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# THE BLUM-HANSON PROPERTY FOR $\mathcal{C}(K)$ SPACES

PASCAL LEFÈVRE AND ÉTIENNE MATHERON

ABSTRACT. We show that if  $K$  is a compact metrizable space, then the Banach space  $\mathcal{C}(K)$  has the so-called Blum-Hanson property exactly when  $K$  has finitely many accumulation points. We also show that the space  $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$  does not have the Blum-Hanson property.

## 1. INTRODUCTION

The following intriguing result is usually referred to as the *Blum-Hanson theorem* (see [3] and [6]): if  $T$  is a linear operator on a Hilbert space  $H$  with  $\|T\| \leq 1$ , and if  $x \in H$  is such that  $T^n x \rightarrow 0$  weakly as  $n \rightarrow \infty$ , then the sequence  $(T^n x)$  is “strongly mixing”, which means that every subsequence of  $(T^n x)$  converges to 0 in the Cesàro sense; in other words,

$$\lim_{K \rightarrow \infty} \left\| \frac{1}{K} \sum_{i=1}^K T^{n_i} x \right\| = 0$$

for any increasing sequence of integers  $(n_i)$ . (The terminology “strongly mixing” comes from [2]).

Accordingly, a Banach space  $X$  is said to have the *Blum-Hanson property* if the Blum-Hanson theorem holds true on  $X$ ; that is, if  $T$  is linear operator on  $X$  such that  $\|T\| \leq 1$ , then every weakly null  $T$ -orbit is strongly mixing. For example, it was shown rather recently in [8] that  $\ell_p(\mathbb{N})$  has the Blum-Hanson property for any  $p \in [1, \infty)$ . On the other hand, it is known since [1] that  $\mathcal{C}(\mathbb{T}^2)$ , the space of all continuous real-valued functions on the torus  $\mathbb{T}^2$ , does not have this property. Further results and references can be found in [7].

In this short note, we address the Blum-Hanson property for  $\mathcal{C}(K)$  spaces. Our main result is the following:

**Theorem 1.1.** *Let  $K$  be a metrizable compact space. Then  $\mathcal{C}(K)$  has the Blum-Hanson property if and only if  $K$  has finitely many accumulation points.*

This will be proved in the next Section. In Section 3, we obtain in much the same way one nonmetrizable result, namely that the space  $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$  fails the Blum-Hanson property. Our two results can be put together to get a single theorem on the Blum-Hanson property for spaces of bounded continuous functions, which is done in Section 4. We conclude the paper by stating explicitly the “general principle” underlying our proofs.

## 2. PROOF OF THEOREM 1.1

For the “if” part of the proof, we will make use of a result from [7] which is stated as Lemma 2.1 below.

Let  $X$  be a Banach space. For any  $x \in X$  and  $t \in \mathbb{R}^+$ , set

$$r_X(t, x) := \sup \left\{ \limsup_{n \rightarrow \infty} \|x + ty_n\| \right\},$$

where the supremum is taken over all weakly null sequences  $(y_n) \subset X$  with  $\|y_n\| \leq 1$ .

Since  $r_X(t, x)$  is 1-Lipschitz with respect to  $t$ , the quantity  $r_X(t, x) - t$  is nonincreasing and hence it has a limit as  $t \rightarrow \infty$ , possibly equal to  $-\infty$ . Actually, this limit is nonnegative if  $X$  does not have the Schur property, i.e. there is at least one weakly null sequence in  $X$  which is not norm null.

For the needs of the present paper only, we shall say that the Banach space  $X$  has *property (P)* if, for every weakly null sequence  $(x_k) \subset X$ , it holds that

$$(1) \quad \lim_{k \rightarrow \infty} \lim_{t \rightarrow \infty} (r_X(t, x_k) - t) = 0.$$

The result we need is the following; for the proof, see the Remark just after Theorem 2.1 in [7].

**Lemma 2.1.** *Property (P) implies the Blum-Hanson property.*

An extreme example of a space with property (P) is  $X := c_0(\mathbb{N})$ . Indeed, if  $x \in c_0$  and if  $(z_n)$  is a weakly null sequence in  $c_0$ , then

$$\limsup_{n \rightarrow \infty} \|x + z_n\|_\infty = \max(\|x\|_\infty, \limsup_{n \rightarrow \infty} \|z_n\|_\infty).$$

It follows that

$$(*) \quad r_{c_0}(t, x) = \max(\|x\|, t),$$

so that  $r_{c_0}(t, x) - t = 0$  whenever  $t \geq \|x\|$ , for any  $x \in c_0$ .

Let us also note the following useful stability property, whose proof is straightforward.

**Remark 2.2.** *If  $X_1, \dots, X_N$  are Banach spaces with property (P), then the  $\ell_\infty$  direct sum  $X_1 \oplus \dots \oplus X_N$  also has (P).*

We can now start the proof of theorem 1.1.

*Proof of Theorem 1.1.* Let us denote by  $K'$  the set of all accumulation points of  $K$ . We may assume that  $K' \neq \emptyset$ , since otherwise  $K$  is finite and hence  $\mathcal{C}(K)$  is finite-dimensional.

(a) Assume first that  $K'$  is finite say  $K' = \{a_1, \dots, a_N\}$ , and let us show that  $X := \mathcal{C}(K)$  has the Blum-Hanson property.

One may write  $K = K_1 \cup \dots \cup K_N$ , where the  $K_i$  are pairwise disjoint compact sets and  $K'_i = \{a_i\}$ . Then  $\mathcal{C}(K)$  is isometric to the  $\ell_\infty$  direct sum  $\mathcal{C}(K_1) \oplus \dots \oplus \mathcal{C}(K_N)$ , and each  $\mathcal{C}(K_i)$  is isometric to the space  $c$  of all convergent sequences of real numbers.

Therefore (by Lemma 2.1 and Remark 2.2) it is enough to show that the space  $c$  has property (P).

We view  $c$  as the space  $\mathcal{C}(\mathbb{N} \cup \{\infty\})$ , so that  $c_0$  is identified with the subspace of all  $f \in \mathcal{C}(\mathbb{N} \cup \{\infty\})$  such that  $f(\infty) = 0$ . We have to show that if  $(f_k)$  is a weakly null sequence in  $c$ , then  $\lim_{k \rightarrow \infty} \lim_{t \rightarrow \infty} (r_c(t, f_k) - t) = 0$ .

Observe first that since  $f_k(\infty) \rightarrow 0$  as  $k \rightarrow \infty$ , one can find a (weakly null) sequence  $(\tilde{f}_k) \subset c$  such that  $\tilde{f}_k \in c_0$  for all  $k$  and  $\|\tilde{f}_k - f_k\|_\infty \rightarrow 0$ : just set  $\tilde{f}_k := f_k - f_k(\infty)\mathbf{1}$ .

Let  $(g_n)$  be a weakly null sequence in  $c$  with  $\|g_n\|_\infty \leq 1$ . As above, choose a (weakly null) sequence  $(\tilde{g}_n) \subset c$  such that  $\|\tilde{g}_n - g_n\|_\infty \rightarrow 0$  and  $\tilde{g}_n \in c_0$  for all  $n$ . Since  $\|g_n\|_\infty \leq 1$ , we may also assume that  $\|\tilde{g}_n\|_\infty \leq 1$  for all  $n$ . Then, since  $f_k$  and the  $\tilde{g}_n$  are living in  $c_0$ , we get from (\*) above that for any  $t \in \mathbb{R}^+$  and for each  $k \in \mathbb{N}$ :

$$\limsup_{n \rightarrow \infty} \|\tilde{f}_k + t\tilde{g}_n\|_\infty \leq r_{c_0}(t, \tilde{f}_k) = \max(\|\tilde{f}_k\|_\infty, t).$$

By the triangle inequality, it follows that

$$\limsup_{n \rightarrow \infty} \|f_k + tg_n\|_\infty \leq \|\tilde{f}_k - f_k\|_\infty + \max(\|\tilde{f}_k\|_\infty, t)$$

for each  $k \in \mathbb{N}$  and all  $t \geq 0$ . This being true for any weakly null sequence  $(g_n)$  with  $\|g_n\|_\infty \leq 1$ , we conclude that

$$\lim_{t \rightarrow \infty} (r_c(f_k, t) - t) \leq \|\tilde{f}_k - f_k\|_\infty$$

for each  $k \in \mathbb{N}$ , and hence that  $\lim_{k \rightarrow \infty} \lim_{t \rightarrow \infty} (r_c(t, f_k) - t) = 0$ .

(b) Now assume that  $K'$  is infinite. Since  $K$  is metrizable, it follows that  $K$  contains a compact set  $S$  of the following form:

$$S = \bigcup_{k=1}^{\infty} \left[ \{s_{i,k}; i \in \mathbb{N}\} \cup \{s_{\infty,k}\} \right] \cup \{s_{\infty,\infty}\},$$

where all the points involved are distinct and

- $s_{i,k} \rightarrow s_{\infty,k}$  as  $i \rightarrow \infty$  for each fixed  $k \geq 1$ ;
- $s_{\infty,k} \rightarrow s_{\infty,\infty}$  as  $k \rightarrow \infty$ ;
- the sets  $S_k := \{s_{i,k}; i \in \mathbb{N}\} \cup \{s_{\infty,k}\}$  “accumulate to  $\{s_{\infty,\infty}\}$ ”, i.e. they are eventually contained in any neighbourhood of  $s_{\infty,\infty}$ .

Thus, we have  $S' = \{s_{\infty,k}; k \geq 1\} \cup \{s_{\infty,\infty}\}$  and  $S'' = \{s_{\infty,\infty}\}$ .

The key point is now to construct a special continuous map  $\theta : S \rightarrow S$  and to consider the associated *composition operator*  $C_\theta$  acting on  $\mathcal{C}(S)$ . This is the same strategy as in [1], in our setting.

**Fact 1.** One can construct a continuous map  $\theta : S \rightarrow S$  such that, denoting by  $\theta^n$  the iterates of  $\theta$ , the following properties hold true.

- (i)  $\theta^n(s) \rightarrow s_{\infty,\infty}$  pointwise on  $S$  as  $n \rightarrow \infty$ ;

(ii) there exists an open neighbourhood  $V$  of  $s_{\infty,\infty}$  in  $S$  such that

$$\sup_{s \in S} \#\{n \in \mathbb{N}; \theta^n(s) \notin V\} = \infty.$$

*Proof.* We define the map  $\theta$  as follows:

$$\begin{cases} \theta(s_{\infty,\infty}) &= s_{\infty,\infty} \\ \theta(s_{i,k}) &= s_{i,k-1} & \text{if } k \geq 2 \\ \theta(s_{\infty,k}) &= s_{\infty,k-1} & \text{if } k \geq 2 \\ \theta(s_{i,1}) &= s_{i-1,i-1} & \text{if } i \geq 2 \\ \theta(s_{\infty,1}) &= s_{\infty,\infty} \\ \theta(s_{1,1}) &= s_{\infty,\infty} \end{cases}$$

It is clear that  $\theta$  is continuous at each point  $s_{\infty,k}$ ,  $k \geq 2$ . Moreover, since  $s_{i-1,i-1} \rightarrow s_{\infty,\infty}$  as  $i \rightarrow \infty$ , the map  $\theta$  is also continuous at  $s_{\infty,1}$  and at  $s_{\infty,\infty}$ . Since all other points of  $S$  are isolated, it follows that  $\theta$  is continuous on  $S$ .

An examination of the orbits of  $\theta$  reveals that for any  $s \in S$ , we have  $\theta^n(s) = s_{\infty,\infty}$  for all but finitely many  $n \in \mathbb{N}$ . Indeed, if  $s = s_{\infty,k}$  for some  $k \in \mathbb{N}$ , then  $\text{Orb}(s, \theta) = \{s_{\infty,k}, s_{\infty,k-1}, \dots, s_{\infty,1}, s_{\infty,\infty}\}$ , whereas if  $s = s_{i,k}$  for some  $(i, k) \in \mathbb{N} \times \mathbb{N}$ , then  $\text{Orb}(s, \theta) = \{s_{i,k}, s_{i,k-1}, \dots, s_{i,1}, s_{i-1,i-1}, \dots, s_{i-1,1}, s_{i-2,i-2}, \dots, s_{1,2}, s_{1,1}, s_{\infty,\infty}\}$ . So property (i) is satisfied.

Set  $V := S \setminus S_1$ , where  $S_1 = \{s_{i,1}; i \in \mathbb{N}\} \cup \{s_{\infty,1}\}$ . This is an open (actually clopen) neighbourhood of  $s_{\infty,\infty}$  in  $S$ . For any  $N \in \mathbb{N}$ , the orbit of  $s_N := s_{N,1}$  contains exactly  $N$  points of  $S \setminus V = S_1$ , namely  $s_{N,1}, s_{N-1,1}, \dots, s_{1,1}$ . So property (ii) is satisfied as well.  $\square$

From Fact 1, it is straightforward to deduce

**Fact 2.** The space  $\mathcal{C}(S)$  does not have the Blum-Hanson property.

*Proof.* Let  $\theta : S \rightarrow S$  be given by Fact 1, and let  $C_\theta : \mathcal{C}(S) \rightarrow \mathcal{C}(S)$  be the composition operator associated with  $\theta$ :

$$C_\theta u = u \circ \theta \quad \text{for all } u \in \mathcal{C}(S).$$

By property (i) above, we see that  $C_\theta^n u \rightarrow u(s_{\infty,\infty})\mathbf{1}$  weakly as  $n \rightarrow \infty$ , for every  $u \in \mathcal{C}(S)$ .

Let us choose a function  $f \in \mathcal{C}(S)$  such that  $f(s_{\infty,\infty}) = 0$  and  $f \equiv 1$  on  $F := S \setminus V$ , where  $V$  satisfies (ii). Then  $C_\theta^n f \rightarrow 0$  weakly. On the other hand, since  $f \equiv 1$  on  $F$  it follows from (ii) that one can find points  $s \in S$  such that  $\#\{n \in \mathbb{N}; C_\theta^n f(s) = 1\}$  is arbitrarily large. So we have

$$\frac{1}{\#I} \left\| \sum_{n \in I} C_\theta^n f \right\|_\infty \geq 1$$

for finite sets  $I \subset \mathbb{N}$  with arbitrarily large cardinality. From this, it is a simple matter to deduce that the sequence  $(C_\theta^n f)$  is not strongly mixing, which concludes the proof of Fact 2.  $\square$

It is now easy to conclude the proof of Theorem 1.1, by using the following trivial observation.

**Fact 3.** Let  $X$  be a Banach space, and let  $Z$  be a closed subspace of  $X$ . Assume that  $Z$  is 1-complemented in  $X$ , i.e. there is a linear projection  $\pi : X \rightarrow Z$  such that  $\|\pi\| = 1$ . If  $Z$  fails the Blum-Hanson property, then so does  $X$ .

*Proof.* If  $T : Z \rightarrow Z$  and  $z \in Z$  witness that  $Z$  fails the Blum-Hanson property, then  $\tilde{T} := T \circ \pi : X \rightarrow Z \subset X$  and  $z$  witness that so does  $X$ .  $\square$

It is well known that since  $K$  is metrizable, there is an isometric linear extension operator  $J : \mathcal{C}(S) \rightarrow \mathcal{C}(K)$ : this is a classical result due to Dugundji [4]. So the space  $\mathcal{C}(S)$  is isometric to a 1-complemented subspace of  $\mathcal{C}(K)$ , namely  $Z := J[\mathcal{C}(S)]$ . By Fact 3, this concludes the proof of Theorem 1.1.  $\square$

*Remark 1.* The above proof shows that the space  $\mathcal{C}(S)$  fails the Blum-Hanson property in a very special way. Namely, there exists a composition operator  $C_\theta$  on  $\mathcal{C}(S)$  all whose orbits are weakly convergent and such that some weakly null orbit is not strongly mixing. As shown in [1], the same is true for the space  $\mathcal{C}(\mathbb{T}^2)$ . On the other hand, it is observed in [7] that this is *not* so in the space  $\mathcal{C}([0, 1])$ , for the following reason: if  $\theta : [0, 1] \rightarrow [0, 1]$  is a continuous map and if the iterates  $\theta^n$  converge pointwise to some continuous map  $\alpha : [0, 1] \rightarrow [0, 1]$ , then the convergence is in fact uniform.

*Remark 2.* Our proof gives in fact the following more precise result: if  $K$  has finitely accumulation points, then  $\mathcal{C}(K)$  has property (P); and otherwise, one can find an operator  $T$  on  $\mathcal{C}(K)$  with  $\|T\| \leq 1$  such that *all*  $T$ -orbits are weakly convergent and some weakly null orbit is not strongly mixing.

### 3. ONE NONMETRIZABLE EXAMPLE

We have been unable to show without the metrizability assumption on  $K$  that  $\mathcal{C}(K)$  fails the Blum-Hanson property if  $K$  has infinitely many accumulation points. Note that metrizability was used twice in the proof of Theorem 1.1: to ensure that if  $K'$  is infinite then  $K$  contains the special compact set  $S$ ; and for the existence of an isometric (linear) extension operator  $J : \mathcal{C}(S) \rightarrow \mathcal{C}(K)$ .

It is known that the linear extension theorem may fail in the nonmetrizable case (see e.g. [9, Remark 2.3]). It may also happen that a compact set  $K$  has infinitely many accumulation points and yet does not contain any compact set like  $S$ . For example, this holds for  $K = \beta\mathbb{N}$  (the Stone-Ćech compactification of  $\mathbb{N}$ ) because there are no nontrivial convergent sequences in  $\beta\mathbb{N}$ . However, in this (very) special case it is possible to adapt the proof of Theorem 1.1 to obtain the following result.

**Proposition 3.1.** *The space  $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$  does not have the Blum-Hanson property.*

*Proof.* It will be more convenient to view  $\ell_\infty$  as  $\ell_\infty(\mathbb{N} \times \mathbb{N}) = \mathcal{C}(\beta(\mathbb{N} \times \mathbb{N}))$ .

Let  $\theta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  be essentially the same map as in the proof of Theorem 1.1 but ignoring the limit points:

$$\begin{cases} \theta(i, k) &= (i, k - 1) & \text{if } k \geq 2 \\ \theta(i, 1) &= (i - 1, i - 1) & \text{if } i \geq 2 \\ \theta(1, 1) &= (1, 1) \end{cases}$$

We denote by  $C_\theta$  the associated composition operator acting on  $\ell_\infty = \ell_\infty(\mathbb{N} \times \mathbb{N})$ , i.e

$$C_\theta f(i, k) = f(\theta(i, k)) \quad \text{for every } (i, k) \in \mathbb{N} \times \mathbb{N}.$$

Set  $f := \mathbf{1}_F \in \ell_\infty(\mathbb{N} \times \mathbb{N})$ , where  $F = \{(i, 1); i \geq 1\} \setminus \{(1, 1)\} = \{(i, 1); i \geq 2\}$ . Exactly as in the proof of Theorem 1.1, one checks that the sequence  $(C_\theta^n f)$  is not strongly mixing in  $\ell_\infty(\mathbb{N} \times \mathbb{N})$ . So it is enough to show that, on the other hand,  $C_\theta^n f \rightarrow 0$  weakly in  $\ell_\infty(\mathbb{N} \times \mathbb{N})$ .

Viewing  $\ell_\infty(\mathbb{N} \times \mathbb{N})$  as  $\mathcal{C}(\beta(\mathbb{N} \times \mathbb{N}))$ , we have to show that  $C_\theta^n f(\mathcal{U}) \rightarrow 0$  for every ultrafilter  $\mathcal{U}$  on  $\mathbb{N} \times \mathbb{N}$ . Let us fix such an ultrafilter  $\mathcal{U}$ .

Since  $C_\theta^n f = C_\theta^n \mathbf{1}_F = \mathbf{1}_{\theta^{-n}(F)}$  when considered as an element of  $\ell_\infty(\mathbb{N} \times \mathbb{N})$ , we have for any  $n \in \mathbb{N}$ :

$$C_\theta^n f(\mathcal{U}) = \begin{cases} 1 & \text{if } \theta^{-n}(F) \in \mathcal{U} \\ 0 & \text{if } \theta^{-n}(F) \notin \mathcal{U} \end{cases}$$

So we need to prove that if  $n$  is large enough, then  $\theta^{-n}(F) \notin \mathcal{U}$ .

Observe first that if we set  $S_1 := \mathbb{N} \times \{1\}$ , then  $\theta^{-n}(S_1) \cap S_1$  is finite for every  $n \in \mathbb{N}$ . This is readily checked from the definition of  $\theta$ . Indeed, for each  $s = (i, 1) \in S_1$ , the first  $n \in \mathbb{N}$  such that  $\theta^n(s) \in S_1$  is at least equal (in fact, exactly equal) to  $i$ ; so for each fixed  $n$  there are at most  $n$  points  $s \in S_1$  such that  $\theta^n(s) \in S_1$ .

Since  $F \subset S_1$  and  $\theta$  is finite-to-one, it follows that  $\theta^{-n}(F) \cap \theta^{-n'}(F)$  is finite whenever  $n \neq n'$ .

Now, assume without loss of generality that  $\theta^{-n}(F) \in \mathcal{U}$  for more than one  $n \in \mathbb{N}$ . Then, by what we have just observed,  $\mathcal{U}$  contain a finite set. Hence,  $\mathcal{U}$  is a principal ultrafilter, defined by some point  $s_0 \in \mathbb{N} \times \mathbb{N}$ . On the other hand, we know from the definition of the map  $\theta$  that  $\theta^n(s_0) = (1, 1)$  for all but finitely many  $n \in \mathbb{N}$ . Since  $(1, 1) \notin F$ , it follows that  $\theta^{-n}(F) \notin \mathcal{U}$  for all but finitely many  $n$ . □

From Proposition 3.1, we immediately deduce

**Corollary 3.2.** *The space  $L_\infty = L_\infty(0, 1)$  does not have the Blum-Hanson property. Likewise, if  $H$  is an infinite-dimensional Hilbert space, then the space  $\mathcal{B}(H)$  of all bounded operators on  $H$  does not have the Blum-Hanson property.*

*Proof.* This is clear since these two spaces contain a 1-complemented isometric copy of  $\ell_\infty$ . □

#### 4. FURTHER REMARKS

For any topological space  $E$ , let us denote by  $\mathcal{C}_b(E)$  the space of all real-valued, bounded continuous functions on  $E$ . Putting together Theorem 1.1 and Proposition 3.1, we obtain the following result.

**Theorem 4.1.** *If  $T$  is a metrizable topological space, then  $\mathcal{C}_b(T)$  has the Blum-Hanson property exactly when  $T$  is compact and has finitely many accumulation points.*

*Proof.* By Theorem 1.1, it is enough to show that if  $\mathcal{C}_b(T)$  has the Blum-Hanson property, then  $T$  is compact. Now, if  $T$  is not compact, it contains a countably infinite closed discrete set  $S$  (thanks to the metrizability assumption). By Dugundji's extension theorem,  $\mathcal{C}_b(T)$  then contains a 1-complemented isometric copy of  $\mathcal{C}_b(S)$ . Since  $\mathcal{C}_b(S)$  is isometric to  $\ell_\infty(\mathbb{N})$ , it follows from Proposition 3.1 that  $\mathcal{C}_b(T)$  does not have the Blum-Hanson property.  $\square$

To conclude this paper, and since this may be useful elsewhere, we isolate the following kind of criterion for detecting the failure of the Blum-Hanson property in  $\mathcal{C}_b(T)$  for a not necessarily metrizable topological space  $T$ .

**Lemma 4.2.** *Let  $T$  be a Hausdorff topological space. Assume that there exists a subset  $S$  of  $T$  which is normal as a topological space, such that the following properties hold true.*

(1) *One can find a continuous map  $\theta : S \rightarrow S$  and a point  $a \in S$  such that*

- (i)  $\theta^n(s) \rightarrow a$  pointwise on  $S$  as  $n \rightarrow \infty$ ;
- (ii) *there exists an open neighbourhood  $V$  of  $a$  such that*

$$\sup_{s \in S} \#\{n \in \mathbb{N}; \theta^n(s) \notin V\} = \infty;$$

- (iii) *there exists a further open neighbourhood  $W$  of  $a$  with  $\overline{W} \subset V$  such that, for any infinite set  $\mathbf{N} \subset \mathbb{N}$ , one can find  $n_1, \dots, n_p \in \mathbf{N}$  such that the set  $\theta^{-n_1}(S \setminus W) \cap \dots \cap \theta^{-n_p}(S \setminus W)$  is finite.*

(2) *There is a linear isometric extension operator  $J : \mathcal{C}_b(S) \rightarrow \mathcal{C}_b(T)$ .*

*Then, one can conclude that the space  $\mathcal{C}_b(T)$  fails the Blum-Hanson property.*

*Proof.* By (2), it is enough to show that  $\mathcal{C}_b(S)$  does not have the Blum-Hanson property. This will of course be done by considering the composition operator  $C_\theta : \mathcal{C}_b(S) \rightarrow \mathcal{C}_b(S)$ .

Since  $\overline{W} \subset V$  by (iii) and  $S$  is normal, one can choose a function  $f \in \mathcal{C}_b(S)$  such that  $f \equiv 0$  on  $\overline{W}$  and  $f \equiv 1$  on  $F := S \setminus V$ . By condition (ii) in (1), the sequence  $(C_\theta^n f)$  is not strongly mixing; so we just need to check that  $C_\theta^n f \rightarrow 0$  weakly in  $\mathcal{C}_b(S)$ .

Being Hausdorff and normal, the space  $S$  is completely regular; so the space  $\mathcal{C}_b(S)$  is canonically isometric with  $\mathcal{C}(\beta S)$ , where  $\beta S$  is the Stone-Ćech compactification of  $S$ . The latter can be described as the space of all  $z$ -ultrafilters on  $S$ , i.e maximal filters of zero sets for functions in  $\mathcal{C}_b(S)$ , or, equivalently (since  $S$  is normal) maximal filters of closed subsets of  $S$ ; see [5]. Therefore, what we have to do is to show that

$$\lim_{n \rightarrow \infty} \left[ \lim_{\mathcal{U}} f(\theta^n(s)) \right] = 0 \quad \text{for any } z\text{-ultrafilter } \mathcal{U} \text{ on } S.$$



If  $\mathcal{U}$  is a “principal”  $z$ -ultrafilter defined by some  $s_0 \in S$ , i.e.  $\mathcal{U}$  is convergent with limit  $s_0$ , then  $\lim_{\mathcal{U}} f(\theta^n(s)) = f(\theta^n(s_0))$  for all  $n$ , so the result is clear since  $f(\theta^n(s_0)) \rightarrow f(a) = 0$  as  $n \rightarrow \infty$  by (i).

Now, let us assume that  $\mathcal{U}$  is not principal. Then  $\mathcal{U}$  does not contain any finite set. Indeed, if a maximal filter of closed sets contains a finite union of closed sets  $F_1 \cup \dots \cup F_N$ , then it has to contain one of the  $F_i$  by maximality; so, if  $\mathcal{U}$  were to contain a finite set, then it would contain a singleton and hence would be principal in a trivial way. By (iii), it follows that  $\theta^{-n}(S \setminus W) \notin \mathcal{U}$  for all but finitely many  $n \in \mathbb{N}$ ; and since  $\mathcal{U}$  is a maximal filter of closed sets, this implies that  $\theta^{-n}(\overline{W}) \in \mathcal{U}$  for all but finitely many  $n$ . Since  $f \equiv 0$  on  $\overline{W}$ , it follows that  $\lim_{\mathcal{U}} f(\theta^n(s)) = 0$  for all but finitely many  $n$ , which concludes the proof.  $\square$

*Remark 1.* This lemma would be much neater if condition (iii) above could be dispensed with; but we don’t know how to prove the lemma without it. The proof of Theorem 1.1 shows that when  $S$  is compact, (i) and (ii) alone are enough for  $\mathcal{C}(S)$  to fail the Blum-Hanson property. At the other extreme, the proof of Proposition 3.1 shows that when  $S$  is discrete (and infinite), one can find a map  $\theta : S \rightarrow S$  satisfying (i), (ii) and a property stronger than (iii).

*Remark 2.* When  $S$  is compact, condition (iii) actually follows from (i). Indeed, let  $W$  be any open neighbourhood of  $a$ , and assume that (iii) fails for  $W$  and some infinite set  $\mathbf{N} \subset \mathbb{N}$ . Then, by compactness we have  $\bigcap_{n \in \mathbf{N}} \theta^{-n}(S \setminus W) \neq \emptyset$ . But if  $s \in \bigcap_{n \in \mathbf{N}} \theta^{-n}(S \setminus W)$  then  $\theta^n(s)$  does not tend to  $a$  as  $n \rightarrow \infty$ , which contradicts (i).

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