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# Maximum likelihood estimation in the context of a sub-ballistic random walk in a parametric random environment

Mikael FALCONNET\*    Arnaud GLOTER\*    Dasha LOUKIANOVA\*

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## Abstract

We consider a one dimensional sub-ballistic random walk evolving in a parametric i.i.d. random environment. We study the asymptotic properties of the maximum likelihood estimator (MLE) of the parameter based on a single observation of the path till the time it reaches a distant site. In that purpose, we adapt the method developed in the ballistic case by Comets et al. (2014) and Falconnet et al. (2013). Using a supplementary assumption due to the specificity of the sub-ballistic regime, we prove consistency and asymptotic normality as the distant site tends to infinity. To emphasize the role of the additional assumption, we investigate the Temkin model with unknown support, and it turns out that the MLE is consistent but, unlike in the ballistic regime, the Fisher information is infinite. We also explore the numerical performance of our estimation procedure.

*Key words* : Asymptotic normality, Sub-ballistic random walk, Confidence regions, Cramér-Rao efficiency, Maximum likelihood estimation, Random walk in random environment. *MSC 2000* : Primary 62M05, 62F12; secondary 60J25.

Let  $\omega = (\omega_x)_{x \in \mathbb{Z}}$  be a collection of independent and identically distributed (i.i.d.)  $(0, 1)$ -valued random variables with distribution  $\nu$ . We suppose that the law  $\nu = \nu_\theta$  depends on some unknown parameter  $\theta \in \Theta$ , where  $\Theta \subset \mathbb{R}^d$  is assumed to be a compact set. Denote by  $\mathbb{P}^\theta = \nu_\theta^{\otimes \mathbb{Z}}$  the law on  $(0, 1)^{\mathbb{Z}}$  of the environment  $\omega$  and by  $\mathbb{E}^\theta$  the expectation under this law.

For fixed environment  $\omega$ , let  $X = (X_t)_{t \in \mathbb{Z}_+}$  be the Markov chain on  $\mathbb{Z}$  starting at  $X_0 = 0$  and with transition probabilities

$$P_\omega(X_{t+1} = y | X_t = x) = \begin{cases} \omega_x & \text{if } y = x + 1, \\ 1 - \omega_x & \text{if } y = x - 1, \\ 0 & \text{otherwise.} \end{cases}$$

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The symbol  $P_\omega$  denotes the measure on the path space of  $X$  given  $\omega$ , usually called *quenched* law. The (unconditional) law of  $X$  is given by

$$\mathbf{P}^\theta(\cdot) = \int P_\omega(\cdot) d\mathbb{P}^\theta(\omega),$$

this is the so-called *annealed* law. We write  $E_\omega$  and  $\mathbf{E}^\theta$  for the corresponding quenched and annealed expectations, respectively. The behaviour of the process  $X$  is related to the ratio sequence

$$\rho_x = \frac{1 - \omega_x}{\omega_x}, \quad x \in \mathbb{Z}, \quad (1)$$

and we refer to Solomon (1975) for the classification of  $X$  between transient or recurrent cases according to whether  $\mathbf{E}^\theta(\log \rho_0)$  is different or not from 0.

The transient case may be further split into two sub-cases, called *ballistic* and *sub-ballistic* that correspond to a linear and a sub-linear speed for the walk, respectively. More precisely, letting  $T_n$  be the first hitting time of the positive integer  $n$ ,

$$T_n = \inf\{t \in \mathbb{N} : X_t = n\}, \quad (2)$$

and assuming  $\mathbf{E}^\theta(\log \rho_0) < 0$  all through, we can distinguish the following cases.

(a1) (Ballistic). If  $\mathbf{E}^\theta(\rho_0) < 1$ , then,  $\mathbf{P}^\theta$ -almost surely,

$$\frac{T_n}{n} \xrightarrow[n \rightarrow \infty]{} \frac{1 + \mathbf{E}^\theta(\rho_0)}{1 - \mathbf{E}^\theta(\rho_0)}. \quad (3)$$

(a2) (Sub-ballistic). If  $\mathbf{E}^\theta(\rho_0) \geq 1$ , then  $T_n/n \rightarrow +\infty$ ,  $\mathbf{P}^\theta$ -almost surely when  $n$  tends to infinity.

Moreover, the fluctuations of  $T_n$  depend in nature on a parameter  $\kappa_\theta \in (0, \infty]$ , which is defined as the unique positive solution of

$$\mathbf{E}^\theta(\rho_0^{\kappa_\theta}) = 1, \quad (4)$$

when such a number exists, and  $\kappa_\theta = +\infty$  otherwise. The sub-ballistic case corresponds to  $\kappa_\theta \leq 1$ . In our statements, the quantity  $\kappa_\theta$  plays a crucial role that we will emphasize when it is implicitly involved, since  $\kappa_\theta$  does not appear explicitly in our assumptions.

Comets et al. (2014) provide a maximum likelihood estimator (MLE) of the parameter of the environment distribution in the specific case of a transient *ballistic* one-dimensional nearest neighbour path. In the latter work, the authors establish the consistency of their estimator while the asymptotic normality of the MLE as well as its asymptotic efficiency (namely, that it asymptotically achieves the Cramér-Rao bound) is investigated in Falconnet et al. (2013). The method

used in these two articles can not be applied directly for a sub-ballistic RWRE, due to the non-integrability of the criterion function, but can be adapted to the sub-ballistic regime. However, unlike in the ballistic regime, the asymptotic behavior of the estimator turns out to be very different when estimating the support of the law of the environment. We illustrate this when we consider the one-parameter Temkin model, a simple framework with finite and unknown support, which already reveals the main features of the estimation problem. One explanation is that in the sub-ballistic regime, due to the existence of deeper local traps of the potential than in the ballistic regime, the walk spends a long time in the bottom of these traps, and the Fisher information of the support parameter becomes infinite. The non-finiteness of the Fisher information suggests that the convergence of  $\hat{\theta}_n$  is faster than  $\sqrt{n}$  and we provide a simulation experiment that supports this. Determining the true rate of convergence is a challenging problem that we leave to further research.

This article is organised as follows. In Section 1, we present our MLE procedure to infer the parameter of the environment distribution inspired from Comets et al. and recall briefly some already known results on an underlying branching process in a random environment related to the RWRE. Then, we state in Section 2 our consistency and asymptotic normality results, and present three examples of environment distributions which are already introduced in Comets et al. (2014) and Falconnet et al. (2013). The MLE is consistent in the three frameworks, but asymptotically normal and efficient only in the first two cases. In the last example, the Fisher information is infinite and one of our assumptions fails. In Section 3, all the proofs are presented, and we conclude with some simulation experiment in Section 4.

## 1 Maximum likelihood estimator in the sub-ballistic transient case

We always assume that  $\Theta$  satisfies the following assumption.

**Assumption I.** For any  $\theta \in \Theta$ ,

$$i) \mathbb{E}^\theta |\log \rho_0| < \infty,$$

$$ii) \mathbb{E}^\theta (\log \rho_0) < 0,$$

$$iii) \mathbb{E}^\theta (\rho_0) \in [1, +\infty).$$

The estimator in Comets et al. (2014) is based on the sequence of the number of left steps performed by the process  $X$  from sites 0 to  $n$  at time  $T_n$  defined by (2). More precisely, their estimator is the maximizer of the criterion function

$$\theta \mapsto \ell_n(\theta) = \sum_{x=0}^{n-1} \phi_\theta(L_{x+1}^n, L_x^n), \quad (5)$$

where  $\phi_\theta$  is the function from  $\mathbb{Z}_+^2$  to  $\mathbb{R}$  defined by

$$\phi_\theta(u, v) = \log \int_0^1 a^{u+1} (1-a)^v dv_\theta(a), \quad (6)$$

and for any  $x \in \{0, \dots, n\}$

$$L_x^n := \sum_{t=0}^{T_n-1} \mathbb{1}\{X_t = x, X_{t+1} = x-1\}. \quad (7)$$

Comets et al. (2014) show that the limiting behavior of the sequential log-likelihood function in the case of ballistic RWRE is equivalent to (5). Recall from Kesten et al. (1975) that for an i.i.d. environment, under the annealed law  $\mathbf{P}^\theta$ , the sequence  $L_n^n, L_{n-1}^n, \dots, L_0^n$  has the same distribution as a branching process with immigration in random environment (BPIRE) denoted  $Z_0, Z_1, \dots, Z_n$  and defined by

$$Z_0 = 0, \quad \text{and for } k = 0, \dots, n-1, \quad Z_{k+1} = \sum_{i=0}^{Z_k} \xi_{k+1,i}, \quad (8)$$

with  $\{\xi_{k,i}\}_{k \in \mathbb{N}; i \in \mathbb{Z}_+}$  independent and

$$\forall m \in \mathbb{Z}_+, \quad P_\omega(\xi_{k,i} = m) = (1 - \omega_k)^m \omega_k.$$

Under point *ii*) of Assumption I, Comets et al. proved that the process  $(Z_n)_{n \in \mathbb{Z}_+}$  is a positive recurrent Markov chain with transition kernel  $Q_\theta$  defined as

$$Q_\theta(u, v) = \binom{u+v}{v} \int_0^1 a^{u+1} (1-a)^v dv_\theta(a) = \binom{u+v}{v} e^{\phi_\theta(v,v)}, \quad \forall u, v \in \mathbb{Z}_+. \quad (9)$$

The unique invariant probability measure  $\pi_\theta$  of the process  $(Z_n)_{n \in \mathbb{Z}_+}$  is defined as

$$\pi_\theta(u) = \mathbb{E}^\theta[S(1-S)^u], \quad \forall u \in \mathbb{Z}_+, \quad (10)$$

where

$$S = \left( \sum_{k=0}^{\infty} \prod_{i=1}^k \rho_i \right)^{-1} = (1 + \rho_1 + \rho_1 \rho_2 + \dots + \rho_1 \dots \rho_k + \dots)^{-1} \in (0, 1). \quad (11)$$

Due to the equality in law between  $(L_n^n, \dots, L_0^n)$  and  $(Z_0, \dots, Z_n)$ , the MLE problem for RWRE is reduced to the one for the irreducible positive recurrent homogeneous Markov chain  $(Z_n)_n$ . Thanks to an ergodic theorem for Markov chains, Comets et al. proved that in the *ballistic* transient case the normalized criterion  $\ell_n(\cdot)/n$  converges in probability to a limiting function  $\ell(\cdot)$  with finite values. The former limiting function identifies the true value of the parameter and consistency follows. In the *sub-ballistic* transient case, Comets et al. prove that the limiting function  $\ell(\cdot)$  still exists but might be infinite everywhere, and hence do not identify the true value of the parameter.

Let us explain briefly where is the problem. Introduce the probability measure  $\tilde{\pi}_\theta$  on  $\mathbb{Z}_+ \times \mathbb{Z}_+$  defined as

$$\tilde{\pi}_\theta(u, v) = \pi_\theta(u)Q_\theta(u, v), \quad (12)$$

and denote  $\tilde{\pi}_\theta(g)$  for any function  $g: \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  such that  $\sum_{x,y} \tilde{\pi}_\theta(x, y)|g(x, y)| < \infty$  with  $\tilde{\pi}_\theta$  the quantity defined as

$$\tilde{\pi}_\theta(g) = \sum_{(x,y) \in \mathbb{N}^2} \tilde{\pi}_\theta(x, y)g(x, y). \quad (13)$$

In Comets et al. (2014), the limiting function  $\ell(\cdot)$  is defined as  $\theta \mapsto \tilde{\pi}_{\theta^*}(\phi_\theta)$  where  $\theta^*$  is the true parameter value, and the integrability of  $\phi_\theta$  with respect to  $\tilde{\pi}_{\theta^*}$  is equivalent to the existence of a first moment for  $\pi_{\theta^*}$ . We will see in Proposition 2.4 that  $\kappa_\theta$  defined by (4) is the upper critical value for the existence of finite moments for  $\pi_\theta$ . Therefore, since in the *sub-ballistic* case, we have  $\kappa_{\theta^*} \leq 1$ , we know that  $\pi_{\theta^*}$  does not have a first moment and  $\ell(\theta)$  is infinite. In the light of this, the natural idea is to consider the difference of two log-likelihood functions.

**Definition 1.1.** Fix  $\theta_0 \in \Theta$ . The criterium function  $\theta \mapsto \ell_n^{\text{sb}}(\theta)$  is defined as

$$\ell_n^{\text{sb}}(\theta) = \sum_{x=0}^{n-1} [\phi_\theta(L_{x+1}^n, L_x^n) - \phi_{\theta_0}(L_{x+1}^n, L_x^n)]. \quad (14)$$

An estimator  $\hat{\theta}_n$  of  $\theta$  is defined as a measurable choice

$$\hat{\theta}_n \in \underset{\theta \in \Theta}{\text{Argmax}} \ell_n^{\text{sb}}(\theta). \quad (15)$$

As soon as the function  $\theta \mapsto \phi_\theta(u, v)$  is continuous on the compact parameter set  $\Theta$  for any pair of integers  $(u, v)$ , the criterion function  $\ell_n^{\text{sb}}(\cdot)$  achieves its maximum, and the estimator  $\hat{\theta}_n$  is well defined as one maximizer of this criterion. However, it is not necessarily unique.

## 2 Consistency and asymptotic normality results

From now on, we assume that the process  $X$  is generated under the true parameter value  $\theta^*$ , an interior point of the parameter space  $\Theta$ , that we aim at estimating. We shorten to  $\mathbf{P}^*$  and  $\mathbf{E}^*$  (resp.  $\mathbb{P}^*$  and  $\mathbb{E}^*$ ) the annealed (resp. the law of the environment ) probability  $\mathbf{P}^{\theta^*}$  (resp.  $\mathbb{P}^{\theta^*}$ ) and corresponding expectation  $\mathbf{E}^{\theta^*}$  (resp.  $\mathbb{E}^{\theta^*}$ ) under parameter value  $\theta^*$ .

### 2.1 Consistency result

Assumption II below ensures that the maximizer of criterion  $\ell_n^{\text{sb}}$  is a consistent estimator of the unknown parameter.

**Assumption II.**

- i) (Continuity). For any  $(x, y) \in \mathbb{N}^2$ , the map  $\theta \mapsto \phi_\theta(x, y)$  is continuous on the parameter set  $\Theta$ .
- ii) (Identifiability). For any  $(\theta, \theta') \in \Theta^2$ ,  $v_\theta \neq v_{\theta'} \iff \theta \neq \theta'$ .
- iii) (Uniform integrability). For any  $\theta \in \Theta$ ,  $\tilde{\pi}_\theta(\sup_{\theta' \in \Theta} |\phi_{\theta'} - \phi_{\theta_0}|) < \infty$ .

We now state our main result.

**Theorem 2.1.** (Consistency). Under Assumptions I and II, for any choice of  $\widehat{\theta}_n$  satisfying (15), we have

$$\lim_{n \rightarrow \infty} \widehat{\theta}_n = \theta^*,$$

in  $\mathbf{P}^*$ -probability.

Theorem 2.1 is a straight application of Theorem 5.7 in van der Vaart (1998). Hence, it suffices to check that the assumptions of the former theorem are fulfilled. The first one is the uniform weak law of large numbers for the renormalized criterion given in Proposition 2.2, and the second one is the statement of Proposition 2.3. Sections 3.1 and 3.2 are dedicated to their respective proof.

**Proposition 2.2.** Under Assumptions I and II, the following uniform convergence holds:

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \ell_n^{\text{sb}}(\theta) - \ell^{\text{sb}}(\theta) \right| \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^* \text{-probability,} \quad (16)$$

with

$$\ell^{\text{sb}}(\theta) = \tilde{\pi}_{\theta^*}(\phi_\theta - \phi_{\theta_0}). \quad (17)$$

**Proposition 2.3.** Under Assumptions I and II, for any  $\varepsilon > 0$ ,

$$\sup_{\theta: \|\theta - \theta^*\| \geq \varepsilon} \ell^{\text{sb}}(\theta) < \ell^{\text{sb}}(\theta^*). \quad (18)$$

From Section 1, point *iii*) of Assumption II is essential to ensure that  $\ell^{\text{sb}}(\cdot)$  takes finite values and therefore prove consistency. This point can be expressed in terms of the growth of  $\dot{\phi}$  and thereby is related to the existence of moments of the probability distribution  $\pi_{\theta^*}$  which are characterized in Proposition 2.4 below. Note that in the ballistic regime, since  $\pi_{\theta^*}$  possesses a finite first moment and the growth of  $\dot{\phi}$  is linear, point *iii*) of Assumption II is automatically satisfied.

**Proposition 2.4.** Let  $\kappa_\theta$  defined by (4) and  $\alpha \in (0, +\infty)$ . Under point *ii*) of Assumption I, the following dichotomy holds:

- i)  $\alpha < \kappa_\theta \implies \sum_{k=0}^{\infty} k^\alpha \pi(k) < \infty$ ;
- ii)  $\alpha \geq \kappa_\theta \implies \sum_{k=0}^{\infty} k^\alpha \pi(k) = \infty$ .

Section 3.3 is dedicated to the proof of 2.4.

## 2.2 Asymptotic normality results

The asymptotic normality result in Falconnet et al. (2013) involves the gradient and the second derivative of  $\ell_n(\cdot)$  with respect to  $\theta$ . Since they are equal to the gradient and the second derivative of  $\ell_n^{\text{sb}}(\cdot)$  with respect to  $\theta$ , their result can be extended to the sub-ballistic case under the same assumptions and without any modification of their proof.

In the following, for any function  $g_\theta$  depending on the parameter  $\theta$ , the symbols  $\dot{g}_\theta$  or  $\partial_\theta g_\theta$  and  $\ddot{g}_\theta$  or  $\partial_\theta^2 g_\theta$  denote the (column) gradient vector and Hessian matrix with respect to  $\theta$ , respectively. Moreover,  $Y^\top$  is the row vector obtained by transposing the column vector  $Y$ .

### Assumption III.

- i) (differentiability). The collection of probability measures  $\{\nu_\theta : \theta \in \Theta\}$  is such that for any  $(x, y) \in \mathbb{N}^2$ , the map  $\theta \mapsto \phi_\theta(x, y)$  is twice continuously differentiable on  $\Theta$ .
- ii) (Regularity conditions). For any  $\theta \in \Theta$ , there exists some  $q > 1$  such that  $\tilde{\pi}_\theta(\|\dot{\phi}_\theta\|^{2q}) < +\infty$ .
- iii) (Invertibility). For any  $u \in \mathbb{Z}_+$ ,  $\sum_{v \in \mathbb{Z}_+} \dot{Q}_\theta(u, v) = \partial_\theta(\sum_{v \in \mathbb{Z}_+} Q_\theta(u, v))$ .
- iv) (Uniform conditions). For any  $\theta \in \Theta$ , there exists some neighborhood  $\mathcal{V}(\theta)$  of  $\theta$  such that  $\tilde{\pi}_\theta(\sup_{\theta' \in \mathcal{V}(\theta)} \|\dot{\phi}_{\theta'}\|^2) < +\infty$  and  $\tilde{\pi}_\theta(\sup_{\theta' \in \mathcal{V}(\theta)} \|\ddot{\phi}_{\theta'}\|) < +\infty$ .
- v) (Fisher information matrix). For any value  $\theta \in \Theta$ , the matrix  $\Sigma_\theta = \tilde{\pi}_\theta(\dot{\phi}_\theta \dot{\phi}_\theta^\top) = -\tilde{\pi}_\theta(\ddot{\phi}_\theta)$  is non singular.

**Theorem 2.5.** Under Assumptions I to III, the score vector sequence  $\dot{\ell}_n^{\text{sb}}(\theta^*)/\sqrt{n}$  is asymptotically normal with mean zero and finite covariance matrix  $\Sigma_{\theta^*}$ .

**Theorem 2.6.** (Asymptotic normality). Under Assumptions I to III, for any choice of  $\hat{\theta}_n$  satisfying (15), the sequence  $\{\sqrt{n}(\hat{\theta}_n - \theta^*)\}_{n \in \mathbb{N}}$  converges in  $\mathbf{P}^*$ -distribution to a centered Gaussian random vector with covariance matrix  $\Sigma_{\theta^*}^{-1}$ .

Note that the limiting covariance matrix of  $\sqrt{n}\hat{\theta}_n$  is exactly the inverse Fisher information matrix of the model. As such, our estimator is efficient.

## 2.3 Examples

We illustrate our results in the same frameworks than the ones presented by Comets et al. (2014) and Falconnet et al. (2013). Note that point *iii*) of Assumption II, which requires integrability of the criterion, is always satisfied in the ballistic regime

whereas it might fail in the sub-ballistic regime. For instance, when  $\sup_{\theta' \in \Theta} |\dot{\phi}_{\theta'}|$  is integrable with respect to  $\pi_{\theta}$ , point *iii*) of Assumption II follows. This occurs in Examples I and II presented below. However, this point is not satisfied in Example III as suggested by point (c) of Proposition 2.9 below. Nevertheless, we show the consistency of the MLE and prove that the Fisher information is infinite in this framework suggesting that the rate of convergence is faster than  $\sqrt{n}$ .

**Example I.** Fix  $a_1 < a_2 \in (0, 1)$  and let  $\nu_p = p\delta_{a_1} + (1-p)\delta_{a_2}$ , where  $\delta_a$  is the Dirac mass located at value  $a$ . Here, the unknown parameter is the proportion  $p \in \Theta \subset [0, 1]$  (namely  $\theta = p$ ). We suppose that  $a_1, a_2$  and  $\Theta$  are such that Assumption I is satisfied.

This example is easily generalized to  $\nu$  having  $m \geq 2$  support points namely  $\nu_{\theta} = \sum_{i=1}^m p_i a_i$ , where  $a_1, \dots, a_m$  are distinct, fixed and known in  $(0, 1)$ , we let  $p_m = 1 - \sum_{i=1}^{m-1} p_i$  and the parameter is now  $\theta = (p_1, \dots, p_{m-1})$ .

In the framework of Example I, we have

$$\phi_p(x, y) = \log[p a_1^{x+1} (1-a_1)^y + (1-p) a_2^{x+1} (1-a_2)^y], \quad (19)$$

**Proposition 2.7.** In the framework of Example I, assuming moreover that  $\Theta \subset (0, 1)$ , Assumptions II and III are satisfied, and hence the MLE of the parameter  $p$  is consistent and asymptotically normal.

**Example II.** We let  $\nu_{\theta}$  be a Beta distribution with parameters  $(\alpha, \beta)$ , namely

$$d\nu_{\theta}(a) = \frac{1}{B(\alpha, \beta)} a^{\alpha-1} (1-a)^{\beta-1} da, \quad B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt.$$

Here, the unknown parameter is  $\theta = (\alpha, \beta) \in \Theta$  where  $\Theta$  is a compact subset of

$$\{(\alpha, \beta) \in (0, +\infty)^2 : \beta < \alpha \leq \beta + 1\}.$$

The inequalities  $\beta < \alpha$  and  $\alpha \leq \beta + 1$  ensures that points *ii*) and *iii*) of Assumption I are satisfied.

In the framework of Example II, we have

$$\phi_{\theta}(x, y) = \log \frac{B(x+1+\alpha, y+\beta)}{B(\alpha, \beta)} \quad (20)$$

**Proposition 2.8.** In the framework of Example II, Assumptions II and III are satisfied, and hence MLE of the parameter  $(\alpha, \beta)$  is consistent and asymptotically normal.

**Example III (Temkin model).** We let  $\nu_{\theta} = p\delta_a + (1-p)\delta_{1-a}$ , where  $p$  is fixed in  $(0, 1/2)$  and the unknown parameter is  $\theta = a \in \Theta$ , where  $\Theta$  is a compact subset of  $(0, p)$ .

The inequalities  $p < 1/2$  and  $a < p$  ensures that points *ii*) and *iii*) of Assumption I are satisfied.

In this framework, we have

$$Q_\theta(u, v) = \binom{u+v}{v} e^{\phi_\theta(u, v)} = pK_a(u, v) + (1-p)K_{1-a}(u, v), \quad (21)$$

with  $K_a(u, v)$  defined as

$$K_a(u, v) = \binom{u+v}{v} a^{u+1} (1-a)^v. \quad (22)$$

**Proposition 2.9.** *In the framework of Example III, the following holds.*

(a) For any  $\alpha > 0$ ,

$$\frac{1}{n} \sum_{x=0}^{n-1} \sup_{\theta \in \mathcal{V}_\alpha^c} [\phi_\theta(L_{x+1}^n, L_x^n) - \phi_{\theta^*}(L_{x+1}^n, L_x^n)] \xrightarrow{n \rightarrow \infty} -\infty, \quad \text{in } \mathbf{P}^* \text{-probability,} \quad (23)$$

where  $\mathcal{V}_\alpha^c$  is the complement of  $\mathcal{V}_\alpha$  defined as  $\mathcal{V}_\alpha = \{a \in \Theta : d_{\text{KL}}(a^* | a) \leq \alpha\}$ , with  $d_{\text{KL}}(\cdot | \cdot)$  is the Kullback-Leibler distance on  $(0, 1) \times (0, 1)$  defined as

$$d_{\text{KL}}(q | q') = q \log \frac{q}{q'} + (1-q) \log \frac{1-q}{1-q'} \geq 0.$$

Therefore, the MLE of the parameter  $a$  is consistent.

(b) The Fisher information is infinite, that is, for any  $\theta$ ,

$$\Sigma_\theta = \mathbb{E}^\theta [(\phi'_\theta)^2] = +\infty. \quad (24)$$

(c) For any  $\theta \neq \theta^*$ ,

$$J = \mathbb{E}^* [|\phi'_\theta|] = +\infty. \quad (25)$$

## 3 Proofs

### 3.1 Proof of Proposition 2.2

First, we establish the weak law of large numbers

$$\frac{1}{n} \ell_n^{\text{sb}}(\theta) \xrightarrow{n \rightarrow \infty} \ell^{\text{sb}}(\theta), \quad \text{in } \mathbf{P}^* \text{-probability.} \quad (26)$$

Since the sequence  $L_n^n, L_{n-1}^n, \dots, L_0^n$  has the same distribution as the BPIRE  $Z_0, Z_1, \dots, Z_n$  defined by (8), we have

$$\ell_n^{\text{sb}}(\theta) \sim \sum_{k=0}^{n-1} [\phi_\theta(Z_k, Z_{k+1}) - \phi_{\theta_0}(Z_k, Z_{k+1})], \quad (27)$$

under  $\mathbf{P}^*$ , where  $\sim$  means equality in distribution. Comets et al. proved that under point *ii*) of Assumption I, the process  $(Z_n, Z_{n+1})_{n \in \mathbb{Z}_+}$  is a positive recurrent homogeneous Markov chain which admits the unique invariant probability measure  $\tilde{\pi}_{\theta^*}$  defined by (12). Hence, according to Theorem 4.2 in Chapter 4 from Revuz (1984), for any function  $g : \mathbb{Z}_+^2 \rightarrow \mathbb{R}^d$  such that  $\tilde{\pi}_{\theta^*}(\|g\|) < \infty$ , the following ergodic theorem holds

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(Z_k, Z_{k+1}) = \tilde{\pi}_{\theta^*}(g), \quad (28)$$

$\mathbf{P}^*$ -almost surely and in  $\mathbb{L}^1(\mathbf{P}^*)$ . Under point *iii*) of Assumption II, we can use (28) with  $g = \phi_\theta - \phi_{\theta_0}$ , and combining with (27), this yields (26).

Now we turn to the local uniform weak law of large numbers. This could be verified by the same arguments as in the proof of the standard uniform law of large numbers (see Theorem 6.10 and its proof in Appendix 6.A in Bierens, 2005) where (26) plays the role of the weak law of large numbers for a random sample in the former reference.

Indeed, under point *i*) of Assumption II, the map  $\theta \mapsto \phi_\theta - \phi_{\theta_0}$  is continuous, and under point *iii*) of Assumption II, we have

$$\tilde{\pi}_{\theta^*} \left( \sup_{\theta \in \Theta} |\phi_\theta - \phi_{\theta_0}| \right) < +\infty,$$

which implies that

$$\tilde{\pi}_{\theta^*} \left( \sup_{\theta \in \Theta} \phi_\theta - \phi_{\theta_0} \right) < +\infty \quad \text{and} \quad \tilde{\pi}_{\theta^*} \left( \inf_{\theta \in \Theta} \phi_\theta - \phi_{\theta_0} \right) > -\infty.$$

Therefore, the proof of Theorem 6.10 in Bierens (2005) can be adapted to our context and this implies (16).  $\square$

### 3.2 Proof of Proposition 2.3

First of all, note that under point *iii*) of Assumption II, the limit  $\ell^{\text{sb}}(\theta)$  is finite for any value  $\theta \in \Theta$ . From (17), we may write

$$\ell^{\text{sb}}(\theta) - \ell^{\text{sb}}(\theta^*) = \tilde{\pi}_{\theta^*}(\phi_\theta - \phi_{\theta^*}).$$

Using (12) and noting that  $Q_\theta(u, v) = \binom{u+v}{u} \exp[\phi_\theta(u, v)]$  yields

$$\ell^{\text{sb}}(\theta) - \ell^{\text{sb}}(\theta^*) = \sum_{u \in \mathbb{Z}_+} \pi_{\theta^*}(u) \left[ \sum_{v \in \mathbb{Z}_+} \log \left( \frac{Q_\theta(u, v)}{Q_{\theta^*}(u, v)} \right) Q_{\theta^*}(u, v) \right].$$

Using Jensen's inequality with respect to the logarithm function and the (conditional) distribution  $Q_{\theta^*}(u, \cdot)$  yields

$$\ell^{\text{sb}}(\theta) - \ell^{\text{sb}}(\theta^*) \leq \sum_{u \in \mathbb{Z}_+} \pi_{\theta^*}(u) \log \left[ \sum_{v \in \mathbb{Z}_+} \frac{Q_\theta(u, v)}{Q_{\theta^*}(u, v)} Q_{\theta^*}(u, v) \right] = 0. \quad (29)$$

The equality in (29) occurs if and only if for any  $u \in \mathbb{Z}_+$ , we have  $Q_\theta(u, \cdot) = Q_{\theta^*}(u, \cdot)$ , which is equivalent to the probability measures  $\nu_\theta$  and  $\nu_{\theta^*}$  having identical moments. Since their supports are included in the bounded set  $(0, 1)$ , these probability measures are then identical (see for instance Shiryaev, 1996, Chapter II, Paragraph 12, Theorem 7). Hence, the equality  $\ell^{\text{sb}}(\theta) = \ell^{\text{sb}}(\theta^*)$  yields  $\nu_\theta = \nu_{\theta^*}$  which is equivalent to  $\theta = \theta^*$  under point *ii*) of Assumption II.

In other words, we proved that  $\ell^{\text{sb}}(\theta) \leq \ell^{\text{sb}}(\theta^*)$  with equality if and only if  $\theta = \theta^*$ . To conclude the proof of Proposition 2.3, it suffices to use that the function  $\theta \mapsto \ell^{\text{sb}}(\theta)$  is continuous.  $\square$

### 3.3 Proof of Proposition 2.4

Let  $\kappa_\theta$  defined by (4) and  $\alpha$  be a positive number. Let  $\Lambda$  be the positive random variable such that

$$1 - S = e^{-\Lambda},$$

where  $S$  is defined by (11). Then, we have

$$\sum_{k=0}^{\infty} k^\alpha \pi_\theta(k) = \mathbb{E}^\theta \left[ S \sum_{k=0}^{\infty} k^\alpha e^{-\Lambda k} \right]. \quad (30)$$

From the fact that for any integer  $k$  and any positive  $\lambda$

$$\int_k^{k+1} x^\alpha e^{-\lambda x} dx \geq e^{-\lambda} k^\alpha e^{-\lambda k} \quad \text{and} \quad \int_k^{k+1} x^\alpha e^{-\lambda x} dx \leq e^\lambda (k+1)^\alpha e^{-\lambda(k+1)},$$

we deduce that

$$(1 - S) \cdot \frac{\Gamma(\alpha + 1)}{\Lambda^{1+\alpha}} \leq \sum_{k=1}^{\infty} k^\alpha e^{-\Lambda k} \leq \frac{1}{1 - S} \cdot \frac{\Gamma(\alpha + 1)}{\Lambda^{1+\alpha}}, \quad (31)$$

where  $\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} dx$ . Using the fact that there exists a constant  $C$  such that

$$S \sum_{k=1}^{\infty} k^\alpha e^{-\Lambda k} \mathbb{1}_{\{S > 1/2\}} \leq C,$$

that  $\Lambda \geq S$  and (31) yields

$$\mathbb{E}^\theta \left[ S \sum_{k=0}^{\infty} k^\alpha e^{-\Lambda k} \right] \leq C + 2\Gamma(\alpha + 1) \mathbb{E}^\theta [S^{-\alpha}]. \quad (32)$$

Kesten (1973) showed that there exists a positive constant  $c_\theta$  such that

$$\mathbb{P}^\theta(S^{-1} > x) \cdot x^{\kappa_\theta} \rightarrow c_\theta, \quad \text{when } x \rightarrow \infty. \quad (33)$$

Combining (30), (33) and (32) implies point *i*) of Proposition 2.4.

Now, we turn to point *ii*) of Proposition 2.4. Using the convexity of the function  $x \mapsto |\log(1-x)|$  on  $(0, 1)$ , we obtain

$$\frac{\mathbb{1}_{\{S < 1/2\}}}{2S \log 2} \leq \frac{\mathbb{1}_{\{S < 1/2\}}}{\Lambda},$$

which combined with (31) yields

$$\mathbb{E}^\theta \left[ S \sum_{k=0}^{\infty} k^\alpha e^{-\Lambda k} \right] \geq \frac{\Gamma(\alpha+1)}{2(2 \log 2)^{\alpha+1}} \mathbb{E}^\theta [S^{-\alpha} \mathbb{1}_{\{S < 1/2\}}],$$

and finally

$$\mathbb{E}^\theta \left[ S \sum_{k=0}^{\infty} k^\alpha e^{-\Lambda k} \right] \geq \frac{\Gamma(\alpha+1)}{2(2 \log 2)^{\alpha+1}} (\mathbb{E}^\theta [S^{-\alpha}] - 2^\alpha). \quad (34)$$

Combining (30), (33) and (34) implies point *ii*) of Proposition 2.4.  $\square$

### 3.4 Proof of Proposition 2.7

Falconnet et al. have already established that points *i*) and *ii*) of Assumption II as well as point *i*) of Assumption III are satisfied. From the latter reference, we also know that the first derivative  $\dot{\phi}_p$  as well as the second derivative  $\ddot{\phi}_p$  are uniformly bounded when  $\Theta \in (0, 1)$ , and this implies that point *iii*) of Assumption II and points *ii*) and *iv*) of Assumption III are satisfied. Points *iii*) and *v*) of Assumption III can be checked exactly as in Falconnet et al. (2013).  $\square$

### 3.5 Proof of Proposition 2.8

Falconnet et al. have already established that points *i*) and *ii*) of Assumption II as well as point *i*) of Assumption III are satisfied.

From the latter reference, we know that there exists a constant  $A_1$  independent of  $\theta$ , such that for any  $u$  and  $v$

$$|\partial_\alpha \phi_\theta(u, v)| \leq A_1 \log(1+v) \quad \text{and} \quad |\partial_\beta \phi_\theta(u, v)| \leq A_1 \log(1+u). \quad (35)$$

Define  $\kappa_\theta \in (0, 1]$  as the unique positive number satisfying  $\mathbb{E}^\theta[\rho_0^{\kappa_\theta}] = 1$ , that is,

$$\Gamma(\alpha - \kappa_\theta) \Gamma(\beta + \kappa_\theta) = \Gamma(\alpha) \Gamma(\beta).$$

Define  $\kappa = \min\{\kappa_\theta : \theta \in \Theta\}$ . From (35), there exists  $A_2 > 0$  and  $A_3 > 0$  independent of  $\theta$ , such that for any  $u$  and  $v$

$$|\partial_\alpha \phi_\theta(u, v)| \leq A_2 v^{\kappa/2} \quad \text{and} \quad |\partial_\beta \phi_\theta(u, v)| \leq A_2 u^{\kappa/2}, \quad (36)$$

and

$$|\partial_\alpha \phi_\theta(u, v)|^4 \leq A_3 v^{\kappa/2} \quad \text{and} \quad |\partial_\beta \phi_\theta(u, v)|^4 \leq A_3 u^{\kappa/2}. \quad (37)$$

Using the fact that  $\mathbb{E}^\theta[\rho_0^{\kappa/2}] < 1$  for any  $\theta \in \Theta$ , Proposition 2.4, the fact that

$$\sum_{k \in \mathbb{Z}_+} k^{\kappa/2} \pi_\theta(k) = \sum_{u, v \in \mathbb{Z}_+} u^{\kappa/2} \tilde{\pi}_\theta(u, v) = \sum_{u, v \in \mathbb{Z}_+} v^{\kappa/2} \tilde{\pi}_\theta(u, v),$$

(36) and (37) yields that point *iii*) of Assumption II is satisfied, as well as point *ii*) of Assumption III with  $q = 2$ .

Now, we turn to point *iii*) of Assumption III. To exchange the order of derivation and summation, it is sufficient to prove that

$$\sum_v \sup_{\theta \in \Theta} \|\dot{Q}_\theta(u, v)\| < \infty, \quad (38)$$

for any integer  $u$ . Define  $\theta' = (\alpha', \beta')$  with

$$\alpha' = \inf(\text{proj}_1(\Theta)) \quad \text{and} \quad \beta' = \inf(\text{proj}_2(\Theta)),$$

where  $\text{proj}_i$ ,  $i = 1, 2$  are the two projectors on the coordinates. Note that  $\theta'$  does not necessarily belong to  $\Theta$ . However, it still belongs to the sub-ballistic region. From Falconnet et al. (2013), we know that there exists a constant  $A_4$  such that

$$Q_\theta(u, v) \leq A_4 Q_{\theta'}(u, v),$$

for any integers  $u$  and  $v$ . Define  $\kappa' \in (0, 1]$  as the unique positive number satisfying  $\mathbb{E}^{\theta'}[\rho_0^{\kappa'}] = 1$ , and recall that  $\dot{Q}_\theta(u, v) = Q_\theta(u, v) \dot{\phi}_\theta(u, v)$ . Hence, using the last inequality and the fact that  $\|\dot{\phi}_\theta\| = O(v^{\kappa'/2})$ , it is sufficient to prove that

$$\sum_v v^{\kappa'/2} Q_{\theta'}(u, v) < \infty, \quad \text{for any integer } u, \quad (39)$$

to get (38). We have

$$\sum_u \left( \sum_v v^{\kappa'/2} Q_{\theta'}(u, v) \right) \pi_{\theta'}(u) = \sum_v v^{\kappa'/2} \pi_{\theta'}(v) < \infty,$$

where the last inequality comes from the fact that  $\mathbb{E}^{\theta'}[\rho_0^{\kappa'/2}] < 1$  and Proposition 2.4. Hence, (39) is satisfied for any integer  $u$  which proves that (38) is satisfied.

The second order derivatives of  $\phi_\theta$  are given by

$$\begin{aligned} \partial_\alpha^2 \phi_\theta(x, y) &= - \sum_{k=0}^x \frac{1}{(k+\alpha)^2} + \sum_{k=0}^{x+y} \frac{1}{(k+\alpha+\beta)^2}, \\ \partial_\alpha \partial_\beta \phi_\theta(x, y) &= \sum_{k=0}^{x+y} \frac{1}{(k+\alpha+\beta)^2}, \end{aligned}$$

and similar formulas for  $\beta$  instead of  $\alpha$ . Thus, the second derivative  $\ddot{\phi}_\theta$  is uniformly bounded on  $\Theta$ , and this implies that point *iv*) of Assumption III is satisfied. Point *v*) of Assumption II can be checked exactly as in Falconnet et al. (2013).  $\square$

### 3.6 Proof of point (a) of Proposition 2.9

Fix  $\alpha > 0$ . To prove (23), we show

$$\text{i) } \tilde{\pi}_{\theta^*}([\sup_{\theta \in \mathcal{V}_\alpha^c} \phi_\theta - \phi_{\theta^*}]^-) = +\infty,$$

$$\text{ii) } \tilde{\pi}_{\theta^*}([\sup_{\theta \in \mathcal{V}_\alpha^c} \phi_\theta - \phi_{\theta^*}]^+) < +\infty.$$

Indeed, under points i) and ii), we can apply an ergodic theorem to  $(Z_n)$  which yields

$$\frac{1}{n} \sum_{x=0}^{n-1} \sup_{\theta \in \mathcal{V}_\alpha^c} [\phi_\theta(Z_x, Z_{x+1}) - \phi_{\theta^*}(Z_x, Z_{x+1})] \xrightarrow[n \rightarrow \infty]{} -\infty, \quad \mathbf{P}^* \text{-almost surely,}$$

and then (23).

We note that  $K_a(u, \cdot)$  defined by (22) is the distribution of a negative binomial random variable  $\text{NB}(u+1, a)$  with probability of success  $1-a$  and number of failures  $u+1$ , that is, the distribution of the number of successes in a sequence of independent Bernoulli trials until  $u+1$  failures has occurred.

We will make use several times of the fact that  $\text{NB}(u+1, a)$  is the sum of  $(u+1)$  i.i.d. geometric random variables  $G_1(a), \dots, G_{u+1}(a)$  with parameter  $1-a$ , that is  $\text{Prob}(G_1(a) = k) = (1-a)^k a$ , whose mean is given by  $\mu = (1-a)/a > 1$ . As a shorthand of notation, we write  $\mu^*$  as the ratio  $(1-a^*)/a^* > 1$ .

Define for any  $\varepsilon > 0$  and any integer  $u$ , the sets

$$A(\varepsilon, u) = \left\{ v \in \mathbb{Z}_+ : \left| \frac{v}{u+1} - \mu^* \right| \leq \varepsilon \right\}, \quad (40)$$

$$B(\varepsilon, u) = \left\{ v \in \mathbb{Z}_+ : \left| \frac{v}{u+1} - \frac{1}{\mu^*} \right| \leq \varepsilon \right\}, \quad (41)$$

$$C(\varepsilon, u) = \mathbb{Z}_+ \setminus (A(\varepsilon) \cup B(\varepsilon)). \quad (42)$$

We have

$$\sum_{v \in A(\varepsilon, u)} K_{a^*}(u, v) = \text{Prob}\left(|\overline{G}_{u+1}(a^*) - \mu^*| \leq \varepsilon\right),$$

with

$$\overline{G}_{u+1}(a^*) = \frac{1}{u+1} \sum_{k=1}^{u+1} G_k(a^*).$$

Using concentration inequalities, there exists a constant  $c_\varepsilon$  such that

$$\sum_{v \in A(\varepsilon, u)} K_{a^*}(u, v) \geq 1 - e^{-c'_\varepsilon(u+1)}. \quad (43)$$

Similarly, there exists a constant  $c'_\varepsilon$  such that

$$\sum_{v \in B(\varepsilon, u)} K_{1-a^*}(u, v) \geq 1 - e^{-c''_\varepsilon(u+1)}, \quad (44)$$

and as a consequence of (21), (43) and (44), there exists a constant  $c_\varepsilon''$  such that

$$\sum_{v \in C(\varepsilon, u)} Q_{\theta^*}(u, v) \leq e^{-c_\varepsilon''(u+1)}. \quad (45)$$

Introduce the quantity  $\beta$  to be used later and defined as

$$\beta = \max \left\{ \sup_{a \in \mathcal{V}_\alpha^c} \left| \log \frac{1-a^*}{1-a} \right|, \sup_{a \in \mathcal{V}_\alpha^c} \left| \log \frac{a^*}{a} \right| \right\}. \quad (46)$$

For any  $0 < \varepsilon < \mu^* - 1$  and for any  $v$  in  $A(\varepsilon, u)$ , we have  $v > u + 1$  and as a consequence, for any  $a \in \Theta$ ,

$$K_a(u, v) > K_{1-a}(u, v).$$

Thus, we deduce that for any  $v$  in  $A(\varepsilon, u)$ ,

$$\begin{aligned} (\phi_\theta - \phi_{\theta^*})(u, v) &= \log \frac{pK_a(u, v) + (1-p)K_{1-a}(u, v)}{pK_{a^*}(u, v) + (1-p)K_{1-a^*}(u, v)} \\ &\leq \log \frac{1}{p} + (u+1) \log \frac{a}{a^*} + v \log \frac{1-a}{1-a^*} \\ &\leq \log \frac{1}{p} - \frac{u+1}{a^*} \left[ d_{\text{KL}}(a^*|a) + \left( \frac{v}{u+1} - \mu^* \right) a^* \log \frac{1-a^*}{1-a} \right] \\ &\leq \log \frac{1}{p} - (u+1) \left( \frac{\alpha}{a^*} - \varepsilon\beta \right). \end{aligned}$$

Similarly, we deduce that for any  $0 < \varepsilon < 1 - 1/\mu^*$  and for any  $v$  in  $B(\varepsilon, u)$ ,

$$(\phi_\theta - \phi_{\theta^*})(u, v) \leq \log \frac{1}{1-p} - (u+1) \left( \frac{\alpha}{1-a^*} - \varepsilon\beta \right).$$

Hence, choosing

$$\varepsilon = \min \left\{ \mu^* - 1, 1 - 1/\mu^*, \frac{\alpha}{2\beta(1-a^*)} \right\}$$

yields the existence of  $u_0$  such that for any  $u \geq u_0$  and any  $v$  in  $A(\varepsilon, u) \cup B(\varepsilon, u)$

$$\left( \sup_{\theta \in \mathcal{V}_\alpha^c} (\phi_\theta - \phi_{\theta^*})(u, v) \right)^+ = 0 \quad \text{and} \quad \left( \sup_{\theta \in \mathcal{V}_\alpha^c} (\phi_\theta - \phi_{\theta^*})(u, v) \right)^- \geq \frac{\alpha}{3(1-a^*)}(u+1). \quad (47)$$

Combining (45) and (47) immediatly yields

$$\tilde{\pi}_{\theta^*} \left( \sup_{\theta \in \mathcal{V}_\alpha^c} [\phi_\theta - \phi_{\theta^*}]^- \right) \geq \frac{\alpha}{3(1-a^*)} \sum_{u \geq u_0} \pi_{\theta^*}(u)(u+1)(1 - e^{-c_\varepsilon''(u+1)}) = +\infty,$$

where the last equality comes from Proposition 2.4. This achieves the proof of point i). To prove point ii), note that there exists a positive constant  $c_1$  such that for any  $u$  and any  $v$

$$\left( \sup_{\theta \in \mathcal{V}_\alpha^c} (\phi_\theta - \phi_{\theta^*})(u, v) \right)^+ \leq \left| \sup_{\theta \in \mathcal{V}_\alpha^c} (\phi_\theta - \phi_{\theta^*})(u, v) \right| \leq c_1(u+1+v).$$

Furthermore, from Cauchy-Schwarz inequality, the fact that  $K_{a^*}(u, \cdot)$  (resp.  $K_{1-a^*}(u, \cdot)$ ) possesses a second moment quadratic with  $u$ , and (45), there exists two positive constants  $c_2$  and  $c_3$  such that

$$\begin{aligned} \sum_{v \geq 0} Q_{\theta^*}(u, v) v \cdot \mathbb{1}_{C(\varepsilon, u)}(v) &\leq \left( \sum_{v \geq 0} Q_{\theta^*}(u, v) v^2 \right)^{1/2} \cdot \left( \sum_{v \geq 0} Q_{\theta^*}(u, v) \mathbb{1}_{C(\varepsilon, u)}(v) \right)^{1/2} \\ &\leq c_2(u+1)e^{-c_3(u+1)}, \end{aligned}$$

Therefore, there exists two positive constants  $c_4$  and  $c_5$  such that

$$\tilde{\pi}_{\theta^*} \left( \sup_{\theta \in \mathcal{I}_a^c} [\phi_\theta - \phi_{\theta^*}]^+ \right) \leq c_4 + c_5 \sum_{u \geq u_0} \pi_{\theta^*}(u) (u+1) e^{-c''(u+1)} < +\infty,$$

which achieves the proof of point ii).

Noting that  $\hat{\theta}_n$  does not depend on the choice of  $\theta_0$  in (14), we can take  $\theta_0 = \theta^*$ . Obviously, we have  $\ell_n^{\text{sb}}(\theta^*) = 0$ , for all integer  $n$ , whereas from (23), we have  $\ell_n^{\text{sb}}(\theta)/n$  which goes to infinity, for any  $\theta$  outside a neighborhood of  $\theta^*$ . Hence, the consistency follows.  $\square$

### 3.7 Proof of point (b) of Proposition 2.9

We have,

$$\phi'_\theta(u, v) \cdot Q_\theta(u, v) = pK_a(u, v) \left( \frac{u+1}{a} - \frac{v}{1-a} \right) - (1-p)K_{1-a}(u, v) \left( \frac{u+1}{1-a} - \frac{v}{a} \right). \quad (48)$$

Recall that

$$\Sigma_\theta = \mathbb{E}^\theta [(\phi'_\theta)^2] = \sum_{u \geq 0} \pi_\theta(u) \sum_{v \geq 0} Q_\theta(u, v) [\phi'_\theta(u, v)]^2,$$

which can be rewritten using (48) as

$$\Sigma_\theta = \sum_{u \geq 0} \pi_\theta(u) \sum_{v \geq 0} \frac{1}{Q_\theta(u, v)} \left[ pK_a(u, v) \left( \frac{u+1}{a} - \frac{v}{1-a} \right) - (1-p)K_{1-a}(u, v) \left( \frac{u+1}{1-a} - \frac{v}{a} \right) \right]^2.$$

Define

$$v(u) = u+1, \quad V(u) = \max\{v \in \mathbb{Z}_+ : v \leq \mu \cdot (u+1) - (1-a)\sqrt{u+1}\},$$

with  $\mu = (1-a)/a$ . From the fact that  $\mu > 1$ , there exists  $u_0$  such that for any  $u \geq u_0$ , we have  $v(u) < V(u)$ . Furthermore, for any  $u \geq u_0$  and any  $v$  in  $[v(u), V(u)]$ , we have

$$K_a(u, v) \geq K_{1-a}(u, v), \quad \frac{u+1}{1-a} - \frac{v}{a} \leq 0 \quad \text{and} \quad \frac{u+1}{a} - \frac{v}{1-a} \geq \sqrt{u+1}.$$

Thus,

$$\begin{aligned}\Sigma_\theta &\geq \sum_{u \geq u_0} \pi_\theta(u) \sum_{v=v(u)}^{V(u)} \frac{1}{Q_\theta(u, v)} \left[ pK_a(u, v) \left( \frac{u+1}{a} - \frac{v}{1-a} \right) \right]^2 \\ &\geq p^2 \sum_{u \geq u_0} (u+1) \pi_\theta(u) \sum_{v=v(u)}^{V(u)} K_a(u, v).\end{aligned}\quad (49)$$

Recall that

$$\begin{aligned}\sum_{v=v(u)}^{V(u)} K_a(u, v) &= \text{Prob}\left(\text{NB}(u+1, a) \in [v(u), V(u)]\right) \\ &= \text{Prob}\left(\sqrt{u+1}(\bar{G}_{u+1} - \mu) \in [\sqrt{u+1}(1-\mu), -(1-a)]\right),\end{aligned}$$

with

$$\bar{G}_{u+1} = \frac{1}{u+1} \sum_{k=1}^{u+1} G_k(a),$$

where  $G_1(a), \dots, G_{u+1}(a)$  are i.i.d. geometric random variables with mean  $\mu$ . From the central limit theorem applied to the sequence  $(G_k(a))$ , there exists  $u_1 \geq u_0$  such that for any  $u \geq u_1$

$$\sum_{v=v(u)}^{V(u)} K_a(u, v) \geq \frac{1}{2} \text{Prob}\left(\mathcal{N}(0, \sigma^2) \in [-2\mu, -(1-a)]\right) = C > 0, \quad (50)$$

where  $\sigma^2 = (1-a)/a^2$  is the variance of  $G_1(a)$  and  $\mathcal{N}(0, \sigma^2)$  a Gaussian random variable with mean 0 and variance  $\sigma^2$ . Injecting (50) in (49) yields

$$\Sigma_\theta \geq Cp^2 \sum_{u \geq u_1} (u+1) \pi_\theta(u).$$

From Proposition 2.4,  $\pi_\theta$  does not possess a finite first moment in the sub-ballistic regime and we deduce that  $\Sigma_\theta = +\infty$ .  $\square$

### 3.8 Proof of point (c) of Proposition 2.9

Assume that  $\theta \neq \theta^*$ . Recall that

$$J = \sum_{u \geq 0} \pi_{\theta^*}(u) \sum_{v \geq 0} Q_{\theta^*}(u, v) |\phi'_\theta(u, v)|,$$

which can be rewritten using (48) and (9)

$$J = \sum_{u \geq 0} \pi_{\theta^*}(u) \sum_{v \geq 0} \frac{Q_{\theta^*}(u, v)}{Q_\theta(u, v)} \left| pK_a(u, v) \left( \frac{u+1}{a} - \frac{v}{1-a} \right) - (1-p)K_{1-a}(u, v) \left( \frac{u+1}{1-a} - \frac{v}{a} \right) \right|.$$

Using the set  $A(\varepsilon, u)$ , we have

$$J \geq pJ_1 - (1-p)J_2, \quad (51)$$

with

$$J_1 = \sum_{u \geq 0} \pi_{\theta^*}(u) \sum_{v \in A(\varepsilon, u)} \frac{Q_{\theta^*}(u, v)}{Q_{\theta}(u, v)} K_a(u, v) \left| \frac{u+1}{a} - \frac{v}{1-a} \right| \quad (52)$$

$$J_2 = \sum_{u \geq 0} \pi_{\theta^*}(u) \sum_{v \in A(\varepsilon, u)} \frac{Q_{\theta^*}(u, v)}{Q_{\theta}(u, v)} K_{1-a}(u, v) \left| \frac{u+1}{1-a} - \frac{v}{a} \right|. \quad (53)$$

Choose  $\varepsilon = \min\{|\mu - \mu^*|/2, \mu^* - 1\}$ . Then, for any  $u$  and any  $v \in A(\varepsilon, u)$ , we have

$$\left| \frac{u+1}{a} - \frac{v}{1-a} \right| \geq \frac{u+1}{1-a} \varepsilon, \quad K_a(u, v) \geq Q_{\theta}(u, v),$$

and as a consequence

$$\begin{aligned} J_1 &\geq \frac{\varepsilon}{1-a} \sum_{u \geq 0} \pi_{\theta^*}(u)(u+1) \sum_{v \in A(\varepsilon, u)} Q_{\theta^*}(u, v) \\ &\geq \frac{p\varepsilon}{1-a} \sum_{u \geq 0} \pi_{\theta^*}(u)(u+1) \sum_{v \in A(\varepsilon, u)} K_{a^*}(u, v). \end{aligned}$$

Using (43) and the fact that  $\pi_{\theta^*}$  does not possess a finite first moment in the sub-ballistic regime, we deduce that  $J_1$  is infinite. On the other hand, we have for any  $u$  and any  $v \in A(\varepsilon, u)$ ,

$$\left| \frac{u+1}{1-a} - \frac{v}{a} \right| \leq (u+1) \left( \frac{1}{1-a} + \frac{\varepsilon + \mu^*}{a} \right),$$

and,

$$\frac{K_{1-a}(u, v)}{Q_{\theta}(u, v)} \leq \frac{K_{1-a}(u, v)}{pK_a(u, v)} \leq \frac{1}{p} \mu^{u+1-v},$$

and as a consequence,

$$J_2 \leq \frac{1}{p} \left( \frac{1}{1-a} + \frac{\varepsilon + \mu^*}{a} \right) \sum_{u \geq 0} \pi_{\theta^*}(u)(u+1) \cdot \gamma^{u+1} \sum_{v \in A(\varepsilon, u)} Q_{\theta^*}(u, v),$$

where

$$\gamma = \mu^{-(\mu^* - 1 - \varepsilon)} < 1.$$

From the fact that  $u \mapsto (u+1)\gamma^{u+1} \sum_{v \in A(\varepsilon, u)} Q_{\theta^*}(u, v)$  is bounded, and therefore integrable against  $\pi_{\theta^*}$ , we deduce that  $J_2$  is finite. This achieves the proof of (25).  $\square$

## 4 Numerical performance

In this section, we explore the numerical performance of our estimation procedure in the frameworks of Example I and the Temkin model. We compare our performance with the performance of the estimator proposed by Adelman and Enriquez

(2004). An explicit description of the form of Adelman and Enriquez's estimator in the particular case of the one-dimensional nearest neighbour path is provided in Section 5.1 of Comets et al. (2014). Therefore, one can estimate  $\theta^*$  by the solution of an appropriate system of equations, as illustrated below.

**Example I (continued).** In this case the parameter  $\theta$  equals  $p$  and we have

$$v = \mathbb{E}^*[\omega_0] = p^* a_1 + (1 - p^*) a_2.$$

Hence, among the visited sites, the proportion of those from which the first move is to the right gives an estimator for  $p^* a_1 + (1 - p^*) a_2$ . Using this observation, we can estimate  $p^*$ .

**Example III (continued).** In this case the parameter  $\theta$  equals  $a$  and we have

$$v = \mathbb{E}^*[\omega_0] = p a^* + (1 - p)(1 - a^*).$$

Hence, among the visited sites, the proportion of those from which the first move is to the right gives an estimator for  $p a^* + (1 - p)(1 - a^*)$ . Using this observation, we can estimate  $a^*$ .

## 4.1 Experiments

We now present the simulation experiment corresponding to Example I and Example III where we include a comparison with Adelman and Enriquez's procedure.

For each of the two simulations, we *a priori* fix a parameter value  $\theta^*$  as given in Table 1 and repeat 1,000 times the procedure described below.

Simulation	Fixed parameter	Estimated parameter
Example I	$(a_1, a_2) = (0.4, 0.7), \kappa = 0.9$	$p^* \approx 0.548$
Example III	$\kappa = 0.9, p \approx 0.41$	$a^* = 0.4$

Table 1: Parameter values for each experiment.

Then, we generate a random environment according to  $\nu_{\theta^*}$  on the set of sites  $\{-10^3, \dots, 10^3\}$ . In fact, we do not use the environment values for all the  $10^3$  negative sites, since only few of these sites are visited by the walk. However the computation cost is very low comparing to the rest of the estimation procedure, and the symmetry is convenient for programming purpose. Then, we run a random walk in this environment and stop it successively at the hitting times  $T_n$  defined by (2), with  $n \in \{10^2 k : 1 \leq k \leq 10\}$ . For each stop, we estimate  $\theta^*$  according to our procedure and Adelman and Enriquez's one. The likelihood optimization procedure was performed as a combination of golden section search and successive parabolic interpolation.

The parameter is chosen such that the RWRE is transient to the right and sub-ballistic. Note that the length of the random walk is not  $n$  but rather  $T_n$ . The fluctuations of  $T_n$  depend in nature on the parameter  $\kappa$ . Under mild additional assumptions, Kesten et al. (1975) proved that if  $\kappa < 1$ , then  $n^{-1/\kappa} \cdot T_n$  has a non-degenerate limit distribution, a stable law with index  $\kappa$ .

In the simulations, the quantity  $T_n$  varies considerably. To avoid too long computations, when  $T_n$  is too large, we fixed a threshold for the number of steps for the walk at  $t_{\max} = 500n^{1/\kappa} \approx 10^6$ . When the threshold is reached, we did not compute our estimator. This case happened for 4.4% (when  $n = 100$ ) and for 41.9% (when  $n = 1000$ ) of the simulation in Example I, and for 0.3% (when  $n = 100$ ) and for 4.9% (when  $n = 1000$ ) of the simulation in Example III.

Figure 1 shows the boxplots of our estimator and Adelman and Enriquez's estimator obtained from 1,000 iterations of the procedures in Example I. First, we shall notify that in order to simplify the visualisation of the results, we removed in the boxplots corresponding to Example I about 1.5% of outliers values (outside 1.5 times the interquartile range above the upper quartile and below the lower quartile) from our estimator. We observe that the accuracies of the procedures increase with the value of  $n$ . We also note that whereas Adelman and Enriquez's seems unbiased our procedure seems to be slightly biased. However, our procedure exhibits a much smaller variance than Adelman and Enriquez's one. One explanation for the worse performance of Adelman and Enriquez's estimator comparing to our procedure is the fact that only a few part of the trajectory is used in the estimation.

Figure 2 shows the boxplots of our estimator and Adelman and Enriquez's estimator obtained from 1,000 iterations of the procedures in Example III. First, we shall notify that in order to simplify the visualisation of the results, we removed in the boxplots corresponding to Example I about 15% of outliers values (outside 1.5 times the interquartile range above the upper quartile and below the lower quartile) from our estimator. We first observe that the accuracies of the procedures increase with the value of  $n$ . We also note that both procedures seem unbiased. However, our procedure exhibits a much smaller variance than Adelman and Enriquez's one, but also a much smaller one than when we were not estimating the support. This suggests that the rate of convergence when estimating the support in the Temkin model is faster than the square root of  $n$ .

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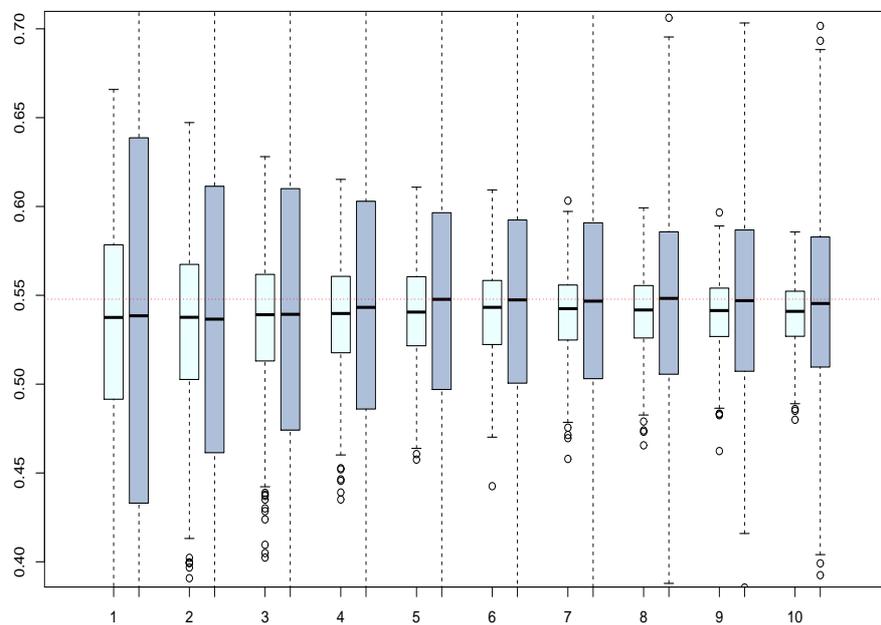


Figure 1: Boxplots of our estimator (left and white) and Adelman and Enriquez's estimator (right and grey) obtained from 1,000 iterations and for values  $n$  ranging in  $\{10^2 k; 1 \leq k \leq 10\}$  ( $x$ -axis indicates the value  $k$ ). The panel displays estimation of  $p^*$  in Example I. The true value is indicated by horizontal lines.

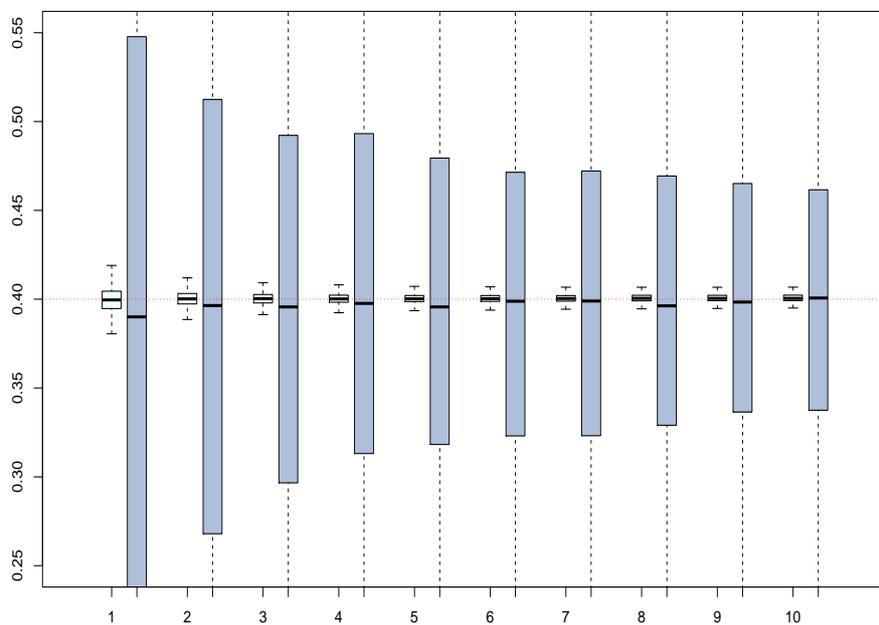


Figure 2: Boxplots of our estimator (left and white) and Adelman and Enriquez's estimator (right and grey) obtained from 1,000 iterations and for values  $n$  ranging in  $\{10^2 k; 1 \leq k \leq 10\}$  ( $x$ -axis indicates the value  $k$ ). The panel displays estimation of  $a^*$  in Example III. The true value is indicated by horizontal lines.