Symmetry in Concurrent Games

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***Abstract*—Behavioural symmetry is introduced into concurrent games. It expresses when plays are essentially the same. A characterization of strategies on games with symmetry is provided. This leads to a bicategory of strategies on games with symmetry. Symmetry helps allay the perhaps overly-concrete nature of games and strategies, and shares many mathematical features with homotopy. In the presence of symmetry we can consider monads for which the monad laws do not hold on the nose but do hold up to symmetry. This broadening of the concept of monad has a dramatic effect on the types concurrent games can support and allows us, for example, to recover the replication needed to express and extend traditional game semantics.**

I. Introduction

In *game semantics* of programming languages a type of a program is represented by a game, and the program itself by a strategy in the game. The approach is very flexible: game semantics has managed to capture accurately a wide variety of features of higher-order programming languages, including state, control, exceptions, and many others. Game semantics follows the methodology of *denotational semantics*, and interprets complex programs compositionally in terms of interpretations of their components. Game semantics is also operational: moves in a game correspond to computation steps. As such it is increasingly exploited to provide a syntax-free operational semantics for programming languages, a much needed tool for the analysis and verification of programs.

Concurrency is a central concern in computer science. Reflecting the traditions of concurrency theory, game semantics for concurrent programs come in two styles: based on *interleavings* and on *partial-orders*. Of these, the historical focus has been on interleaving, giving models of various programming languages [1], [2], even a model-checking tool [3]. Partial-order methods have the strength of supporting reasoning about dependency directly, but are at a more preliminary stage. Abramsky and Melliès proposed in [4] a framework for concurrent games based on closure operators, re-understood later by Melliès and Mimram in terms of asynchronous transition systems [5], [6]; Faggian and Piccolo have also presented strategies as partial orders [7]. In [8], Rideau and Winskel gave a new foundation based on *event structures*, generalizing all previous approaches and allowing nondeterminism. It is this framework we refer to as *concurrent games*.

When developing a game semantics, one has to deal with the low-level aspect of games—some identities that hold operationally do not hold automatically in games. An important example of such phenomena occurs in the replication of resources in programming languages: whereas two accesses to the same resource might be indistinguishable operationally, they can correspond to different and unrelated events in the game. In this sense, games are overly-concrete. In the history of game semantics [9], this has been alleviated by introducing *symmetry* into games. Informally, symmetry in a game concerns when one play of a game is essentially the same as another. Our treatment of symmetry in concurrent games, where plays can be highly-distributed, stems from earlier work on symmetry in event structures [10] and makes use of a general method of open maps for defining bisimulation in a variety of models [11]. Briefly, a symmetry in a game is expressed as a bisimulation equivalence (given as a span of open maps) that says when two plays are similar according to the symmetry. This feature considerably enhances the mathematical theory of concurrent games. Symmetry comes to share many features with homotopy—symmetric plays are like homotopic paths—which plays a role in its mathematical development.

a) Contributions.: Firstly, we introduce *concurrent games with symmetry*. This involves a new definition of the copycat strategy and of composition, which now have to respect symmetry. As in [8], we characterize strategies, for which copycat behaves like an identity w.r.t. composition. This leads to the construction of a bicategory (up to symmetry) of concurrent games with symmetry and symmetry-respecting strategies. Secondly, we give two illustrations of how this framework can be used to model logics and programming languages. The first is a presentation within concurrent games of the construction of [12], using an adaptation of AJM games [9] to model classical linear logic (CLL). The second is a concurrent games presentation of HO games [13], giving a concurrent and non-deterministic notion of innocent strategies (in the sense of Hyland and Ong), and a new proof that standard innocent strategies are stable under composition.

b) Related work.: In sequential games, the notion of symmetry that is closest to ours is that of AJM games [9], and in particular its variant in [12]. In asynchronous games, Melliès expressed symmetry by giving groups acting on the game [14], reindexing the events. Restricting to a polarized deterministic setting, it should be possible to reformulate Melliès' approach in terms of concurrent games with symmetry—left however for future work.

c) Outline.: In Section II, we present event structures, and their extension with symmetry and polarity. In Section III, we give the main contribution of this paper, the bicategory of concurrent games with symmetry and concurrent strategies. In Section IV we give two important applications: concurrent generalizations of AJM games and HO games, showing how concurrent games with symmetry extend the games of traditional game semantics.

II. Preliminaries

A. Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}
$$

The (finite) *configurations*, $\mathcal{C}(E)$, of an event structure E consist of those finite subsets $x \subseteq E$ which are

Consistent: $x \in \mathrm{Con}$, and

Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We say an event structure is *elementary* when the consistency relation consists of all finite subsets of events. For $X \subseteq E$ we write $[X]_E$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X]_E \in \mathrm{Con}$ is a configuration. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play a very important role. For configurations x, y, we use $x \mathrel{-\!\!\!\subset} y$ to mean y covers x, *i.e.* $x \subset y$ with nothing in between, and $x \overset{e}{\longrightarrow\!\!\!\!\subset} y$ to mean $x \cup \{e\} = y$ for an event $e \notin x$. We sometimes use $x \overset{e}{\longrightarrow\!\!\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{\longrightarrow\!\!\!\!\subset} y$ for some y.

A (partial) map of event structures $f : E \to E'$ is a partial function $f : E \rightharpoonup E'$ from the events of E to the events of E' such that for all configurations $x \in \mathcal{C}(E)$,

$$
fx \in \mathcal{C}(E') \;\&\; (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.
$$

Maps compose as functions. We say a map is *total* when it is total as a function; then f restricts to a bijection $x \cong fx$ on $x \in \mathcal{C}(E)$. Say a total map of event structures is *rigid* when it preserves causal dependency. We write $\mathcal{E}$ (resp. $\mathcal{E}_p$) for the category of event structures with total (resp. partial). These two categories have coproducts, binary products and pullbacks.

Proposition 1. *Finite configurations of a product $A \times B$ in $\mathcal{E}$ correspond to* secured *bijections $\theta : x \cong y$ between configurations $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$, such that the order generated on θ by taking $(a, b) \leq (a', b')$ if $a \leq_A a$ or $b \leq_B b'$ is a partial order. The correspondence respects inclusion.*

Individual configurations inherit an order from the ambient event structure and can themselves be regarded as finite elementary event structures. Viewed this way, an inclusion $x \subseteq y$ between configurations induces a rigid map $x \hookrightarrow y$ between the configurations regarded as event structures. The configurations of an event structure form a non-empty family of finite partial orders closed under rigid inclusions. Conversely, given such a *rigid family*, we can build an event structure:

Proposition 2. *Let $\mathcal{Q}$ be a rigid family, a non-empty family of finite partial orders closed under rigid inclusions, i.e. if $q \in \mathcal{Q}$ and $q' \hookrightarrow q$ is a rigid inclusion (regarded as a map of event structures) then $q' \in \mathcal{Q}$. The family $\mathcal{Q}$ determines an event structure $\mathrm{Pr}(\mathcal{Q}) =_{\mathrm{def}} (P, \leq, \mathrm{Con})$:*

- *the events P are the finite partial orders in $\mathcal{Q}$ with a top element (the primes),*
- *the causal dependency relation $p' \leq p$ holds when there is a rigid inclusion from $p' \hookrightarrow p$;*
- *for any finite $X \subseteq P$, $X \in \mathrm{Con}$ iff there is $q \in \mathcal{Q}$ and rigid inclusions $p \hookrightarrow q$ for all $p \in X$.*

If $x \in \mathcal{C}(P)$ then $\bigcup x$, the union of the partial orders in x, is in $\mathcal{Q}$. The function $x \mapsto \bigcup x$ is an order-isomorphism from $\mathcal{C}(P)$, ordered by inclusion, to $\mathcal{Q}$, ordered by rigid inclusions.

B. Symmetry

We endow event structures with symmetry. A relation of symmetry on an event structure expresses when two configurations are essentially the same and is expressed as a form of bisimulation equivalence, based on open maps [11].

1) Open maps: Open maps are a generalisation of functional bisimulations, known from transition systems. Let $\mathcal{C}$ be a category with a distinguished subcategory $\mathcal{P}$ of path objects with path-extension maps. A map $f : A \to B$ in $\mathcal{C}$ is *open* if it satisfies a path-lifting property:

$$
\begin{tikzcd}
p \arrow[r] \arrow[d, "m"'] & A \arrow[d, "f"] \\
q \arrow[r] \arrow[ur, dashed] & B.
\end{tikzcd}
$$

Any commuting square, with $m : p \to q$ in $\mathcal{P}$, factors into two commuting triangles as shown. Bisimulation is then expressed as a span of open maps.

2) Event structures with symmetry: The way we equip event structures with symmetry is an instance of the following general construction. Let X be an object of a category $\mathcal{C}$. Recall that a relation on X is an object $\widetilde{X}$ and pair of maps $l_X, r_X : \widetilde{X} \to X$—so forming a span—which are jointly-monic. A map between objects with relations $f : (X, \widetilde{X}) \to (Y, \widetilde{Y})$ is a map $f : X \to Y$ in $\mathcal{C}$ for which there is a necessarily unique map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\mathcal{C}$ such that $f l_X = l_Y \widetilde{f}$ and $f r_X = r_Y \widetilde{f}$. If $\mathcal{C}$ has products, then a relation on X can equivalently be given by a mono $\widetilde{X} \rightarrowtail X \times X$. If $\mathcal{C}$ has pullbacks we can formulate diagrammatically the requirement that $\widetilde{X}$ be an equivalence relation—see [10].

If $\mathcal{C}$ is equipped with a class of open maps, we say that a relation $l_X, r_X : \widetilde{X} \to X$ is a *symmetry* in X if it is an equivalence relation with both l_X and r_X open; this amounts to $\widetilde{X}$ being a bisimulation equivalence. A map $f : (X, \widetilde{X}) \to (Y, \widetilde{Y})$ necessarily preserves symmetry. We obtain a category $\mathcal{CS}$ of objects with symmetry and symmetry-preserving maps. Given two maps $f, g : (X, \widetilde{X}) \to (Y, \widetilde{Y})$ they are equal *up to symmetry*, written $f \sim g$, if there is a necessarily unique map $h : X \to \widetilde{Y}$ in $\mathcal{C}$ such that $f = l_Y h$ and $g = r_Y h$. The category $\mathcal{CS}$ is more fully described as enriched in the category of equivalence relations and so, because equivalence relations are a degenerate form of category, as a 2-category in which the 2-cells are instances of the equivalence. This view informs the constructions in such categories which are often very simple examples of the (pseudo- and bi-) constructions of 2-categories. In particular, objects A and B are *equivalent*, written $A \simeq B$, if there are maps $f : A \to B$ and $g : B \to A$ such that $fg \sim \mathrm{id}_B$ and $gf \sim \mathrm{id}_A$.

We can take advantage of the concrete nature of event structures to give an explicit description of symmetries there, a characterization which is independent of whether the category of event structures carries rigid, total or partial maps, though for definiteness assume the category is $\mathcal{E}$. A symmetry in an event structure E determines a mono $\widetilde{E} \rightarrowtail E \times E$, thus a subset of $\mathcal{C}(E \times E)$, and so by Proposition 1 a family of bijections between finite configurations of E. In this way, a symmetry in an event structure E corresponds to an *isomorphism family* comprising a non-empty family of bijections $\theta : x \cong_E y$ between pairs of finite configurations of E such that:

(i) for all identities $\mathrm{id}_x : x \cong_E x$, where $x \in \mathcal{C}(E)$; if $\theta : x \cong_E y$, then the inverse $\theta^{-1} : y \cong_E x$; and if $\theta : x \cong_E y$ and $\varphi : y \cong_E z$, then their composition $\varphi \circ \theta : x \cong_E z$.
(ii) for $\theta : x \cong_E y$ whenever $x' \subseteq x$ with $x' \in \mathcal{C}(E)$, then there is a (necessarily unique) $y' \in \mathcal{C}(E)$ with $y' \subseteq y$ such that the restriction $\theta' : x' \cong_E y'$.
(iii) for $\theta : x \cong_E y$ whenever $x \subseteq x'$ for $x' \in \mathcal{C}(E)$, there is an extension of θ to θ' so $\theta' : x' \cong_E y'$ for some (not necessarily unique) $y' \in \mathcal{C}(E)$ with $y \subseteq y'$.

The isomorphism family makes precise the sense in which a symmetry expresses when two configurations are essentially the same. Note that (ii) implies that the bijections in the isomorphism family respect the partial order of causal dependency on configurations inherited from E; the bijections in an isomorphism family are isomorphisms between the configurations regarded as elementary event structures.

An event structure with symmetry A corresponds to an isomorphism family $\cong_A$ of the underlying event structure of A [10]. There are straightforward reformulations of what it means for a map to preserve symmetry or for two maps to be equal up to symmetry in terms of isomorphism families. A total map $f : A \to B$ preserving symmetry amounts to $x \overset{\theta}{\cong}_A y$ implying $fx \overset{\widetilde{f}\theta}{\cong}_B fy$, where $\widetilde{f}\theta$ is the composite bijection $fx \cong x \overset{\theta}{\cong}_A y \cong fy$; while $f \sim g$, for two total maps $f, g : A \to B$ preserving symmetry, iff $fx \overset{\varphi_x}{\cong}_B gx$ for all $x \in \mathcal{C}(A)$, where φ_x is the composite bijection $fx \cong x \cong gx$.

We define the category $\mathcal{ES}$ (resp. $\mathcal{ES}_p$) to consist of event structures with symmetry, with total (resp. partial) maps.

Proposition 3. *Any map $f : A \to B$ in $\mathcal{ES}_p$ has a partial-total factorization as a composite*

$$A \xrightarrow{p_V} (A{\downarrow}V) \xrightarrow{f_1} B$$

where: $V =_{\mathrm{def}} \{a \in A \mid f(a) \text{ is defined}\}$ is the domain of definition of f; $A{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$ with $v \leq_V v'$ iff $v \leq_A v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}_A$ & $X \subseteq V$; its isomorphism family is given by $\theta : x \cong_{A{\downarrow}V} y$ iff θ extends to $\theta' : [x]_A \cong_A [y]_A$; the map $p_V : A \to A{\downarrow}V$ is the partial map acting as identity on V and undefined elsewhere; and f_1 is a total map acting as f on V. If $f \sim g : A \to B$ in $\mathcal{ES}_p$, then the domains of definition of f and g are the same, V say, and $f_1 \sim g_1$ in their partial-total factorizations $f = f_1 p_V$ and $g = g_1 p_V$.

Through the addition of symmetry event structures can represent a much richer class of categories than mere partial orders. The finite configurations of an event structure with symmetry can be extended by inclusion or rearranged bijectively under an isomorphism allowed by the symmetry. In this way an event structure with symmetry determines, in general, a *category* of finite configurations with maps those injections obtained by repeatedly composing the inclusions and allowed isomorphisms. By property (ii) in the definition of isomorphism family any such map factors uniquely as an isomorphism of the symmetry followed by an inclusion.

3) Constructions: We review from [10] important constructions in $\mathcal{ES}$.

a) Products: First, $\mathcal{ES}$ has binary products.

Theorem 4. *Let A and B be objects in $\mathcal{ES}$. Their product in $\mathcal{ES}$ is given by $(A \times B; l_A \times l_B, r_A \times r_B : \widetilde{A} \times \widetilde{B} \to A \times B)$, based on the product $A \times B$ in $\mathcal{E}$, and sharing the same projections, $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$. Let $f, f' : C \to A$ and $g, g' : C \to A$ in $\mathcal{ES}$. If $f \sim f'$ and $g \sim g'$, then $\langle f, g \rangle \sim \langle f', g' \rangle$.*

b) Simple parallel composition: The operation of simple parallel composition of event structures $A \| B$ which juxtaposes the two event structures A and B—with a finite set of events consistent if its overlaps with A and B are consistent—and acts similarly on maps, extends to a functor on $\mathcal{ES}$. The symmetry on $A \| B$ is $\widetilde{A} \| \widetilde{B}$ with left and right maps $l_A \| l_B$ and $r_A \| r_B$.

c) Pseudo pullbacks: It will have great bearing on this paper that, while $\mathcal{ES}$ does not have equalizers or pullbacks in general, it does have *pseudo equalizers* and *pseudo pullbacks*. We give their definition and refer the reader to [10] for proofs.

Maps $f, g : B \to C$ have a *pseudo equalizer* $i : E \to B$, *i.e.* $fi \sim gi$ and for any map $i' : E' \to A$ such that $fi' \sim gi'$, there is a unique map $h : E' \to E$ such that $i' = ih$. The object E can be described as having configurations those $x \in \mathcal{C}(B)$ for which $fx \overset{\varphi_x}{\cong}_C gx$, where φ_x is the bijection induced by x; its isomorphism family is the restriction of that on B.

Maps $f : A \to C$, $g : B \to C$ have a pseudo pullback given as the pseudo equalizer of $f\pi_1, g\pi_2 : A \times B \to C$. We summarize its properties: The *pseudo pullback* comprises an object D and maps $\pi_1 : D \to A$ and $\pi_2 : D \to B$ such that $f\pi_1 \sim g\pi_2$ which satisfies the further property that for any object D' and maps $p_1 : D' \to A$ and $p_2 : D' \to B$ such that $fp_1 \sim gp_2$, there is a unique map $h : D' \to D$ such that $p_1 = \pi_1 h$ and $p_2 = \pi_2 h$. The pseudo pullback is defined up to isomorphism, and sometimes written $A \times_C B$.

Concretely, a configuration in $\mathcal{C}(A \times_C B)$ corresponds to a a triple

$$x,\ fx \overset{\theta}{\cong}_C gy,\ y$$

where the composite $x \cong fx \overset{\theta}{\cong}_C gy \cong y$ is a secured bijection between $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$ in the sense of Proposition 1.

4) Homotopy: We remark that the category $\mathcal{ES}$ has the structure of a homotopy category. In particular it has path objects. From the pseudo pullback

$$\begin{tikzcd} & \widetilde{A} \arrow[dl, "l_A"'] \arrow[dr, "r_A"] & \\ A \arrow[dr, "\mathrm{id}_A"'] & \sim & A \arrow[dl, "\mathrm{id}_A"] \\ & A & \end{tikzcd}$$

we recover $\widetilde{A}$, the symmetry on A, but as an object in $\mathcal{ES}$ *itself equipped with symmetry* $\widetilde{\widetilde{A}}$. This universal property is that associated with a *path object* generally written A^I, where I stands for (a generalization of) the unit interval: asserting $\theta : x \cong_A y$, that a bijection between two configurations is in the isomorphism family of A, is analogous to specifying a path from x to y. (There are also *cylinder objects* in $\mathcal{ES}$.)

Later, in defining the copycat strategy we shall make essential use of the fact that in $(A, \widetilde{A})$, an event structure with symmetry, the symmetry $\widetilde{A}$ itself possesses a symmetry $\widetilde{\widetilde{A}}$. Concretely, a configuration of $\widetilde{A}$ corresponds to an element of the isomorphism family $\theta : x \cong_A y$ and a configuration of $\widetilde{\widetilde{A}}$ to a pair φ_1, φ_2 in a commuting square mediating between θ_1 and θ_2 in the isomorphism family of A:

$$\begin{array}{ccc} x_1 & \overset{\theta_1}{\cong_A} & y_1 \\ \varphi_1 \wr\!\Vert_A & & {}_A\Vert\!\wr\, \varphi_2 \\ x_2 & \overset{\theta_2}{\cong_A} & y_2 \end{array}$$

Lemma 5. *Let $f : A \to B$ and $g : B \to C$ be maps in $\mathcal{ES}$ with pseudo pullback P with maps $\pi_1 : P \to A$ and $\pi_2 : P \to B$. Then, $\widetilde{P}$, as an object in $\mathcal{ES}$, with maps $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ is a pseudo pullback of $\widetilde{f}$ and $\widetilde{g}$ in $\mathcal{ES}$.*

C. Adding polarity

Games and strategies will be represented in terms of event structures where events are moves of Player or Opponent, signified by events carrying a polarity, + for Player and − for Opponent. Formally, an event structure with polarity comprises an event structure A with a function $pol_A : A \to \{+, -\}$ assigning a polarity to each event. Maps are assumed to preserve polarity.

The addition of polarity to an event structure A means that the $\subseteq$-order on its finite configurations is now obtained as compositions $(\subseteq^- \cup \subseteq^+)^+$ of two more fundamental orders, where for $x, y \in \mathcal{C}^\infty(A)$,

$$x \subseteq^- y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{-\}, \text{ and}$$
$$x \subseteq^+ y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{+\}.$$

More surprisingly there is a new partial order, the *Scott* order, between configurations which is intimately related to copycat strategies. The Scott order $\sqsubseteq_A$ is defined to be $(\supseteq^- \cup \subseteq^+)^+$. (We use $\supseteq^-$ for the converse order to $\subseteq^-$.) The Scott order possesses a unique factorization: for $x, y \in \mathcal{C}^\infty(A)$,

$$y \sqsubseteq_A x \iff \exists! z \in \mathcal{C}^\infty(A).\ y \supseteq^- z \subseteq^+ x$$

—it is an easy exercise to show that z is necessarily $x \cap y$. Not only do the configurations of copycat strategies of A correspond to pairs of configurations in the Scott order, but also strategies in A correspond to certain, simply-described, discrete fibrations over $(\mathcal{C}(A), \sqsubseteq_A)$—see [15] for the full story. Given this we can expect a variation on the Scott order to play a role in strategies in games with symmetry.

An event structure with polarity and symmetry (henceforth an e.p.s.) consists of $(E, \widetilde{E})$, an event structure with polarity also endowed with a symmetry $\widetilde{E}$. The categories $\mathcal{EPS}$ and $\mathcal{EPS}_p$ of such objects have maps preserving both symmetry and polarity, which are respectively total and partial.

The addition of polarity and symmetry brings a new richness to the configurations of an event structure. The Scott order becomes a *Scott category*, $Scott(A)$, of an e.p.s. A, where now maps between configurations are obtained as compositions of (partial injections) $\supseteq^-$, $\subseteq^+$ and the isomorphism family $\cong_A$. Maps from y to x in $Scott(A)$ have a unique factorization

$$y \supseteq^- y_0 \overset{\theta}{\cong}_A x_0 \subseteq^+ x\,.$$

The unique factorization follows from property (ii) of isomorphism families and uniqueness of the factorization of the Scott order. A map $f : A \to B$ in $\mathcal{EPS}_p$ preserves $\supseteq^-$, $\subseteq^+$ and isomorphism families so extends directly to a functor $f : Scott(A) \to Scott(B)$ making $Scott$ a functor to $\mathbf{Cat}$.

III. Games with symmetry

A *concurrent game with symmetry* is represented by an e.p.s. A. A *pre-strategy* in A is a total map $\sigma : S \to A$ in $\mathcal{EPS}$. A *map between pre-strategies*, from $\sigma : S \to A$ to $\sigma' : S' \to A$, is a map $f : S \to S'$ in $\mathcal{EPS}$ such that

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

commutes. We say the two pre-strategies σ and σ' are *equivalent*, and write $\sigma \simeq \sigma'$, and sometimes $f : \sigma \simeq \sigma'$, when there are maps f from σ to σ', and g from σ' to σ determining an equivalence $S \simeq S'$, *i.e.* such that $gf \sim \mathrm{id}_S$ and $fg \sim \mathrm{id}_{S'}$; isomorphism $\sigma \cong \sigma'$ occurs when $gf = \mathrm{id}_S$ and $fg = \mathrm{id}_{S'}$. A *weak* map from σ to σ' is a map $f : S \to S'$ such that the triangle above commutes up to $\sim$ and analogously say σ and σ' are *weakly equivalent* when there are weak maps f and g making $S \simeq S'$.

Simple parallel composition extends directly to a functor $A \| B$ on A and B in $\mathcal{EPS}$. The *dual* of an e.p.s. A, written $A^\perp$, has the same underlying event structure with symmetry but with a reversal of polarities.

Following Joyal [16], a pre-strategy *from A to B* is a pre-strategy in the game $A^\perp \| B$. In refining the notion of pre-strategy to that of *strategy* we shall follow the guiding principle of [8]: a strategy is a pre-strategy for which copycat is an identity w.r.t. composition. The next few sections make this precise and culminate in the definition of strategy between concurrent games with symmetry.

A. Copycat

Let A be an e.p.s.. Configurations of $\mathbb{CC}_A$ will correspond to maps in $Scott(A)$. Recall a map from y to x in $Scott(A)$ can be put into a unique form

$$y \supseteq^- y_0 \overset{\theta}{\cong}_A x_0 \subseteq^+ x\,.$$

Define $q(x,\theta,y)$ to be the partial order with underlying set $x\|y =_{\mathrm{def}} (\{1\}\times x)\cup(\{2\}\times y)$, causal dependency that inherited from $A^\perp\|A$ with additional causal dependencies

$$\{(a_1,a_2)\in x_0\times y_0 \mid pol_A(a_2)=+\ \&\ \theta(a_1)=a_2\}\cup$$
$$\{(a_2,a_1)\in y_0\times x_0 \mid pol_{A^\perp}(a_1)=+\ \&\ \theta(a_1)=a_2\}\,.$$

That $q(x,\theta,y)$ is indeed a partial order follows as in [8]. The set of all such partial orders forms a rigid family $\mathcal{Q}$. We define the event structure of $\mathbb{CC}_A$ to be $\mathrm{Pr}(\mathcal{Q})$. Because $Scott$ is a functor, the operation $\mathbb{CC}_A$ is functorial in A. We define the symmetry on $\mathbb{CC}_A$ as

$$\mathbb{CC}_A \xleftarrow{\mathbb{CC}_{l_A}} \mathbb{CC}_{\widetilde{A}} \xrightarrow{\mathbb{CC}_{r_A}} \mathbb{CC}_A$$

Note, that in the construction of $\mathbb{CC}_{\widetilde{A}}$ we are using the fact that $\widetilde{A}$ possesses a symmetry $\widetilde{\widetilde{A}}$—see Section II-B4. The map $\gamma_A : \mathbb{CC}_A \to A^\perp\|A$ takes a prime order to its top event.

Lemma 6. *The construction* $(\mathbb{CC}_A,\ \mathbb{CC}_{l_A}, \mathbb{CC}_{r_A} : \mathbb{CC}_{\widetilde{A}} \to \mathbb{CC}_A)$ *is an e.p.s. and* γ_A *a pre-strategy.*

In future we shall overload $\mathbb{CC}_A$ and write $\mathbb{CC}_A$ for the e.p.s. of Lemma 6. From the definitions, $\widetilde{\mathbb{CC}_A} = \mathbb{CC}_{\widetilde{A}}$.

B. Composition

Let $\sigma : S \to A^\perp\|B$ and $\tau : T \to B^\perp\|C$ be pre-strategies between games A,B,C. To define their composition first form the pseudo pullback

$$\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\vee\ \sim"] & \\
S\|C \arrow[dr, "\sigma\|C"'] & & A\|T \arrow[dl, "A\|\tau"] \\
 & A\|B\|C &
\end{tikzcd}$$

of the maps on the underlying event structures with symmetry, ignoring polarities, *viz.* $\sigma : S \to A\|B$ and $\tau : T \to B\|C$. There is an obvious partial map of event structures $A\|B\|C \to A\|C$ undefined on B and acting as identity on A and C. The partial maps from P to $A\|C$, given by following the diagram either way round the pseudo pullback,

$$\begin{tikzcd}
 & A \parallel T \arrow[dr, "A\|\tau"] & & \\
P \arrow[ur, "\pi_2"] \arrow[dr, "\pi_1"'] \arrow[rr, phantom, "> \quad \sim"] & & A \parallel B \parallel C \arrow[r, rightharpoonup] & A \parallel C \\
 & S \parallel C \arrow[ur, "\sigma\|C"'] & &
\end{tikzcd}$$

are defined on a common subset $V \subseteq P$—Proposition 3. Forming the partial-total factorization of either map (for preciseness take the left/lowest) we obtain

$$P \rightharpoonup P{\downarrow}V \to A\|C\,.$$

The resulting total map gives us the composition

$$\tau\odot\sigma : T\odot S =_{\mathrm{def}} P{\downarrow}V \to A^\perp\|C$$

once we reinstate polarities to make $\tau\odot\sigma$ a map in $\mathcal{EPS}$. The *projection* operation $P\downarrow V$ hides all events not visible in A and C.

From Lemma 5, regarding *e.g.* $\widetilde{T}$ as itself an e.p.s., it follows that $\widetilde{T\odot S} \cong \widetilde{T}\odot\widetilde{S}$ and $\widetilde{\tau\odot\sigma} \cong \widetilde{\tau}\odot\widetilde{\sigma}$.

C. Strategies

We are interested in necessary and sufficient conditions on a pre-strategy $\sigma : S \to A$ to ensure $\sigma \simeq \gamma_A\odot\sigma$. As we shall see σ should be equivalent to a pre-strategy which is strong-receptive, innocent and saturated. (If a pre-strategy $\sigma : S \to A^\perp\|B$ satisfies these conditions, then $\sigma \simeq \gamma_B\odot\sigma\odot\gamma_A$.)

1) Necessity: We show that for any pre-strategy $\sigma : S \to A$ the pre-strategy $\gamma_A\odot\sigma$ is necessarily strong-receptive, innocent and saturated.

A pre-strategy σ is *receptive* iff for all $x \in \mathcal{C}(S)$, $\sigma x \overset{a}{\longrightarrow\!\subset}\ \&\ pol_A(a) = -$ implies $\exists! s \in S.\ x \overset{s}{\longrightarrow\!\subset}\ \&\ \sigma(s) = a$. The pre-strategy $\sigma : S \to A$, as a map in $\mathcal{EPS}$ preserves symmetry, so is associated with a map $\widetilde{\sigma} : \widetilde{S} \to \widetilde{A}$. Say σ is *strong-receptive* if $\widetilde{\sigma}$, and so also σ, is receptive. A pre-strategy σ is *innocent* when $s \rightarrowtriangle s'$ and ($pol(s) = +$ or $pol(s') = -$) implies $\sigma(s) \rightarrowtriangle \sigma(s')$.

To specify when a pre-strategy $\sigma : S \to A$ is *saturated* we need some background. Form the pseudo pullback

$$\begin{tikzcd}
 & S\times_A A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\vee\ \sim"] & \\
S \arrow[dr, "\sigma"'] & & A \arrow[dl, "\mathrm{id}_A"] \\
 & A\,. &
\end{tikzcd}$$

The operation taking σ to its saturation π_2 is part of a monad on pre-strategies in A. Clearly

$$\begin{tikzcd}
 & S \arrow[dl, "\mathrm{id}_S"'] \arrow[dr, "\sigma"] & \\
S \arrow[dr, "\sigma"'] & & A \arrow[dl, "\mathrm{id}_A"] \\
 & A &
\end{tikzcd}$$

commutes. Hence there is a unique map $\eta : S \to S\times_A A$ such that $\pi_1\eta = \mathrm{id}_S$ and $\pi_2\eta = \sigma$—thus η is a map from σ to π_2. We say σ is *saturated* when η is part of an equivalence $\sigma \simeq \pi_2$, *i.e.* there is a map $act : S\times_A A \to S$ such that

$$\begin{tikzcd}
S\times_A A \arrow[r, "act"] \arrow[dr, "\pi_2"'] & S \arrow[d, "\sigma"] \\
 & A
\end{tikzcd}$$

commutes and $act\,\eta \sim \mathrm{id}_S$ with $\eta\, act \sim \mathrm{id}_{S\times_A A}$. Concretely, a finite configuration of $S\times_A A$ can be identified with a pair (x,θ) where $x \in \mathcal{C}(S)$ and $\theta : \sigma x \cong_A y$. The action of act is to transport the configuration x across θ it to a configuration $x' =_{\mathrm{def}} act(x,\theta)$ with $\varphi : x \cong_S x'$ and $\widetilde{\sigma}\varphi = \theta$.

Certain compositions are automatically saturated:

Proposition 7. . *The composition $\tau\odot\sigma$ of pre-strategies $\sigma : \varnothing \rightarrow\!\!\!\!\rightarrow B$ and $\tau : B \rightarrow\!\!\!\!\rightarrow C$ is saturated.*

Proof. Adopt the notation of the diagram defining the composition $\tau\odot\sigma$ in Section III-B, with $A = \varnothing$. Let $u \in \mathcal{C}(T\odot S)$. Its down-closure $[u]$ is a configuration in the pseudo pullback P, and so corresponds to a secured bijection $x\|v \overset{\varphi}{\cong} y$ where $x \in \mathcal{C}(S)$, $v \in \mathcal{C}(C)$ and $y \in \mathcal{C}(T)$. Because we define $\tau\odot\sigma$ to be got via the left way round the pseudo pullback square the configuration u is sent to v via $\tau\odot\sigma$. Consequently, given $\theta : v \cong_C v'$ we can define $act(u,\theta)$ to be that configuration $u' \in \mathcal{C}(T\odot S)$ with down-closure $x\|v' \overset{x\|\theta}{\cong} x\|v \overset{\varphi}{\cong} y$ in the pseudo pullback. □

Lemma 8. *γ_A is strong-receptive, innocent and saturated.*

Proof. The construction of $\mathbb{CC}_A$ directly ensures the innocence and receptivity of γ_A. The way symmetry of copycat is obtained from $\mathbb{CC}_{\widetilde{A}}$ makes $\widetilde{\gamma_A}$ equal $\gamma_{\widetilde{A}}$ so receptive, guaranteeing strong-receptivity of γ_A. To see γ_A is saturated we require a map $act : \mathbb{CC}_A \times_{A\|A} A\|A \to \mathbb{CC}_A$. A configuration of $\mathbb{CC}_A \times_{A\|A} A\|A$ corresponds to a configuration of $\mathbb{CC}_A$, so a map in $Scott(A)$

$$x \supseteq^+ x_0 \overset{\theta}{\cong}_A y_0 \subseteq^- y\,,$$

and a configuration of $A\|A$, so a pair v, w, for which $\varphi_1 : x \cong_A v$ and $\varphi_2 : y \cong_A w$. This data and the factorization properties in $Scott(A)$ yield:

$$\begin{array}{ccccccc} x & \supseteq^+ & x_0 & \overset{\theta}{\cong}_A & y_0 & \subseteq^- & y \\ {\scriptstyle\varphi_1}\wr\!\|{\scriptstyle A} & & \wr\!\|{\scriptstyle A} & & {\scriptstyle A}\|\!\wr & & {\scriptstyle A}\|\!\wr{\scriptstyle\varphi_2} \\ v & \supseteq^+ & v_0 & \overset{\theta'}{\cong}_A & w_0 & \subseteq^- & w \end{array}$$

We take the configuration of $\mathbb{CC}_A$ got via act to be that corresponding to

$$v \quad \supseteq^+ \quad v_0 \quad \overset{\theta'}{\cong}_A \quad w_0 \quad \subseteq^- \quad w\,.$$

The map act together with $\eta : \mathbb{CC}_A \to \mathbb{CC}_A \times_{A\|A} A\|A$ establishes the equivalence needed for γ_A to be saturated. □

Lemma 9 (Necessity)**.** *$\gamma_A\odot\sigma$ is strong-receptive, innocent and saturated for any pre-strategy σ in A.*

Proof. The composition $\gamma_A\odot\sigma$ inherits innocence and receptivity directly from that of γ_A. Now $\widetilde{\gamma_A\odot\sigma} \cong \widetilde{\gamma_A}\odot\widetilde{\sigma} \cong \gamma_{\widetilde{A}}\odot\widetilde{\sigma}$ whence $\widetilde{\gamma_A\odot\sigma}$ inherits receptivity from that of $\gamma_{\widetilde{A}}$, making $\gamma_A\odot\sigma$ strong-receptive. The composition is saturated for general reasons—Proposition 7. □

2) Sufficiency: We show the conditions strong-receptive, innocent and saturated are sufficient to ensure that a pre-strategy σ is equivalent to its composition with copycat $\gamma_A\odot\sigma$.

Lemma 10. *Let $\sigma : S \to A$ be a pre-strategy. There is a map $I : S \to \mathbb{CC}_A\odot S$ in $\mathcal{EPS}$, unique up to symmetry, such that $\sigma = (\gamma_A\odot\sigma)I$.*

Proof. We sketch the existence part of the proof by describing how I acts on configurations. Given $x \in \mathcal{C}(S)$, there is a secured bijection $x\|\sigma x \cong q(\sigma x, \mathrm{id}_{\sigma x}, \sigma x)$—the bijection is that given by $x \cong \sigma x$ between left components and $\mathrm{id}_{\sigma x}$ between the right. The secured bijection corresponds to a configuration z of the pseudo pullback $(S\|A)\times_{A\|A}\mathbb{CC}_A$. The result of the projection operation is to hide all those events not above the right component in $A\|A$, so from z yields a configuration of $\mathbb{CC}_A\odot S$ with image σx under $\gamma_A\odot\sigma$. □

Lemma 11. *If a pre-strategy σ is strong-receptive, innocent and saturated, then $I : \sigma \simeq \gamma_A\odot\sigma$ is an equivalence.*

Proof. We require a map $K : \mathbb{CC}_A\odot S \to S$ that with I establishes an equivalence $\gamma_A\odot\sigma \simeq \sigma$. Let $u \in \mathcal{C}(\mathbb{CC}_A\odot S)$. Its down-closure $[u]$ is a configuration in the pseudo pullback $(S\|A)\times_{A\|A}\mathbb{CC}_A$:

$$\begin{tikzcd} & (S\|A)\times_{A\|A}\mathbb{CC}_A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\vee"] & \\ S\|A \arrow[dr, "\sigma\|A"'] & \sim & \mathbb{CC}_A \arrow[dl, "\gamma_A"] \\ & A\|A & \end{tikzcd}$$

As a configuration of the pseudo pullback, $[u]$ corresponds to a triple, a configuration in $S\|A$ and a configuration in $\mathbb{CC}_A$, mediated by an element of the isomorphism family of $A\|A$—see Section II-B3c. A configuration of $S\|A$ corresponds to a pair of configurations x of S and w of A; a configuration of $\mathbb{CC}_A$ to configurations of A in the relations $z_1 \supseteq^+ z_1' \cong_A z_2' \subseteq^- z_2$; and the mediating element of the isomorphism family to a pair $\sigma x \cong_A z_1$ and $w \cong_A z_2$. From this data we obtain the composite map

$$\sigma x \cong_A z_1 \supseteq^+ z_1' \cong_A z_2' \subseteq^- z_2 \cong_A w$$

in $Scott(A)$. This factors uniquely into

$$\sigma x \supseteq^+ y_1 \overset{\theta}{\cong}_A y_2 \subseteq^- w\,.$$

From innocence it follows (see Lemma 1 of [15]) that there is a unique $x_1 \in \mathcal{C}(S)$ for which $x \supseteq^+ x_1$ and $\sigma x_1 = y_1$. Now, the pair x_1 and $\theta : y_1 \cong_A y_2$ can be identified with a configuration of $S\times_A A$. Hence we can apply act to obtain $act(x_1,\theta) \in \mathcal{C}(S)$ with $\sigma\, act(x_1,\theta) = y_2$.

Define $p(u) =_{\mathrm{def}} act(x_1,\theta)$. By considering how it acts on isomorphism families, p extends to a monotonic function

$$\widetilde{p} : \mathcal{C}(\widetilde{\mathbb{CC}_A\odot S}) \to \mathcal{C}(\widetilde{S})$$

such that

$$\forall u \in \mathcal{C}(\mathbb{CC}_A\odot S).\ \widetilde{\sigma}\,\widetilde{p}(u) \subseteq^- \widetilde{\gamma_A\odot\sigma}\, u\,.$$

Using Lemma 23 of [8] and the strong-receptivity of σ, we obtain a unique total map $K : \mathbb{C}_S \odot S \to S$ such that $\forall u \in \mathcal{C}(\mathbb{C}_A \odot S).\ p(u) \subseteq^- Ku$ and $\gamma_A \odot \sigma = \sigma K$. On checking $KI \sim \mathrm{id}_S$ and $IK \sim \mathrm{id}_{\mathbb{C}_A \odot S}$ we have the desired equivalence. □

Define a *strategy* to be a pre-strategy which is strong-receptive, innocent and saturated.

A strategy $\sigma : S \to A$ induces a fibration $\sigma : Scott(S) \to Scott(A)$. In fact, $\supseteq^-$ and $\subseteq^+$ maps in $Scott(A)$ have cartesian liftings again as $\supseteq^-$ and $\subseteq^+$ maps, respectively, in $Scott(S)$ because σ is receptive and innocent [15]—with *strong*-receptivity ensuring the appropriate uniqueness—while $\cong_A$ maps have cartesian liftings in $\cong_S$ because σ is saturated.

Lemma 12. *If σ and τ are strategies, so is $\tau \odot \sigma$.*

Proof. The composition inherits innocence and receptivity from σ and τ. Because $\widetilde{\tau \odot \sigma} \cong \tilde{\tau} \odot \tilde{\sigma}$ it also inherits strong receptivity. Its saturation obtains via Proposition 7 and the saturation of τ. □

3) A $\sim$-bicategory of games with symmetry: Combining the above results, we do not quite obtain a bicategory but rather $\mathcal{S}trat$, a "bicategory up to symmetry" in the following sense:

- its objects are e.p.s.'s—the games;
- its arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!> B$ related by maps of pre-strategies—$\mathcal{S}trat(A, B)$ is thus a category enriched with equivalence relations $\sim$;
- horizontal composition is given by composition of strategies $\odot$, which extends to functors $\mathcal{S}trat(B, C) \times \mathcal{S}trat(A, B) \to \mathcal{S}trat(A, C)$ via the universality of pseudo pullback;
- there is a natural isomorphism (derived from the universality of pseudo pullback) to express the associativity of composition, but only natural equivalences (derived from the equivalence of Lemma 11) for left and right identity laws;
- of the usual coherence axioms for bicategories, that for identity only commutes up to $\sim$.

Because categories of event structures with symmetry are degenerate 2-categories, the above describes a special case of weak 3-category, which we call a *$\sim$-bicategory* as it is morally a bicategory but where the axioms hold up to $\sim$.

$\mathcal{S}trat$ is rich in structure. Observe the duality: a strategy $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!> B$ corresponds to a strategy $\sigma^\perp : B^\perp \rightarrow\!\!\!\!\!+\!\!\!\!> A^\perp$. There is a bijective correspondence between strategies $A \| B \rightarrow\!\!\!\!\!+\!\!\!\!> C$ and strategies $A \rightarrow\!\!\!\!\!+\!\!\!\!> (B^\perp \| C)$, making $\mathcal{S}trat$ monoidal closed, and in fact compact closed, in an extended sense.

To be more precise, and to relate to standard game semantics, we can quotient out the higher-dimensional structure to obtain a category. The category $\mathcal{S}trat/\simeq$ has e.p.s.'s as objects, and $\simeq$-equivalence classes of strategies $\sigma : S \to A^\perp \parallel B$ as morphisms from A to B. The $\sim$-bicategorical structure ensures that equivalence of strategies is preserved by composition, so we get a category $\mathcal{S}trat/\simeq$.

Proposition 13. *The category $\mathcal{S}trat/\simeq$ is compact closed, with tensor product $\|$ and dual $(-)^\perp$.*

The compact closed structure of $\mathcal{S}trat/\simeq$ is not so surprising: the category is defined in a similar fashion to Joyal's category of Conway games [16], which is compact closed as well. As compact closed categories, they are $*$-autonomous and hence models of Multiplicative Linear Logic (MLL) [17].

4) Weak strategies: A *weak strategy* is a pre-strategy which is weakly equivalent to $\gamma_A \odot \sigma$; so, directly from this definition, any weak strategy is weakly equivalent to a strategy. (We do not have a direct characterization of weak strategies.)

As we shall see, it is sometimes convenient to work with weak strategies (which need not be saturated) and then compose with copycat to obtain the strategies they represent.

Another potential advantage of weak strategies is that they are closed under a more general composition than that of strategies. We can build a $\sim$-bicategory of weak strategies—biequivalent to the $\sim$-bicategory of strategies—in which instead composition is based on a choice of *bipullbacks* rather than pseudo-pullbacks. (See [10] for the definition of bipullbacks of event structures with symmetry.) This extra latitude in the choice of definition of composition is likely to have technical advantages when working with sub $\sim$-bicategories of games, for which the saturation of strategies seems unnecessary or unnatural.

Lemma 14. *If two saturated pre-strategies are weakly equivalent they are equivalent. A fortiori, if two strategies are weakly equivalent they are equivalent.*

IV. Applications

Once we have symmetry in games we can support a rich repertoire of (pseudo) monads on e.p.s.'s, and *e.g.* all the monads of [10] are undisturbed by the presence of polarity. Monads to support copying w.r.t. maps of e.p.s.'s can often translate to monads w.r.t. strategies, and so by duality also to comonads w.r.t. strategies. Following Girard's work on linear logic [18], this opens up the possibility of modelling programming languages that are not resource-sensitive, in that (copies of) the same resource can be used multiple times. We describe, in particular, how AJM games [9] and HO games [13] generalize and can be recovered from concurrent games. The (co)monads involved rely pivotally on the presence of symmetry. Their structure lifts from simple structural maps.

A. Maps as strategies

A *structural pair* $f = (f^L, f^R) : A \to B$ comprises

- $f^L : A \to B$, a total map of e.p.s.'s, as left component, and
- $f^R : B \to A$, an injective, partial map of event structures, not necessarily preserving symmetry, as right component,

such that $f^R \circ f^L = id_A$. (Such pairs correspond to Kahn and Plotkin's *rigid embeddings* if we ignore symmetry.)

A structural pair $f : A \to B$, lifts to a strategy $\overline{f} : S(f) \to A^\perp \| B$, obtained via a rigid family $\mathcal{Q}$. Whenever

$$x \supseteq^+ x' \overset{f^L}{\mapsto} f^L\, x' \overset{\theta}{\cong}_B y' \subseteq^- y\,,$$

with $x, x' \in \mathcal{C}(A)$ and $y, y' \in \mathcal{C}(B)$, define a typical $q(x, \theta, y) \in \mathcal{Q}$ to have underlying set $x \parallel y$, and causal dependency that inherited from $A^\perp \parallel B$ with additional causal dependencies

$$\{(a, \theta(f^L a)) \in x' \times y' \mid pol_{A^\perp}(a) = -\} \cup$$
$$\{(\theta(f^L a), a) \in y' \times x' \mid pol_{A^\perp}(a) = +\}.$$

The event structure $S(f)$ is then defined as $\Pr(\mathcal{Q})$, and the strategy-as-map $\overline{f} : S(f) \to A^\perp \| B$ by $\overline{f}(q) = a$, where a is the top element of the prime $q \in S(f)$. Elements of the isomorphism family of $S(f)$ correspond to isomorphisms

$$\begin{array}{ccccccccc} x_1 & \supseteq^+ & x_1' & \overset{f}{\mapsto} & f\,x_1' & \overset{\theta}{\cong}_B & y_1' & \subseteq^- & y_1 \\ \varphi_1 \wr\!\wr_A & & \varphi_2 \wr\!\wr_A & & f\ \varphi_2 \wr\!\wr_B & & \varphi_3 \wr\!\wr_B & & \varphi_4 \wr\!\wr_B \\ x_2 & \supseteq^+ & x_2' & \overset{f}{\mapsto} & f\,x_2' & \overset{\theta}{\cong}_B & y_2' & \subseteq^- & y_2\,. \end{array}$$

This induces an isomorphism family because f^L preserves symmetry, which by construction is preserved by $\overline{f}$. Additionally, one can check that $\overline{f}$ is strong-receptive, innocent and saturated, so a strategy. Note that this lifting operation also preserves equivalence: if $f \sim g$ then it is easy to show that $\overline{f}$ and $\overline{g}$ are weakly equivalent, so equivalent by Lemma 14.

The following lemma relates composition of maps to that of their lifts as strategies.

Lemma 15. *Let $\sigma : T \to A$ be a strategy and $f : A \to B$ be a structural pair. Then the pre-strategy $f^L \circ \sigma : T \to B$ is weakly equivalent to the strategy $\overline{f} \odot \sigma : S(f) \odot T \to B$.*

From this lemma it follows easily that lifting is functorial. We also need to examine the composition of a strategy with the dual of one lifted from a structural pair. A right map $f^R : B \to A$ of a structural pair does not necessarily preserve symmetry, but it does *preserve a sub-symmetry*, in the sense that the set of isomorphisms $x \overset{\theta}{\cong}_B y$, such that $f^R x \overset{f^R\theta}{\cong}_A f^R y$, forms an isomorphism family. Then B can be restricted to make f^R a total map preserving symmetry. Write $(B \upharpoonright f^R)$ for the event structure with events those on which f^R is defined and with isomorphism family that part preserved by f^R. Obviously, $f^R : (B \upharpoonright f^R) \to A$ is a total map preserving symmetry.

Lemma 16. *Let $\sigma : T \to B$ be a strategy and $f : A \to B$ a structural pair. Then $f^R \circ \sigma$ preserves a sub-symmetry, and $f^R \circ \sigma : (T \upharpoonright f^R \circ \sigma) \to A$ is a strategy equivalent to the strategy $(\overline{f})^\perp \odot \sigma : S(f) \odot T \to A$.*

B. AJM games

We have not assumed that games are *polarized*, *i.e.* that initial moves share the same polarity, a condition imposed in most presentations of games. Non-polarized games are useful because they permit an account of negation as just polarity-reversal, and hence model directly the involutive negation of Classical Linear Logic. In contrast, polarized games lose involutive negation and are restricted to modelling "polarized" logics, such as Intuitionistic Linear Logic (ILL). Through concurrent games we can give a concurrent version of the construction in [12] of a non-polarized adaptation of AJM games [9] to model CLL.

A categorical model of Multiplicative Exponential Linear Logic (MELL) is a $\star$-autonomous category $(\mathcal{C}, \otimes)$ (such as $\mathcal{S}trat/\simeq$) with a *linear exponential comonad*, *i.e.* a monoidal comonad $(!, \epsilon, \delta, m)$ with monoidal natural transformations $e_A : !A \to 1$ and $d_A : !A \to !A \otimes !A$ such that each $(!A, e_A, d_A)$ is a commutative comonoid, e_A and d_A are coalgebra maps and any coalgebra map between free coalgebras is also a comonoid morphism. We aim to build this structure on $\mathcal{S}trat/\simeq$. There is one proviso however. Just as in [12], the absence of polarization means that the naturality of weakening $e_A : !A \twoheadrightarrow I$ will be missing, so we model CLL in the sense of [12] and not quite MELL. If we restrict to negatively polarized games the naturality of e_A is recovered at the cost of self-duality, yielding a model of ILL.

From a game A, we form the game $!A$ comprising ω similar copies of A. Its events are pairs (i, a) where $a \in A$ and $i \in \omega$, with causal dependency

$$(i_1, a_1) \le (i_2, a_2) \Leftrightarrow i_1 = i_2 \ \&\ a_1 \le_A a_2$$

and consistency relation

$$\mathrm{Con}_{!A} = \bigcup_{i \in I} \{i\} \times X_i\,,$$

where I is a finite subset of ω, and for each $i \in I$, $X_i \in \mathrm{Con}_A$. Polarity is inherited from A. We describe its symmetry as an isomorphism family. If $x, y \in \mathcal{C}(!A)$, we have $x \overset{\theta}{\cong}_{!A} y$ if, writing $x = \bigcup_{i \in I} \{i\} \times x_i$ and $y = \bigcup_{j \in J} \{j\} \times y_j$ with each x_i and y_j nonempty, there is a bijection $\pi : I \cong J$ so $x_i \overset{\theta_i}{\cong}_A y_{\pi(i)}$ for each $i \in I$, where $\theta((i, a)) = (\pi(i), \theta_i(a))$ for all $(i, a) \in x$.

The construction extends to a functor on $\mathcal{EPS}$: if $f : A \to B$ then $!f : !A \to !B$ sends (i, a) to $(i, f(a))$. It is convenient to investigate the monad/monoid structure of $!$ in $\mathcal{EPS}$ first, their duals will be eventually deduced by duality in $\mathcal{S}trat$. In $\mathcal{EPS}$, we have:

$$\begin{array}{lcl} \eta_A & : & A \to !A \\ \mu_A & : & !!A \to !A \\ q_{A,B} & : & !(A \parallel B) \to !A \parallel !B \end{array} \qquad \begin{array}{lcl} e_A & : & 1 \to !A \\ m_A & : & !A \parallel !A \to !A \end{array}$$

where η_A sends any event a to $(0, a)$; μ_A tracks an arbitrary bijection between $\omega \times \omega$ and ω; $q_{A,B}$ is the obvious distribution map; e_A is the empty map; and m_A tracks an arbitrary bijection between $\omega + \omega$ and ω. All these maps are natural in their parameters in the category $\mathcal{EPS}$. In particular, $(!, \eta, \mu)$ forms a *monad* on $\mathcal{EPS}$.

The functors $\parallel$ and $!$ on $\mathcal{EPS}$ extend to functors on strategies written $\parallel^{\mathcal{S}}$ and $!^{\mathcal{S}}$ for disambiguation: given strategies $\sigma_1 : S_1 \to A_1^\perp \parallel B_1$ and $\sigma_2 : S_2 \to A_2^\perp \parallel B_2$, we have $\sigma_1 \parallel^{\mathcal{S}} \sigma_2 : S_1 \parallel S_2 \to (A_1 \parallel A_2)^\perp \parallel (B_1 \parallel B_2)$ given by the obvious map of $\mathcal{EPS}$. To define $!^{\mathcal{S}}\sigma$, the obvious choice is the composition $q_{A_1^\perp, B_1} \circ !\sigma : !S \to !A_1^\perp \parallel !B_1$, which we must then compose with copycat to obtain a strategy. These yield a bifunctor $\parallel^{\mathcal{S}}$ and a functor $!^{\mathcal{S}}$ on strategies, using the fact that $!$ and $\parallel$ preserve pseudo-pullbacks.

The natural tranformations above all have adjoints together with which they form structural pairs. By the techniques of Section IV-A, lifting maps to strategies, they become (apart from e_A) natural as families of strategies:

Lemma 17. *The strategies $\overline{\epsilon_A}, \overline{\delta_A}, \overline{d_A}$ and $\overline{q_{A,B}}$ are natural in A, B in the category $\mathcal{S}trat/\simeq$.*

Proof. Each of these maps f has an adjoint $f^\bullet$ making $(f, f^\bullet)$ a structural pair. By Lemmas 15 and 16 the compositions involved in the naturality squares can be computed by simple composition of maps, and the equivalences then amount to elementary verifications. □

The monoid and monad laws follow from those in $\mathcal{EPS}$ by functoriality of lifting. The fact that algebra morphisms between free algebras $(!A, \mu_A)$ satisfy the laws needed to be morphisms of monoids follows from a simple diagram chase, using that μ_A is an invertible monoid morphism and that this property remains true of $\overline{\mu_A}$.

We have established that the natural transformations above define the dual of a linear exponential comonad, short of the naturality of e_A. By self-duality of $\mathcal{S}trat/\simeq$, we have:

Theorem 18. *The category $\mathcal{S}trat/\simeq$ with $!$ is a model of CLL in the sense of [12].*

C. HO games and HO-innocence

Finally, we sketch an application of our setting to construct a notion of concurrent games with pointers, obtaining a concurrent generalization of HO games. An e.p.s. A is *negative* when all its minimal events have negative polarity.

1) Concurrent games with pointers: An *arena* is a countable forest $(A, \leq_A, pol_A)$ with polarities (but without conflict or symmetry), which is also negative, and *alternating* in the sense that if $a_1 \rightarrowtriangle a_2$ then $pol_A(a_1) \neq pol_A(a_2)$.

From an arena A, we now define a game $?A$ understood as "A with pointers". Its definition requires the notion of an *exponential slice*: a slice for an event a is a function $\alpha : [a] \to \omega$, giving a copy index to each event on which a depends. For two slices $\alpha : [a] \to \omega$ and $\alpha' : [a'] \to \omega$, we write $\alpha \sqsubseteq \alpha'$ when $a \leq a'$ and $\alpha(b) = \alpha'(b)$ for all $b \leq a$. The game $?A$ has:

- events, pairs (α, a) where $\alpha : [a] \to \omega$,
- causal dependency, $(\alpha, a) \leq (\alpha', a')$ iff $\alpha \sqsubseteq \alpha'$,
- consistency, all finite subsets of $?A$,
- polarity, inherited from A, and
- isomorphism family, order-isos $\theta : x \cong y$ such that for all $(\alpha, a) \in x$ there is $\alpha' : [a] \to \omega$ with $\theta(\alpha, a) = (\alpha', a)$.

To convey the similarity with the plays with pointers of HO games, we could say that (α, a) "points to" (α', a') when $\alpha' \sqsubseteq \alpha$ and $a' \rightarrowtriangle a$. However $?A$ leaves duplicated events causally unrelated whereas they would appear in some chronological order in a play with pointers. Configurations of $?A$ are more closely related to the *thick subtrees* of Boudes [19].

Games of the form $?A$ are comonoids in $\mathcal{S}trat/\simeq$. Indeed, we have maps of event structures $m_A : ?A^\perp \parallel ?A^\perp \to ?A^\perp$ and $e_A : 1 \to ?A^\perp$ satisfying monoid laws up to symmetry, and the lifting operation yields strategies $\overline{m_A}^\perp : ?A \rightarrow\!\!\!\!\!\rightarrow ?A \parallel ?A$ and $\overline{e_A}^\perp : ?A \rightarrow\!\!\!\!\!\rightarrow 1$ in $\mathcal{S}trat/\simeq$, satisfying comonoid laws. We will now describe a subcategory of $\mathcal{S}trat/\simeq$ whose morphisms preserve this comonoid structure.

2) A cartesian closed category: A strategy $\sigma : S \to ?A^\perp \parallel ?B$ is *single-threaded* iff *(1)* for all $s \in S$, $[s]$ has exactly one minimal event, which is negative, and *(2)* there is a map $\langle -, - \rangle : S \parallel S \to S$ such that $\sigma \circ \langle -, - \rangle = (m_A^\perp \parallel m_B) \circ (\sigma \parallel^S \sigma)$, and such that for all $x_1, x_2 \in \mathcal{C}(S)$ disjoint and compatible, the composite bijection $x_1 \cup x_2 \cong x_1 \parallel x_2 \cong \langle x_1, x_2 \rangle$ is in the isomorphism family of S. Single-threaded strategies are stable under composition, and lifting yields single-threaded strategies (so copycat is single-threaded).

Condition *(1)* expresses that causality is local to *threads* (*i.e.* configurations with a unique minimal event), it will be used to decompose configurations of S into threads. On the other hand, condition *(2)* allows us to merge a set of threads back into a configuration; *(1)* and *(2)* together allow us to prove the following key lemma.

Lemma 19. *Single-threaded strategies are comonoid morphisms for the comonoids $(?A, \overline{m_A}^\perp, \overline{e_A}^\perp)$ in $\mathcal{S}trat/\simeq$.*

Since we want a category of comonoids and comonoid morphisms, we will now restrict our attention to single-threaded strategies between games of the form $?A$. An *HO-strategy* from arena A to arena B is a single-threaded strategy $\sigma : S \to ?A^\perp \parallel ?B$. There is a category $\mathcal{HOS}trat$ having arenas as objects, and HO-strategies as morphisms.

Arenas are closed under $\parallel$, and there are obvious isomorphisms $?(A \parallel B) \cong ?A \parallel ?B$ and $?1 \cong 1$. Using these, $\mathcal{HOS}trat$ inherits from $\mathcal{S}trat/\simeq$ its symmetric monoidal structure and terminal object. Moreover, arenas also inherit a comonoid structure that HO-strategies preserve by Lemma 19, hence (see *e.g.* Corollary 17 in [17]) $\mathcal{HOS}trat$ is cartesian.

We also need $\mathcal{HOS}trat$ to be monoidal closed. For that we introduce the usual arrow arena construction [13], that (following [20]) we denote by $A \multimap\!\!\boxplus B$:

- events, $(\parallel_{b \in \min(B)} A) \parallel B$ where $\min(B)$ denotes the set of minimal events of B, and
- causality, that of $\parallel_{b \in \min(B)} A^\perp \parallel B$ enriched with $\{((2, b), (1, (b, a))) \mid a \in A \ \&\ b \in \min(B)\}$.
- polarity, inherited in B and reversed in A.

HO-strategies from A to B can be represented as HO-strategies on $A \multimap\!\!\boxplus B$ through the following proposition.

Proposition 20. *For any arenas A, B, C there is a bijection (up to equivalence) $\Phi_{A,B,C}$ between single-threaded $\sigma : ?C \rightarrow\!\!\!\!\!\rightarrow ?A^\perp \parallel ?B$ and single-threaded $\sigma' : ?C \rightarrow\!\!\!\!\!\rightarrow ?(A \multimap\!\!\boxplus B)$. Moreover Φ preserves composition, in the sense that for all $\tau : ?D \rightarrow\!\!\!\!\!\rightarrow ?C$ we have $\Phi(\sigma \odot \tau) = \Phi(\sigma) \odot \tau$.*

Using this, the category $\mathcal{HOS}trat$ inherits the closed structure of $\mathcal{S}trat/\simeq$. Since its monoidal product is actually cartesian, it is cartesian closed, hence a model of the simply-typed λ-calculus. It has in fact much more structure and we hope to be able to recast and generalize within it various games models of programming languages. In this paper though, we will only show that it contains (a nondeterministic generalization of) the usual HO category of innocent strategies.

3) Sequential HO-innocence: An HO-strategy $\sigma : S \to ?A$ is *sequential HO-innocent* (or an HOIS-strategy) if for any

$s \in S$, *(1)* the prime $[s]$ is a total order, and *(2)* if $[s]$ extends by distinct positive events s_1, s_2, then $[s] \cup \{s_1, s_2\} \notin \mathrm{Con}_S$.

Intuitively, a prime $[s]$ corresponds to a P-view: just like a P-view, a prime configuration of an HOIS-strategy cannot contain two negative events s_1, s_2 whose mapping to $?A$ "point" to the same event (α, a): by innocence s_1 and s_2 would have to be concurrent in $[s]$, which is impossible by condition (1). So similarly to standard HO innocence the condition (1) expresses that the strategy σ is blind to Opponent reopenings: every Player event contains in its causal history at most one Opponent event pointing to a specific Player event.

Proposition 21. *Arenas and sequential HO-innocent strategies form a subCCC, $\mathcal{HOISS}trat$ of $\mathcal{HOS}trat$.*

Note in passing that condition (1) is stable under composition on its own, but is too restrictive to be a satisfactory notion of *concurrent* HO-innocence as it forbids natural concurrent strategies such as that for the parallel or. We leave for future work the design and study of a more satisfactory notion of concurrent HO-innocence. We now go on and isolate a deterministic subcategory of $\mathcal{HOISS}trat$, isomorphic to the usual category of arenas and innocent strategies [13].

An HOIS-strategy $\sigma : S \to ?A$ is *deterministic* iff whenever $x \overset{\varphi}{\cong}_S y$ and $x \overset{+}{\longrightarrow\!\subset} x'$ and $y \longrightarrow\!\subset y'$, then $x' \subseteq x''$ and $y' \subseteq y''$ for some $x'', y'' \in \mathcal{C}(S)$ with $\varphi' : x'' \cong_S y''$ such that $\varphi \subseteq \varphi'$—this is a generalization of the notion of deterministic concurrent strategies [21] in the presence of symmetry.

Theorem 22. *The subcategory of $\mathcal{HOISS}trat$ having arenas as objects and deterministic sequential HO-innocent strategies as morphisms is isomorphic to the usual category of arenas and innocent strategies.*

Sketch. Let $\sigma : S \to ?A$ be a deterministic HOIS-strategy. For $s \in S$ positive, the prime configuration $[s]$ needs to be a chain

$$s_0^- \rightarrowtriangle s_1^+ \rightarrowtriangle s_2^- \rightarrowtriangle \ldots \rightarrowtriangle s^+ ,$$

by condition (1) of HOIS-strategies along with innocence and the alternation and negativity conditions of arenas. This induces a sequence of moves of A:

$$\pi_2\sigma(s_0)\pi_2\sigma(s_1)\ldots\pi_2\sigma(s)$$

In turn, immediate dependency in A equips this sequence with *pointers* such that (by innocence) Opponent always points to the previous move. In other words, this is a *P-view*.

Applying this to all primes in S we get a set of P-views that is O-branching by receptivity, condition (2) of HOIS-strategies, the extension operation on $\widetilde{S}$ and determinism; so a deterministic innocent strategy in the sense of HO games.

This operation has an inverse up to equivalence of strategies and preserves composition, yielding an isomorphism of categories with standard innocent strategies. □

V. Conclusion

Concurrent games with symmetry have proved versatile enough to accommodate and extend in a single framework two radically different fundamental games models: the saturated AJM games model of CLL and the HO innocent games model of the simply-typed λ-calculus, with a generalization to nondeterminism—in the past, providing a notion of nondeterministic HO innocence has proved challenging [22]. The framework is grounded in the mathematically-versatile setting of event structures with symmetry, with potentially fruitful connections to homotopy.

In future, we intend to exploit the framework introduced here to develop concurrent-games models for various programming languages. Its versatility and its proximity to traditional game semantics suggests that it is adequate to give precise partial-order semantics to complex concurrent programming language, including such features as higher-order procedures and shared memory—features which, to our knowledge, have only been modelled through interleaving.

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References

[1] D. R. Ghica and A. S. Murawski, "Angelic semantics of fine-grained concurrency," in *FOSSACS'04*. LNCS 2987, Springer, 2004.

[2] J. Laird, "A game semantics of the asynchronous *pi*-calculus," in *CONCUR*, ser. LNCS, M. Abadi and L. de Alfaro, Eds., vol. 3653. Springer, 2005, pp. 51–65.

[3] D. R. Ghica and A. S. Murawski, "Compositional model extraction for higher-order concurrent programs," in *TACAS*, ser. LNCS, H. Hermanns and J. Palsberg, Eds., vol. 3920. Springer, 2006, pp. 303–317.

[4] S. Abramsky and P.-A. Melliès, "Concurrent games and full completeness," in *LICS '99*. IEEE Computer Society, 1999.

[5] P.-A. Melliès, "Asynchronous games 2: The true concurrency of innocence," *Theor. Comput. Sci.*, vol. 358, no. 2-3, pp. 200–228, 2006.

[6] P.-A. Melliès and S. Mimram, "Asynchronous games: Innocence without alternation," in *CONCUR*, ser. LNCS, L. Caires and V. T. Vasconcelos, Eds., vol. 4703. Springer, 2007, pp. 395–411.

[7] C. Faggian and M. Piccolo, "Partial orders, event structures and linear strategies," in *TLCA '09*, ser. LNCS, vol. 5608. Springer, 2009.

[8] S. Rideau and G. Winskel, "Concurrent strategies," in *LICS 2011*. IEEE Computer Society, 2011.

[9] S. Abramsky, R. Jagadeesan, and P. Malacaria, "Full abstraction for PCF," *Inf. Comput. 163(2): 409-470*, 2000.

[10] G. Winskel, "Event structures with symmetry," *Electr. Notes Theor. Comput. Sci. 172: 611-652*, 2007.

[11] A. Joyal, M. Nielsen, and G. Winskel, "Bisimulation from open maps," *Inf. Comput.*, vol. 127, no. 2, pp. 164–185, 1996.

[12] P. Baillot, V. Danos, T. Ehrhard, and L. Regnier, "Believe it or not, AJM's games model is a model of classical linear logic," in *LICS*. IEEE Computer Society, 1997, pp. 68–75.

[13] J. M. E. Hyland and C.-H. L. Ong, "On full abstraction for PCF: I, II, and III." *Inf. Comput. 163(2): 285-408*, 2000.

[14] P.-A. Melliès, "Asynchronous games 1: A group-theoretic formulation of uniformity," *Manuscript, Available online*, 2003.

[15] G. Winskel, "Strategies as profunctors," in *FOSSACS 2013*. LNCS, Springer, 2013.

[16] A. Joyal, "Remarques sur la théorie des jeux à deux personnes," *Gazette des sciences mathématiques du Québec, 1(4)*, 1997.

[17] P.-A. Melliès, "Categorical semantics of linear logic," *Panoramas et synthèses*, vol. 27, pp. 1–196, 2009.

[18] J.-Y. Girard, "Linear logic," *TCS*, vol. 50, pp. 1–102, 1987.

[19] P. Boudes, "Thick subtrees, games and experiments," in *TLCA*, ser. LNCS, P.-L. Curien, Ed., vol. 5608. Springer, 2009, pp. 65–79.

[20] R. Harmer, M. Hyland, and P.-A. Melliès, "Categorical combinatorics for innocent strategies," in *LICS '07*. IEEE Computer Society, 2007.

[21] G. Winskel, "Deterministic concurrent strategies," *Formal Asp. Comput.*, vol. 24, no. 4-6, pp. 647–660, 2012.

[22] R. Harmer, *Games and full abstraction for non-deterministic languages*, 1999, PhD thesis, University of London.