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# Complex free field and Eulerian circuits

Yves Le Jan

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## 1 The loop ensemble and the free field

We adopt the framework described in [4]. Given a graph  $(X, E)$ , a set of non negative conductances  $C_{x,y} = C_{y,x}$  indexed by  $E$  and a non negative killing measure  $\kappa$  on  $X$ , we can associate with them an energy (or Dirichlet form)  $\mathcal{E}$ , we will assume to be positive definite, which is a transience assumption. There is a duality measure  $\lambda$  defined by  $\lambda_x = \sum_y C_{x,y} + \kappa_x$ . Let  $G_{x,y}$  be the symmetric Green's function associated with  $\mathcal{E}$ . It is assumed that  $\sum_x G_{x,x} \lambda_x$  is finite.

The symmetric Markov process associated with it can be obtained from the Markov chain defined by the transition matrix  $P_{x,y} = \frac{C_{x,y}}{\lambda_y}$  by adding independent exponential holding times of mean 1 before each jump. The complex (real) free field is a the complex (real) Gaussian field on  $X$  whose covariance function is  $G$ . We will denote it by  $\varphi$  ( $\varphi_{\mathbb{R}}$ ).

We denote by  $\mu$  the loop measure associated with the symmetric Markov process ( $\mu$  can be also viewed as a shift invariant measure on based loops). The Poissonian loop ensemble  $\mathcal{L}$  is the Poisson process of loops of intensity  $\mu$ . Recall that it can be constructed by the following procedure:

- Splitting the set of based loops erased around each vertex in the construction of the random spanning tree associated with the continuous time Markov chain through Wilson's algorithm, by dividing the local time at their base points according to independent Poisson-Dirichlet random variables (Cf: [4], [1]).
- Mapping this set of based loops to their equivalence class by shift (this is the definition of loops).

## 2 Occupation field and Isomorphisms

Given any vertex  $x$  of the graph, denote by  $\hat{\mathcal{L}}^x$  the total time spent in  $x$  by the loops, normalized by  $\lambda_x$ .  $\hat{\mathcal{L}}$  is known as the occupation field of  $\mathcal{L}$ .

Recall that as a Poisson process,  $\mathcal{L}$  is infinitely divisible. We denote by  $\mathcal{L}_{\frac{1}{2}}$  the Poisson process of loops of intensity  $\frac{\mu}{2}$  and by  $\hat{\mathcal{L}}_{\frac{1}{2}}$  the associated occupation field. It has been shown in [3] ( see also [4]) that the fields  $\hat{\mathcal{L}}$  ( $\hat{\mathcal{L}}_{\frac{1}{2}}$ ) and  $\frac{1}{2}\varphi^2$  ( $\frac{1}{2}\varphi_{\mathbb{R}}^2$ ) have the same distribution.

We can use the second identity in law to give a simple proof of a result known as generalized second Ray-Knight theorem ([2], [7], [5]) Let  $x_0$  be a point of  $X$ , and assume that  $\kappa$  is supported by Let  $x_0$ . Set  $D = X - \{x_0\}$ . Then

$$\varphi_{\mathbb{R}} = \varphi_{\mathbb{R}}^D + \varphi_{\mathbb{R}}(x_0)$$

and  $\varphi_{\mathbb{R}}^D$  (the real free field associated with the restriction of  $\mathcal{E}$  to  $D$ ) is independent of  $\varphi_{\mathbb{R}}(x_0)$ . On the other hand,

$$\hat{\mathcal{L}}_{\frac{1}{2}} = \hat{\mathcal{L}}_{\frac{1}{2}}^D + \hat{\mathcal{L}}_{\frac{1}{2}}^{(x_0)}$$

where  $\hat{\mathcal{L}}_{\frac{1}{2}}^{(x_0)}$  denotes the occupation field of the set of loops of  $\mathcal{L}_{\frac{1}{2}}$  hitting  $x_0$  and  $\hat{\mathcal{L}}_{\frac{1}{2}}^D$  denotes the occupation field of the set of loops of  $\mathcal{L}_{\frac{1}{2}}$  contained in  $D$ .

The two terms of the decomposition are clearly independent. Moreover, given that its value in  $x_0$  is  $\rho$ , the field  $\hat{\mathcal{L}}_{\frac{1}{2}}^{(x_0)}$  has the same distribution as the occupation field  $\hat{\gamma}_{\tau_\rho}$  of an independent copy of the Markov chain started at  $x_0$  and stopped when the local time at  $x_0$  equals  $\rho$ . The identity in law which is valid between  $\hat{\mathcal{L}}_{\frac{1}{2}}^D$  and  $\frac{1}{2}(\varphi_{\mathbb{R}}^D)^2$  as well as between  $\hat{\mathcal{L}}_{\frac{1}{2}}$  and  $\frac{1}{2}\varphi_{\mathbb{R}}^2$  can be desintegrated taking  $\hat{\mathcal{L}}_{\frac{1}{2}}^{x_0} = \frac{1}{2}\varphi_{\mathbb{R}}^2(x_0) = \rho$ . Noting finally that the sign  $\eta$  of  $\varphi_{\mathbb{R}}$  at  $x_0$  is independent of the other variables we get that

$$\frac{1}{2}\varphi_{\mathbb{R}}^2 + \hat{\gamma}_{\tau_\rho} \stackrel{(d)}{=} \hat{\mathcal{L}}_{\frac{1}{2}}^D + \hat{\gamma}_{\tau_\rho} \stackrel{(d)}{=} \frac{1}{2}(\varphi_{\mathbb{R}}^D + \eta\sqrt{2\rho})^2$$

but we have also, by symmetry of  $\varphi_{\mathbb{R}}^D$ ,

$\frac{1}{2}(\varphi_{\mathbb{R}}^D + \eta\sqrt{2\rho})^2 \stackrel{(d)}{=} \frac{1}{2}(\varphi_{\mathbb{R}}^D + \sqrt{2\rho})^2 \stackrel{(d)}{=} \frac{1}{2}(\varphi_{\mathbb{R}}^D - \sqrt{2\rho})^2$ . so that finally we have proved that:

$$\frac{1}{2}\varphi_{\mathbb{R}}^2 + \hat{\gamma}_{\tau_\rho} \stackrel{(d)}{=} \frac{1}{2}(\varphi_{\mathbb{R}}^D + \sqrt{2\rho})^2$$

### 3 Eulerian networks

Given any oriented edge  $(x, y)$  of the graph, denote by  $N_{x,y}(l)$  the total number of jumps made from  $x$  to  $y$  by the loop  $l$  and by  $N_{x,y}$  the total number of jumps made from  $x$  to  $y$  by the loops of  $\mathcal{L}$ . We define a network to be a matrix  $k$  with  $\mathbb{N}$ -valued coefficient which vanishes on the diagonal and on entries  $(x, y)$  such that  $\{x, y\}$  is not an edge of the graph. We say that  $k$  is Eulerian if

$$\sum_y k_{x,y} = \sum_y k_{y,x}$$

For any Eulerian network  $k$ , we define  $k_x$  to be  $\sum_y k_{x,y} = \sum_y k_{y,x}$ . The matrix  $N$  defines a random network which verifies the Eulerian property.

Let  $Z$  be any Hermitian matrix indexed by pairs of vertices such that  $\forall x, y, 0 < |Z_{x,y}| \leq 1$ , and such that all but a finite set of entries indexed by  $K \times K$ , with  $K$  finite, are equal to 1.

Denote by  $P_{x,y}^Z$  the matrix  $P_{x,y} Z_{x,y}$

The content of the following lemma appeared already in [4].

**Lemma 3.1** *we have:*

$$E\left(\prod_{x \neq y} Z_{x,y}^{N_{x,y}}\right) = E\left(e^{\sum_{x \neq y} (\frac{1}{2} C_{x,y} (Z_{x,y} - 1) \varphi_x \bar{\varphi}_y)}\right)$$

Indeed on one hand, the left side can be expressed as

$$\exp\left(\int \left(\prod_{x,y} Z_{x,y}^{N_{x,y}(l)} - 1\right) \mu(dl)\right) \text{ and}$$

$$\int \left(\prod_{x,y} Z_{x,y}^{N_{x,y}(l)} - 1\right) \mu(dl) = \int \left(\prod_{m=1}^{p(l)} Z_{\xi_m-1, \xi_m} - 1\right) \mu(dl) - \mu(\{\text{non trivial loops}\})$$

which equals  $\sum_{n=1}^{\infty} \frac{1}{n} (\text{Tr}([P^Z]^n) - \text{Tr}(P^n))$ .

On the other hand, the right hand side equals  $\frac{\det(I-P)}{\det(I-P^Z)}$ .

and the lemma follows from the identity  $\log(\det) = \text{Tr}(\log)$ .

The distribution of the random network is given in the following:

**Theorem 3.1** *For any Eulerian network  $k$ ,*

$$P(N = k) = \det(I - P) \frac{\prod_x k_x!}{\prod_{x,y} k_{x,y}!} \prod_{x,y} P_{x,y}^{k_{x,y}}$$

Let  $\mathfrak{N}$  be the additive semigroup of networks and  $\mathfrak{E}$  be the additive semigroup of Eulerian networks. On one hand, note that:

$$E\left(\prod_{x,y} Z_{x,y}^{N_{x,y}}\right) = \sum_{k \in \mathfrak{E}} P(N = k) \prod_{x,y} Z_{x,y}^{k_{x,y}}$$

On the other hand,

$$\begin{aligned}
& E(e^{\sum_{x,y} (\frac{1}{2} C_{x,y} (Z_{x,y} - 1) \varphi_x \bar{\varphi}_y)}) \\
&= \frac{1}{(2\pi)^d \det(G)} \int e^{-\frac{1}{2} (\sum_x \lambda_x \varphi_x \bar{\varphi}_x - \sum_{(x,y) \in K \times K} C_{x,y} Z_{x,y} \varphi_x \bar{\varphi}_y)} \prod_x \frac{1}{2i} d\varphi_x \wedge d\bar{\varphi}_x \\
&= \frac{1}{(2\pi)^d \det(G)} \int_0^\infty \int_0^{2\pi} e^{-\frac{1}{2} (\sum_x \lambda_x r_x^2 - \sum_{x,y} C_{x,y} Z_{x,y} r_x r_y e^{i(\theta_x - \theta_y)})} \prod_x r_x dr_x d\theta_x \\
&= \frac{1}{(2\pi)^d \det(G)} \int_0^\infty \int_0^{2\pi} e^{-\frac{1}{2} \sum_x \lambda_x r_x^2} \sum_{n \in \mathfrak{N}} \prod_{x,y \in K} \frac{1}{n_{x,y}!} (C_{x,y} (\frac{1}{2} Z_{x,y} r_x r_y e^{i(\theta_x - \theta_y)})^{n_{x,y}} \prod_x r_x dr_x d\theta_x
\end{aligned}$$

Integrating in  $\theta$  and using the definition of Eulerian networks, it equals

$$\begin{aligned}
& \frac{1}{\det(G)} \int_0^\infty e^{-\frac{1}{2} \sum_x \lambda_x r_x^2} \sum_{n \in \mathfrak{E}} \prod_{(x,y) \in K \times K} \frac{1}{n_{x,y}!} (\frac{1}{2} C_{x,y} Z_{x,y} r_x r_y)^{n_{x,y}} \prod_x r_x dr_x \\
&= \frac{1}{\det(G) \prod \lambda_x} \sum_{n \in \mathfrak{E}} \prod_{x \in K} n_x! \prod_{(x,y) \in K \times K} \frac{1}{n_{x,y}!} (\frac{C_{x,y}}{\lambda_x} Z_{x,y})^{n_{x,y}} \\
&= \det(I - P) \sum_{n \in \mathfrak{E}} \prod_{x \in K} n_x! \prod_{(x,y) \in K \times K} \frac{1}{n_{x,y}!} (P_{x,y} Z_{x,y})^{n_{x,y}}
\end{aligned}$$

## 4 Additional remarks

### 4.1 A determinant formula

Lemma 3.1 can be stated in a more general form (cf [4] (6-4)).

$$E\left(\prod_{x \neq y} Z_{x,y}^{N_{x,y}} \prod_x Z_{x,x}^{-(N_x+1)}\right) = E\left(e^{\sum_{x \neq y} (\frac{1}{2} C_{x,y} (Z_{x,y} - 1) \varphi_x \bar{\varphi}_y) + \sum_x (\frac{1}{2} \lambda_x (1 - Z_{x,x}) \varphi_x \bar{\varphi}_x)}\right)$$

A consequence is that is that for any set  $(x_i, y_i)$  of distinct oriented edges, and any set  $z_l$  of distinct vertices,

$$E\left(\prod_i N_{(x_i, y_i)} \prod_l (N_{z_l} + 1)\right) = E\left(\prod_i \varphi_{x_i} \bar{\varphi}_{y_i} C_{(x_i, y_i)} \prod_l \lambda_{z_l} \varphi_{z_l} \bar{\varphi}_{z_l}\right)$$

In particular, assuming  $X$  is finite, if we set  $\check{N}_{(x,y)} = 0$  for  $x \neq y$  and  $\check{N}_{(x,x)} = 1 + N_x$ , for all  $\chi \geq \lambda$

$$E(\det(M_\chi \check{N} - N)) = \det(M_\varphi (M_\chi - C) M_{\bar{\varphi}}) = \det(M_\chi - C) \text{Per}(G)$$

## 4.2 An application of the BEST theorem.

The BEST theorem (Cf [6]) allows to determine the measure induced on Eulerian networks by the restriction of  $\mu$  to non trivial loops. If  $k$  is a Eulerian network let  $\tilde{k}$  be the oriented Eulerian graph associated with it. Its set of vertices is  $X$  and it has  $k_{x,y}$  oriented edges from  $x$  to  $y$ . Let  $|k|$  be the total number of edges in  $\tilde{k}$ . Noting that all pointed loops inducing the same network  $k \in \mathfrak{E}$  have the same measure  $\frac{1}{|k|} \prod_{x,y} P_{x,y}^{k_{x,y}}$ , the measure of  $k$  is given by  $\frac{N(k)}{|k|} \prod_{x,y} \frac{P_{x,y}^{k_{x,y}}}{k_{x,y}!}$  where  $N(k)$  is the number of Eulerian tours of  $\tilde{k}$ , i.e. of directed closed paths visiting each edge of  $\tilde{k}$  exactly once. It is given by the BEST theorem:

$$N(k) = |k| \det(Q) \prod_x (k_x - 1)!$$

where  $\det(Q)$  is the number of arborescence of  $\tilde{k}$  given by the matrix-tree theorem (Cf [6] ) and the factor  $|k|$  takes into account the choice of the first oriented edge in the Eulerian tour. Hence,

$$\mu(k) = \det(Q) \prod_x (k_x - 1)! \prod_{x,y} \frac{P_{x,y}^{k_{x,y}}}{k_{x,y}!}$$

for  $k$  non zero. We now already that the total  $\mu$  measure of non zero networks is  $-\log(\det(I - P))$ .

Our probability on Eulerian networks is therefore the sum the the convolutions powers of this measure with Poissonian weights.

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