. In this case, there is then at least one $B_i \neq 0$.

Proof. Let $i \in \{1, \dots, p\}$. Firstly, $d > \lambda$ implies that $d\text{Id} - A_i$ is invertible and $z_i(d) = (d\text{Id} - A_i)^{-1}B_i$ is the unique invariant point of f_i . Secondly, $d > \lambda$ implies that $(K_n)_n$ converges to L_d . Thus, the invariant point $z_i(d)$ of $\frac{1}{d}f_i$ belongs to L_d (see [28]). Therefore $\{z_1(d), \dots, z_p(d)\} \subset L_d$, and, by monotonicity, $\rho(\{z_1(d), \dots, z_p(d)\}) \leq \rho(L_d) = 1$. That gives (15).

Now, if all the A_i 's are homotheties, Lemma 2.5 applied to the IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_p\}$ gives $\text{ch}(\{z_1(d), \dots, z_p(d)\}) = \text{ch}(L_d)$. Since $\rho(\text{ch}(K)) = \rho(K)$ for all $K \in \mathcal{K}$, we obtain

$$1 = \rho(L_d) = \rho(\text{ch}(L_d)) = \rho(\text{ch}(\{z_1(d), \dots, z_p(d)\})) = \rho(\{z_1(d), \dots, z_p(d)\}),$$

hence (15) becomes an equality. Finally, if $B_i = 0$ for all $i \in \{1, \dots, p\}$ then the lhs of (15) is zero, what is not possible. \square

Notice that using the stability property of ρ , (15) gives (13).

2.4. Case of the homotheties.

We can give a complete answer when all the A_i 's are homotheties: the sequence $(K_n)_n$ always converges and its limit may be explicitated. First, we find the exact value of d when $d > \lambda$: it is the upper bound given by Lemma 2.3.

Lemma 2.7. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = \alpha_i x + B_i$ with $\alpha_i \geq 0$. If $(d_n)_n$ converges to d with $d > \lambda$, then*

$$d = \max_{1 \leq i \leq p} \{\alpha_i + \|B_i\|\}.$$

Proof. Applying Proposition 2.6 we see that d is a solution of $\max_{1 \leq i \leq p} \frac{\|B_i\|}{|t - \alpha_i|} = 1$ and we can consider only the $B_i \neq 0$. Then, since $d > \lambda$ and the functions $t \mapsto \frac{\|B_i\|}{|t - \alpha_i|}$ are strictly decreasing on $(\lambda, +\infty)$, the unique solution is $d = \max_{1 \leq i \leq p} \{\alpha_i + \|B_i\|\}$. \square

Theorem 2.8. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = \alpha_i x + B_i$ with $\alpha_i \geq 0$. Let $\alpha_k + \|B_k\| = \max_{1 \leq i \leq p} \{\alpha_i + \|B_i\|\}$ and $\alpha_j = \max_{1 \leq i \leq p} \{\alpha_i\}$. Then, for all $K_0 \in \mathcal{K}$, the sequence $(H_\rho^n(K_0))_n$ converges to a set $K \in \mathcal{K}$. Precisely,*

(i) *If $B_j \neq 0$ then*

- (a) *Either $\alpha_j - \|B_j\| > 0$, $f_i(-\frac{1}{\|B_j\|}B_j) = (\alpha_j - \|B_j\|)(-\frac{1}{\|B_j\|}B_j)$ for all $i \in \{1, \dots, p\}$ and $K_0 = \{-\frac{1}{\|B_j\|}B_j\}$, and then $K = K_0 = \{-\frac{1}{\|B_j\|}B_j\}$,*
- (b) *Or else K does not depend on K_0 , it is the attractor L_d with $d = \alpha_k + \|B_k\|$ and $\frac{1}{\|B_k\|}B_k \in L_d$;*

(ii) *If $B_j = 0$ then*

- (a) *Either $\alpha_j \geq \alpha_k + \|B_k\|$ and then $K = \text{cl}\left(\bigcup_{n \geq 1} H_\rho^n(K_0)\right)$,*
- (b) *Or else $\alpha_j < \alpha_k + \|B_k\|$ and then K does not depend on K_0 , it is the attractor L_d with $d = \alpha_k + \|B_k\|$ and $\frac{1}{\|B_k\|}B_k \in L_d$.*

Proof. First we claim that $(d_n)_{n \geq 1}$ always converges. For all $n \geq 1$ we can find $x_n \in K_n$ and $i_n \in \{1, \dots, p\}$ such that $d_n = \|\alpha_{i_n} x_n + B_{i_n}\|$. Then, $u_n = \frac{1}{d_n}(\alpha_{i_n} x_n + B_{i_n})$ satisfies $u_n \in K_{n+1}$ and $\|u_n\| = 1$. Since $\|x_n\| \leq 1$ we obtain

$$d_{n+1} \geq \|\alpha_{i_n} u_n + B_{i_n}\| \geq \|\alpha_{i_n} u_n + d_n u_n\| - \|d_n u_n - B_{i_n}\| = (\alpha_{i_n} + d_n)\|u_n\| - \alpha_{i_n}\|x_n\| \geq d_n.$$

Thus $(d_n)_{n \geq 1}$ is increasing and bounded (see (9)), so it converges. Let d be its limit.

(i) Assume that $B_j \neq 0$. If the hypotheses of (a) hold, it is clear that $K_n = K_0$ for all $n \geq 1$, hence the result. Thus, let us assume in the sequel that they are not satisfied. Let us suppose that $d \leq \alpha_j$. Applying Proposition 2.4 (11), we get $0 < \|B_j\| \leq |d - \alpha_j|$. Thus $d \leq \alpha_j - \|B_j\| < \alpha_j$. In particular we must have $\alpha_j - \|B_j\| > 0$. Now, let us consider the

sequence $(x_n)_{n \geq 1}$ defined by $x_1 \in K_1$ and $x_{n+1} = \frac{1}{d_n}(\alpha_j x_n + B_j)$ for all $n \geq 1$. We have $x_n \in K_n$ so that $\|x_n\| \leq 1$. Moreover

$$1 \geq \|x_{n+1}\| = \frac{1}{d_n} \|\alpha_j x_n + B_j\| \geq \frac{1}{d} \|\alpha_j x_n + B_j\| \geq \frac{1}{d} (\|\alpha_j x_n\| - \|B_j\|) = \frac{\alpha_j}{d} \|x_n\| - \frac{1}{d} \|B_j\|.$$

By induction one obtains

$$\forall n \geq 1, \quad 1 \geq \|x_n\| \geq \left(\frac{\alpha_j}{d}\right)^{n-1} \left(\|x_1\| - \frac{\|B_j\|}{\alpha_j - d}\right) + \frac{\|B_j\|}{\alpha_j - d}. \quad (16)$$

Since $d < \alpha_j$ we must have $\|x_1\| - \frac{\|B_j\|}{\alpha_j - d} \leq 0$. Now we choose x_1 such that $\|x_1\| = 1$. Then, we get $d \geq \alpha_j - \|B_j\|$ and finally $d = \alpha_j - \|B_j\|$. Substituting this value into (16) we obtain $\|x_n\| = 1$ for all $n \geq 1$. Thus, $1 = \|x_{n+1}\| = \frac{1}{d_n} \|\alpha_j x_n + B_j\|$ and $d_n = \|\alpha_j x_n + B_j\|$. Therefore,

$$\forall n \geq 1, \quad d \geq d_n \geq \|\alpha_j x_n\| - \|B_j\| = \alpha_j \|x_n\| - \|B_j\| = \alpha_j - \|B_j\| = d,$$

i.e. the sequence $(d_n)_{n \geq 1}$ is constant. Now x_1 is any point in K_1 . We have $x_{n+1} = \frac{\alpha_j}{d} x_n + \frac{1}{d} B_j$ for all $n \geq 1$. Let us introduce $u = -\frac{1}{\|B_j\|} B_j$. One has $x_{n+1} - u = \frac{\alpha_j}{d} (x_n - u)$ so that, by induction,

$$\forall n \geq 1, \quad 2 \geq \|x_n - u\| = \left(\frac{\alpha_j}{d}\right)^{n-1} \|x_1 - u\|.$$

Since $d < \alpha_j$ we must have $\|x_1 - u\| = 0$. It follows that $K_1 = \{u\}$. Let $x_0 \in K_0$. Then $f_i(x_0) = du$ for all $i \in \{1, \dots, p\}$, in particular for $i = j$. Since $\alpha_j \neq 0$ the function $\frac{1}{d} f_j$ is bijective so $f_j(u) = du$ implies $x_0 = u$. Finally, $K_0 = \{u\}$ and $f_i(u) = du$ for all $i \in \{1, \dots, p\}$. But these conditions are not fulfilled by hypotheses. This contradiction yields $d > \alpha_j$, then $d = \alpha_k + \|B_k\|$ by Lemma 2.7 and, finally, $K_n \rightarrow L_d$ by Theorem 2.2. Since L_d is the attractor of the IFS $\{\frac{1}{d} f_1, \dots, \frac{1}{d} f_p\}$, it contains the invariant point z_k of $\frac{1}{d} f_k$ which is $z_k = \frac{1}{\|B_k\|} B_k$.

(ii) Assume that $B_j = 0$. Let $u_1 \in K_1$ such that $\|u_1\| = 1$. We have

$$\forall n \geq 1, \quad d \geq d_n \geq d_1 \geq \|f_j(u_1)\| = \|\alpha_j u_1\| = \alpha_j.$$

Thus $d \geq \alpha_j$ and if $d = \alpha_j$ then $d_n = \alpha_j$ for all $n \geq 1$.

(a) Assume that $\alpha_j \geq \alpha_k + \|B_k\|$. It follows from Lemma 2.3 that $d = \alpha_j$. Thus $K_n = H_d^{n-1}(K_1)$ for all $n \geq 1$ and the result follows from Lemma 2.1.

(b) Assume that $\alpha_j < \alpha_k + \|B_k\|$. Let us suppose that $d = \alpha_j$ and consider the sequence $(x_n)_{n \geq 1}$ defined by $x_1 \in K_1$ and $x_{n+1} = \frac{1}{d_n}(\alpha_k x_n + B_k) = \frac{\alpha_k}{\alpha_j} x_n + \frac{1}{\alpha_j} B_k$ for all $n \geq 1$. We have $x_n \in K_n$ so that $\|x_n\| \leq 1$. Thus we can find a subsequence $(x_{n_m})_m$ such that $x_{n_m} \rightarrow x$ with $\|x\| \leq 1$. Moreover, $x_{n+2} - x_{n+1} = \frac{\alpha_k}{\alpha_j} (x_{n+1} - x_n)$ yields

$$\forall n \geq 1, \quad \|x_{n+1} - x_n\| = \left(\frac{\alpha_k}{\alpha_j}\right)^{n-1} \|x_2 - x_1\|. \quad (17)$$

We have $0 \leq \alpha_k \leq \alpha_j < \alpha_k + \|B_k\|$, so that $B_k \neq 0$. Applying Proposition 2.4 (11) we get $d \neq \alpha_k$. Hence $\alpha_k < \alpha_j$ and (17) implies that $x_{n_m+1} \rightarrow x$. It follows that $x = \frac{1}{\alpha_j}(\alpha_k x + B_k)$ i.e. $x = \frac{1}{\alpha_j - \alpha_k} B_k$. Since $\|x\| \leq 1$ we obtain $\alpha_j \geq \alpha_k + \|B_k\|$ which is a contradiction with the hypotheses. Therefore $d > \alpha_j$ and we conclude as the end of the part (i) above. \square

When $D = 1$, since the unit sphere is finite, the sequence $(d_n)_n$ is always stationary.

Example 2.9. Let us consider the IFS $\{f_1, f_2, f_3\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by $f_i(x) = 2x + B_i$ with

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Then, for all $K_0 \in \mathcal{K}$, the sequence $(H_\rho^n(K_0))_n$ converges to the attractor of the IFS $\{\frac{1}{4}f_1, \frac{1}{4}f_2, \frac{1}{4}f_3\}$. It is a classical Sierpinski gasket (see Figure 2).

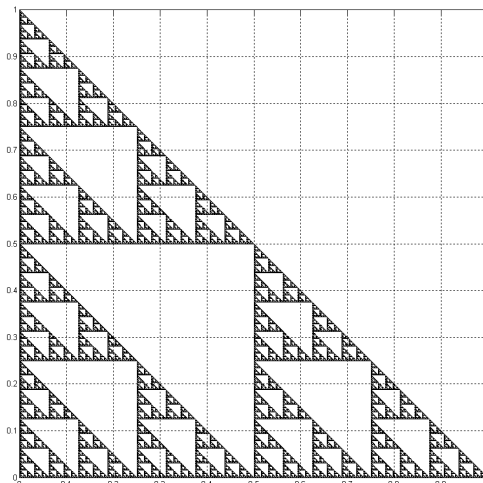


FIGURE 2. The limit set K obtained by renormalizing the IFS $\{f_1, f_2, f_3\}$ with the radius function ρ . Maps f_1, f_2, f_3 are given in Example 2.9.

Proof. We apply Theorem 2.8 with $\max_{1 \leq i \leq 3} \{\alpha_i + \|B_i\|\} = 4$ and $\max_{1 \leq i \leq 3} \{\alpha_i\} = 2$ (notice here that indices k and j are not unique). Whatever is the choice of k and j , we are in the case (b). We have $d = 4$ and $(1, 0) \in L_d$, $(0, 1) \in L_d$. \square

Example 2.10. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by

$$f_1(x) = 4x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad f_2(x) = 2x.$$

Then, the sequence $(H_\rho^n(K_0))_n$ converges to $\{(-1, 0)\}$ if $K_0 = \{(-1, 0)\}$, and to $[0, 1] \times \{0\}$ if $K_0 \neq \{(-1, 0)\}$.

Proof. We apply Theorem 2.8 with $\max_{1 \leq i \leq 2} \{\alpha_i + \|B_i\|\} = 6$ and $\max_{1 \leq i \leq 2} \{\alpha_i\} = 4$ (here $j = k = 1$). We are in the case (i). The subcase (a) is possible (with $\alpha_1 - \|B_1\| = 2$), that gives the result when $K_0 = \{(-1, 0)\}$. Otherwise, we are in the subcase (b) (with $\alpha_1 + \|B_1\| = 6$) and $(H_\rho^n(K_0))_n$ converges to the attractor $[0, 1] \times \{0\}$ of the IFS $\{\frac{1}{6}f_1, \frac{1}{6}f_2\}$. \square

2.5. Trivial renormalization.

We examine here the case when $(H_\rho^n(K_0))_n = (H_d^n(K_0))_n$ for a $d > 0$, i.e when the renormalization only consists in considering a classic IFS. In particular, if $d = 1$ the renormalization has no effect on the original H -orbit of K_0 .

Theorem 2.11. *Let $p \geq 1$ maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ and $d = \max_{1 \leq i \leq p} \{\rho(f_i(B(0,1)))\}$. Assume that $d > \lambda$ and that there exists $z_j \in \mathbb{R}^D$ such that $f_j(z_j) = dz_j$ and $\|z_j\| = 1$, for a certain $j \in \{1, \dots, p\}$. Then, the sequence $(H_\rho^n(K_0))_n$ converges to the attractor L_d for all $K_0 \in \mathcal{K}$ such that $\{z_j\} \subset K_0 \subset B(0,1)$. Moreover,*

- (i) *If $K_0 = \{z_j\}$ then $(H_\rho^n(K_0))_n$ is increasing and $L_d = \text{cl}(\bigcup_{n \geq 0} H_\rho^n(K_0))$,*
- (ii) *If $K_0 = B(0,1)$ then $(H_\rho^n(K_0))_n$ is decreasing and $L_d = \bigcap_{n \geq 0} H_\rho^n(K_0)$.*

Proof. Since $z_j \in K_0$, the monotonicity of ρ implies

$$\rho(\{f_j(z_j)\}) \leq \rho\left(\bigcup_{i=1}^p f_i(K_0)\right) \leq \max_{1 \leq i \leq p} \{\rho(f_i(B(0,1)))\}.$$

We have $\rho(\{f_j(z_j)\}) = \|f_j(z_j)\| = \|dz_j\| = d$. It follows that $d_0 = d$. Therefore $K_1 = \frac{1}{d} \bigcup_{i=1}^p f_i(K_0) = H_d(K_0)$ and, since $\frac{1}{d} f_j(z_j) = z_j$, $z_j \in K_1$. One can then prove by induction that $K_n = H_d^n(K_0)$ for all $n \geq 0$. Since $d > \lambda$, the IFS $\{\frac{1}{d}f_1, \dots, \frac{1}{d}f_p\}$ is hyperbolic so that $(H_d^n(K_0))_n$ converges to L_d . We have proved above that $z_j \in K_1$. Thus, if $K_0 = \{z_j\}$ one has $K_0 \subset K_1$. It follows by induction that $K_n \subset K_{n+1}$ for all $n \geq 0$, and the result. In the similar way, if $K_0 = B(0,1)$ then $K_1 = H_d(K_0) \subset K_0$. It follows by induction that $K_{n+1} \subset K_n$ for all $n \geq 0$, and the result. \square

If f_j is an affine function then $z_j = (d\text{Id} - A_j)^{-1}B_j$ and one hypothesis is nothing but $\|(d\text{Id} - A_j)^{-1}B_j\| = 1$ (see (13) in Proposition 2.4 and (15) in Proposition 2.6).

Notice that such sequences provide examples of constructions from the inside of the attractor L_d taking $K_0 = \{z_j\}$ (see [30]), and from the outside taking $K_0 = B(0,1)$ (see [44]).

Example 2.12. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by

$$f_1(x) = \frac{1}{2} \begin{bmatrix} b+a & b-a \\ b-a & b+a \end{bmatrix} x + c \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad f_2(x) = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} x + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$$

with $0 \leq a \leq b < 1$ and $c \geq 1 - b$. Then, for all $K_0 \in \mathcal{K}$ such that $\{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\} \subset K_0 \subset B(0,1)$, the sequence $(H_\rho^n(K_0))_n$ converges to L_d with $d = b+c$ (see Figure 3). In particular, if $b+c = 1$ then L_d is the attractor of the original hyperbolic IFS $\{f_1, f_2\}$ and the renormalization has no effect on the sequence $(H^n(K_0))_n$.

Proof. One easily calculates $\rho(f_1(B(0,1))) = b+c$ and $\rho(f_2(B(0,1))) = \frac{11}{12}$. Thus $d = b+c \geq 1$. Moreover, $\|f_1\| = b$ and $\|f_2\| = \frac{2}{3}$, hence $\lambda = \max\{b, \frac{2}{3}\} < 1 \leq d$. Finally, if $z_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ then $\|z_1\| = 1$ and $f_1(z_1) = dz_1$. Thus the result follows from Theorem 2.11. \square

We can generalize the previous result when $f_j = \alpha_j \text{Id}$, $\alpha_j > 0$. In this case $d = \alpha_j = \lambda$. This may be deduced from the next general theorem dealing with the case $d = \lambda$.

Theorem 2.13. *Let $p \geq 2$ maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that $f_p = \text{Id}$ and $f_i(B(0,1)) \subset B(0,1)$ for all $i \in \{1, \dots, p-1\}$. Then, for all $K_0 \in \mathcal{K}$ such that $\rho(K_0) = 1$, the sequence $(H_\rho^n(K_0))_n$ converges to the set*

$$L = \text{cl}\left(\bigcup_{n \geq 0} H_\rho^n(K_0)\right).$$

Moreover, assuming that $\lambda_i < 1$ for all $i \in \{1, \dots, p-1\}$ and denoting by L' the attractor of the hyperbolic IFS $\{f_1, \dots, f_{p-1}\}$, we have $L = L'$ as soon as $K_0 \subset L'$.

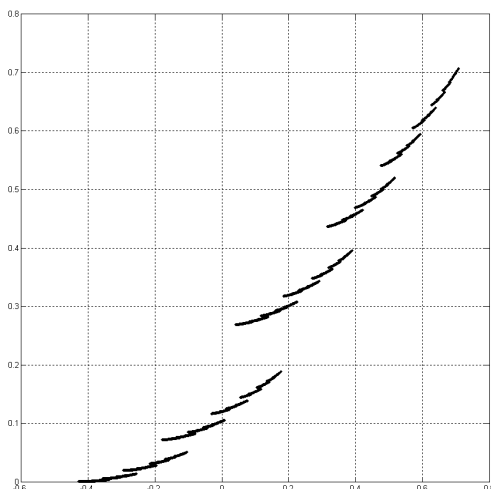


FIGURE 3. The limit set K obtained by renormalizing the IFS $\{f_1, f_2\}$ with the radius function ρ . Maps f_1, f_2 are given in Example 2.12 with parameters $a = \frac{2}{3}$, $b = \frac{3}{4}$ and $c = \frac{1}{2}$. The starting set is $K_0 = \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$.

Proof. Let $u \in K_0$ such that $\|u\| = 1$. The monotonicity of ρ implies

$$\rho(\{f_p(u)\}) \leq \rho\left(\bigcup_{i=1}^p f_i(K_0)\right) \leq \rho\left(\bigcup_{i=1}^p f_i(B_0)\right) \leq \rho(B(0, 1)).$$

Since $\rho(\{f_p(u)\}) = \|u\|$, we have $d_0 = 1$. Therefore $K_1 = \bigcup_{i=1}^p f_i(K_0) = H(K_0)$ and $u = f_p(u) \in K_1$. One can then prove by induction that $K_n = H^n(K_0)$ for all $n \geq 0$. Thus the result follows from Lemma 2.1. \square

To end this section, we give an example of a very particular situation when one has a trivial renormalization which cannot be deduced from Theorem 2.11 or Theorem 2.13. The maps f_i 's are affine with 'identity-block' matrices. Such matrices will be studied in the linear case (see Theorem 3.6).

Example 2.14. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are given by

$$f_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} x \quad \text{and} \quad f_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} x + \begin{bmatrix} 0 \\ (1-a)\frac{\sqrt{2}}{2} \end{bmatrix}$$

with $0 \leq a < 1$. Then, for all $K_0 \in \mathcal{K}$ such that $\{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\} \subset K_0 \subset [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \times [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, the sequence $(H_p^n(K_0))_n$ converges to $L = p_x(K_0) \times \tilde{L}$ where p_x is the projection onto $\text{Span}\{(1, 0)\}$ and \tilde{L} is the Cantor set Γ_a when $0 < a < \frac{1}{2}$ and the interval $[0, \frac{\sqrt{2}}{2}]$ when $\frac{1}{2} \leq a < 1$. Moreover, the renormalization has no effect on the sequence $(H^n(K_0))_n$.

Proof. Let us write $S = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \times [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $z_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. One has $f_2(z_2) = z_2$ with $\|z_2\| = 1$, and, for all $i \in \{1, 2\}$, $f_i(S) \subset S$ with $\rho(S) = 1$. Since $z_2 \in K_0 \subset S$, it follows by induction that $H_p^n(K_0) = H^n(K_0)$ for all $n \geq 0$.

We also prove by induction that

$$\forall n \geq 0, \quad H^n(K_0) = \{(x_0, \tilde{H}^n(\{y_0\})) : (x_0, y_0) \in K_0\},$$

where \tilde{H} is the Hutchinson operator associated with the IFS $\{\tilde{f}_1, \tilde{f}_2\}$ where the $\tilde{f}_i : \mathbb{R} \rightarrow \mathbb{R}$ are defined by $\tilde{f}_1(y) = ay$ and $\tilde{f}_2(y) = ay + (1-a)\frac{\sqrt{2}}{2}$. This latter IFS is hyperbolic with attractor \tilde{L} so that we have $\tilde{H}^n(\{y_0\}) \rightarrow \tilde{L}$ with the estimate $d_H(\tilde{L}, \tilde{H}^n(\{y_0\})) \leq a^n d_H(\tilde{L}, \{y_0\})$. It follows that

$$\begin{aligned} d_H(p_x(K_0) \times \tilde{L}, H^n(K_0)) &= d_H\left(\bigcup_{(x_0, y_0) \in K_0} (x_0, \tilde{L}), \bigcup_{(x_0, y_0) \in K_0} (x_0, \tilde{H}^n(\{y_0\}))\right) \\ &\leq \sup_{(x_0, y_0) \in K_0} d_H((x_0, \tilde{L}), (x_0, \tilde{H}^n(\{y_0\}))) \\ &\leq \sup_{(x_0, y_0) \in K_0} d_H(\tilde{L}, \tilde{H}^n(\{y_0\})) \\ &\leq a^n \sup_{(x_0, y_0) \in K_0} d_H(\tilde{L}, \{y_0\}). \end{aligned}$$

Since $0 \leq a < 1$ we obtain the result letting n goes to ∞ . \square

3. THE LINEAR CASE

In all this section we suppose that we have $p \geq 1$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x$. We denote by $\mathcal{M} = \{A_1, \dots, A_p\}$ the set of their canonical matrices and recall that $\lambda = \max_{1 \leq i \leq p} \{\|A_i\|\}$. Let $\alpha(A_i)$ be the spectral radius of A_i . Since $\|A_i\| \geq \alpha(A_i)$, A_i is not a contraction when $\alpha(A_i) \geq 1$.

Since ρ is homogeneous and the f_i 's are linear, we have

$$\forall n \geq 1, \quad K_n = H_\rho^n(K_0) = \frac{1}{d_0 \cdots d_{n-1}} H^n(K_0) \quad (18)$$

where H still denotes the Hutchinson operator associated with the family $\{f_1, \dots, f_p\}$ (see (1)).

3.1. General strategy.

When all the f_i 's are homotheties, the convergence is stated thanks Theorem 2.8 (ii) (a). In particular, $(d_n)_n$ converges to $d = \lambda$. In the general case, if $(d_n)_n$ converges to d , it follows from Lemma 2.3 that $d \leq \lambda$. Thus we cannot apply Theorem 2.2 and we must find another strategy to study the convergence of $(K_n)_n$.

Since ρ is homogeneous and the f_i 's are linear, the sequence $(K_n)_n$ obtained by iterating H_ρ is the same when one replaces f_i with $\frac{1}{d} f_i$. Moreover, it is possible to not divide $\bigcup_{i=1}^p f_i(K_n)$ by d_n at each step but *a posteriori*. The next result will make clear these key observations.

Lemma 3.1. *Let $K_0 \in \mathcal{K}$. Assume that there exists $d > 0$ such that $(H_d^n(K_0))_n$ converges to a set $L \in \mathcal{K}$ with $\rho(L) \neq 0$. Then, $(d_n)_n$ converges to d (in particular d is unique, it only depends on K_0) and $(H_\rho^n(K_0))_n$ converges to the set $K = \frac{1}{\rho(L)} L$.*

Proof. Since ρ is homogeneous and $\rho(K_n) = 1$, it follows from (18) that $\rho(H^n(K_0)) = d_0 \cdots d_{n-1}$. Using linearity we observe that $H_d^n(K_0) = \frac{1}{d^n} H^n(K_0)$ for all $d > 0$. Thus $\rho(H_d^n(K_0)) = \prod_{k=0}^{n-1} \frac{d_k}{d}$. By hypothesis, this last sequence is a proper convergent product, hence $\frac{d_k}{d} \rightarrow 1$. Now, using linearity again, we have, for all $d > 0$,

$$\forall n \geq 1, \quad K_n = \frac{1}{\rho(H^n(K_0))} H^n(K_0) = \frac{1}{\rho(H_d^n(K_0))} H_d^n(K_0). \quad (19)$$

Hypotheses and continuity of ρ allow us to take the limit in the rhs of (19). That gives the last part of the lemma. \square

Thus, to study the convergence of $(K_n)_n$ it is enough to find a ‘good’ d such that one can apply Lemma 3.1. We must have $d \leq \lambda$ otherwise $L = \{0\}$. Intuitively, one should exactly take $d = \lambda$. Actually, by linearity and homogeneity, the problem is nothing but to find ‘classic’ IFS, made up with linear maps, that are not hyperbolic but provide convergent orbits. We will examine in the sequel some particular classes of such IFS’s.

3.2. LCP sets of matrices.

Here we look for a result of convergence of $(H^n(K_0))_n$ valid for all sets $K_0 \in \mathcal{K}$. We say that \mathcal{M} is a *Left Convergent Product set of matrices* (LCP set in short) if the infinite products $A_{i_n} \cdots A_{i_1}$ converge for all sequences $(i) = (i_1, i_2, \dots) \in \mathcal{I} = \{1, \dots, p\}^\infty$. In this case, we set $A_{(i)} = \lim_{n \rightarrow \infty} A_{i_n} \cdots A_{i_1}$ (see [14, 27]). The theory of LCP sets was popularized in the 90s (see [14]) and it is still of interest nowadays (see [27]), in particular for the study of inhomogeneous Markov chains (see e.g. [42]). One can always associate a canonical IFS with a LCP set. The next result gives sufficient conditions to obtain its convergence.

Lemma 3.2. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that*

- (i) \mathcal{M} is a LCP set,
- (ii) *There exists a sequence $(\varepsilon_n)_n$ of positive numbers such that $\varepsilon_n \rightarrow 0$ and*

$$\forall (i) = (i_1, i_2, \dots) \in \mathcal{I}, \quad \forall n \geq 1, \quad \|A_{(i)} - A_{i_n} \cdots A_{i_1}\| \leq \varepsilon_n. \quad (20)$$

Then, $(H^n(K_0))_n$ converges for all $K_0 \in \mathcal{K}$ to the limit set

$$L = \text{cl} \left(\bigcup_{(i) \in \mathcal{I}} A_{(i)}(K_0) \right). \quad (21)$$

Proof. Let us write $K'_n = H^n(K_0)$ and $L' = \bigcup_{(i) \in \mathcal{I}} A_{(i)}(K_0)$. Hypothesis (i) implies that \mathcal{M} is product bounded (see [7]), then there exists $R > 0$ such that $\|A_{(i)}\| \leq R$ for all $(i) \in \mathcal{I}$. Since K_0 is compact, it follows that L' is bounded, hence L is compact. We claim that $d_H(K'_n, L') \leq C\varepsilon_n$ for all $n \geq 1$, $C > 0$. Let $n \geq 1$ be fixed. We have

$$K'_n = \{A_{i_n} \cdots A_{i_1}(x_0) : x_0 \in K_0 \text{ and } 1 \leq i_1, \dots, i_n \leq p\}.$$

Let $x' \in L'$. One has $x' = A_{(i)}(x_0)$ with $x_0 \in K_0$ and $(i) = (i_1, i_2, \dots, i_n, \dots) \in \mathcal{I}$. Let $x = A_{i_n} \cdots A_{i_1}(x_0)$. One has $x \in K'_n$ and $\|x' - x\| \leq \|A_{(i)} - A_{i_n} \cdots A_{i_1}\| \|x_0\| \leq C\varepsilon_n$ where $C = \rho(K_0)$. Thus $L' \subset K'_n(C\varepsilon_n)$. We prove in a similar way that $K'_n \subset L'(C\varepsilon_n)$, hence $d_H(K'_n, L') \leq C\varepsilon_n$. It follows that $d_H(K'_n, L') \rightarrow 0$ and, since $d_H(K'_n, L') = d_H(K'_n, L)$, that $K'_n \rightarrow L$. \square

Hypothesis (20) implies that the so-called limit function $\mathcal{A} : (i) \mapsto A_{(i)}$ is continuous, endowing \mathcal{I} with the usual distance $d((i), (j)) = 2^{-\inf\{k \geq 1 \mid i_k \neq j_k\}}$. This function is linked to the usual code map in IFS theory (see e.g. [28, 1]). Notice that this lemma includes the contractive case. Indeed, if all the f_i ’s are contractive then \mathcal{M} is a LCP set with $\mathcal{A} = 0$. Hence we can apply Lemma 3.2 with $\varepsilon_n = \lambda^n$ and obtain that $H^n(K_0) \rightarrow L = \{0\}$.

We illustrate the previous result with the family of positive stochastic matrices in \mathbb{R}^2 . In this case, we can give a precise description of the limit set L . Let us recall that a positive stochastic matrix A is a matrix whose rows consist of positive real numbers, with each row summing to 1. Notice that such a matrix satisfies $\alpha(A) = 1$ hence is never contractive.

Proposition 3.3. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form $f_i(x) = A_i x$ where A_i is a positive stochastic matrix. Then, for all $K_0 \in \mathcal{K}$, the sequence $(H^n(K_0))_n$ converges to the set*

$$L = \text{cl} \left(\bigcup_{v_0 \in K_0} \{(x, x) : x \in h_{v_0}(\Gamma)\} \right)$$

where $h_{v_0} : \mathbb{R} \rightarrow \mathbb{R}$ is an affine map which depends on v_0 and f_i , and Γ is the attractor of an IFS $\{g_1, \dots, g_p\}$ where $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is an affine map which only depends on f_i .

Proof. We will apply Lemma 3.2. First, since each product $A_{i_n} \cdots A_{i_1}$ contains a positive stochastic matrix then it converges (see [9]) and \mathcal{M} is a LCP set. Next, there exists a matrix P of the form

$$P = \begin{bmatrix} 1 & u \\ 1 & v \end{bmatrix} \text{ with } u, v \in \mathbb{R},$$

such that, for all $i \in \{1, \dots, p\}$, $A_i = P T_i P^{-1}$ where T_i is a matrix of the form

$$T_i = \begin{bmatrix} 1 & a_i \\ 0 & b_i \end{bmatrix} \text{ with } a_i \in \mathbb{R} \text{ and } b_i \in [0, 1).$$

Notice that it is also proved in [14] that $\{T_1, \dots, T_p\}$ is a LCP set with a continuous limit function. Here we want more and describe precisely the limit set of matrices. Let us define $g_i : \mathbb{R} \rightarrow \mathbb{R}$ by $g_i(x) = b_i x + a_i$ and set $b = \max_{1 \leq i \leq p} b_i < 1$. We obtain by induction that, for all sequences of indices i_1, \dots, i_n , $n \geq 2$,

$$T_{i_n} \cdots T_{i_1} = \begin{bmatrix} 1 & (g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_{n-1}})(a_{i_n}) \\ 0 & b_{i_1} \cdots b_{i_n} \end{bmatrix}.$$

Hence, considering the contractive Hutchinson operator G associated with the IFS $\{g_1, \dots, g_p\}$, its attractor Γ , and the orbit $(G^n(A))_n$ of the compact set $A = \{a_1, \dots, a_p\}$, we obtain, for all $n \geq 2$,

$$\{T_{i_n} \cdots T_{i_1}\}_{1 \leq i_1, \dots, i_n \leq p} = \left\{ \begin{bmatrix} 1 & c_{i_1 \dots i_n} \\ 0 & b(c_{i_1 \dots i_n}) \end{bmatrix}, c_{i_1 \dots i_n} \in G^{n-1}(A) \right\}$$

with $0 \leq b(c_{i_1 \dots i_n}) \leq b^n$ and $d(c_{i_1 \dots i_n}, \Gamma) \leq C b^n$ for a constant $C > 0$ (see [28]).

It follows first that, for all $(i) \in \mathcal{I}$ and all $n \geq 1$,

$$\|A_{(i)} - A_{i_n} \cdots A_{i_1}\| = \|P(T_{(i)} - T_{i_n} \cdots T_{i_1})P^{-1}\| \leq \|P\| \|T_{(i)} - T_{i_n} \cdots T_{i_1}\| \|P^{-1}\| \leq C' b^n$$

with $C' > 0$. Hence (20) and all the hypotheses of Lemma 3.2 are satisfied with $\varepsilon_n = C' b^n$.

Moreover, letting n goes to ∞ we obtain the following set of limit matrices:

$$\{T_{(i)}\}_{(i) \in \mathcal{I}} = \left\{ \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}, c \in \Gamma \right\}.$$

Therefore, if $v_0 = (x_0, y_0) \in K_0$ we get

$$A_{(i)}(v_0) = \left(P \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} P^{-1} \right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \frac{y_0 - x_0}{v - u} c + \frac{x_0 v - y_0 u}{v - u} \\ \frac{y_0 - x_0}{v - u} c + \frac{x_0 v - y_0 u}{v - u} \end{bmatrix}.$$

The result follows by taking $h_{v_0}(x) = \frac{y_0 - x_0}{v - u} x + \frac{x_0 v - y_0 u}{v - u}$ and using (21). \square

Notice that if $K_0 \subset \text{Span}\{(1, 1)\}$ then $f_i(v_0) = v_0$ and $L = K_0$. Actually, $\text{Span}\{(1, 1)\}$ is an invariant space for all the f_i 's so that $K_n = K_0$ for all $n \geq 0$.

Example 3.4. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \frac{1}{4} \begin{bmatrix} 1 + 3a & 3 - 3a \\ 1 - a & 3 + a \end{bmatrix} \quad \text{and} \quad A_2 = \frac{1}{2} \begin{bmatrix} 1 + a & 1 - a \\ 1 - a & 1 + a \end{bmatrix}$$

with $0 < a < 1$. With $P = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ one obtains

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 1 & 1 - a \\ 0 & a \end{bmatrix}.$$

Thus $g_1(x) = ax$, $g_2(x) = ax + (1 - a)$ and Γ is the Cantor set Γ_a when $0 < a < \frac{1}{2}$ and the interval $[0, 1]$ when $\frac{1}{2} \leq a < 1$. The limit L of the sequence $(H^n(K_0))_n$ depends on the starting set K_0 . One has

$$L = \text{cl} \left(\bigcup_{(x_0, y_0) \in K_0} \{(x, x) : x \in h_{(x_0, y_0)}(\Gamma)\} \right)$$

where $h_{(x_0, y_0)} : \mathbb{R} \rightarrow \mathbb{R}$ is the affine map defined by $h_{(x_0, y_0)}(x) = \frac{x_0 - y_0}{4}x + \frac{3y_0 + x_0}{4}$. For example, when $a = \frac{1}{3}$, $L = \left(-\frac{1}{4}\Gamma_{\frac{1}{3}} + \frac{7}{4}, -\frac{1}{4}\Gamma_{\frac{1}{3}} + \frac{7}{4}\right)$ if $K_0 = \{(1, 2)\}$ (see Figure 4(a)) whereas $L = \left(-\frac{1}{4}\Gamma_{\frac{1}{3}} + \frac{7}{4}, -\frac{1}{4}\Gamma_{\frac{1}{3}} + \frac{7}{4}\right) \cup \left(\frac{1}{2}\Gamma_{\frac{1}{3}} + \frac{1}{2}, \frac{1}{2}\Gamma_{\frac{1}{3}} + \frac{1}{2}\right)$ if $K_0 = \{(1, 2), (2, 0)\}$ (see Figure 4(b)). More K_0 contains points then more complicated is the limit set L , with unions of overlapping Cantor sets.

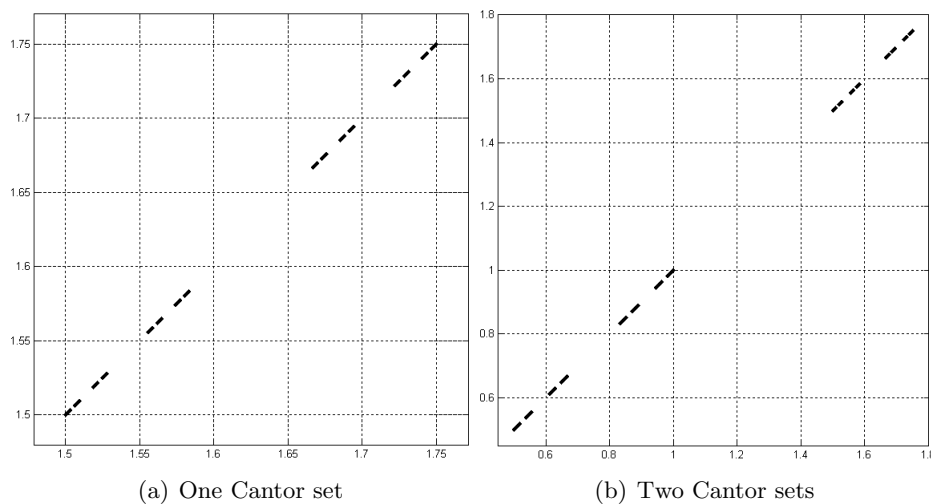


FIGURE 4. The limit set L of $(H^n(K_0))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.4 with parameter $a = \frac{1}{3}$. Figure (a): the starting set is $K_0 = \{(1, 2)\}$ and L is a Cantor set. Figure (b): the starting set is $K_0 = \{(1, 2), (2, 1)\}$ and L is a union of two disjoint Cantor sets.

Several necessary and sufficient conditions for a finite set of matrices to be a LCP set have been given (see [9, 8, 23] and [27] for a survey). However, such conditions, involving existence of appropriate norms, evaluation of the joint spectral radius or determination of the generalized eigenspaces of the A_i 's, are difficult to state. Moreover, they do not provide a geometric

description of the limit set $\{A_{(i)}\}_{(i) \in \mathcal{I}}$ which can be used to describe the limit of $(H^n(K_0))_n$. Furthermore, very simple LCP sets may have a discontinuous function \mathcal{A} preventing from applying Lemma 3.2. This situation happens for example adding the matrix Id to a LCP set with a continuous limit function (see [14]).

3.3. Identity-block matrices.

Even if the set \mathcal{M} does not fulfill hypotheses of Lemma 3.2, the corresponding IFS may provide convergent H -orbits. We deal now with such situations when some matrices A_i have an identity-block. Let us begin with a first basic result.

Lemma 3.5. *Let $p \geq 2$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x$ such that $A_p = \text{Id}$ and $\|A_i\| \leq 1$ for all $i \in \{1, \dots, p-1\}$. Then, for all $K_0 \in \mathcal{K}$, the sequence $(H^n(K_0))_n$ converges to the set*

$$L = \text{cl} \left(\bigcup_{n \geq 0} H^n(K_0) \right).$$

Proof. Let $R > 0$ be large enough to ensure $K_0 \subset B(0, R)$. Then, $f_i(B(0, R)) \subset B(0, R)$ for all $i \in \{1, \dots, p\}$. Therefore $H^n(K_0) \subset B(0, R)$ for all $n \geq 0$. Hence the result follows from Lemma 2.1. \square

The next result deals with a more general situation.

Theorem 3.6. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that there exists two subspaces $V, W \subset \mathbb{R}^D$ satisfying $V \oplus W = \mathbb{R}^D$ and, for all $i \in \{1, \dots, p\}$,*

- (i) $(f_i - \text{Id})(V) \subset W$,
- (ii) $f_i(W) \subset W$ and the linear function $\tilde{f}_i : W \rightarrow W$ induced by f_i is a contraction.

Then, $(H^n(K_0))_n$ converges for all $K_0 \in \mathcal{K}$ to the set

$$L = \text{cl} \left(\bigcup_{v_0 \in p_{V,W}(K_0)} v_0 + \check{L}(v_0) \right)$$

where $p_{V,W}$ is the projection onto V along W and $\check{L}(v_0)$ is the attractor of the IFS $\{\check{f}_1, \dots, \check{f}_p\}$ where $\check{f}_i : W \rightarrow W$ is defined by $\check{f}_i(w) = \tilde{f}_i(w) + (f_i - \text{Id})(v_0)$.

Proof. Let us write $V_0 = p_{V,W}(K_0)$, $W_0 = p_{W,V}(K_0)$ where $p_{W,V}$ is the projection onto W along V , and set $\tilde{\lambda} = \max_{1 \leq i \leq p} \|\tilde{f}_i\| = \max_{1 \leq i \leq p} \|\check{f}_i\| < 1$. The proof is in the same vein as the one of Example 2.14.

(a) First let $z_0 \in K_0$ with $z_0 = v_0 + w_0 \in V_0 + W_0 \subset V \oplus W$. We have

$$\forall i \in \{1, \dots, p\}, \quad f_i(z_0) = v_0 + ((f_i(v_0) - v_0) + f_i(w_0)) = v_0 + \check{f}_i(w_0) \in V_0 + W.$$

We then prove by induction that

$$\forall n \geq 0, \quad H^n(\{z_0\}) = v_0 + \check{H}^n(\{w_0\})$$

where \check{H} is the Hutchinson operator associated with the hyperbolic IFS $\{\check{f}_1, \dots, \check{f}_p\}$. Therefore the sequence $(\check{H}^n(w_0))_n$ converges to a set $\check{L}(v_0)$ which only depends on v_0 . It follows that $(H^n(\{z_0\}))_n$ converges to $v_0 + \check{L}(v_0)$ with the estimate

$$d_H(v_0 + \check{L}(v_0), v_0 + \check{H}^n(w_0)) = d_H(\check{L}(v_0), \check{H}^n(w_0)) \leq \tilde{\lambda}^n d_H(\check{L}(v_0), w_0).$$

(b) Let

$$R = \frac{1}{1 - \tilde{\lambda}} \max_{1 \leq i \leq p} \{\rho((f_i - \text{Id})(V_0))\}.$$

We have $\check{f}_i(B(0, R) \cap W) \subset B(0, R) \cap W$ for all $i \in \{1, \dots, p\}$, so that $\check{H}(B(0, R) \cap W) \subset B(0, R) \cap W$. It follows that $\check{L}(v_0) \subset B(0, R) \cap W$ for all $v_0 \in V_0$. Thus

$$\forall v_0 \in V_0, \quad \rho(v_0 + \check{L}(v_0)) \leq \|v_0\| + R \leq \rho(V_0) + R.$$

We then have proved that the set $\bigcup_{v_0 \in V_0} v_0 + \check{L}(v_0)$ is bounded, i.e. $L \in \mathcal{K}$.

Furthermore, since $\check{L}(v_0)$ does not depend on w_0 , we can write

$$\bigcup_{v_0 \in V_0} v_0 + \check{L}(v_0) = \bigcup_{v_0 + w_0 \in K_0} v_0 + \check{L}(v_0).$$

(c) Finally, writing $H^n(K_0) = \bigcup_{v_0 + w_0 \in K_0} v_0 + \check{H}^n(w_0)$ and using (a) and (b), we obtain

$$\begin{aligned} d_H(L, H^n(K_0)) &= d_H\left(\bigcup_{v_0 + w_0 \in K_0} v_0 + \check{L}(v_0), \bigcup_{v_0 + w_0 \in K_0} v_0 + \check{H}^n(w_0)\right) \\ &\leq \sup_{v_0 + w_0 \in K_0} d_H(v_0 + \check{L}(v_0), v_0 + \check{H}^n(w_0)) \\ &\leq \tilde{\lambda}^n \sup_{v_0 + w_0 \in K_0} d_H(\check{L}(v_0), w_0). \end{aligned}$$

Finally, since this latter supremum is finite, we obtain the result letting n goes to ∞ . \square

Notice here that hypotheses of Theorem 3.6 mean that, for all $i \in \{1, \dots, p\}$, the maps f_i 's have a block matrix with respect to the sum $V \oplus W$ of the form

$$A_i = \begin{bmatrix} \text{Id} & 0 \\ M_i & \tilde{A}_i \end{bmatrix}$$

with \tilde{A}_i contractive and some matrix M_i .

Example 3.7. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$$

with $a, b \in \mathbb{R}$ and $0 < c < 1$. Then, for all $K_0 \in \mathcal{K}$, the sequence $(H^n(K_0))_n$ converges to the set

$$L = \text{cl}\left(\bigcup_{(x_0, y_0, z_0) \in K_0} \left\{(x_0, y_0, \frac{ax_0 + by_0}{1-c} L_c)\right\}\right)$$

where L_c is the Cantor set Γ_c if $0 < c < \frac{1}{2}$ and the interval $[0, 1]$ if $\frac{1}{2} \leq c < 1$.

As an example, L is shown in Figure 5 when K_0 is the unit circle $\{(\cos t, \sin t, 0) : t \in [0, 2\pi]\}$ and parameters $(a, b, c) = (1, 1, \frac{1}{4})$.

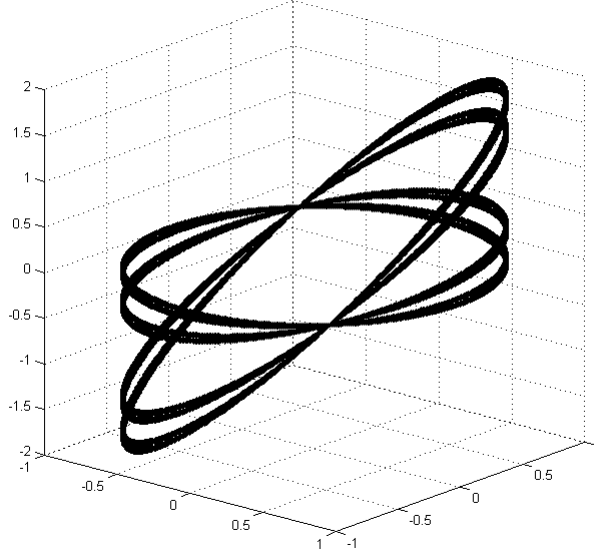


FIGURE 5. The limit set L of $(H^n(K_0))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.7 with parameters $(a, b, c) = (1, 1, \frac{1}{4})$. The starting set is the circle $K_0 = \{(\cos t, \sin t, 0) : t \in [0, 2\pi]\}$.

Proof. We can apply Theorem 3.6 with $V = \text{Span}\{(1, 0, 0), (0, 1, 0)\}$ and $W = \text{Span}\{(0, 0, 1)\}$. Thus, $(H^n(K_0))_n$ converges to the set

$$L = \text{cl} \left(\bigcup_{(x_0, y_0, z_0) \in K_0} (x_0, y_0, 0) + L(x_0, y_0) \right)$$

where $L(x_0, y_0)$ is the attractor of the IFS $\{\check{f}_1, \check{f}_2\}$ with, for all $w = (0, 0, z) \in W$,

$$\check{f}_1(w) = \check{f}_1(0, 0, z) = (0, 0, cz + ax_0 + by_0) \text{ and } \check{f}_2(w) = \check{f}_2(0, 0, z) = (0, 0, cz).$$

By uniqueness, we check that this attractor is the one announced.

Assume that $(a, b, c) = (1, 1, \frac{1}{4})$ and $K_0 = \{(\cos t, \sin t) : t \in [0, 2\pi]\} \times \{0\}$. We have

$$L = \text{cl} \left(\bigcup_{t \in [0, 2\pi]} (\cos t, \sin t, \frac{4}{3}(\cos t + \sin t)\Gamma_{\frac{1}{4}}) \right).$$

Then L is the closure of a union of circles drawn on the cylinder $\{(\cos t, \sin t) : t \in [0, 2\pi]\} \times \{\frac{4\sqrt{2}}{3}t : t \in [-1, 1]\}$. Each intersection with a generatrice is homothetic with the Cantor set $\Gamma_{\frac{1}{4}}$ (excepted for the two special values $t = \frac{3\pi}{4}$ and $t = \frac{7\pi}{4}$). \square

3.4. Orbit of the unit ball.

One of the difficulties occurring in the linear case lives in the different behaviors induced by the particular linear subspaces of the A_i 's associated with the eigenvalue 1. To avoid these degenerate behaviors, we propose to take into account all the directions of \mathbb{R}^D by focusing on the H -orbit of the unit ball $B(0, 1)$. We begin with a simple general result.

Lemma 3.8. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x$. Assume that there exists $N \geq 1$ indices $i_1, \dots, i_N \in \{1, \dots, p\}$ such that the matrix $A_{i_N} \cdots A_{i_1}$ has eigenvalue 1 and denote by V the corresponding eigenspace. Then, for all $K_0 \in \mathcal{K}$ such that*

$H(K_0) \subset K_0$ and $K_0 \cap (V \setminus \{0\}) \neq \emptyset$, the sequence $(H^n(K_0))_n$ is decreasing and converges to a set $L \neq \{0\}$.

Proof. Since $H(K_0) \subset K_0$, the sequence $(H^n(K_0))_n$ is decreasing and converges to the compact set $L = \bigcap_{n \geq 0} H^n(K_0)$. Let $v \neq 0$ in $K_0 \cap V$. Since $A_{i_N} \cdots A_{i_1} v = v$ one has $v \in H^{ki_N}(K_0)$ for all $k \geq 0$, thus $v \in L$ and $L \neq \{0\}$. \square

When $K_0 = B(0, 1)$ we obtain the following result for $(H_\rho^n(K_0))_n$.

Proposition 3.9. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x$ such that*

- (i) *For all $i \in \{1, \dots, p\}$, $\|A_i\| \leq 1$,*
- (ii) *There exists $N \geq 1$ indices $i_1, \dots, i_N \in \{1, \dots, p\}$ such that the matrix $A_{i_N} \cdots A_{i_1}$ has eigenvalue 1.*

Then, the sequence $(H_\rho^n(B(0, 1)))_n$ is decreasing and converges to the set

$$L = \bigcap_{n \geq 0} H_\rho^n(B(0, 1)). \quad (22)$$

Moreover, $H_\rho^n(B(0, 1)) = H^n(B(0, 1))$ for all $n \geq 0$.

Proof. Hypothesis (i) implies that $H(B(0, 1)) \subset B(0, 1)$ and we have $B(0, 1) \cap (V \setminus \{0\}) \neq \emptyset$. Thus $(H^n(B(0, 1)))_n$ is decreasing and converges to L from Lemma 3.8. Moreover, choosing $v \in B(0, 1)$ with $\|v\| = 1$ and such that $A_{i_N} \cdots A_{i_1} v = v$ we obtain $v \in L$ and then $v \in H^n(B(0, 1))$ for all $n \geq 0$. Since $\|v\| = 1$, one has $\rho(H^n(B(0, 1))) = 1$ so that $H_\rho^n(B(0, 1)) = H^n(B(0, 1))$. \square

Notice that the two hypotheses imply that $\alpha(A_{i_N} \cdots A_{i_1}) = \|A_{i_N} \cdots A_{i_1}\| = 1$. In particular, the IFS $\{f_1, \dots, f_p\}$ is not hyperbolic. Notice also that \mathcal{M} need not to be a LCP set. One can for example consider matrices of rotations or symmetries.

Example 3.10. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$$

with $a \geq 0$. Then, the sequence $(H_\rho^n(B(0, 1)))_n$ is decreasing and converges to the set L given by (22). Moreover, the point

$$u = \left(\frac{1}{\sqrt{2}} \frac{1 + \sqrt{1 + 4a^2}}{\sqrt{1 + 4a^2 + \sqrt{1 + 4a^2}}}, \frac{1}{\sqrt{2}} \frac{2a}{\sqrt{1 + 4a^2 + \sqrt{1 + 4a^2}}} \right)$$

belongs to L and satisfies $\|u\| = 1$. As an example, L is shown in Figure 6 when $a = 1$.

Proof. Since $\|A_1\|^2 = \|A_2\|^2 = a^2 + \frac{1}{2}(1 + \sqrt{1 + 4a^2}) \geq 1$, we cannot apply directly Proposition 3.9. Thus we consider the normalized matrices $A'_i = \frac{1}{d} A_i$ with $d = \|A_1\| = \|A_2\|$ so that hypothesis (i) is satisfied. One checks that the matrix

$$A'_1 A'_2 = \frac{1}{a^2 + \frac{1}{2}(1 + \sqrt{1 + 4a^2})} \begin{bmatrix} 1 + a^2 & a \\ a & a^2 \end{bmatrix}$$

has eigenvalue 1. Hence one can apply Proposition 3.9. It follows from its proof that u , one of the unit eigenvectors associated with 1, belongs to L . Notice also that one can easily prove that L is symmetric with respect to the origin and the line $y = x$. \square

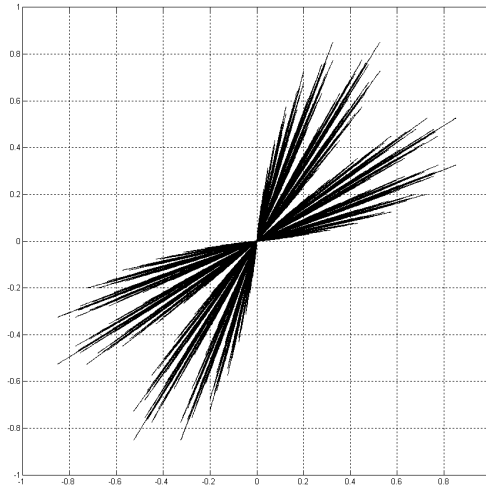


FIGURE 6. The limit set L of $(H^n(B(0, 1)))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.10 with parameter $a = 1$.

Notice that the previous method may be generally applied when two matrices of \mathcal{M} are symmetric each other, or when one of them is symmetric.

3.5. Description of the limits in \mathbb{R}^2 .

When $(H^n(K_0))_n$ converges but $\{f_1, \dots, f_p\}$ is not hyperbolic, the limit set L does depend on the starting set K_0 . Hence, various complicated situations may occur and it seems rather difficult to obtain a complete inventory of all these possible limit sets. However, in this section, we can go further by considering only the space \mathbb{R}^2 and particular K_0 , and focusing on the ‘angular structure’ of L .

Let $\mathbb{P} = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Every point $(x, y) \in \mathbb{P} \setminus \{0\}$ may be written with polar coordinates as $(R \cos \theta, R \sin \theta)$ with $R > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. For a compact set $K \subset \mathbb{P}$, we define its *set of slopes* $S \subset [-\infty, \infty]$ by

$$S = \{ \tan \theta \text{ such that there exists } (R \cos \theta, R \sin \theta) \in K \setminus \{0\} \}$$

with the convention $\tan(\pm \frac{\pi}{2}) = \pm \infty$.

When $(H^n(K_0))_n$ converges to $L \subset \mathbb{P}$ with $L \neq \{0\}$, its set of slopes is a well-defined non-empty set. The aim of this section is to describe this set. Notice that L and $K = \frac{1}{\rho(L)}L$ share the same set of slopes, so that it will also give the description of the limit of the renormalized sequence $(H_\rho^n(K_0))_n$.

Proposition 3.11. *Let $p \geq 1$ linear maps $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by their canonical matrices $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ such that $\det(A_i) \neq 0$ and $f_i(\mathbb{P}) \subset \mathbb{P}$. Let $K_0 \subset \mathbb{P}$ be a compact set such that $(H^n(K_0))_n$ converges to $L \neq \{0\}$. Then, the set of slopes of L is a non-empty invariant set of the operator $\widehat{H} = \bigcup_{i=1}^p \widehat{f}_i$ where $\widehat{f}_i : [-\infty, \infty] \rightarrow [-\infty, \infty]$ is the homographic function defined by*

$$\widehat{f}_i(z) = \frac{d_i z + c_i}{b_i z + a_i}.$$

Proof. One has $H(K_0) \subset H(\mathbb{P}) \subset \mathbb{P}$, hence $L \subset \mathbb{P}$. Since $L \neq \{0\}$, its set of slopes S is not empty. Writing $L \setminus \{0\} = \{(R \cos \theta, R \sin \theta) : R > 0, \theta \in \Theta\}$ we have $S = \tan \Theta$. Since the f_i 's are bijective, $H(L \setminus \{0\}) = \bigcup_{i=1}^p f_i(L \setminus \{0\})$. Moreover,

$$A_i \begin{bmatrix} R \cos \theta \\ R \sin \theta \end{bmatrix} = R \begin{bmatrix} a_i \cos \theta + b_i \sin \theta \\ c_i \cos \theta + d_i \sin \theta \end{bmatrix} \neq 0,$$

thus the set of slopes of $H(L)$ is the set of all the points of the form

$$\frac{c_i \cos \theta + d_i \sin \theta}{a_i \cos \theta + b_i \sin \theta} = \widehat{f}_i(\tan \theta), \quad \theta \in \Theta,$$

i.e. the set $\bigcup_{i=1}^p \widehat{f}_i(S)$. Since $L = H(L)$, they have the same set of slopes S . Hence the result. \square

In the situation of the previous theorem, the angular structure of L is known as soon as we can describe the invariant sets of the Hutchinson operator \widehat{H} . This operator may have several invariant sets S , not necessarily closed. However, there is a useful way to determine them. Indeed, if \widehat{H} is contractive then every bounded invariant set S of \widehat{H} satisfies $\text{cl}(S) = \widehat{L}$ where \widehat{L} is the attractor associated with \widehat{H} . It is possible to determine such attractors of IFS made up with homographic functions (see for example [24] page 136). Notice that $[-\infty, \infty]$ is always a compact set and then would be an invariant set. Thus, it will be often necessary to consider a restriction of \widehat{H} to obtain a contractive operator.

Example 3.12. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \frac{b}{1-a} & 0 \\ b & \frac{ab}{1-a} \end{bmatrix}$$

with $0 < a < 1$, $0 < b < 1$ and $a + b \leq 1$. Then, the sequence $(H^n(K_0))_n$ starting at $K_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$ converges to a set $L \in \mathcal{K}$ whose set of slopes S satisfies $\text{cl}(S) = \Gamma_a$ if $0 < a < \frac{1}{2}$ and $\text{cl}(S) = [0, 1]$ if $\frac{1}{2} \leq a \leq 1$ (see Figure 7).

Proof. One checks that $H(K_0) \subset K_0$ with the choice of a and b . Since A_1 has eigenvalue 1 with $V = \text{Span}\{(1, 0)\}$, the convergence of $(H^n(K_0))_n$ is given by Lemma 3.8. Moreover, according to Proposition 3.11, the set of slopes S of the limit L is an invariant set of the operator $\widehat{H} = \widehat{f}_1 \cup \widehat{f}_2$ where $\widehat{f}_1(z) = az$ and $\widehat{f}_2(z) = az + (1 - a)$. Since $L \subset K_0$, one has $S \subset [0, 1]$. One checks that \widehat{H} is contractive from $[0, 1]$ into $[0, 1]$, implying $\text{cl}(S) = \Gamma_a$ if $0 < a < \frac{1}{2}$ and $\text{cl}(S) = [0, 1]$ if $\frac{1}{2} \leq a < 1$. \square

The fact that in Example 3.12 the set S is not the whole Cantor set Γ_a comes from the contraction f_2 . Actually, all the orbits of points $z \in K_0$ which are associated with contractions correspond to a 'slope' $s = \tan \theta$ for which $R = R(\theta) = 0$. Therefore, if one wants to see the whole attractor of the IFS $\{\widehat{f}_1, \dots, \widehat{f}_p\}$, then one has to not consider contractions. Such a situation is presented in the following example.

Example 3.13. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1-a & a \end{bmatrix}$$

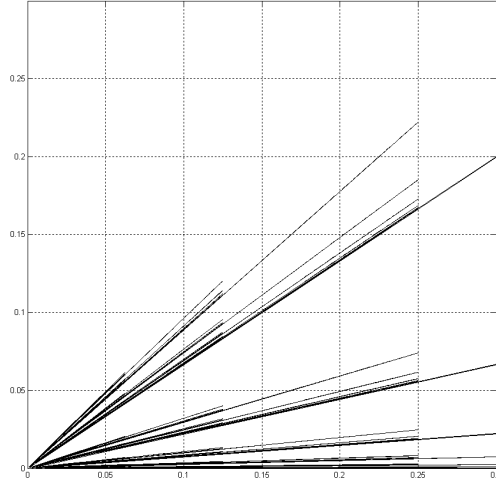


FIGURE 7. Zoom in the limit set L of $(H^n(K_0))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.12 with parameters $a = b = \frac{1}{3}$. The starting set is $K_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$. The set of slopes is dense in the triadic Cantor set.

with $0 < a < 1$. Then, the sequence $(H^n(K_0))_n$ starting at $K_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$ converges to a set $L \in \mathcal{K}$ whose set of slopes S satisfies $\text{cl}(S) = S = \Gamma_a$ if $0 < a < \frac{1}{2}$ and $\text{cl}(S) = S = [0, 1]$ if $\frac{1}{2} \leq a < 1$ (see Figure 8).

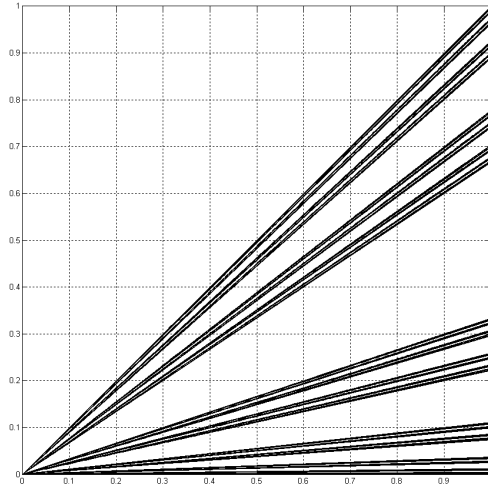


FIGURE 8. The limit set L of $(H^n(K_0))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.13 with parameter $a = \frac{1}{3}$. The starting set is $K_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$. The set of slopes is the triadic Cantor set.

Proof. Actually this example is a particular case of Example 3.12 taking $b = 1 - a$. To obtain a more precise description of the limit set L we apply Theorem 3.6 with $V = \text{Span}\{(1, 0)\}$ and $W = \text{Span}\{(0, 1)\}$. Let $z_0 = (x_0, y_0) \in K_0$. We have $p_{V,W}(z_0) = (x_0, 0)$. It follows that $(H^n(K_0))_n$ converges to the set $L = \text{cl}(\bigcup_{x_0 \in [0,1]} (x_0, 0) + L(x_0))$ where $L(x_0)$ is the attractor

of the IFS $\{\check{f}_1, \check{f}_2\}$ with, for all $w = (0, y) \in W$,

$$\check{f}_1(w) = \check{f}_1(0, y) = (0, ay) \quad \text{and} \quad \check{f}_2(w) = \check{f}_2(0, y) = (0, ay + (1 - a)x_0).$$

By uniqueness, this attractor is $L(x_0) = (0, x_0 L_a)$ with $L_a = \Gamma_a$ if $a < \frac{1}{2}$ and $L_a = [0, 1]$ if $a \geq \frac{1}{2}$. The limit set is then

$$L = \text{cl} \left(\bigcup_{x_0 \in [0, 1]} (x_0, x_0 L_a) \right).$$

Here we can directly see that the set of slopes of L is L_a . One can check that, according to Proposition 3.11, this set of slopes is the only invariant set $S \subset [0, 1]$ of the operator $\widehat{H} = \widehat{f}_1 \cup \widehat{f}_2$ where $\widehat{f}_1(z) = az$ and $\widehat{f}_2(z) = az + (1 - a)$. \square

Example 3.14. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the linear maps given by their canonical matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ a & a \end{bmatrix}$$

with $0 < a \leq \frac{1}{2}$. Then, the sequence $(H^n(K_0))_n$ starting at $K_0 = [0, 1] \times \{0\}$ converges to a set $L \in \mathcal{K}$ whose set of slopes is the countable set $\{1\} \cup \{1 - a^n : n \geq 0\}$ (see Figure 9).

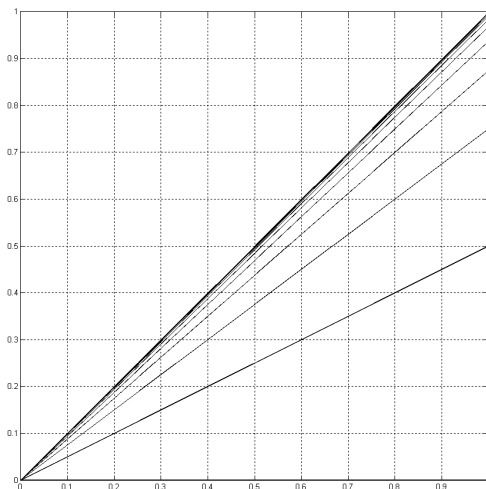


FIGURE 9. The limit set L of $(H^n(K_0))_n$ where H is the Hutchinson operator associated with the IFS $\{f_1, f_2\}$. Maps f_1, f_2 are given in Example 3.14 with parameter $a = \frac{1}{2}$. The starting set is $K_0 = [0, 1] \times \{0\}$. The set of slopes is the countable set $\{1\} \cup \{1 - a^n : n \geq 0\}$.

Proof. Applying Lemma 3.5, since $\|A_2\| = 1$, we obtain that $(H^n(K_0))_n$ converges to $L = \text{cl}(\bigcup_{n \geq 0} H^n(K_0))$. Moreover, according to Proposition 3.11, the set of slopes of L is an invariant set of the operator $\widehat{H} = \widehat{f}_1 \cup \widehat{f}_2$ where $\widehat{f}_1(z) = z$ and $\widehat{f}_2(z) = az + (1 - a)$. The operator \widehat{H} is not contractive. However, iterating f_2 , one can easily see that L is the closure of a countable union of segments, with $L \subset \mathbb{T}$, and show that the invariant set to consider is $\{1\} \cup \{1 - a^n : n \geq 0\}$. \square

The limit sets found in Example 3.13 and in Example 3.14 are respectively a ‘Cantor fan’ and a ‘geometric fan’ (see [10, 11]). Such compact sets arise in various contexts (see [41, 17]).

4. POSSIBLE GENERALIZATIONS

We decided to renormalize the sets $H^n(K_0)$ by dividing them by their radius. As we mentioned in the introduction, one generally uses the diameter to renormalize a sequence of compact sets. Thus, one can wonder what a such renormalization would give here. We present in this section some possible generalizations considering functions φ more general than the radius ρ and their associated operators H_φ .

4.1. Renormalization with a size function.

We are interested in functions φ that describe the size of a compact set and the way it occupies the space. Following the example of the radius function ρ , we will say that a function $\varphi : \mathcal{K} \rightarrow [0, +\infty)$ is a *size function* if it is continuous with respect to d_H , monotonic and homogeneous (see Section 2.2). For example, the max-radius function ρ_∞ and the diameter δ respectively defined on \mathcal{K} by

$$\rho_\infty(K) = \max_{(x_1, \dots, x_D) \in K} \{|x_j| : 1 \leq j \leq D\} \quad \text{and} \quad \delta(K) = \max\{\|x - y\| : x, y \in K\}$$

are two size functions.

We can then define an associated operator H_φ in the same way as (3) setting

$$\forall K \in \mathcal{K}, \quad H_\varphi(K) = \frac{1}{\varphi(\bigcup_{i=1}^p f_i(K))} \bigcup_{i=1}^p f_i(K) \quad (23)$$

and consider the H_φ -orbit of some set $K_0 \in \mathcal{K}$. Let us keep all the notation of the previous sections, easily adapted by replacing ρ with φ . In particular $K_n = H_\varphi^n(K_0)$ (see (7)) and $d_n = \varphi(\bigcup_{i=1}^p f_i(K_n))$ (see the first equality in (8)). We will assume that K_n is always well-defined, i.e. $\varphi(K_n) \neq 0$.

We are still interested in the convergence of the sequence $(K_n)_n$ and the description of its limit. The key-points are Lemma 2.1, Theorem 2.2 and Lemma 3.1 which provide general conditions of convergence. We summarize here the main results which still hold for any size functions.

Theorem 4.1. *Let $p \geq 1$ affine maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$.*

- (i) *If $(d_n)_n$ converges to $d > \lambda$, then $(H_\varphi^n(K_0))_n$ converges to the attractor L_d and d satisfies the inequality*

$$\varphi(\{(d\text{Id} - A_1)^{-1}B_1, \dots, (d\text{Id} - A_p)^{-1}B_p\}) \leq 1.$$

- (ii) *Assume that $B_i = 0$ for all $i \in \{1, \dots, p\}$. If $(H^n(K_0))_n$ converges to a set $L \in \mathcal{K}$ such that $\varphi(L) \neq 0$, then $(K_n)_n$ converges to $K = \frac{1}{\varphi(L)}L$.*

In particular, when all the f_i 's are linear, the choice of φ is not important. The problem is exactly the same one as for the radius case, that is to find non hyperbolic IFS but giving convergent orbits. Therefore, all the results of Section 3 providing the convergence of the sequence $(H^n(K_0))_n$ to a set L (e.g. Lemma 3.2, Lemma 3.5, Theorem 3.6) and the description of its limit (Proposition 3.11) may be used. In case of convergence, the limit set K will only depend on φ through the factor $\varphi(L)$.

4.2. Renormalization with the max-radius function.

We consider here the size function ρ_∞ , which is nothing but the ‘radius function’ associated with the usual maximum norm $\|\cdot\|_\infty$. In particular, Property (6), Lemma 2.3 and Proposition 2.4 still work *mutadis mutandis*. Therefore it should be possible to obtain a result similar to Theorem 2.8. Since the Euclidean norm is isotropic we assumed in Theorem 2.8 the A_i ’s were homotheties. Here, the maximum norm allows us to deal with more general diagonal matrices. However, many complicated particular situations may happen. Thus, for the sake of simplicity, we give here a simpler result with an additional hypothesis avoiding these special behaviors.

Proposition 4.2. *Let $p \geq 1$ maps $f_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ of the form $f_i(x) = A_i x + B_i$ where $A_i = \text{diag}(a_{i,1}, \dots, a_{i,D})$ is a diagonal matrix with non-negative entries and $B_i = (b_{i,1}, \dots, b_{i,D})$. Let $K_0 \in \mathcal{K}$. Assume that*

$$\max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq D}} \{|a_{i,j} x_j + b_{i,j}| : x = (x_1, \dots, x_D) \in H_{\rho_\infty}(K_0)\} > \max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq D}} \{|a_{i,j}|\}. \quad (24)$$

Then, the sequence $(H_{\rho_\infty}^n(K_0))_n$ converges to L_d where $d = \lim_{n \rightarrow \infty} d_n$.

Proof. The proof is very similar to that of Theorem 2.8. Firstly we claim that $(d_n)_{n \geq 1}$ always converges. For all $n \geq 1$ we can find $x_n \in K_n$, $i_n \in \{1, \dots, p\}$ and $j_n \in \{1, \dots, D\}$ such that $d_n = |a_{i_n, j_n} x_{n, j_n} + b_{i_n, j_n}|$. Then, $u_n = \frac{1}{d_n}(A_{i_n} x_n + B_{i_n})$ satisfies $u_n \in K_{n+1}$ and $|u_{n, j_n}| = 1$. Since $|x_{n, j_n}| \leq 1$ we obtain

$$\begin{aligned} d_{n+1} &\geq \max_{1 \leq j \leq D} \{|a_{i_n, j} u_{n, j} + b_{i_n, j}|\} \geq \max_{1 \leq j \leq D} \{|a_{i_n, j} u_{n, j} + d_n u_{n, j}| - |d_n u_{n, j} - b_{i_n, j}|\} \\ &\geq |a_{i_n, j_n} u_{n, j_n} + d_n u_{n, j_n}| - |d_n u_{n, j_n} - b_{i_n, j_n}| \\ &\geq (a_{i_n, j_n} + d_n) |u_{n, j_n}| - a_{i_n, j_n} |x_{n, j_n}| \\ &\geq d_n + a_{i_n, j_n} (1 - |x_{n, j_n}|) \geq d_n. \end{aligned}$$

Hence, $(d_n)_{n \geq 1}$ is increasing and bounded so it converges. Let d be its limit. Notice that the lhs of (24) is d_1 and the rhs of (24) is λ . Thus one has $d \geq d_1 > \lambda$ and the result follows from Theorem 2.2. \square

4.3. Renormalization with the diameter function.

We consider here the diameter function δ . The situation is more complicated than the previous ones, even if the matrices A_i ’s are homotheties. The stability property of ρ was a key-point in the proof of Theorem 2.8 but unfortunately, properties (6) and (10) are no longer satisfied with δ . We will only deal with the one dimensional case $D = 1$.

Proposition 4.3. *Let $p \geq 1$ affine maps $f_i : \mathbb{R} \rightarrow \mathbb{R}$ of the form $f_i(x) = \alpha_i x + B_i$ with $\alpha_i \geq 0$. Let us define the function*

$$F(x) = \frac{\min_{1 \leq i \leq p} \{\alpha_i x + B_i\}}{\max_{1 \leq i \leq p} \{\alpha_i(x+1) + B_i\} - \min_{1 \leq i \leq p} \{\alpha_i x + B_i\}}.$$

Assume that the sequence $(F^n(u))_n$ starting at $u = \min(H_\delta(K_0))$ converges to a number $c \in \mathbb{R}$. Then,

- (i) $(d_n)_n$ converges to $d = \max_{1 \leq i \leq p} \{\alpha_i(c+1) + B_i\} - \min_{1 \leq i \leq p} \{\alpha_i c + B_i\}$,
- (ii) If $d > \max_{1 \leq i \leq p} \{\alpha_i\}$ then $(H_\delta^n(K_0))_n$ converges to L_d whose convex hull is $[c, c+1]$.

Proof. Let us write $\text{ch}(K_n) = [a_n, a_n + 1]$ for all $n \geq 0$.

(i) Let $d(x) = \max_{1 \leq i \leq p} \{\alpha_i(x+1) + \beta_i\} - \min_{1 \leq i \leq p} \{\alpha_i x + \beta_i\}$. Since $\alpha_i \geq 0$ one checks that $d_n = d(a_n)$ and $a_{n+1} = F(a_n)$ where F is given by (4.3). The result follows.

(ii) Since $\lambda = \max_{1 \leq i \leq p} \{\alpha_i\}$ the result is obtained by applying Theorem 2.2. Finally, since $\text{ch}(L_d) = \lim_{n \rightarrow \infty} \text{ch}(K_n)$, we get the last part of the assertion. \square

This Proposition gives a very simple and practical tool to prove the convergence of $(K_n)_n$. It is enough to study the sequence $(a_n)_n$ i.e. the iteration of F which is just a piecewise homographic function.

Example 4.4. Let us consider the IFS $\{f_1, f_2, f_3\}$ where the $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$f_1(x) = 2x + 1, \quad f_2(x) = 3x - 4 \quad \text{and} \quad f_3(x) = x + 2.$$

Then, for all $K_0 \in \mathcal{K}$, the sequence $(H_\delta^n(K_0))_n$ converges to the attractor L_d with $d = 5 + 2\sqrt{3}$. In particular its convex hull is the interval $[1 - \sqrt{3}, 2 - \sqrt{3}]$ (see Figure 10).

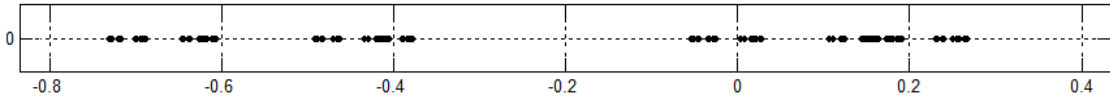


FIGURE 10. The limit set K obtained by renormalizing the IFS $\{f_1, f_2, f_3\}$ with the diameter function δ . Maps f_1, f_2, f_3 are given in Example 4.4. The limit set is a Cantor set.

Proof. First we determine the function F . One has

$$F(x) = \left(\frac{3x-4}{7-2x}\right) \mathbb{1}_{(-\infty, 0]}(x) + \left(\frac{3x-4}{7-x}\right) \mathbb{1}_{(0, 3]}(x) + \left(\frac{x+2}{x+1}\right) \mathbb{1}_{(3, 4]}(x) + \left(\frac{x+2}{2x-3}\right) \mathbb{1}_{(4, +\infty)}(x).$$

Equation $F(x) = x$ has a unique solution $c = 1 - \sqrt{3}$ and one checks that the sequence $(F^n(u))_n$ converges to c for every $u \in \mathbb{R}$. Thus $(d_n)_n$ converges to $d = f_3(c+1) - f_2(c) = 5 + 2\sqrt{3}$. Since $d > \max\{\alpha_1, \alpha_2, \alpha_3\} = 5$, the result follows from Proposition 4.3. \square

Example 4.5. Let $p \geq 1$ affine maps $f_i : \mathbb{R} \rightarrow \mathbb{R}$ of the form $f_i(x) = \alpha x + B_i$ with $\alpha \geq 0$ and $B_1 < \dots < B_p$. Then, for all $K_0 \in \mathcal{K}$, the sequence $(H_\delta^n(K_0))_n$ converges to L_d with $d = \alpha + (B_p - B_1)$.

Proof. One has

$$F(x) = \frac{\alpha x + B_1}{\alpha + (B_p - B_1)}.$$

Thus F is a contraction, $(a_n)_{n \geq 1}$ converges to the invariant point $c = \frac{B_1}{B_p - B_1}$ and $(d_n)_{n \geq 1}$ is constant to $d = \alpha + (B_p - B_1)$. Since $d > \alpha = \lambda$, the result follows from Proposition 4.3. \square

We end this section with an example showing that the sequence $(H_\delta^n(K_0))_n$ may diverge even for very basic affine maps f_i 's. The convergence may depend on the starting set K_0 . Thus, it seems not possible to state a theorem as general as Theorem 2.8.

Example 4.6. Let us consider the IFS $\{f_1, f_2\}$ where the $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$f_1(x) = 2 \quad \text{and} \quad f_2(x) = x + 1.$$

Let $K_0 \in \mathcal{K}$. Then, the sequence $(H_\delta^n(K_0))_n$ converges if and only if the set $\text{ch}(K_0)$ takes one of the following forms:

$$\text{ch}(K_0) = \begin{cases} [a_0, \frac{1+a_0(1+\sqrt{2})}{\sqrt{2}}] & \text{with } \frac{\sqrt{2}-1}{\sqrt{2}+1} \leq a_0 \leq 1, \\ [a_0, 1 + \sqrt{2}] & \text{with } 1 \leq a_0 \leq 1 + \sqrt{2}, \\ [\frac{\sqrt{2}-1}{\sqrt{2}+1}, b_0] & \text{with } \frac{\sqrt{2}-1}{\sqrt{2}+1} \leq b_0 \leq 1. \end{cases}$$

In each case, the attractor is L_d with $d = \sqrt{2}$. In all other cases, the sequence $(H_\delta^n(K_0))_n$ diverges. For example, if $K_0 = \{2\}$ then $(H_\delta^n(K_0))_n$ has two different accumulation points:

$$K = \{3\} \cup \{3 - (\frac{1}{2})^n : n \geq 0\} \text{ and } K' = \{2\} \cup \{2 - (\frac{1}{2})^n : n \geq 0\}.$$

Proof. First one has

$$F(x) = \left(\frac{x+1}{1-x}\right) \mathbb{1}_{(-\infty, 0]}(x) + (x+1) \mathbb{1}_{(0, 1]}(x) + \left(\frac{2}{x}\right) \mathbb{1}_{(1, +\infty)}(x).$$

One checks that the unique invariant point of F is $\sqrt{2}$ and that it is a repulsive point. Thus, the sequence $(F^n(u))_n$ converges if and only if $u = \sqrt{2}$. If $\text{ch}(K_0) = [a_0, b_0]$ then easy computations lead to the three possible cases given above. In these cases we obtain $a_n = d_n = \sqrt{2}$ for all $n \geq 1$. Since $\sqrt{2} > \lambda = 1$, the convergence of $(K_n)_n$ follows from Proposition 4.3.

Assume now that $K_0 = \{2\}$. Then one proves by induction that, for all $m \geq 1$,

$$K_{2m-1} = \{3\} \cup \{3 - (\frac{1}{2})^n : 0 \leq n \leq m-1\} \text{ and } K_{2m} = \{2\} \cup \{2 - (\frac{1}{2})^n : 0 \leq n \leq m\}.$$

The sequences $(K_{2m-1})_m$ and $(K_{2m})_m$ are bounded and increasing so they converge to K and K' respectively. \square

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MODAL'X - EA 3454, UNIVERSITÉ PARIS OUEST NANTERRE LA DÉFENSE, 200 AVENUE DE LA RÉPUBLIQUE, 92000 NANTERRE, FRANCE

E-mail address: yann.demichel@u-paris10.fr