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# Directed and multi-directed animals on the King's lattice

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**Abstract.** We define the *directed King's lattice* to be the square lattice with diagonal (next nearest neighbor) bonds and with the preferred directions  $\{\leftarrow, \nearrow, \uparrow, \searrow, \rightarrow\}$ . We enumerate directed animals on this lattice using a bijection with Viennot's heaps of pieces. We also define and enumerate a superclass of directed animals, the elements of which are called multi-directed animals. This follows Bousquet-Mélou and Rechnitzer's work on the directed triangular and square lattices. Our final results show that directed and multi-directed animals asymptotically behave similarly to the ones on the triangular and square lattices.

## 1 Introduction

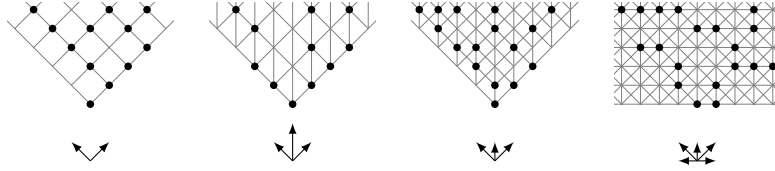
An animal on a lattice is a finite and connected set of vertices. The enumeration of animals (up to a translation) is a longstanding problem in statistical physics and combinatorics. The problem, however, is extremely difficult, and little progress has been made [16, 12]. A more realistic goal, therefore, is to enumerate natural subclasses of animals.

The class of *directed animals* is one of the most classical of these subclasses. Directed animals have been enumerated in a variety of lattices; let us cite, non-exhaustively, the square and triangular lattices [18, 14, 9, 11, 2], Bousquet-Mélou and Conway's lattices  $\mathcal{L}_n$  [4, 8], and the "strange" or  $n$ -decorated lattices [7, 3] (Figure 1). Unsolved lattices include, notably, the honeycomb lattice [13].

The class of *multi-directed animals* is a superclass of directed animals, first introduced by Klarner [15] on the square and triangular lattices. Bousquet-Mélou and Rechnitzer [5] clarified Klarner's definition and introduced a variant class on the square lattice. Moreover, they gave closed expressions for the generating function of multi-directed animals and showed that it is not D-finite.

The goal of this paper is to enumerate directed and multi-directed animals on a new lattice. We call *King's lattice* the square lattice with added diagonal bonds, or next nearest neighbor bonds. We also consider the preferred orientations  $\{\leftarrow, \nearrow, \uparrow, \searrow, \rightarrow\}$  (Figure 1, right). Directed animals on the King's lattice are a superclass of directed animals on Bousquet-Mélou and Conway's lattice  $\mathcal{L}_3$ , which has arcs  $\{\searrow, \uparrow, \nearrow\}$  [4].

Several techniques have been used to enumerate directed animals on the various lattices. Among them are direct bijections with other combinatorial objects [11], comparison with gas models [9, 3, 17, 1] and the use of Viennot's theory of heaps of pieces [20, 2, 8, 5, 21]. In this paper, we use the last method; we show that directed animals on the King's lattice are in bijection with heaps of segments, already defined in [6].



**Fig. 1.** Directed animals on a selection of lattices. From left to right: the square lattice, the triangular lattice, the lattice  $\mathcal{L}_3$ , and the King's lattice.

## 2 Animals on the King's lattice and heaps of segments

### 2.1 Definitions

**Definition 1.** We call *segment* a closed real interval of the form  $[i, j]$ , where  $i$  and  $j$  are integers such that  $j > i$ . Two segments are called *concurrent* if they intersect, even by a point. A *heap of segments* is a finite sequence of segments, considered up to commutation of non-concurrent segments.

The heaps of segments described here are the same as in [6], except that the segment reduced to a point is not allowed. More information on heaps of pieces in general can be found in [20]. Graphically, a heap is built by dropping segments in succession; a segment either falls on the ground or on another segment concurrent to it. Examples are shown in Figures 2 and 3.

### 2.2 Directed animals and pyramids of segments

Let  $A$  be an animal; we say that a site  $t$  of  $A$  is *connected* to another site  $s$  if there exists a directed path (*i.e.* respecting the preferred directions of the lattice) from  $s$  to  $t$  visiting only sites of  $A$ . We say that the animal  $A$  is *directed* of source  $s$  if every site  $t$  of  $A$  is connected to  $s$ . The source  $s$  is not unique; it may be any of the bottommost sites of  $A$  (see Figure 2, left). By convention, we call *source* of  $A$  the leftmost bottom site.

In Figure 2 is illustrated a bijection between directed animals and *pyramids of segments* (or heaps with only one segment lying on the ground). This bijection works identically to the classical bijection between directed animals on the square lattice and strict pyramids of dimers [20, 2].



**Fig. 2.** Left: a directed animal on the King's lattice (represented, for clarity, as a polyomino on the dual lattice) with its source circled. Right: the pyramid of segments obtained by replacing each maximal sequence of  $\ell$  consecutive sites by a segment of length  $\ell$ .

### 2.3 Multi-directed animals and connected heaps of segments

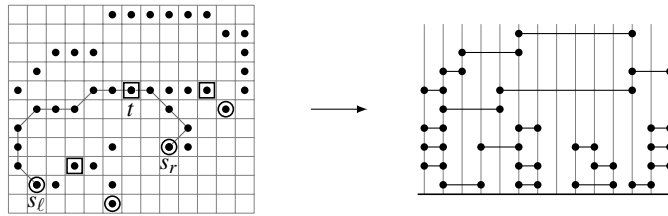
Let  $A$  be an animal. For any abscissa  $i$ , we denote by  $b(i)$  the ordinate of the bottommost site of  $A$  at abscissa  $i$  (or  $b(i) = +\infty$  if there is no site of  $A$  at abscissa  $i$ ). We call *source* of  $A$  a site that realizes a local minimum of  $b$  and *keystone* of  $A$

a site that realizes a local maximum. In case several consecutive sites realize a minimum or maximum, the source or keystone is the leftmost one (Figure 3, left). This is a purely arbitrary choice that does not alter the definition.

**Definition 2.** Let  $A$  be an animal. The animal  $A$  is said *multi-directed* if it satisfies the two conditions:

- for every site  $t$  of  $A$ , there exists a source  $s$  such that  $t$  is connected to  $s$ ;
- for every keystone  $t$  of  $A$ , there exist two sources  $s_\ell$  and  $s_r$ , to the left and to the right of  $t$  respectively, such that  $t$  is connected to both  $s_\ell$  and  $s_r$ . Moreover, the directed paths connecting  $t$  to  $s_\ell$  and  $s_r$  do not go through a keystone at the same height as  $t$ .

As a directed animal has only one source and no keystone, every directed animal is multi-directed. Multi-directed animals are in bijection with *connected heaps of segments* (or heaps without an empty column). A multi-directed animal and its corresponding heap are depicted in Figure 3.



**Fig. 3.** Left: a multi-directed animal with four sources (circled) and three keystones (boxed). The directed paths connecting one keystone, denoted by  $t$ , to the sources  $s_\ell$  and  $s_r$  are shown. Right: the corresponding connected heap of segments, with four minimal pieces (one for each source of the animal).

Definition 2 can be adapted in the directed square and triangular lattices; the animals thus defined are in bijection with *connected heaps of dimers*. Bousquet-Mélou and Rechnitzer also defined multi-directed animals in bijection with connected heaps of dimers in [5], in a slightly different way. Our definition of multi-directed animals has the advantages of being more intrinsic and of having a vertical symmetry.

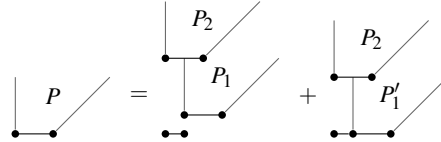
### 3 Enumeration of directed animals

In this section, we use the bijection with pyramids of segments to enumerate directed animals on the King's lattice. We call *half-animal* a directed animal with no site on the left side of its source. The associated pyramids are called *half-pyramids*. We adapt Bétréma and Penaud's methods [2] to decompose the pyramids of segments, which yields the following result.

**Theorem 3.** The generating functions  $S(t)$  and  $D(t)$  of half-animals and animals are:

$$S(t) = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{4t}; \quad D(t) = \frac{1}{4} \left( \frac{1+t}{\sqrt{1 - 6t + t^2}} - 1 \right).$$

The decomposition of the half-pyramids is sketched in Figure 4. Interestingly, the generating function  $S(t)$  is already known in combinatorics. Its coefficients are the little Schröder numbers, **A001003** in the OEIS [19]. The coefficients of  $D(t)$  also appear as **A047781**. This is remindful of the triangular lattice, where the half-animals are enumerated by the Catalan numbers [2].



**Fig. 4.** Sketch of the two cases in the decomposition of half-pyramids. The generating function of the possible heaps  $P_1$  and  $P_2$  is  $1 + S(t)$ , while the generating function of the possible heaps  $P_1'$  is  $S(t)$ . This shows the identity  $S = t(1 + S)^2 + tS(1 + S)$ , from which we derive the value of  $S(t)$ .

#### 4 Enumeration of multi-directed animals

In this section, we enumerate multi-directed animals, or, equivalently, connected heaps of segments. To do this, we adapt the *Nordic decomposition*, invented by Viennot to enumerate connected heaps of dimers [21].

**Theorem 4.** Let  $M = M(t)$  be the generating function of multi-directed animals. Let  $S = S(t)$ ,  $D = D(t)$  be the power series defined in Theorem 3,  $R = S + t(1 + S)$  and  $Q = 2(1 - t)S - t$ . The generating function  $M$  is given by:

$$M = \frac{D}{1 - \sum_{k \geq 0} S(1 + S)^k \frac{QR^k}{1 - QR^k}}.$$

#### 5 Asymptotic results

Finally, we derive asymptotic results from Theorems 3 and 4.

**Theorem 5.** Let  $D_n$  and  $M_n$  be the number of directed and multi-directed animals of area  $n$ , respectively. As  $n$  tends to infinity, we have:

$$D_n \sim \kappa(3 + \sqrt{8})^n n^{-1/2}; \quad M_n \sim \lambda \mu^n,$$

with  $\mu = 6.475\dots$ . The average width of directed animals grows like  $\sqrt{n}$ , while the average width of multi-directed animals grows like  $n$ . Finally, the series  $M(t)$  is not  $D$ -finite.

The results on directed animals are a straightforward application of singularity analysis [10, Theorem VI.4]. The results on multi-directed animals are more involved. Similar results already exist on the square and triangular lattices, including the non- $D$ -finiteness of the series  $M(t)$  [5].

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