# POTENTIAL ESTIMATES AND QUASILINEAR PARABOLIC EQUATIONS WITH MEASURE DATA 

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## To cite this version:

Quoc Hung Nguyen. POTENTIAL ESTIMATES AND QUASILINEAR PARABOLIC EQUATIONS WITH MEASURE DATA. 2014. hal-00989464v2

## HAL Id: hal-00989464 <br> https://hal.science/hal-00989464v2

Preprint submitted on 10 Sep 2014

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# POTENTIAL ESTIMATES AND QUASILINEAR PARABOLIC EQUATIONS WITH MEASURE DATA 

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September 10, 2014


#### Abstract

In this paper, we study the existence and regularity of the quasilinear parabolic equations: $$
u_{t}-\operatorname{div}(A(x, t, \nabla u))=B(u, \nabla u)+\mu
$$ in $\mathbb{R}^{N+1}, \mathbb{R}^{N} \times(0, \infty)$ and a bounded domain $\Omega \times(0, T) \subset \mathbb{R}^{N+1}$. Here $N \geq 2$, the nonlinearity $A$ fulfills standard growth conditions and $B$ term is a continuous function and $\mu$ is a radon measure. Our first task is to establish the existence results with $B(u, \nabla u)= \pm|u|^{q-1} u$, for $q>1$. We next obtain global weighted-Lorentz, LorentzMorrey and Capacitary estimates on gradient of solutions with $B \equiv 0$, under minimal conditions on the boundary of domain and on nonlinearity $A$. Finally, due to these estimates, we solve the existence problems with $B(u, \nabla u)=|\nabla u|^{q}$ for $q>1$.


MSC: primary $35 \mathrm{~K} 55,35 \mathrm{~K} 58,35 \mathrm{~K} 59,31 \mathrm{E} 05$; secondary $35 \mathrm{~K} 67,42 \mathrm{~B} 37$
Keywords: quasilinear parabolic equations; renormalized solutions; Wolff parabolic potential; Riesz parabolic potential; Bessel parabolic potential; maximal potential; heat kernel; Radon measures; uniformly thick domain; Reifenberg flat domain; decay estimates; Lorentz spaces; Riccati type equations; capacity

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## 1 Introduction

In this article, we study a class of quasilinear parabolic equations:

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=B(x, t, u, \nabla u)+\mu \tag{1.1}
\end{equation*}
$$

in $\mathbb{R}^{N+1}$ or $\mathbb{R}^{N} \times(0, \infty)$ or a bounded domain $\Omega_{T}:=\Omega \times(0, T) \subset \mathbb{R}^{N+1}$. Where $N \geq 2$, $A: \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is a Carathéodory function which satisfies

$$
\begin{align*}
|A(x, t, \zeta)| & \leq \Lambda_{1}|\zeta| \text { and }  \tag{1.2}\\
\langle A(x, t, \zeta) & -A(x, t, \lambda), \zeta-\lambda\rangle \geq \Lambda_{2}|\zeta-\lambda|^{2} \tag{1.3}
\end{align*}
$$

for every $(\lambda, \zeta) \in \mathbb{R}^{N} \times \mathbb{R}^{N}$ and a.e. $(x, t) \in \mathbb{R}^{N} \times \mathbb{R}$, here $\Lambda_{1}$ and $\Lambda_{2}$ are positive constants, $B: \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ is also a Carathéodory function and $\mu$ is a Radon measure.

The existence and regularity theory, the Wiener criteria and Harnack inequalities, Blowup at a finite time associated with above parabolic quasilinear operator was studied and developed intensely over the past 50 years, one can found in $[58,44,30,48,49,25,50$, $60,83,75,73]$. Moreover, we also refer to [19]-[22] for $L^{p}$-gradient estimates theory in non-smooth domains and [63] Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

First, we are specially interested in the existence of solutions to quasilinear parabolic equations with absorption, source terms and data measure:

$$
\begin{align*}
& u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\mu,  \tag{1.4}\\
& u_{t}-\operatorname{div}(A(x, t, \nabla u))=|u|^{q-1} u+\mu, \tag{1.5}
\end{align*}
$$

in $\mathbb{R}^{N+1}$ and

$$
\begin{array}{ll}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\mu, & u(0)=\sigma, \\
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|u|^{q-1} u+\mu, & u(0)=\sigma, \tag{1.7}
\end{array}
$$

in $\mathbb{R}^{N} \times(0, \infty)$ or a bounded domain $\Omega_{T} \subset \mathbb{R}^{N+1}$, where $q>1$ and $\mu, \sigma$ are Radon measures.
The linear case $A(x, t, \nabla u)=\nabla u$ was studied in detail by Fujita, Brezis and Friedman, Baras and Pierre.

In [18], showed that if $\mu=0$ and $\sigma$ is a Dirac mass in $\Omega$, the problem (1.6) in $\Omega_{T}$ (with Dirichlet boundary condition) admits a (unique) solution if and only if $q<(N+2) / N$. Then, optimal results had been considered in [5], for any $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $\sigma \in \mathfrak{M}_{b}(\Omega)$ :
there exists a (unique) solution of (1.6) in $\Omega_{T}$ if and only if $\mu, \sigma$ are absolutely continuous with respect to the capacity $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{2 / q}, q^{\prime}}$ (in $\Omega_{T}, \Omega$ ) respectively, for simplicity we write $\mu \ll \operatorname{Cap}_{2,1, q^{\prime}}$ and $\sigma \ll \operatorname{Cap}_{\mathbf{G}_{2 / q}, q^{\prime}}$, with $q^{\prime}$ is the conjugate exponent of $q$, i.e $q^{\prime}=\frac{q}{q-1}$. Where these two capacities will be defined in section 2 .

For source case, in [6], showed that for any $\mu \in \mathfrak{M}_{b}^{+}\left(\Omega_{T}\right)$ and $\sigma \in \mathfrak{M}_{b}^{+}(\Omega)$, the problem (1.7) in bounded domain $\Omega_{T}$ has a nonnegative solution if

$$
\mu(E) \leq C \operatorname{Cap}_{2,1, q^{\prime}}(E) \text { and } \sigma(O) \leq C \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}(O)
$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$ here $C=C(N$, $\operatorname{diam}(\Omega), T)$ is small enough. Conversely, the existence holds then for compact subset $K \subset \subset \Omega$, one find $C_{K}>0$ such that

$$
\mu(E \cap(K \times[0, T])) \leq C_{K} \operatorname{Cap}_{2,1, q^{\prime}}(E) \text { and } \sigma(O \cap K) \leq C_{K} \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}(O)
$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$. In unbounded domain $\mathbb{R}^{N} \times(0, \infty)$, in [30] asserted that an inequality

$$
\begin{equation*}
u_{t}-\Delta u \geq u^{q}, u \geq 0 \text { in } \mathbb{R}^{N} \times(0, \infty) \tag{1.8}
\end{equation*}
$$

i. if $q<(N+2) / N$ then the only nonnegative global (in time) solution of above inequality is $u \equiv 0$,
ii. if $q>(N+2) / N$ then there exists global positive solution of above inequality.

More general, see [6], for $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$, (1.7) has a nonnegative solution in $\mathbb{R}^{N} \times(0, \infty)$ (with $\left.A(x, t, \nabla u)=\nabla u\right)$ if and only if

$$
\begin{equation*}
\mu(E) \leq C \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \quad \text { and } \quad \sigma(O) \leq C \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}(O) \tag{1.9}
\end{equation*}
$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$, here $C=C(N, q)$ is small enough, two capacities $\operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}, \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}$ will be defined in section 2 . Note that a necessary and sufficient condition for (1.9) holding with $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N} \times(0, \infty)\right) \backslash\{0\}$ or $\sigma \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ is $q \geq(N+2) / N$. In particular, (1.8) has a (global) positive solution if and only if $q \geq$ $(N+2) / N$. It is known that conditions for data $\mu, \sigma$ in problems with absorption are softer than source. Recently, in exponential case, i.e $|u|^{q-1} u$ is replaced by $P(u) \sim \exp \left(a|u|^{q}\right)$, for $a>0$ and $q \geq 1$ was established in [61].

We consider (1.6) and (1.7) in $\Omega_{T}$ with Dirichlet boundary conditions when $\operatorname{div}(A(x, t, \nabla u))$ is replaced by $\Delta_{p} u:=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ for $p \in(2-1 / N, N)$. In [66], showed that for any $q>p-1$, (1.6) admits a (unique renormalized) solution provided $\sigma \in L^{1}(\Omega)$ and $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ is diffuse measure i.e absolutely continuous with respect to $C_{p}$-parabolic capacity in $\Omega_{T}$ defined on a compact set $K \subset \Omega_{T}$ :

$$
C_{p}\left(K, \Omega_{T}\right)=\inf \left\{\|\varphi\|_{X}: \varphi \geq \chi_{K}, \varphi \in C_{c}^{\infty}\left(\Omega_{T}\right)\right\}
$$

where $X=\left\{\varphi: \varphi \in L^{p}\left(0, T ; W_{0}^{1, p}(\Omega)\right), \varphi_{t} \in L^{p^{\prime}}\left(0, T ; W^{-1, p^{\prime}}(\Omega)\right)\right\}$ endowed with norm $\|\varphi\|_{X}=\|\varphi\|_{L^{p}\left(0, T ; W_{0}^{1, p}(\Omega)\right)}+\left\|\varphi_{t}\right\|_{L^{p^{\prime}}\left(0, T ; W^{-1, p^{\prime}}(\Omega)\right)}$ and $\chi_{K}$ is the characteristic function of $K$. An improving result was presented in [14] for measures that have good behavior in time, it is based on results of [16] relative to the elliptic case. That is, (1.6) has a (renormalized) solution for $q>p-1$ if $\sigma \in L^{1}(\Omega)$ and $|\mu| \leq f+\omega \otimes F$, where $f \in L_{+}^{1}\left(\Omega_{T}\right), F \in L_{+}^{1}((0, T))$ and $\omega \in \mathfrak{M}_{b}^{+}(\Omega)$ is absolutely continuous with respect to $\operatorname{Cap}_{\mathbf{G}_{p}, \frac{q}{q-p+1}}$ in $\Omega$. Also, (1.7) has a (renormalized) nonnegative solution if $\sigma \in L_{+}^{\infty}(\Omega), 0 \leq \mu \leq \omega \otimes \chi_{(0, T)}$ with $\omega \in \mathfrak{M}_{b}^{+}(\Omega)$ and

$$
\omega(E) \leq C_{1} \operatorname{Cap}_{\mathbf{G}_{p}, \frac{q}{q-p+1}}(E) \forall \text { compact } E \subset \mathbb{R}^{N}, \quad\|\sigma\|_{L^{\infty}(\Omega)} \leq C_{2}
$$

for some $C_{1}, C_{2}$ small enough. Another improving results are also stated in [15], especially if $q>p-1, p>2, \mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $\sigma \in \mathfrak{M}_{b}(\Omega)$ are absolutely continuous with respect to $\operatorname{Cap}_{2,1, q^{\prime}}$ in $\Omega_{T}$ and $\operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}$ in $\Omega$ then (1.6) has a distribution solution.

In [15], we also obtain the existence of solutions for porous medium equation with absorption and data measure: for $q>m>\frac{N-2}{N}$, a sufficient condition for existence solution to the problem

$$
u_{t}-\Delta\left(|u|^{m-1} u\right)+|u|^{q-1} u=\mu \text { in } \Omega_{T}, \quad u=0 \text { on } \partial \Omega \times(0, T), \quad \text { and } \quad u(0)=\sigma \text { in } \Omega,
$$

is $\mu \ll \operatorname{Cap}_{2,1, q^{\prime}}, \sigma \ll \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}$ if $m \geq 1$ and $\mu \ll \operatorname{Cap}_{\mathcal{G}_{2}, \frac{2 q}{2(q-1)+N(1-m)}}, \sigma \ll$ $\operatorname{Cap}_{\mathbf{G}_{\underline{2-N(1-m)}}^{q}}, \frac{2 q}{2(q-1)+N(1-m)}$ if $\frac{N-2}{N}<m \leq 1$. A necessary condition is $\mu \ll \operatorname{Cap}_{2,1, \frac{q}{q-\max \{m, 1\}}}$ and $\sigma \ll \operatorname{Cap}_{\mathbf{G}_{\frac{2 \max \{m, 1\}}{},}^{q-\max \{m, 1\}}}$. Moreover, if $\mu=\mu_{1} \otimes \chi_{[0, T]}$ with $\mu_{1} \in \mathfrak{M}_{b}(\Omega)$ and $\sigma \equiv 0$ then a condition $\mu_{1} \ll \operatorname{Cap}_{\mathbf{G}_{2}, \frac{q}{q-m}}$ is not only a sufficient but also a necessary for existence of solutions to above problem.

We would like to make a brief survey of quasilinear elliptic equations with absorption, source terms and data measure:

$$
\begin{align*}
& -\Delta_{p} u+|u|^{q-1} u=\omega  \tag{1.10}\\
& -\Delta_{p} u=|u|^{q-1} u+\omega, u \geq 0 \tag{1.11}
\end{align*}
$$

in $\Omega$ with Dirichlet boundary conditions where $1<p<N, q>p-1$. In [16], we proved that the existence solution of equation (1.10) holds if $\omega \in \mathfrak{M}_{b}(\Omega)$ is absolutely continuous with respect to $\operatorname{Cap}_{\mathbf{G}_{p}, \frac{q}{q-p+1}}$. Moreover, a necessary condition for existence was also showed in $[10,11]$. For problem with source term, it was solved in [68] (also see [69]). Exactly, if $\omega \in \mathfrak{M}_{b}^{+}(\Omega)$ has compact support in $\Omega$, then a sufficient and necessary condition for the existence of solutions of problem (1.11) is

$$
\omega(E) \leq C \operatorname{Cap}_{\mathbf{G}_{p}, \frac{q}{q-p+1}}(E) \quad \text { for all compact set } E \subset \Omega
$$

where $C$ is a constant only depending on $N, p, q$ and $d(\operatorname{supp}(\omega), \partial \Omega)$. Their construction is based upon sharp estimates of solutions of the problem

$$
-\Delta_{p} u=\omega \quad \text { in } \Omega, \quad u=0 \quad \text { on } \partial \Omega
$$

for nonnegative Radon measures $\omega$ in $\Omega$ and a deep analysis of the Wolff potential.
Corresponding results in case that $u^{q}$ term is changed by $P(u) \approx \exp \left(a u^{\lambda}\right)$ for $a>0, \lambda>0$, was given in [16, 62].

In [27], Duzaar and Mingione gave a local pointwise estimate from above of solutions to equation

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\mu \tag{1.12}
\end{equation*}
$$

in $\Omega_{T}$ involving the Wolff parabolic potential $\mathbb{I}_{2}[|\mu|]$ defined by

$$
\mathbb{I}_{2}[|\mu|](x, t)=\int_{0}^{\infty} \frac{|\mu|\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N}} \frac{d \rho}{\rho} \quad \text { for all } \quad(x, t) \in \mathbb{R}^{N+1}
$$

here $\tilde{Q}_{\rho}(x, t):=B_{\rho}(x) \times\left(t-\rho^{2} / 2, t+\rho^{2} / 2\right)$. Specifically if $u \in L^{2}\left(0, T ; H^{1}(\Omega)\right) \cap C\left(\Omega_{T}\right)$ is a weak solution to above equation with data $\mu \in L^{2}\left(\Omega_{T}\right)$, then

$$
\begin{equation*}
|u(x, t)| \leq C f_{\tilde{Q}_{R}(x, t)}|u| d y d s+C \int_{0}^{2 R} \frac{|\mu|\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N}} \frac{d \rho}{\rho} \tag{1.13}
\end{equation*}
$$

for any $Q_{2 R}(x, t):=B_{2 R}(x) \times\left(t-(2 R)^{2}, t\right) \subset \Omega_{T}$, where a constant $C$ only depends on $N$ and the structure of operator $A$. Moreover, in this paper we show that if $u \geq 0, \mu \geq 0$ we also have local pointwise estimate from below:

$$
\begin{equation*}
u(y, s) \geq C^{-1} \sum_{k=0}^{\infty} \frac{\mu\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}} \tag{1.14}
\end{equation*}
$$

for any $Q_{r}(y, s) \subset \Omega_{T}$, see section 5 , where $r_{k}=4^{-k} r$.
From preceding two inequalities, we obtain global pointwise estimates of solution to (1.12). For example, if $\mu \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ with $\mathbb{I}_{2}[|\mu|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$ then there exists a distribution solution to (1.12) in $\mathbb{R}^{N+1}$ such that

$$
\begin{equation*}
-K \mathbb{I}_{2}\left[\mu^{-}\right](x, t) \leq u(x, t) \leq K \mathbb{I}_{2}\left[\mu^{+}\right](x, t) \quad \text { for a.e }(x, t) \in \mathbb{R}^{N+1} \tag{1.15}
\end{equation*}
$$

and we emphasize that if $u \geq 0, \mu \geq 0$ then

$$
u(x, t) \geq K^{-1} \sum_{k=-\infty}^{\infty} \frac{\mu\left(Q_{2^{-2 k-3}}\left(x, t-35 \times 2^{-4 k-7}\right)\right)}{2^{-2 N k}} \quad \text { for a.e }(x, t) \in \mathbb{R}^{N+1}
$$

and for $q>1$,

$$
\|u\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \approx\left\|\mathbb{I}_{2}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}
$$

Where a constant $K$ only depends on $N$ and the structure of operator $A$.
Our first aim is to verify that
i. problems (1.4) and (1.6) have solutions if $\mu, \sigma$ are absolutely continuous with respect to the capacity $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}$ respectively,
ii. problems (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^{N} \times(0, \infty)$ with data signed measure $\mu, \sigma$ admit a solution if

$$
\begin{equation*}
|\mu|(E) \leq C \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \text { and }|\sigma|(O) \leq C \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}, q^{\prime}}}(O) \tag{1.16}
\end{equation*}
$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$. Also, the equation (1.7) in a bounded domain $\Omega_{T}$ has a solution if (1.16) holds where capacities $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}$ are exploited instead of $\operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}, \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}$.
It is worth mention that solutions obtained of (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^{N} \times(0, \infty)$ obey

$$
\int_{E}|u|^{q} d x d t \leq C \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \quad \text { for all compact } E \subset \mathbb{R}^{N+1}
$$

and we also have an analogous estimate for a solution of (1.7) in $\Omega_{T}$;

$$
\int_{E}|u|^{q} d x d t \leq C \operatorname{Cap}_{2,1, q^{\prime}}(E) \quad \text { for all compact } E \subset \mathbb{R}^{N+1}
$$

for some a constant $C>0$.
In case $\mu \equiv 0$, solutions (1.7) in $\mathbb{R}^{N} \times(0, \infty)$ and $\Omega_{T}$ are accepted the decay estimate

$$
-C t^{-\frac{1}{q-1}} \leq \inf _{x} u(x, t) \leq \sup _{x} u(x, t) \leq C t^{-\frac{1}{q-1}} \text { for any } t>0
$$

The strategy for establishment above results that is, we rely upon the combination some techniques of quasilinear elliptic equations in two articles $[16,68]$ with the global pointwise
estimate (1.15), delicate estimates on Wolff parabolic potential and the stability theorem see [13], Proposition 3.17 of this paper. They will be demonstrated in section 6 .

We next are interested in global regularity of solutions to quasilinear parabolic equations

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\mu \text { in } \Omega_{T}, u=0 \text { on } \partial \Omega \times(0, T) \text { and } u(0)=\sigma \text { in } \Omega, \tag{1.17}
\end{equation*}
$$

where domain $\Omega_{T}$ and nonlinearity $A$ are as mentioned at the beginning.
Our aim is to achieve minimal conditions on the boundary of $\Omega$ and on nonlinearity $A$ so that the following statement holds

$$
\left\|\left|\nabla u\left\|_{\mathcal{K}} \leq C| | \mathbb{M}_{1}[\omega]\right\|_{\mathcal{K}} .\right.\right.
$$

Here $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$ and $\mathbb{M}_{1}$ is the first order fractional Maximal parabolic potential defined by

$$
\mathbb{M}_{1}[\omega](x, t)=\sup _{\rho>0} \frac{\omega\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+1}} \forall(x, t) \in \mathbb{R}^{N+1}
$$

a constant $C$ does not depend on $u$ and $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$ and $\mathcal{K}$ is a function space. The same question is as above for the elliptic framework studied by N. C. Phuc in [70, 71, 72].

First, we take $\mathcal{K}=L^{p, s}\left(\Omega_{T}\right)$ for $1 \leq p \leq \theta$ and $0<s \leq \infty$ under a capacity density condition on the domain $\Omega$ where $L^{p, s}\left(\Omega_{T}\right)$ is the Lorentz space and a constant $\theta>2$ depends on the structure of this condition and of nonlinearity $A$. It follows the recent result in [7], see remark 2.18. The capacity density condition is that, the complement of $\Omega$ satisfies uniformly 2 -thick, see section 2 . We remark that under this condition, the Sobolev embedding $H_{0}^{1}(\Omega) \subset L^{\frac{2 N}{N-2}}(\Omega)$ for $N>2$ is valid and it is fulfilled by any domain with Lipschitz boundary, or even of corkscrew type. This condition was used in two papers [70, 72]. Also, it is essentially sharp for higher integrability results, presented in [41, Remark 3.3]. Furthermore, we also assert that if $\frac{\gamma}{\gamma-1}<p<\theta, 2 \leq \gamma<N+2,0<s \leq \infty$ and $\sigma \equiv 0$ then

$$
\|\mid \nabla u\|\left\|_{L_{*}^{p, s ;(\gamma-1) p}\left(\Omega_{T}\right)} \leq C\right\| \mu \|_{L_{*}} \frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\Omega_{T}\right)
$$

for some a constant $C$ where $L_{*}^{p, s ;(\gamma-1) p}\left(\Omega_{T}\right), L_{*}^{\frac{(\gamma-1) p}{\gamma}}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\Omega_{T}\right)$ are the LorentzMorrey spaces involving "calorie" introduced in section 2. We would like to refer to [55] as the first paper where Lorentz-Morrey estimates for solutions of quasilinear elliptic equations via fractional operators have been obtained.

Next, in order to obtain shaper results, we take $\mathcal{K}=L^{q, s}\left(\Omega_{T}, d w\right)$, the weighted Lorentz spaces with weight in the Muckenhoupht class $A_{\infty}$ for $q \geq 1,0<s \leq \infty$, we require some stricter conditions on the domain $\Omega$ and nonlinearity $A$. A condition on $\Omega$ is flat enough in the sense of Reifenberg, essentially, that at boundary point and every scale the boundary of domain is between two hyperplanes at both sides (inside and outside) of domain by a distance which depends on the scale. Conditions on $A$ are that BMO type of $A$ with respect to the $x$-variable is small enough and the derivative of $A(x, t, \zeta)$ with respect to $\zeta$ is uniformly bounded. By choosing an appropriate weight we can establish the following important estimates:
a. The Lorentz-Morrey estimates involving "calorie" for $0<\kappa \leq N+2$ is obtained

$$
\||\nabla u|\|_{L_{*}^{q, s ; \kappa}\left(\Omega_{T}\right)} \leq C| | \mathbb{M}_{1}[|\omega|] \mid \|_{L_{*}^{q, s ; \kappa}\left(\Omega_{T}\right)}
$$

b. Another Lorentz-Morrey estimates is also obtained for $0<\vartheta \leq N$

$$
\left\|\left.\mathbb{M}(|\nabla u|)\right|_{L_{* *}^{q, s ; \theta}\left(\Omega_{T}\right)} \leq C| | \mathbb{M}_{1}[|\omega|]\right\|_{L_{* *}^{q, s ; \theta}\left(\Omega_{T}\right)}
$$

where $L_{* *}^{q, s ; \vartheta}\left(\Omega_{T}\right)$ is introduced in section 2. This estimate implies global Holder-estimate in space variable and $L^{q}$-estimate in time, that is for all ball $B_{\rho} \subset \mathbb{R}^{N}$

$$
\left(\int_{0}^{T}\left|\operatorname{osc}_{B_{\rho} \cap \bar{\Omega}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq C \rho^{1-\frac{\vartheta}{q}} \|\left.\left|\mathbb{M}_{1}[|\omega|]\right|\right|_{L_{* *}^{q ; \vartheta}\left(\Omega_{T}\right)} \text { provided } 0<\vartheta<\min \{q, N\}
$$

In particular, there hold

$$
\left(\int_{0}^{T}\left|\operatorname{osc}_{B_{\rho} \cap \bar{\Omega}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq C \rho^{1-\frac{\vartheta}{q}}\|\sigma\|_{L^{\frac{\vartheta q}{\vartheta+2-q} ; \vartheta}(\Omega)}+C \rho^{1-\frac{\vartheta}{q}}\|\mu\|_{L^{\left(\vartheta+2+q q_{1} q_{1}-2 q\right.} ; \vartheta}{ }_{\left(\Omega, L^{q_{1}}((0, T))\right)}
$$

provided

$$
\begin{aligned}
& 1<q_{1} \leq q<2 \\
& \max \left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2 q}{q_{1}}\right)\right\}<\vartheta \leq N
\end{aligned}
$$

Where $L^{\frac{\vartheta q}{\vartheta+2-q} ; \vartheta}(\Omega)$ is the standard Morrey space and

$$
\|\mu\|_{L^{q_{2} ; \vartheta}\left(\Omega, L^{q_{1}}((0, T))\right)}=\sup _{\rho>0, x \in \Omega} \rho^{\frac{\vartheta-N}{q_{2}}}\left(\int_{B_{\rho}(x) \cap \Omega}\left(\int_{0}^{T}|\mu(y, t)|^{q_{1}} d t\right)^{\frac{q_{2}}{q_{1}}} d y\right)^{\frac{1}{q_{2}}}
$$

with $q_{2}=\frac{\vartheta q q_{1}}{(\vartheta+2+q) q_{1}-2 q}$. Besides, we also find

$$
\left(\int_{0}^{T}\left|\operatorname{osc}_{B_{\rho} \cap \bar{\Omega}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq C \rho^{1-\frac{\vartheta}{q}}\|\mu\|_{L^{\frac{\vartheta q+2+q q_{1}}{\left(\vartheta q_{1}-2 q\right.}} ; \vartheta}\left(\Omega, L^{q_{1}}((0, T))\right)
$$

provided

$$
\begin{aligned}
& \sigma \equiv 0, \quad q \geq 2,1<q_{1} \leq q \\
& \frac{1}{q-1}\left(2+q-\frac{2 q}{q_{1}}\right)<\vartheta \leq N
\end{aligned}
$$

c. A global capacitary estimate is also given

$$
\sup _{\substack{\operatorname{compact} \\ \operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)>0}}\left(\frac{\int_{K}|\nabla u|^{q} d x d t}{\operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)}\right) \leq C \underset{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ \operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)>0}}{ }\left(\frac{|\omega|(K)}{\operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)}\right)^{q}
$$

To obtain this estimate we employ profound techniques in nonlinear potential theory, see section 4 and Theorem 2.22.

We utilize some ideas (in the quasilinear elliptic framework) in articles of N.C. Phuc [70, 72, 71] during we establish above estimates.

We would like to emphasize that above estimates is also true for solutions to equation (1.17) in $\mathbb{R}^{N+1}$ with data $\mu$ (of course still true for (1.17) in $\mathbb{R}^{N} \times(0, \infty)$ ) with data $\mu$ provided $\mathbb{I}_{2}[|\mu|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$ see Theorem 2.25 and 2.27. Moreover, a global pointwise estimates of gradient of solutions is obtained when $A$ is independent of space variable x , that is

$$
|\nabla u(x, t)| \leq C \mathbb{I}_{1}[|\mu|](x, t) \quad \text { a.e }(x, t) \in \mathbb{R}^{N+1}
$$

see Theorem 2.5.

Our final aim is to obtain existence results for the quasilinear Riccati type parabolic problems (1.1) where $B(x, t, u, \nabla u)=|\nabla u|^{q}$ for $q>1$. The strategy we use in order to prove these existence results is that using Schauder Fixed Point Theorem and all above estimates and the stability Theorem see [13], Proposition 3.17 in section 3. They will be carried out in section 9. By our methods in the paper, we can treat general equations (1.1), where

$$
|B(x, t, u, \nabla u)| \leq C_{1}|u|^{q_{1}}+C_{2}|\nabla u|^{q_{2}}, q_{1}, q_{2}>1
$$

with constant coefficients $C_{1}, C_{2}>0$.

## Acknowledgements:

The author wishes to express his deep gratitude to his advisors Professor Laurent Véron and Professor Marie-Françoise Bidaut-Véron for encouraging, taking care and giving many useful comments during the preparation of the paper. Besides the author would like to thank Nguyen Phuoc Tai for many interesting comments.

## 2 Main Results

Throughout the paper, we assume that $\Omega$ is a bounded open subset of $\mathbb{R}^{N}, N \geq 2$ and $T>0$. Besides, we always denote $\Omega_{T}=\Omega \times(0, T), T_{0}=\operatorname{diam}(\Omega)+T^{1 / 2}$ and $Q_{\rho}(x, t)=$ $B_{\rho}(x) \times\left(t-\rho^{2}, t\right) \tilde{Q}_{\rho}(x, t)=B_{\rho}(x) \times\left(t-\rho^{2} / 2, t+\rho^{2} / 2\right)$ for $(x, t) \in \mathbb{R}^{N+1}$ and $\rho>0$. We always assume that $A: \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is a Caratheodory vector valued function, i.e. $A$ is measurable in $(x, t)$ and continuous with respect to $\nabla u$ for each fixed $(x, t)$ and satisfies (1.2) and (1.3). This article is divided into three parts. First part, we study the existence problems for the quasilinear parabolic equations with absorption and source terms

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\mu \text { in } \Omega_{T},  \tag{2.1}\\
u=0 \quad \text { on } \partial \Omega \times(0, T), \\
u(0)=\sigma \quad \text { in } \Omega
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|u|^{q-1} u+\mu \text { in } \Omega_{T}  \tag{2.2}\\
u=0 \quad \text { on } \partial \Omega \times(0, T) \\
u(0)=\sigma \quad \text { in } \Omega
\end{array}\right.
$$

where $q>1$, and $\mu, \sigma$ are Radon measures.
In order to state our results, let us introduce some definitions and notations. If $D$ is either a bounded domain or whole $\mathbb{R}^{l}$ for $l \in \mathbb{N}$, we denote by $\mathfrak{M}(D)$ (resp. $\mathfrak{M}_{b}(D)$ ) the set of Radon measure (resp. bounded Radon measures) in $D$. Their positive cones are $\mathfrak{M}^{+}(D)$ and $\mathfrak{M}_{b}^{+}(D)$ respectively. For $R \in(0, \infty]$, we define the $R$-truncated Riesz parabolic potential $\mathbb{I}_{\alpha}$ and Fractional Maximal parabolic potential $\mathbb{M}_{\alpha}, \alpha \in(0, N+2)$, on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ by

$$
\begin{equation*}
\mathbb{I}_{\alpha}^{R}[\mu](x, t)=\int_{0}^{R} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha}} \frac{d \rho}{\rho} \text { and } \mathbb{M}_{\alpha}^{R}[\mu](x, t)=\sup _{0<\rho<R} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha}} \tag{2.3}
\end{equation*}
$$

for all $(x, t)$ in $\mathbb{R}^{N+1}$. If $R=\infty$, we drop it in expressions of (2.3).
We denote by $\mathcal{H}_{\alpha}$ the Heat kernel of order $\alpha \in(0, N+2)$ :

$$
\mathcal{H}_{\alpha}(x, t)=C_{\alpha} \frac{\chi_{(0, \infty)}(t)}{t^{(N+2-\alpha) / 2}} \exp \left(-\frac{|x|^{2}}{4 t}\right) \text { for }(x, t) \text { in } \mathbb{R}^{N+1}
$$

and $\mathcal{G}_{\alpha}$ the parabolic Bessel kernel of order $\alpha>0$ :

$$
\mathcal{G}_{\alpha}(x, t)=C_{\alpha} \frac{\chi_{(0, \infty)}(t)}{t^{(N+2-\alpha) / 2}} \exp \left(-t-\frac{|x|^{2}}{4 t}\right) \text { for }(x, t) \text { in } \mathbb{R}^{N+1}
$$

see [4], where $C_{\alpha}=\left((4 \pi)^{N / 2} \Gamma(\alpha / 2)\right)^{-1}$. It is known that $\mathcal{F}\left(\mathcal{H}_{\alpha}\right)(x, t)=\left(|x|^{2}+i t\right)^{-\alpha / 2}$ and $\mathcal{F}\left(\mathcal{G}_{\alpha}\right)(x, t)=\left(1+|x|^{2}+i t\right)^{-\alpha / 2}$. We define the parabolic Riesz potential $\mathcal{H}_{\alpha}$ of a measure $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ by

$$
\mathcal{H}_{\alpha}[\mu](x, t)=\left(\mathcal{H}_{\alpha} * \mu\right)(x, t)=\int_{\mathbb{R}^{N+1}} \mathcal{H}_{\alpha}(x-y, t-s) d \mu(y, s) \text { for any } \quad(x, t) \text { in } \mathbb{R}^{N+1}
$$

the parabolic Bessel potential $\mathcal{G}_{\alpha}$ of a measure $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ by

$$
\mathcal{G}_{\alpha}[\mu](x, t)=\left(\mathcal{G}_{\alpha} * \mu\right)(x, t)=\int_{\mathbb{R}^{N+1}} \mathcal{G}_{\alpha}(x-y, t-s) d \mu(y, s) \text { for any }(x, t) \text { in } \mathbb{R}^{N+1}
$$

We also define $\mathbf{I}_{\alpha}, \mathbf{G}_{\alpha}, 0<\alpha<N$ the Riesz, Bessel potential of a measure $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$ by

$$
\mathbf{I}_{\alpha}[\mu](x)=\int_{0}^{\infty} \frac{\mu\left(B_{\rho}(x)\right)}{\rho^{N-\alpha}} \frac{d \rho}{\rho} \text { and } \mathbf{G}_{\alpha}[\mu](x)=\int_{\mathbb{R}^{N}} \mathbf{G}_{\alpha}(x-y) d \mu(y) \text { for any } x \text { in } \mathbb{R}^{N}
$$

where $\mathbf{G}_{\alpha}$ is the Bessel kernel of order $\alpha$, see [2].
Several different capacities will be used over the paper. For $1<p<\infty$, the ( $\left.\mathcal{H}_{\alpha}, p\right)$-capacity, $\left(\mathcal{G}_{\alpha}, p\right)$-capacity of a Borel set $E \subset \mathbb{R}^{N+1}$ are defined by

$$
\begin{aligned}
& \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)=\inf \left\{\int_{\mathbb{R}^{N+1}}|f|^{p} d x d t: f \in L_{+}^{p}\left(\mathbb{R}^{N+1}\right), \mathcal{H}_{\alpha} * f \geq \chi_{E}\right\} \text { and } \\
& \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)=\inf \left\{\int_{\mathbb{R}^{N+1}}|f|^{p} d x d t: f \in L_{+}^{p}\left(\mathbb{R}^{N+1}\right), \mathcal{G}_{\alpha} * f \geq \chi_{E}\right\}
\end{aligned}
$$

The $W_{p}^{2,1}$-capacity of compact set $E \subset \mathbb{R}^{N+1}$ is defined by

$$
\operatorname{Cap}_{2,1, p}(E)=\inf \left\{\|\varphi\|_{W_{p}^{2,1}\left(\mathbb{R}^{N+1}\right)}^{p}: \varphi \in S\left(\mathbb{R}^{N+1}\right), \varphi \geq 1 \text { in a neighborhood of } E\right\}
$$

where
$\|\varphi\|_{W_{p}^{2,1}\left(\mathbb{R}^{N+1}\right)}=\|\varphi\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}+\left\|\frac{\partial \varphi}{\partial t}\right\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}+\|\nabla \varphi\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}+\sum_{i, j=1,2, \ldots, N}\left\|\frac{\partial^{2} \varphi}{\partial x_{i} \partial x_{j}}\right\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}$.
We remark that thanks to Richard J. Bagby's result (see [4]) we obtain the equivalent of capacities $\mathrm{Cap}_{2,1, p}$ and $\mathrm{Cap}_{\mathcal{G}_{2}, p}$, i.e, for any compact set $K \subset \mathbb{R}^{N+1}$ there holds

$$
C^{-1} \operatorname{Cap}_{2,1, p}(K) \leq \operatorname{Cap}_{\mathcal{G}_{2}, p}(K) \leq C \operatorname{Cap}_{2,1, p}(K)
$$

for some $C=C(N, p)$, see Corollary (4.18) in section 4.
The $\left(\mathbf{I}_{\alpha}, p\right)$-capacity, $\left(\mathbf{G}_{\alpha}, p\right)$-capacity of a Borel set $O \subset \mathbb{R}^{N}$ are defined by

$$
\begin{aligned}
& \operatorname{Cap}_{\mathbf{I}_{\alpha}, p}(O)=\inf \left\{\int_{\mathbb{R}^{N}}|g|^{p} d x: g \in L_{+}^{p}\left(\mathbb{R}^{N}\right), \mathbf{I}_{\alpha} * g \geq \chi_{O}\right\} \text { and } \\
& \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(O)=\inf \left\{\int_{\mathbb{R}^{N}}|g|^{p} d x: g \in L_{+}^{p}\left(\mathbb{R}^{N}\right), \mathbf{G}_{\alpha} * g \geq \chi_{O}\right\} .
\end{aligned}
$$

In our first three Theorems, we present global pointwise potential estimates for solutions to quasilinear parabolic problems

$$
\begin{gather*}
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\mu \text { in } \Omega_{T}, \\
u=0 \text { on } \partial \Omega \times(0, T), \\
u(0)=\sigma \text { in } \Omega
\end{array}\right.  \tag{2.4}\\
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\mu \text { in } \mathbb{R}^{N} \times(0, \infty), \\
u(0)=\sigma \text { in } \mathbb{R}^{N},
\end{array}\right. \tag{2.5}
\end{gather*}
$$

and

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\mu \text { in } \mathbb{R}^{N+1} \tag{2.6}
\end{equation*}
$$

Theorem 2.1 There exists a constant $K$ depending on $N, \Lambda_{1}, \Lambda_{2}$ such that for any $\mu \in$ $\mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$ there is a distribution solution $u$ of (2.4) which satisfies

$$
\begin{equation*}
-K \mathbb{I}_{2}^{2 T_{0}}\left[\mu^{-}+\sigma^{-} \otimes \delta_{\{t=0\}}\right] \leq u \leq K \mathbb{I}_{2}^{2 T_{0}}\left[\mu^{+}+\sigma^{+} \otimes \delta_{\{t=0\}}\right] \text { in } \Omega_{T} \tag{2.7}
\end{equation*}
$$

Remark 2.2 Since $\sup _{x \in \mathbb{R}^{N}} \mathbb{I}_{\alpha}\left[\sigma^{ \pm} \otimes \delta_{\{t=0\}}\right](x, t) \leq \frac{\sigma^{ \pm}(\Omega)}{(N+2-\alpha)(2|t|)^{\frac{N+2-\alpha}{2}}}$ for any $t \neq 0$ with $0<\alpha<N+2$. Thus, if $\mu \equiv 0$, then we obtain the decay estimate:

$$
-\frac{K \sigma^{-}(\Omega)}{N(2 t)^{\frac{N}{2}}} \leq \inf _{x \in \Omega} u(x, t) \leq \sup _{x \in \Omega} u(x, t) \leq \frac{K \sigma^{+}(\Omega)}{N(2 t)^{\frac{N}{2}}} \text { for any } 0<t<T
$$

Theorem 2.3 There exists a constant $C$ depending on $N, \Lambda_{1}, \Lambda_{2}$ such that for any $\mu \in$ $\mathfrak{M}_{b}^{+}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}^{+}(\Omega)$, there is a distribution solution $u$ of (2.4) satisfying for a.e $(y, s) \in \Omega_{T}$ and $B_{r}(y) \subset \Omega$

$$
\begin{equation*}
u(y, s) \geq C \sum_{k=0}^{\infty} \frac{\mu\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}+C \sum_{k=0}^{\infty} \frac{\left(\sigma \otimes \delta_{\{t=0\}}\right)\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}} \tag{2.8}
\end{equation*}
$$

where $r_{k}=4^{-k} r$.
Remark 2.4 The Theorem 2.3 is also true when we replace the assumption (1.3) by a weaker one

$$
\langle A(x, t, \zeta), \zeta\rangle \geq \Lambda_{2}|\zeta|^{2}, \quad\langle A(x, t, \zeta)-A(x, t, \lambda), \zeta-\lambda\rangle>0,
$$

for every $(\lambda, \zeta) \in \mathbb{R}^{N} \times \mathbb{R}^{N}, \lambda \neq \zeta$ and a.e. $(x, t) \in \mathbb{R}^{N} \times \mathbb{R}$.
Theorem 2.5 Let $K$ be the constant in Theorem 2.1. Let $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ such that $I_{2}[|\omega|]\left(x_{0}, t_{0}\right)<$ $\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$. Then, there is a distribution solution $u$ to (2.6) with data $\mu=\omega$ satisfying

$$
\begin{equation*}
-K \mathbb{I}_{2}\left[\omega^{-}\right] \leq u \leq K \mathbb{I}_{2}\left[\omega^{+}\right] \text {in } \mathbb{R}^{N+1} \tag{2.9}
\end{equation*}
$$

such that the following statements hold.
a. If $\omega \geq 0$, there exists $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}\right)$ such that for a.e $(x, t) \in \mathbb{R}^{N+1}$

$$
\begin{equation*}
u(x, t) \geq C_{1} \sum_{k=-\infty}^{\infty} \frac{\omega\left(Q_{2-2 k-3}\left(x, t-35 \times 2^{-4 k-7}\right)\right)}{2^{-2 N k}} \tag{2.10}
\end{equation*}
$$

In particular, for any $q>\frac{N+2}{N}$

$$
\begin{equation*}
C_{2}^{-1}\left\|\mathcal{H}_{2}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\|u\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mathcal{H}_{2}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \tag{2.11}
\end{equation*}
$$

with $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}\right)$.
b. If $A$ is independent of space variable $x$ and satisfies (2.27), then there exists $C_{2}=$ $C_{2}\left(N, \Lambda_{1}, \Lambda_{2}\right)$ such that

$$
\begin{equation*}
|\nabla u| \leq C_{2} \mathbb{I}_{1}[|\omega|] \quad \text { in } \mathbb{R}^{N+1} \tag{2.12}
\end{equation*}
$$

c. If $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.5).

Remark 2.6 For $q>\frac{N+2}{N}$, we alway have the following claim:

$$
\left\|\mathcal{H}_{2}\left[\mu+\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \approx\left\|\mathcal{H}_{2}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}+\left\|\mathbf{I}_{2 / q}[\sigma]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}
$$

for every $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$.

Remark 2.7 For $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right), 0<\alpha<N+2$ if $\mathbb{I}_{\alpha}[\omega]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in$ $\mathbb{R}^{N+1}$ then for any $0<\beta \leq \alpha, \mathbb{I}_{\beta}[\omega] \in L_{\text {loc }}^{s}\left(\mathbb{R}^{N+1}\right)$ for any $0<s<\frac{N+2}{N+2-\beta}$. However, for $0<\beta<\alpha<N+2$, one can find $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ such that $\mathbb{I}_{\alpha}[\omega] \equiv \infty$ and $\mathbb{I}_{\beta}[\omega]<\infty$ in $\mathbb{R}^{N+1}$, see Appendix section.
The next four theorems provide the existence of solutions to quasilinear parabolic equations with absorption and source terms. For convenience, we always denote by $q^{\prime}$ the conjugate exponent of $q \in(1, \infty)$ i.e $q^{\prime}=\frac{q}{q-1}$.
Theorem 2.8 Let $q>1, \mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $\sigma \in \mathfrak{M}_{b}(\Omega)$. Suppose that $\mu, \sigma$ are absolutely continuous with respect to the capacities $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}$ in $\Omega_{T}, \Omega$ respectively. Then there exists a distribution solution $u$ of (2.1) satisfying

$$
-K \mathbb{I}_{2}\left[\mu^{-}+\sigma^{-} \otimes \delta_{\{t=0\}}\right] \leq u \leq K \mathbb{I}_{2}\left[\mu^{+}+\sigma^{+} \otimes \delta_{\{t=0\}}\right] \quad \text { in } \Omega_{T}
$$

Here the constant $K$ is in Theorem 2.1.
Theorem 2.9 Let $K$ be the constant in Theorem 2.1. Let $q>1, \mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $\sigma \in$ $\mathfrak{M}_{b}(\Omega)$. There exists a constant $C_{1}=C_{1}\left(N, q, \Lambda_{1}, \Lambda_{2}, \operatorname{diam}(\Omega), T\right)$ such that if

$$
\begin{equation*}
|\mu|(E) \leq C_{1} \operatorname{Cap}_{2,1, q^{\prime}}(E) \quad \text { and } \quad|\sigma|(O) \leq C_{1} \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}(O) \tag{2.13}
\end{equation*}
$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$, then the problem (2.2) has a distribution solution u satisfying

$$
\begin{equation*}
-\frac{K q}{q-1} \mathbb{I}_{2}\left[\mu^{-}+\sigma^{-} \otimes \delta_{\{t=0\}}\right] \leq u \leq \frac{K q}{q-1} \mathbb{I}_{2}\left[\mu^{+}+\sigma^{+} \otimes \delta_{\{t=0\}}\right] \quad \text { in } \Omega_{T} \tag{2.14}
\end{equation*}
$$

Besides, for every compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$
\begin{equation*}
\int_{E}|u|^{q} d x d t \leq C_{2} C a p_{2,1, q^{\prime}}(E) \tag{2.15}
\end{equation*}
$$

where $C_{2}=C_{2}\left(N, q, \Lambda_{1}, \Lambda_{2}, T_{0}\right)$.
Remark 2.10 From (2.15) we get if $q>\frac{N+2}{N}$,

$$
\int_{\tilde{Q}_{\rho}(y, s)}|u|^{q} d x d t \leq C \rho^{N+2-2 q^{\prime}} \quad \text { for any } \quad \tilde{Q}_{\rho}(y, s) \subset \mathbb{R}^{N+1}
$$

if $q=\frac{N+2}{N}$,

$$
\int_{\tilde{Q}_{\rho}(y, s)}|u|^{q} d x d t \leq C(\log (1 / \rho))^{-\frac{1}{q-1}} \quad \text { for any } \quad \tilde{Q}_{\rho}(y, s) \subset \mathbb{R}^{N+1}, 0<\rho<1 / 2
$$

for some $C=C\left(N, q, \Lambda_{1}, \Lambda_{2}, T_{0}\right)$, see Remark 4.14.
Remark 2.11 In the sub-critical case $1<q<\frac{N+2}{N}$, since the capacity $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}$ of a single are positive thus the conditions (2.13) hold for some constant $C_{1}>0$ provided $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$. Moreover, in the super-critical case $q>\frac{N+2}{N}$, we have

$$
\operatorname{Cap}_{2,1, q^{\prime}}(E) \geq c_{1}|E|^{1-\frac{2 q^{\prime}}{N+2}} \quad \text { and } \quad \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}(O) \geq c_{2}|O|^{1-\frac{2}{(q-1) N}}
$$

for every Borel sets $E \subset \mathbb{R}^{N+1}, O \subset \mathbb{R}^{N}$, thus if $\mu \in L^{\frac{N+2}{2 q^{\prime}}, \infty}\left(\Omega_{T}\right)$ and $\sigma \in L^{\frac{(q-1) N}{2}}, \infty(\Omega)$ then (2.13) holds for some constant $C_{1}>0$. In addition, if $\mu \equiv 0$, then (2.14) implies for any $0<t<T$,

$$
-c_{3}\left(T_{0}\right) t^{-\frac{1}{q-1}} \leq \inf _{x \in \Omega} u(x, t) \leq \sup _{x \in \Omega} u(x, t) \leq c_{3}\left(T_{0}\right) t^{-\frac{1}{q-1}}
$$

since $|\sigma|\left(B_{\rho}(x)\right) \leq c_{4}\left(T_{0}\right) \rho^{N-\frac{2}{q-1}}$ for all $x \in \mathbb{R}^{N}, 0<\rho<2 T_{0}$.

Theorem 2.12 Let $K$ be the constant in Theorem 2.1 and $q>1$. If $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ is absolutely continuous with respect to the capacity Cap $p_{2,1, q^{\prime}}$ in $\mathbb{R}^{N+1}$, then there exists a distribution solution $u \in L_{\text {loc }}^{\gamma}\left(\mathbb{R} ; W_{\text {loc }}^{1, \gamma}\left(\mathbb{R}^{N}\right)\right)$ for any $1 \leq \gamma<\frac{2 q}{q+1}$ to problem

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\omega \text { in } \mathbb{R}^{N+1} \tag{2.16}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
-K \mathbb{I}_{2}\left[\omega^{-}\right] \leq u \leq K \mathbb{I}_{2}\left[\omega^{+}\right] \quad \text { in } \mathbb{R}^{N+1} \tag{2.17}
\end{equation*}
$$

Furthermore, when $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$, $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$ then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to problem

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\mu \text { in } \mathbb{R}^{N} \times(0, \infty),  \tag{2.18}\\
u(0)=\sigma \quad \text { in } \mathbb{R}^{N} .
\end{array}\right.
$$

Remark 2.13 The measure $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity Cap ${ }_{2,1, q^{\prime}}$ in $\mathbb{R}^{N+1}$ if and only if $\mu, \sigma$ are absolutely continuous with respect to the capacities Cap $_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}}, q^{\prime}}$ in $\mathbb{R}^{N+1}, \mathbb{R}^{N}$ respectively.

Existence result of the problem (2.2) on $\mathbb{R}^{N+1}$ or on $\mathbb{R}^{N} \times(0, \infty)$ is similar to Theorem 2.9 presented in the following Theorem, where the capacities $\operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}, \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}$ are used in place of respectively $\operatorname{Cap}_{2,1, q^{\prime}}, \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}$.

Theorem 2.14 Let $K$ be the constant in Theorem 2.1 and $q>\frac{N+2}{N}, \omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$. There exists a constant $C_{1}=C_{1}\left(N, q, \Lambda_{1}, \Lambda_{2}\right)$ such that if

$$
\begin{equation*}
|\omega|(E) \leq C_{1} C_{a p_{\mathcal{H}_{2}, q^{\prime}}}(E) \tag{2.19}
\end{equation*}
$$

for every compact set $E \subset \mathbb{R}^{N+1}$, then the problem

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|u|^{q-1} u+\omega \text { in } \mathbb{R}^{N+1} \tag{2.20}
\end{equation*}
$$

has a distribution solution $u \in L_{\text {loc }}^{\gamma}\left(\mathbb{R} ; W_{\text {loc }}^{1, \gamma}\left(\mathbb{R}^{N}\right)\right)$ for any $1 \leq \gamma<\frac{2 q}{q+1}$ satisfying

$$
\begin{equation*}
-\frac{K q}{q-1} \mathbb{I}_{2}\left[\omega^{-}\right] \leq u \leq \frac{K q}{q-1} \mathbb{I}_{2}\left[\omega^{+}\right] \text {in } \mathbb{R}^{N+1} \tag{2.21}
\end{equation*}
$$

Moreover, when $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$, $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$ then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to problem

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|u|^{q-1} u+\mu \text { in } \mathbb{R}^{N} \times(0, \infty),  \tag{2.22}\\
u(0)=\sigma \quad \text { in } \mathbb{R}^{N} .
\end{array}\right.
$$

In addition, for any compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$
\begin{equation*}
\int_{E}|u|^{q} d x d t \leq C_{2} \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \tag{2.23}
\end{equation*}
$$

for some $C_{2}=C_{2}\left(N, q, \Lambda_{1}, \Lambda_{2}\right)$.
Remark 2.15 The measure $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ satisfies (2.19) if and only if

$$
|\mu|(E) \leq C \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \quad \text { and } \quad|\sigma|(O) \leq C C a p_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}(O),
$$

for every compact sets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^{N}$, where $C=C_{3} C_{1}, C_{3}=C_{3}(N, q)$.

Remark 2.16 If $\omega \in L^{\frac{N+2}{2 q^{\prime}}, \infty}\left(\mathbb{R}^{N+1}\right)$ then (2.19) holds for some constant $C_{1}>0$. Moreover, if $\omega=\sigma \otimes \delta_{\{t=0\}}$ with $\sigma \in \mathfrak{M}_{b}\left(\mathbb{R}^{N}\right)$, then from (2.21) we get the decay estimate:

$$
-c_{1} t^{-\frac{1}{q-1}} \leq \inf _{x \in \mathbb{R}^{N}} u(x, t) \leq \sup _{x \in \mathbb{R}^{N}} u(x, t) \leq c_{1} t^{-\frac{1}{q-1}} \text { for any } t>0,
$$

since $|\sigma|\left(B_{\rho}(x)\right) \leq c_{2} \rho^{N-\frac{2}{q-1}}$ for any $B_{\rho}(x) \subset \mathbb{R}^{N}$.
Second part, we establish global regularity in weighted-Lorentz and Lorentz-Morrey on gradient of solutions to problem (2.4). For this purpose, we need a capacity density condition imposed on $\Omega$. That is, the complement of $\Omega$ satisfies uniformly $p$-thick with constants $c_{0}, r_{0}$, i.e, for all $0<r \leq r_{0}$ and all $x \in \mathbb{R}^{N} \backslash \Omega$ there holds

$$
\begin{equation*}
\operatorname{Cap}_{p}\left(\overline{B_{r}(x)} \cap\left(\mathbb{R}^{N} \backslash \Omega\right), B_{2 r}(x)\right) \geq c_{0} \operatorname{Cap}_{p}\left(\overline{B_{r}(x)}, B_{2 r}(x)\right) \tag{2.24}
\end{equation*}
$$

where the involved capacity of a compact set $K \subset B_{2 r}(x)$ is given as follows

$$
\begin{equation*}
\operatorname{Cap}_{p}\left(K, B_{2 r}(x)\right)=\inf \left\{\int_{B_{2 r}(x)}|\nabla \phi|^{p} d y: \phi \in C_{c}^{\infty}\left(B_{2 r}(x)\right), \phi \geq \chi_{K}\right\} \tag{2.25}
\end{equation*}
$$

In order to obtain better regularity we need a stricter condition on $\Omega$ which is expressed in the following way. We say that $\Omega$ is a $\left(\delta, R_{0}\right)$-Reifenberg flat domain for $\delta \in(0,1)$ and $R_{0}>0$ if for every $x_{0} \in \partial \Omega$ and every $r \in\left(0, R_{0}\right]$, there exists a system of coordinates $\left\{z_{1}, z_{2}, \ldots, z_{n}\right\}$, which may depend on $r$ and $x_{0}$, so that in this coordinate system $x_{0}=0$ and that

$$
\begin{equation*}
B_{r}(0) \cap\left\{z_{n}>\delta r\right\} \subset B_{r}(0) \cap \Omega \subset B_{r}(0) \cap\left\{z_{n}>-\delta r\right\} \tag{2.26}
\end{equation*}
$$

We remark that this class of flat domains is rather wide since it includes $C^{1}$, Lipschitz domains with sufficiently small Lipschitz constants and fractal domains. Besides, it has many important roles in the theory of minimal surfaces and free boundary problems, this class was first appeared in a work of Reifenberg (see [74]) in the context of a Plateau problem. Its properties can be found in [37, 38, 78].

On the other hand, it is well-known that in general, conditions (1.2) and (1.3) on the nonlinearity $A(x, t, \zeta)$ are not enough to ensure higher integral of gradient of solutions to problem (2.4), we need to assume that $A$ satisfies

$$
\begin{equation*}
\left\langle A_{\zeta}(x, t, \zeta) \xi, \xi\right\rangle \geq \Lambda_{2}|\xi|^{2}, \quad\left|A_{\zeta}(x, t, \zeta)\right| \leq \Lambda_{1} \tag{2.27}
\end{equation*}
$$

for every $(\xi, \zeta) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \backslash\{(0,0)\}$ and a.e $(x, t) \in \mathbb{R}^{N} \times \mathbb{R}$, where $\Lambda_{1}, \Lambda_{2}$ are constants in (1.2) and (1.3). We also require that the nonlinearity $A$ satisfies a smallness condition of BMO type in the $x$-variable. We say that $A(x, t, \zeta)$ satisfies a $\left(\delta, R_{0}\right)$-BMO condition for some $\delta, R_{0}>0$ with exponent $s>0$ if

$$
[A]_{s}^{R_{0}}:=\sup _{(y, s) \in \mathbb{R}^{N} \times \mathbb{R}, 0<r \leq R_{0}}\left(f_{Q_{r}(y, s)}\left(\Theta\left(A, B_{r}(y)\right)(x, t)\right)^{s} d x d t\right)^{\frac{1}{s}} \leq \delta
$$

where

$$
\Theta\left(A, B_{r}(y)\right)(x, t):=\sup _{\zeta \in \mathbb{R}^{N} \backslash\{0\}} \frac{\left|A(x, t, \zeta)-\bar{A}_{B_{r}(y)}(t, \zeta)\right|}{|\zeta|},
$$

and $\bar{A}_{B_{r}(y)}(t, \zeta)$ is denoted the average of $A(t, ., \zeta)$ over the cylinder $B_{r}(y)$, i.e,

$$
\bar{A}_{B_{r}(y)}(t, \zeta):=f_{B_{r}(y)} A(x, t, \zeta) d x=\frac{1}{\left|B_{r}(y)\right|} \int_{B_{r}(y)} A(x, t, \zeta) d x
$$

The above condition was appeared in [21]. It is easy to see that the $\left(\delta, R_{0}\right)-\mathrm{BMO}$ condition on $A$ is satisfied when $A$ is continuous or has small jump discontinuities with respect to $x$.

In this paper, $\mathbb{M}$ denotes the Hardy-Littlewood maximal function defined for each locally integrable function $f$ in $\mathbb{R}^{N+1}$ by

$$
\mathbb{M}(f)(x, t)=\sup _{\rho>0} f_{\tilde{Q}_{\rho}(x, t)}|f(y, s)| d y d s \quad \forall(x, t) \in \mathbb{R}^{N+1}
$$

We verify that $\mathbb{M}$ is bounded operator from $L^{1}\left(\mathbb{R}^{N+1}\right)$ to $L^{1, \infty}\left(\mathbb{R}^{N+1}\right)$ and $L^{s}\left(\mathbb{R}^{N+1}\right)$ $\left(L^{s, \infty}\left(\mathbb{R}^{N+1}\right)\right)$ to itself for $s>1$, see [76, 77].

We recall that a positive function $w \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{N+1}\right)$ is called an $A_{\infty}$ if there are two positive constants $C$ and $\nu$ such that

$$
w(E) \leq C\left(\frac{|E|}{|Q|}\right)^{\nu} w(Q)
$$

for all cylinder $Q=\tilde{Q}_{\rho}(x, t)$ and all measurable subsets $E$ of $Q$. The pair $(C, \nu)$ is called the $A_{\infty}$ constant of $w$ and is denoted by $[w]_{A_{\infty}}$.

For a weight function $w \in A_{\infty}$, the weighted Lorentz spaces $L^{q, s}(D, d w)$ with $0<q<\infty$, $0<s \leq \infty$ and a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions $g$ on $D$ such that

$$
\|g\|_{L^{q, s}(D, d w)}:=\left\{\begin{array}{l}
\left(q \int_{0}^{\infty}\left(\rho^{q} w(\{(x, t) \in D:|g(x, t)|>\rho\})\right)^{\frac{s}{q}} \frac{d \rho}{\rho}\right)^{1 / s}<\infty \text { if } s<\infty \\
\sup _{\rho>0} \rho w(\{(x, t) \in D:|g(x, t)|>\rho\})^{1 / q}<\infty \quad \text { if } s=\infty
\end{array}\right.
$$

Here we write $w(E)=\int_{E} w(x, t) d x d t$ for a measurable set $E \subset \mathbb{R}^{N+1}$. Obviously, $\|g\|_{L^{q, q}(D, d w)}=$ $\|g\|_{L^{q}(D, d w)}$, thus we have $L^{q, q}(D, d w)=L^{q}(D, d w)$. As usual, when $w \equiv 1$ we simply write $L^{q, s}(D)$ instead of $L^{q, s}(D, d w)$.

We now state the next results of the paper.
Theorem 2.17 Let $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, $\sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data $\mu$ and $\sigma$ such that if $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly $2-$ thick with constants $c_{0}, r_{0}$ then for any $1 \leq p<\theta$ and $0<s \leq \infty$,

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L^{p, s}\left(\Omega_{T}\right)} \leq C_{1}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{p, s}(Q)} \tag{2.28}
\end{equation*}
$$

Here $\theta=\theta\left(N, \Lambda_{1}, \Lambda_{1}, c_{0}\right)>2$ and $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, c_{0}, T_{0} / r_{0}\right)$ and $Q=B_{\operatorname{diam}(\Omega)}\left(x_{0}\right) \times$ $(0, T)$ which $\Omega \subset B_{\operatorname{diam}(\Omega)}\left(x_{0}\right)$.
Especially, when $1<p<2$, then

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L^{p}\left(\Omega_{T}\right)} \leq C_{2}\left(\left\|\mathcal{G}_{1}[|\mu|]\right\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}+\left\|\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]\right\|_{L^{p}\left(\mathbb{R}^{N}\right)}\right) \tag{2.29}
\end{equation*}
$$

where $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}, p, c_{0}, T_{0} / r_{0}\right)$.
Remark 2.18 If $\frac{N+2}{N+1}<p<2$, there hold

$$
\left\|\mathcal{G}_{1}[|\mu|]\right\|_{L^{p}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\|\mu\|_{L^{\frac{p(N+2)}{N+2+p}}\left(\Omega_{T}\right)} \quad \text { and } \quad\left\|\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]\right\|_{L^{p}\left(\mathbb{R}^{N}\right)} \leq C_{1}\|\sigma\|_{L^{\frac{p N}{N+2-p}}(\Omega)}
$$

for some $C_{1}=C_{1}(N, p)$. From (2.29) we obtain

$$
\||\nabla u|\|_{L^{p}\left(\Omega_{T}\right)} \leq C_{2}\|\mu\|_{L^{\frac{p(N+2)}{N+2+p}}\left(\Omega_{T}\right)}+C_{2}\|\sigma\|_{L^{\frac{p N}{N+2-p}}(\Omega)} \text { provided } \frac{N+2}{N+1}<p<2 .
$$

We should mention that if $\sigma \equiv 0$, then

$$
\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{p, s}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\|\mu\|_{L^{\frac{q(N+2)}{N+2+q}, s}\left(\Omega_{T}\right)}
$$

and we get [7, Theorem 1.2] from estimate (2.28).

In order to state the next results, we need to introduce Lorentz-Morrey spaces $L_{*}^{q, s ; \theta}(D)$ involving "calorie" with a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions $g$ on $D$ such that

$$
\|g\|_{L_{*}^{q, s ; \kappa}(D)}:=\sup _{0<\rho<\operatorname{diam}(D),(x, t) \in D} \rho^{\frac{\kappa-N-2}{q}}\|g\|_{L^{q, s}\left(\tilde{Q}_{\rho}(x, t) \cap D\right)}<\infty
$$

where $0<\kappa \leq N+2,0<q<\infty, 0<s \leq \infty$. Clearly, $L_{*}^{q, s ; N+2}(D)=L^{q, s}(D)$. Moreover, when $q=s$ the space $L_{*}^{q, s ; \theta}(D)$ will be denoted by $L_{*}^{q ; \theta}(D)$.
The following theorem provides an estimate on gradient in Lorentz-Morrey spaces.
Theorem 2.19 Let $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, $\sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data $\mu$ and $\sigma$ such that if $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly $2-$ thick with constants $c_{0}$, $r_{0}$ then for any $1 \leq p<\theta$ and $0<s \leq \infty, 2-\gamma_{0}<\gamma<N+2$, $\gamma \leq \frac{N+2}{p}+1$,

$$
\begin{align*}
& \|\mathbb{M}(|\nabla u|)\|_{L_{*}^{p, s ; p(\gamma-1)}\left(\Omega_{T}\right)} \leq C_{1}\left\|\mathbb{M}_{\gamma}[\omega]\right\|_{L^{\infty}\left(\Omega_{T}\right)} \\
& \quad+C_{2} \sup _{0<R \leq T_{0},\left(y_{0}, s_{0}\right) \in \Omega_{T}}\left(R^{\frac{p(\gamma-1)-N-2}{p}}\left\|\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)} \omega\right]\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)}\right) . \tag{2.30}
\end{align*}
$$

Here $\theta$ is in Theorem 2.17, $\gamma_{0}=\gamma_{0}\left(N, \Lambda_{1}, \Lambda_{1}, c_{0}\right) \in(0,1 / 2]$ and $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, \gamma\right.$, $\left.c_{0}, T_{0} / r_{0}\right), C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, \gamma, c_{0}\right)$. Besides, if $\frac{\gamma}{\gamma-1}<p<\theta, 2-\gamma_{0}<\gamma<N+2$, $0<s \leq \infty$ and $\mu \in L_{*}^{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p}\left(\Omega_{T}\right), \sigma \equiv 0$, then $u$ is a unique renormalized solution satisfied

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L_{*}^{p, s ;(\gamma-1) p}\left(\Omega_{T}\right)} \leq C_{3}\|\mu\|_{L_{*}}{ }_{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p}^{\left(\Omega_{T}\right)}{ }^{\prime} \tag{2.31}
\end{equation*}
$$

where $C_{3}=C_{3}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, \gamma, c_{0}, T_{0} / r_{0}\right)$.
Theorem 2.20 Suppose that A satisfies (2.27). Let $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, $\sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+$ $|\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data $\mu, \sigma$ such that the following holds. For any $w \in A_{\infty}, 1 \leq q<\infty, 0<s \leq \infty$ we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s,[w]_{A_{\infty}}\right) \in(0,1)$ and $s_{0}=s_{0}\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$ such that if $\Omega$ is $\left(\delta, R_{0}\right)$-Reifenberg flat domain $\Omega$ and $[A]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ then

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L^{q, s}\left(\Omega_{T}, d w\right)} \leq C\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s}\left(\Omega_{T}, d w\right)} \tag{2.32}
\end{equation*}
$$

Here $C$ depends on $N, \Lambda_{1}, \Lambda_{2}, q, s,[w]_{A_{\infty}}$ and $T_{0} / R_{0}$.
Next results are actually consequences of Theorem 2.20 . For our purpose, we introduce another Lorentz-Morrey spaces $L_{* *}^{q, s ; \theta}\left(O_{1} \times O_{2}\right)$, is the set of measurable functions $g$ on $O_{1} \times O_{2}$ such that

$$
\|g\|_{L_{* *}^{q, s ; \vartheta}\left(O_{1} \times O_{2}\right)}:=\sup _{0<\rho<\operatorname{diam}\left(O_{1}\right), x \in O_{1}} \rho^{\frac{\vartheta-N}{q}}\|g\|_{\left.L^{q, s}\left(\left(B_{\rho}(x) \cap O_{1}\right) \times O_{2}\right)\right)}<\infty
$$

where $O_{1}, O_{2}$ are Borel sets in $\mathbb{R}^{N}$ and $\mathbb{R}$ respectively, $0<\vartheta \leq N, 0<q<\infty, 0<s \leq \infty$. Obviously, $L_{* *}^{q, s ; N}(D)=L^{q, s}(D)$. For simplicity of notation, we write $L_{* *}^{q ; \vartheta}(D)$ instead of $L_{* *}^{q, s ; \vartheta}(D)$ when $q=s$. Moreover,

$$
\|g\|_{L_{* *}^{q, q ; \vartheta}\left(O_{1} \times O_{2}\right)}=\|G\|_{L^{q ; \vartheta}\left(O_{1}\right)},
$$

where $G(x)=\|g(x, .)\|_{L^{q}\left(O_{1}\right)}$ and $L^{q ; \vartheta}\left(O_{1}\right)$ is the usual Morrey space, i.e the spaces of all measurable functions $f$ on $O_{1}$ with

$$
\|f\|_{L^{q ; \vartheta}\left(O_{1}\right)}:=\sup _{0<\rho<\operatorname{diam}\left(O_{1}\right), y \in O_{1}} \rho^{\frac{\vartheta-N}{q}}\|f\|_{L^{q}\left(B_{\rho}(y) \cap O_{1}\right)}<\infty
$$

Theorem 2.21 Suppose that $A$ satisfies (2.27). Let $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=$ $|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. Let $s_{0}$ be in Theorem 2.20. There exists a distribution solution of (2.4) with data $\mu, \sigma$ such that the following holds.
a. For any $1 \leq q<\infty, 0<s \leq \infty$ and $0<\kappa \leq N+2$ we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s, \kappa\right) \in(0,1)$ such that if $\Omega$ is $\left(\delta, R_{0}\right)$-Reifenberg flat domain $\Omega$ and $[A]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ then

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L_{*}^{q, s ; \kappa}\left(\Omega_{T}\right)} \leq C_{1}\left\|\mathbb{M}_{1}[|\omega|]\right\|_{L_{*}^{q, s ; \kappa}\left(\Omega_{T}\right)} \tag{2.33}
\end{equation*}
$$

Here $C_{1}$ depends on $N, \Lambda_{1}, \Lambda_{2}, q, s, \kappa$ and $T_{0} / R_{0}$.
b. For any $1 \leq q<\infty, 0<s \leq \infty$ and $0<\vartheta \leq N$ we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s, \vartheta\right) \in(0,1)$ such that if $\Omega$ is $\left(\delta, R_{0}\right)$-Reifenberg flat domain $\Omega$ and $[A]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ then

$$
\begin{equation*}
\|\mathbb{M}(|\nabla u|)\|_{L_{* *}^{q, s ; \vartheta}\left(\Omega_{T}\right)} \leq C_{2}\left\|\mathbb{M}_{1}[|\omega|]\right\|_{L_{* *}^{q, s ; \vartheta}\left(\Omega_{T}\right)} \tag{2.34}
\end{equation*}
$$

for some $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}, q, s, \vartheta, T_{0} / R_{0}\right)$. Especially, when $q=s$ and $0<\vartheta<$ $\min \{N, q\}$, there holds for any ball $B_{\rho} \subset \mathbb{R}^{N}$

$$
\begin{equation*}
\left(\int_{0}^{T}\left|o s c_{B_{\rho} \cap \bar{\Omega}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq C_{3} \rho^{1-\frac{\vartheta}{q}}| | \mathbb{M}_{1}[|\omega|]| |_{L_{* *}^{q ; \vartheta}\left(\Omega_{T}\right)} \tag{2.35}
\end{equation*}
$$

for some $C_{3}=C_{3}\left(N, \Lambda_{1}, \Lambda_{2}, q, \vartheta, T_{0} / R_{0}\right)$.
The following global capacitary estimates on gradient.
Theorem 2.22 Suppose that A satisfies (2.27). Let $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, $\sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=$ $|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. Let $s_{0}$ be in Theorem 2.20. There exists a distribution solution of (2.4) with data $\mu, \sigma$ such that following holds. For any $1<q<\infty$, we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q\right) \in(0,1)$ such that if $\Omega$ is a $\left(\delta, R_{0}\right)$ - Reifenberg flat domain and $[A]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ then

$$
\begin{equation*}
\sup _{\substack{\text { compact } \\ \text { Cap }_{\mathcal{G}_{1}, q^{\prime}}(K)>0}}\left(\frac{\int_{K \cap \mathbb{R}_{T}}(\nabla+1}{}|\nabla u|^{q} d x d t\right) \leq C_{1} \sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ \operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)>0}}\left(\frac{\omega(K)}{\operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)}\right)^{q}, \tag{2.36}
\end{equation*}
$$

and if $q>\frac{N+2}{N+1}$,

$$
\begin{equation*}
\sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ \text { Cap }_{\mathcal{H}_{1}, q^{\prime}}(K)>0}}\left(\frac{\int_{K \cap \Omega_{T}}|\nabla u|^{q} d x d t}{\operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(K)}\right) \leq C_{2} \sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ \operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(K)>0}}\left(\frac{\omega(K)}{\operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(K)}\right)^{q} \tag{2.37}
\end{equation*}
$$

Where $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, q, T_{0} / R_{0}, T_{0}\right)$ and $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}, q, T_{0} / R_{0}\right)$.
Remark 2.23 We have if $1<q<2$, then

$$
\begin{aligned}
C^{-1} \sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\
\text { Cappop} \mathcal{G}_{1}, q^{\prime}}}(K)>0
\end{aligned}\left(\frac{\left(|\sigma| \otimes \delta_{\{t=0\}}\right)(K)}{\operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}(K)}\right) \leq \sup _{\substack{\text { compact } O \subset \mathbb{R}^{N} \\
\operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}-1}, q^{\prime}}(O)>0}}\left(\frac{|\sigma|(O)}{\operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}-1}, q^{\prime}}(O)}\right)
$$

for $C=C(N, q)$, if $\frac{N+2}{N+1}<q<2$, then above estimate is true when two capacities Cap $_{\mathcal{G}_{1}, q^{\prime}}$, , Cap $_{\mathbf{G}_{\frac{2}{q}-1}, q^{\prime}}$ are replaced by Cap $_{\mathcal{H}_{1}, q^{\prime}}, \operatorname{Cap}_{\mathbf{I}_{\frac{2}{q}-1}, q^{\prime}}$ respectively, see Remark 4.34.

Remark 2.24 Above results also hold when $[A]_{s}^{R_{0}}$ is replaced by $\{A\}_{s}^{R_{0}}$ :

$$
\{A\}_{s}^{R_{0}}:=\sup _{(y, s) \in \mathbb{R}^{N} \times \mathbb{R}, 0<r \leq R_{0}}\left(f_{Q_{r}(y, s)}\left(\Theta\left(A, Q_{r}(y, s)\right)(x, t)\right)^{s} d x d t\right)^{\frac{1}{s}} \leq \delta
$$

where

$$
\Theta\left(A, Q_{r}(y, s)\right)(x, t):=\sup _{\zeta \in \mathbb{R}^{N} \backslash\{0\}} \frac{\left|A(x, t, \zeta)-\bar{A}_{Q_{r}(y, s)}(\zeta)\right|}{|\zeta|}
$$

and $\bar{A}_{Q_{r}(y, s)}(\zeta)$ is denoted the average of $A(., ., \zeta)$ over the cylinder $Q_{r}(y, s)$, i.e,

$$
\bar{A}_{Q_{r}(y, s)}(\zeta):=f_{Q_{r}(y, s)} A(x, t, \zeta) d x d t=\frac{1}{\left|Q_{r}(y, s)\right|} \int_{Q_{r}(y, s)} A(x, t, \zeta) d x d t
$$

Next results are corresponding estimates of gradient for domain $\mathbb{R}^{N} \times(0, \infty)$ or whole $\mathbb{R}^{N+1}$.
Theorem 2.25 Let $\theta \in(2, N+2)$ be in Theorem 2.17 and $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$. There exists $a$ distribution solution $u$ of (2.6) with data $\mu=\omega$ such that the following statements hold
a. For any $\frac{N+2}{N+1}<p<\theta$ and $0<s \leq \infty$,

$$
\begin{equation*}
\||\nabla u|\|_{L^{p, s}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\left\|\mathbb{M}_{1}[|\omega|]\right\|_{L^{p, s}\left(\mathbb{R}^{N+1}\right)} \tag{2.38}
\end{equation*}
$$

for some $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, p, s\right)$.
b. For any $\frac{N+2}{N+1}<p<\theta$ and $0<s \leq \infty, 2-\gamma_{0}<\gamma<N+2$ and $\gamma \leq \frac{N+2}{p}+1$,

$$
\begin{align*}
& \left\|\|\nabla u \mid\|_{L_{*}^{p, s ; p(\gamma-1)}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\right\| \mathbb{M}_{\gamma}[|\omega|] \|_{L^{\infty}\left(\mathbb{R}^{N+1}\right)} \\
& \quad+C_{2} \sup _{R>0,\left(y_{0}, s_{0}\right) \in \mathbb{R}^{N+1}}\left(R^{\frac{p(\gamma-1)-N-2}{p}}\left\|\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)}|\omega|\right]\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)}\right) \tag{2.39}
\end{align*}
$$

provided $\mathbb{I}_{2}[|\omega|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$.
Also, if $\omega \in L_{*}^{\frac{(\gamma-1) p}{\gamma}}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\mathbb{R}^{N+1}\right)$ with $p>\frac{\gamma}{\gamma-1}$ then

$$
\begin{equation*}
\|\mid \nabla u\|_{L_{*}^{p, s ;(\gamma-1) p}\left(\mathbb{R}^{N+1}\right)} \leq C_{3}\|\omega\|_{L_{*}} \frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\mathbb{R}^{N+1}\right), \tag{2.40}
\end{equation*}
$$

for some $\gamma_{0}=\gamma_{0}\left(N, \Lambda_{1}, \Lambda_{2}\right) \in\left(0, \frac{1}{2}\right]$ and $C_{i}=C_{i}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, \gamma\right), i=2,3$.
c. The statement $\boldsymbol{c}$ in Theorem 2.5 is true.

Remark 2.26 Let $s>1$. For $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, $\mathbb{I}_{1}[\omega] \in L^{s, \infty}\left(\mathbb{R}^{N+1}\right)$ implies $\mathbb{I}_{2}[|\omega|]<\infty$ a.e in $\mathbb{R}^{N+1}$ if and only if $s \leq N+2$.

Theorem 2.27 Suppose that $A$ satisfies (2.27). Let $s_{0}$ be in Theorem 2.20. Let $\omega \in$ $\mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ with $\mathbb{I}_{2}[|\omega|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$. There exists a distribution solution of (2.6) with data $\mu=\omega$ such that following statements hold,
a. For any $w \in A_{\infty}, 1 \leq q<\infty, 0<s \leq \infty$ we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s,[w]_{A_{\infty}}\right) \in(0,1)$ such that if $[A]_{s_{0}}^{\infty} \leq \delta$ then

$$
\begin{equation*}
\left\|\|\nabla u\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)} \leq C_{1}\right\| \mathbb{M}_{1}[|\omega|] \|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)} \tag{2.41}
\end{equation*}
$$

Here $C_{1}$ depends on $N, \Lambda_{1}, \Lambda_{2}, q, s,[w]_{A_{\infty}}$.
b. For any $\frac{N+2}{N+1}<q<\infty, 0<s \leq \infty$ and $0<\kappa \leq N+2$ we find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s, \kappa\right) \in$ $(0,1)$ such that if $[A]_{s_{0}}^{\infty} \leq \delta$ then

$$
\begin{equation*}
\||\nabla u|\|_{L_{*}^{q, s ; \kappa}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mathbb{M}_{1}[|\omega|]\right\|_{L_{*}^{q, s ; \kappa}\left(\mathbb{R}^{N+1}\right)} \tag{2.42}
\end{equation*}
$$

Here $C_{2}$ depends on $N, \Lambda_{1}, \Lambda_{2}, q, s, \kappa$.
c. For any $\frac{N+2}{N+1}<q<\infty, 0<s \leq \infty$ and $0<\vartheta \leq N$ one find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q, s, \vartheta\right) \in$ $(0,1)$ such that if $[A]_{s_{0}}^{\infty} \leq \delta$ then

$$
\begin{equation*}
\left\|\left||\nabla u|\left\|_{L_{* *}^{q, s ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq C_{3}\right\|\right| \mathbb{M}_{1}[|\omega|]\right\|_{L_{* *}^{q, s ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \tag{2.43}
\end{equation*}
$$

Here $C_{3}$ depends on $N, \Lambda_{1}, \Lambda_{2}, q, s, \vartheta$. Especially, when $q=s$ and $0<\vartheta<\min \{N, q\}$, there holds for any ball $B_{\rho} \subset \mathbb{R}^{N}$

$$
\begin{equation*}
\left(\int_{\mathbb{R}}\left|o s c_{B_{\rho}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq C_{4} \rho^{1-\frac{\vartheta}{q}} \|\left.\left|\mathbb{M}_{1}[|\omega|]\right|\right|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \tag{2.44}
\end{equation*}
$$

for some $C_{4}=C_{4}\left(N, \Lambda_{1}, \Lambda_{2}, q, \vartheta\right)$.
d. For any $\frac{N+2}{N+1}<q<\infty$, one find $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q\right) \in(0,1)$ such that if $[\mathcal{A}]_{s_{0}}^{\infty} \leq \delta$ then

$$
\begin{equation*}
\sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ \text { Cap }_{\mathcal{H}_{1}, q^{\prime}}(K)>0}}\left(\frac{\int_{K}|\nabla u|^{q} d x d t}{\operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(K)}\right) \leq C_{5} \sup _{\substack{\text { compact } K \subset \mathbb{R}^{N+1} \\ C_{\text {Cap }}^{\mathcal{H}_{1}, q^{\prime}}(K)>0}}\left(\frac{|\omega|(K)}{\operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(K)}\right)^{q} \tag{2.45}
\end{equation*}
$$

for some $C_{5}=C_{5}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$.
e. The statement $\boldsymbol{c}$ in Theorem 2.5 is true.

The following some estimates for norms of $\mathbb{M}_{1}[\omega]$ in $L_{*}^{q ; \kappa}\left(\mathbb{R}^{N+1}\right)$ and $L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)$
Proposition 2.28 Let $1<\kappa \leq N+2,0<\vartheta \leq N$ and $q, q_{1}>1$. Suppose that $\mu \in$ $\mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$. Then $\mathbb{M}_{1}[\mu] \leq 2^{N+2} \mathbb{I}_{1}[\mu]$ and
a. If $q>\frac{\kappa}{\kappa-1}$ then

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu]\right\|_{L_{*}^{q ; \kappa}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\|\mu\|_{L_{*}^{\frac{q \kappa}{q+\kappa} ; \kappa}\left(\mathbb{R}^{N+1}\right)} . \tag{2.46}
\end{equation*}
$$

Here $C_{1}$ depends on $N, q, \kappa$.
b. If $1<q<2$ then

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu](x, .)\right\|_{L^{q}(\mathbb{R})} \leq \mathbf{I}_{\frac{2}{q}-1}\left[\mu_{1}\right](x) \tag{2.47}
\end{equation*}
$$

where $\mu_{1}$ is a nonnegative radon measure in $\mathbb{R}^{N}$ defined by $\mu_{1}(A)=\mu(A \times \mathbb{R})$ for every Borel set $A \subset \mathbb{R}^{N}$. In particular,

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu]\right\|_{L_{* *\left(\mathbb{R}^{q+1}\right)}} \leq\left\|\mathbf{I}_{\frac{2}{q}-1}\left[\mu_{1}\right]\right\|_{L^{q ; \vartheta}\left(\mathbb{R}^{N}\right)} \tag{2.48}
\end{equation*}
$$

and if $\vartheta>\frac{2-q}{q-1}$ there holds

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu]\right\|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mu_{1}\right\|_{L^{\frac{\vartheta q}{\vartheta+2-q} ; \vartheta}\left(\mathbb{R}^{N}\right)}, \tag{2.49}
\end{equation*}
$$

for some $C_{2}=C_{2}(N, q, \vartheta)$.
c. If $\frac{2 q}{q+2}<q_{1} \leq q$ then

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu](x, .)\right\|_{L^{q}(\mathbb{R})} \leq \mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_{1}}}\left[\mu_{2}\right](x), \tag{2.50}
\end{equation*}
$$

where $d \mu_{2}(x)=\|\mu(x, .)\|_{L^{q_{1}}(\mathbb{R})} d x$. In particular,

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[\mu]\right\|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_{1}}}\left[\mu_{2}\right]\right\|_{L^{q ; \vartheta}\left(\mathbb{R}^{N}\right)} \tag{2.51}
\end{equation*}
$$

and if $\vartheta>\frac{1}{q-1}\left(2+q-\frac{2 q}{q_{1}}\right)$ there holds

$$
\begin{equation*}
\left.\left\|\mathbb{I}_{1}[\mu]\right\|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq C_{3}\left\|\mu_{2}\right\|_{L^{\frac{\vartheta q q_{1}}{(\vartheta+2+q) q_{1}-2 q}} ; \vartheta}^{\left(\mathbb{R}^{N}\right)} \right\rvert\,=C_{3}\|\mu\|_{L^{\frac{\vartheta q q_{1}}{(\vartheta+2+q) q_{1}-2 q} ; \vartheta}\left(\mathbb{R}^{N}, L^{q_{1}}(\mathbb{R})\right)}, \tag{2.52}
\end{equation*}
$$

for some $C_{3}=C_{3}(N, q, \vartheta)$.
The proof of Proposition 2.28 will performed at the end of section 8 .
Remark 2.29 Let $1<q<2,0<\vartheta \leq N$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$. From (2.48) and (2.49) in Proposition 2.28 we assert that

$$
\left\|\mathbb{I}_{1}\left[|\sigma| \otimes \delta_{\{t=0\}}\right]\right\|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\left.\mathbf{I}_{\frac{2}{q}-1}[|\sigma|] \right\rvert\,\right\|_{L^{q ; \vartheta}\left(\mathbb{R}^{N}\right)}
$$

and

$$
\left\|\mathbb{I}_{1}\left[|\sigma| \otimes \delta_{\{t=0\}}\right]\right\|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\|\sigma\|_{L^{\frac{\vartheta q}{\vartheta+2-q} ; \vartheta}\left(\mathbb{R}^{N}\right)} \text { if } \vartheta>\frac{2-q}{q-1},
$$

for some $C_{1}=C_{1}(N, q, \vartheta)$.
Furthermore, from preceding inequality and (2.52) in Proposition 2.28 we can state that

$$
\left\|\left.\mathbb{I}_{1}\left[|\sigma| \otimes \delta_{\{t=0\}}+|\mu|\right]\right|_{L_{* *}^{q ; \vartheta}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\right\| \sigma\left\|_{L^{\frac{\vartheta q}{\vartheta+2-q} ; \vartheta}\left(\mathbb{R}^{N}\right)}+C_{2}\right\| \mu \|_{L^{\frac{\vartheta}{\left(\vartheta+2+q q_{1}\right.}}{ }^{\vartheta q q_{1}-2 q} ; \vartheta\left(\mathbb{R}^{N}, L^{q_{1}}(\mathbb{R})\right)},
$$

provided

$$
\begin{aligned}
& 1<q_{1} \leq q<2 \\
& \max \left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2 q}{q_{1}}\right)\right\}<\vartheta \leq N
\end{aligned}
$$

for some $C_{2}=C_{2}(N, q, \vartheta)$. Where

$$
\|\mu\|_{L^{q_{2} ; \vartheta}\left(\mathbb{R}^{N}, L^{q_{1}}(\mathbb{R})\right)}=\sup _{\rho>0, x \in \mathbb{R}^{N}} \rho^{\frac{\vartheta-N}{q_{2}}}\left(\int_{B_{\rho}(x)}\left(\int_{\mathbb{R}}|\mu(y, t)|^{q_{1}} d t\right)^{\frac{q_{2}}{q_{1}}} d y\right)^{\frac{1}{q_{2}}}
$$

with $q_{2}=\frac{\vartheta q q_{1}}{(\vartheta+2+q) q_{1}-2 q}$.
Final part, we prove the existence solutions for the quasilinear Riccati type parabolic problems

$$
\begin{gather*}
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|\nabla u|^{q}+\mu \text { in } \Omega_{T}, \\
u=0 \quad \text { on } \partial \Omega \times(0, T), \\
u(0)=\sigma \quad \text { in } \Omega,
\end{array}\right.  \tag{2.53}\\
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|\nabla u|^{q}+\mu \text { in } \mathbb{R}^{N} \times(0, \infty), \\
u(0)=\sigma \text { in } \mathbb{R}^{N},
\end{array}\right. \tag{2.54}
\end{gather*}
$$

and

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|\nabla u|^{q}+\mu \text { in } \mathbb{R}^{N+1} \tag{2.55}
\end{equation*}
$$

where $q>1$.
The following result is considered in subcritical case this means $1<q<\frac{N+2}{N+1}$, to obtain existence solutions in this case we need data $\mu, \sigma$ to be finite measures and small enough.

Theorem 2.30 Let $1<q<\frac{N+2}{N+1}$ and $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, $\sigma \in \mathfrak{M}_{b}(\Omega)$. There exists $\varepsilon_{0}=$ $\varepsilon_{0}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$ such that if

$$
\left|\Omega_{T}\right|^{-1+\frac{q^{\prime}}{N+2}}\left(|\mu|\left(\Omega_{T}\right)+|\omega|(\Omega)\right) \leq \varepsilon_{0}
$$

the problem (2.53) has a distribution solution u, satisfied

$$
\left|\|\nabla u \mid\|_{L^{N+2}{ }^{\frac{N+1}{N+1}, \infty}\left(\Omega_{T}\right)} \leq C\left(|\mu|\left(\Omega_{T}\right)+|\omega|(\Omega)\right)\right.
$$

for some $C=C\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$.
In the next results are concerned in critical and supercritical case.
Theorem 2.31 Suppose that $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}, r_{0}$. Let $\theta$ be in Theorem 2.17, $q \in\left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right), \mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $\sigma \in \mathfrak{M}_{b}(\Omega)$. Assume that $\sigma \equiv 0$ when $q \geq \frac{N+4}{N+2}$. There exists $\varepsilon_{0}=\varepsilon_{0}\left(N, \Lambda_{1}, \Lambda_{2}, q, c_{0}, T_{0} / r_{0}\right)>0$ such that if

$$
\left\|\mathbb{I}_{1}[|\mu|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+\left\|\mathbf{I}_{\frac{2}{(N+2)(q-1)}}-1[|\sigma|]\right\|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)} \leq \varepsilon_{0}
$$

then the problem (2.53) has a distribution solution u satisfying
$\left|\left||\nabla u| \|_{L^{(q-1)(N+2), \infty\left(\Omega_{T}\right)}} \leq C\right|\right| \mathbb{I}_{1}[|\mu|]| |_{L^{(N+2)(q-1), \infty\left(\mathbb{R}^{N+1}\right)}}+C| | \mathbf{I}_{(N+2)(q-1)}-\left.1[|\sigma|]\right|_{L^{(N+2)(q-1)\left(\mathbb{R}^{N}\right)}}$,
for some $C=C\left(N, \Lambda_{1}, \Lambda_{2}, q, c_{0}, T_{0} / r_{0}\right)$.
We remark that a necessary condition for existence $\sigma \in \mathfrak{M}_{b}(\Omega) \backslash\{0\}$ with $\mathbb{M}_{1}\left[|\sigma| \otimes \delta_{\{t=0\}}\right] \in$ $L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)$ is $\frac{N+2}{N+1} \leq q<\frac{N+4}{N+2}$.

Theorem 2.32 Suppose that A satisfies (2.27). Let $s_{0}$ be the constant in Theorem 2.20. Let $q \geq \frac{N+2}{N+1}$ and $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. There exists $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q\right) \in(0,1)$ such that $\Omega$ is $\left(\delta, R_{0}\right)$-Reifenberg flat domain $\Omega$ and $[A]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ and the following holds. The problem (2.53) has a distribution solution $u$ if one of the following three cases is true:

Case a. $A$ is a linear operator and

$$
\begin{equation*}
\omega(K) \leq C_{1} \text { Cap }_{\mathcal{G}_{1}, q^{\prime}}(K) \quad \text { for every compact subset } K \subset \mathbb{R}^{N+1} \tag{2.57}
\end{equation*}
$$

with a constant $C_{1}$ small enough.
Case b. there holds

$$
\begin{equation*}
\omega(K) \leq C_{2} \text { Cap }_{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}(K) \quad \text { for every compact subset } K \subset \mathbb{R}^{N+1} \tag{2.58}
\end{equation*}
$$

where $\varepsilon>0$ and $C_{2}$ is a constant small enough.
Case c. $\left\{\begin{array}{l}q>\frac{N+2}{N+1}, \\ q \geq \frac{N+4}{N+2} \quad \text { if } \sigma \equiv 0, \\ \left\|\mathbb{I}_{1}[|\mu|] \mid\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)},\left\|\mathbf{I}_{\frac{2}{(N+2)(q-1)}}-1[|\sigma|]\right\|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)} \\ \quad \text { is small enough. }\end{array}\right.$
A solution u corresponds to Case $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ satisfying

$$
\begin{gathered}
\int_{K}|\nabla u|^{q} d x d t \leq C_{3} C_{1}^{q} C a p_{\mathcal{G}_{1}, q^{\prime}}(K) \quad \text { for every compact subset } K \subset \mathbb{R}^{N+1}, \\
\int_{K}|\nabla u|^{q+\varepsilon} d x d t \leq C_{4} C_{2}^{q+\varepsilon} C a p_{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}(K) \quad \text { for every compact subset } K \subset \mathbb{R}^{N+1},
\end{gathered}
$$

and

$$
\begin{aligned}
& \left.\|\nabla \nabla u\|\left\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \quad \leq C_{5}\right\| \mathbb{I}_{1}[|\mu|]\right|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+C_{5}\left\|\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]\right\|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

respectively. Where $C_{3}, C_{4}, C_{5}$ are constants depended on $N, \Lambda_{1}, \Lambda_{2}, q, \varepsilon, T_{0} / R_{0}$, besides $C_{3}, C_{4}$ also depend on $T_{0}$.

Since $\operatorname{Cap}_{\mathcal{G}_{1}, s}\left(B_{r}(0) \times\{t=0\}\right)=0$ for all $r>0$ and $0<s \leq 2$, see Remark 4.13 thus if there is $\sigma \in \mathfrak{M}_{b}(\Omega) \backslash\{0\}$ satisfying $\left(|\sigma| \otimes \delta_{\{t=0\}}\right)(E) \leq \operatorname{Cap}_{\mathcal{G}_{1}, s}(E)$ for every compact subsets $E \subset \mathbb{R}^{N+1}$ then we must have $s>2$.

The above results are not sharp in the case A is a nonlinear operator. However, if $A$ is Holder continuous with respect to $x$ we can prove that problem (2.53) has a distribution solution with data having compact support in $\Omega_{T}$.
Theorem 2.33 Let $\Omega$ be a bounded open subset in $\mathbb{R}^{N}$ such that the boundary of $\Omega$ is in $C^{1, \beta}$ with $\beta \in(0,1)$. Suppose that $A$ satisfies (2.27) and

$$
\begin{equation*}
|A(x, t, \zeta)-A(y, t, \zeta)| \leq \Lambda_{3}|x-y|^{\beta}|\zeta| \tag{2.59}
\end{equation*}
$$

for every $x, y \in \Omega$ and $t>0, \zeta \in \mathbb{R}^{N}$. Let $\Omega^{\prime} \subset \subset \Omega$ and set $d=\operatorname{dist}\left(\Omega^{\prime}, \Omega\right)>0$. Then, there exist $C=C\left(N, q, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \beta, d, \Omega, T\right)>0$ and $\Lambda=\Lambda\left(N, q, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \beta, d, \Omega, T\right)>0$ such that for any $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$ with $\operatorname{supp}(\mu) \subset \Omega^{\prime} \times[0, T]$, $\operatorname{supp}(\sigma) \subset \Omega^{\prime}$, the problem (2.53) has a distribution solution u, satisfying

$$
\begin{equation*}
|\nabla u(x, t)| \leq \Lambda \mathbb{I}_{1}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right](x, t) \quad \text { a.e }(x, t) \in \Omega_{T} \tag{2.60}
\end{equation*}
$$

provided that one of the following two cases is true:
Case a. $1<q<2$ and

$$
\begin{equation*}
|\mu|(E) \leq C C_{\mathcal{G}_{1}, q^{\prime}}(E) \text { and }|\sigma|(O) \leq C \operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}-1}, q^{\prime}}(O) \tag{2.61}
\end{equation*}
$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^{N}$.

Case b. $q \geq 2$ and $\sigma \equiv 0$,

$$
\begin{equation*}
|\mu|(E) \leq C C a p_{\mathcal{G}_{1}, q^{\prime}}(E) \tag{2.62}
\end{equation*}
$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$.

Remark 2.34 If $q>\frac{N+2}{N+1}, \mu \equiv 0$ and Case $\boldsymbol{a}$ is satisfied then (2.60) gives the decay estimate:

$$
\sup _{x \in \Omega}|\nabla u(x, t)| \leq c_{1} t^{-\frac{1}{2(q-1)}} \forall 0<t<T
$$

since $|\sigma|\left(B_{\rho}(x)\right) \leq c_{2}\left(T_{0}\right) \rho^{N-\frac{2-q}{q-1}}$ for any $B_{\rho}(x) \subset \mathbb{R}^{N}$.
We have an important Proposition.
Proposition 2.35 All the existence results considered the bounded domain $\Omega_{T}$ have recently been presented in above Theorems, if $\sigma \in L^{1}(\Omega)$ then the solutions obtained in those Theorems are renormalized solutions.

Theorem 2.36 Let $\theta \in(2, N+2)$ be in Theorem 2.17, $q \in\left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ and $\omega \in$ $\mathfrak{M}\left(\mathbb{R}^{N+1}\right)$. There exists $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$ such that if

$$
\|\left.\mathbb{I}_{1}[|\omega|]\right|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}
$$

then the problem (2.55) has a distribution solution $u \in L_{\text {loc }}^{1}\left(\mathbb{R} ; W_{\text {loc }}^{1,1}\left(\mathbb{R}^{N}\right)\right)$ such that

$$
\begin{equation*}
\||\nabla u|\|_{L^{(q-1)(N+2), \infty}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mathbb{I}_{1}[|\omega|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)} \tag{2.63}
\end{equation*}
$$

for some $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$. Furthermore, when $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times\right.$ $(0, \infty))$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$ then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to problem (2.54).
Theorem 2.37 Suppose that A satisfies (2.27). Let $q>\frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ such that $\mathbb{I}_{2}[|\omega|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$. Let $s_{0}$ be the constant in Theorem 2.20, $\delta$ in Theorem 2.32. There exists $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$ such that if $[A]_{s_{0}}^{\infty} \leq \delta$ and

$$
\begin{equation*}
\left\|\mathbb{I}_{1}[|\omega|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)} \leq C_{1} \tag{2.64}
\end{equation*}
$$

then the problem (2.55) has a distribution solution u satisfying (2.63). Furthermore, when $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$ then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to problem (2.54).
From Remark 2.26, we see that if $q \leq 2$ then (2.64) follows the assumption $\mathbb{I}_{2}[|\omega|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$.

When $A$ is independent of space variable, we can improve the result of Theorem 2.37 as follows:

Theorem 2.38 Suppose that $A$ is independent of space variable and satisfies (2.27). Let $q>\frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$. Assume that $\mathbb{I}_{2}[|\omega|]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$. There exist constants $\Lambda=\Lambda\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$ and $C=C\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$ such that the problem

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(t, \nabla u))=|\nabla u|^{q}+\omega \text { in } \mathbb{R}^{N+1} \tag{2.65}
\end{equation*}
$$

has a distribution solution $u$, satisfying

$$
\begin{equation*}
|\nabla u| \leq \Lambda \mathbb{I}_{1}[\omega] \quad \text { in } \quad \mathbb{R}^{N+1} \tag{2.66}
\end{equation*}
$$

provided that for all compact subset $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
|\omega|(E) \leq C \operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(E) \tag{2.67}
\end{equation*}
$$

Furthermore, when $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$ then $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$ and $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to problem

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(t, \nabla u))=|\nabla u|^{q}+\mu \text { in } \mathbb{R}^{N} \times(0, \infty),  \tag{2.68}\\
u(0)=\sigma \quad \text { in } \quad \mathbb{R}^{N}
\end{array}\right.
$$

Remark 2.39 If $\frac{N+2}{N+1}<q<2, \omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ satisfies (2.67) if and only if

$$
\begin{equation*}
|\mu|(E) \leq C^{\prime} C \operatorname{Cap}_{\mathcal{H}_{1}, q^{\prime}}(E) \text { and }|\sigma|(O) \leq C^{\prime} C C a p_{\mathbf{I}_{\frac{2}{q}-1}, q^{\prime}}(O) \tag{2.69}
\end{equation*}
$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^{N}$, where $C^{\prime}=C^{\prime}(N, q)$.
Remark 2.40 If $\omega=\sigma \otimes \delta_{\{t=0\}}$ then (2.66) follows the decay estimate:

$$
\sup _{x \in \mathbb{R}^{N}}|\nabla u(x, t)| \leq c_{1} t^{-\frac{1}{2(q-1)}} \forall 0<t<T
$$

since $|\sigma|\left(B_{\rho}(x)\right) \leq c_{2} \rho^{N-\frac{2-q}{q-1}}$ for any $B_{\rho}(x) \subset \mathbb{R}^{N}$.

## 3 The notion of solutions and some properties

Although the notion of renormalized solutions becomes more and more familiar in the theory of quasilinear parabolic equations with measure data, it is still necessary to present below some main aspects concerning this notion. Let $\Omega$ be a bounded domain in $\mathbb{R}^{N}$, $(a, b) \subset \subset \mathbb{R}$. If $\mu \in \mathfrak{M}_{b}(\Omega \times(a, b))$, we denote by $\mu^{+}$and $\mu^{-}$respectively its positive and negative part. We denote by $\mathfrak{M}_{0}(\Omega \times(a, b))$ the space of measures in $\Omega \times(a, b)$ which are absolutely continuous with respect to the $C_{2}$-capacity defined on a compact set $K \subset \Omega \times(a, b)$ by

$$
\begin{equation*}
C_{2}(K, \Omega \times(a, b))=\inf \left\{\|\varphi\|_{W}: \varphi \geq \chi_{K}, \varphi \in C_{c}^{\infty}(\Omega \times(a, b))\right\} . \tag{3.1}
\end{equation*}
$$

where $W=\left\{z: z \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right), z_{t} \in L^{2}\left(a, b, H^{-1}(\Omega)\right)\right\}$ endowed with norm $\|\varphi\|_{W}=$ $\|\varphi\|_{L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)}+\left\|\varphi_{t}\right\|_{L^{2}\left(a, b, H^{-1}(\Omega)\right)}$ and $\chi_{K}$ is the characteristic function of $K$.

We also denote $\mathfrak{M}_{s}(\Omega \times(a, b))$ the space of measures in $\Omega \times(a, b)$ with support on a set of zero $C_{2}$-capacity. Classically, any $\mu \in \mathfrak{M}_{b}(\Omega \times(a, b))$ can be written in a unique way under the form $\mu=\mu_{0}+\mu_{s}$ where $\mu_{0} \in \mathfrak{M}_{0}(\Omega \times(a, b)) \cap \mathfrak{M}_{b}(\Omega \times(a, b))$ and $\mu_{s} \in \mathfrak{M}_{s}(\Omega \times(a, b))$. We recall that any $\mu_{0} \in \mathfrak{M}_{0}(\Omega \times(a, b)) \cap \mathfrak{M}_{b}(\Omega \times(a, b))$ can be decomposed under the form $\mu_{0}=f-\operatorname{div} g+h_{t}$ where $f \in L^{1}(\Omega \times(a, b)), g \in L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$ and $h \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ and $(f, g, h)$ is said to be decomposition of $\mu_{0}$. Set $\widehat{\mu_{0}}=\mu_{0}-h_{t}=f-\operatorname{div} g$. In the general case $\widehat{\mu_{0}} \notin \mathfrak{M}(\Omega \times(a, b))$, but we write, for convenience,

$$
\int_{\Omega \times(a, b)} w d \widehat{\mu_{0}}:=\int_{\Omega \times(a, b)}(f w+g . \nabla w) d x d t, \quad \forall w \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right) \cap L^{\infty}(\Omega \times(a, b)) .
$$

However, for $\sigma \in \mathfrak{M}_{b}(\Omega)$ and $t_{0} \in(a, b)$ then $\sigma \otimes \delta_{\left\{t=t_{0}\right\}} \in \mathfrak{M}_{0}(\Omega \times(a, b))$ if and only if $\sigma \in L^{1}(\Omega)$, see [26]. We also have that for $\sigma \in \mathfrak{M}_{b}(\Omega), \sigma \otimes \chi_{[a, b]} \in \mathfrak{M}_{0}(\Omega \times(a, b))$ if and only if $\sigma$ is absolutely continuous with respect to the $\operatorname{Cap}_{\mathbf{G}_{1}, 2^{2}}$-capacity, see [26].

For $k>0$ and $s \in \mathbb{R}$ we set $T_{k}(s)=\max \{\min \{s, k\},-k\}$. We recall that if $u$ is a measurable function defined and finite a.e. in $\Omega \times(a, b)$, such that $T_{k}(u) \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ for any $k>0$, there exists a measurable function $v: \Omega \times(a, b) \rightarrow \mathbb{R}^{N}$ such that $\nabla T_{k}(u)=\chi_{|u| \leq k} v$ a.e. in $\Omega \times(a, b)$ and for all $k>0$. We define the gradient $\nabla u$ of $u$ by $v=\nabla u$.

We recall the definition of a renormalized solution given in [65].
Definition 3.1 Suppose that $B \in C\left(\mathbb{R} \times \mathbb{R}^{N}, \mathbb{R}\right)$. Let $\mu=\mu_{0}+\mu_{s} \in \mathfrak{M}_{b}(\Omega \times(a, b))$ and $\sigma \in L^{1}(\Omega)$. A measurable function $u$ is a renormalized solution of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=B(u, \nabla u)+\mu \text { in } \Omega \times(a, b),  \tag{3.2}\\
u=0 \quad \text { on } \partial \Omega \times(a, b), \\
u(a)=\sigma \quad \text { in } \Omega,
\end{array}\right.
$$

if there exists a decomposition $(f, g, h)$ of $\mu_{0}$ such that

$$
\begin{gather*}
v=u-h \in L^{s}\left(a, b, W_{0}^{1, s}(\Omega)\right) \cap L^{\infty}\left(a, b, L^{1}(\Omega)\right) \forall s \in\left[1, \frac{N+2}{N+1}\right) \\
T_{k}(v) \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right) \forall k>0, B(u, \nabla u) \in L^{1}(\Omega \times(a, b)) \tag{3.3}
\end{gather*}
$$

and:
(i) for any $S \in W^{2, \infty}(\mathbb{R})$ such that $S^{\prime}$ has compact support on $\mathbb{R}$, and $S(0)=0$,
$-\int_{\Omega} S(\sigma) \varphi(a) d x-\int_{\Omega \times(a, b)} \varphi_{t} S(v) d x d t+\int_{\Omega \times(a, b)} S^{\prime}(v) A(x, t, \nabla u) \nabla \varphi d x d t$
$+\int_{\Omega \times(a, b)} S^{\prime \prime}(v) \varphi A(x, t, \nabla u) . \nabla v d x d t=\int_{\Omega \times(a, b)} S^{\prime}(v) \varphi B(u, \nabla u) d x d t+\int_{\Omega \times(a, b)} S^{\prime}(v) \varphi d \widehat{\mu_{0}}$,
for any $\varphi \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right) \cap L^{\infty}(\Omega \times(a, b))$ such that $\varphi_{t} \in L^{2}\left(a, b, H^{-1}(\Omega)\right)+L^{1}(\Omega \times(a, b))$ and $\varphi(., b)=0$;
(ii) for any $\phi \in C(\bar{\Omega} \times[a, b])$,

$$
\begin{align*}
& \lim _{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq v<2 m\}} \phi A(x, t, \nabla u) \nabla v d x d t=\int_{\Omega \times(a, b)} \phi d \mu_{s}^{+} \quad \text { and }  \tag{3.5}\\
& \lim _{m \rightarrow \infty} \frac{1}{m} \int_{\{-m \geq v>-2 m\}} \phi A(x, t, \nabla u) \nabla v d x d t=\int_{\Omega \times(a, b)} \phi d \mu_{s}^{-} \tag{3.6}
\end{align*}
$$

Remark 3.2 If $\mu \in L^{1}(\Omega \times(a, b))$, then we have the following estimates:

$$
\begin{aligned}
& \|u\|_{L^{\frac{N+2}{N}, \infty}(\Omega \times(a, b))} \leq C_{1}\left(\|\sigma\|_{L^{1}(\Omega)}+|\mu|(\Omega \times(a, b))\right) \quad \text { and } \\
& \|\nabla u\|_{L^{\frac{N+2}{N+1}, \infty}(\Omega \times(a, b))} \leq C_{1}\left(\|\sigma\|_{L^{1}(\Omega)}+|\mu|(\Omega \times(a, b))\right),
\end{aligned}
$$

where $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}\right)$, see [13, Remark 4.9].
In particular,

$$
\begin{aligned}
& \|u\|_{L^{1}(\Omega \times(a, b))} \leq C_{2}\left(\operatorname{diam}(\Omega)+(b-a)^{1 / 2}\right)^{2}\left(\|\sigma\|_{L^{1}(\Omega)}+|\mu|(\Omega \times(a, b))\right) \quad \text { and } \\
& \|\nabla u\|_{L^{1}(\Omega \times(a, b))} \leq C_{2}\left(\operatorname{diam}(\Omega)+(b-a)^{1 / 2}\right)\left(\|\sigma\|_{L^{1}(\Omega)}+|\mu|(\Omega \times(a, b))\right),
\end{aligned}
$$

where $C_{2}=C_{2}\left(N, \Lambda_{1}, \Lambda_{2}\right)$.
Remark 3.3 It is easy to see that $u$ is a weak solution of problem (3.2) in $\Omega \times(a, b)$ with $\mu \in L^{2}(\Omega \times(a, b)), \sigma \in H_{0}^{1}(\Omega)$ and $B \equiv 0$ then $U=\chi_{[a, b]}$ u is a unique renormalized solution of

$$
\left\{\begin{array}{l}
U_{t}-\operatorname{div}(A(x, t, \nabla U))=\chi_{(a, b)} \mu+\left(\chi_{[a, b)} \sigma\right)_{t} \text { in } \Omega \times(c, b), \\
U=0 \\
U(c)=0 \quad \text { on } \partial \Omega \times(c, b), \\
\text { in } \Omega
\end{array}\right.
$$

for any $c<a$.
Remark 3.4 Let $\Omega^{\prime} \subset \subset \Omega$ and $a<a^{\prime}<b^{\prime}<b$. For a nonnegative function $\eta \in C_{c}^{\infty}\left(\Omega^{\prime} \times\right.$ $\left.\left(a^{\prime}, b^{\prime}\right)\right)$, from (3.4) we have

$$
\begin{aligned}
& (\eta S(v))_{t}-\eta_{t} S(v)+S^{\prime}(v) A(x, t, \nabla u) \nabla \eta-\operatorname{div}\left(S^{\prime}(v) \eta A(x, t, \nabla u)\right) \\
& \quad+S^{\prime \prime}(v) \eta A(x, t, \nabla u) \nabla v=S^{\prime}(v) \eta f+\nabla\left(S^{\prime}(v) \eta\right) \cdot g-\operatorname{div}\left(S^{\prime}(v) \eta g\right)
\end{aligned}
$$

in $\mathcal{D}^{\prime}\left(\Omega^{\prime} \times\left(a^{\prime}, b^{\prime}\right)\right)$ Thus, $(\eta S(v))_{t} \in L^{2}\left(a^{\prime}, b^{\prime}, H^{-1}\left(\Omega^{\prime}\right)\right)+L^{1}(D)$ and we have the following estimate

$$
\begin{align*}
& \left\|(\eta S(v))_{t}\right\|_{L^{2}\left(a^{\prime}, b^{\prime}, H^{-1}\left(\Omega^{\prime}\right)\right)+L^{1}(D)} \leq C| | S \|_{W^{2, \infty}(\mathbb{R})}\left(\left\|\eta_{t} v\right\|_{L^{1}(D)}\right. \\
& \quad+\| \| \nabla u\|\nabla \eta\|\left\|_{L^{1}(D)}+\right\| \eta|\nabla u| \chi_{|v| \leq M}\left\|_{L^{2}(D)}+\right\| \eta\left|\nabla u\left\|\nabla v \mid \chi_{|v| \leq M}\right\|_{L^{2}(D)}\right. \\
& \left.\quad+\|\eta f\|_{L^{1}(D)}+\left\|\eta|\nabla u|^{2} \chi_{|v| \leq M \mid}\right\|_{L^{1}(D)}+\left\|\eta|g|^{2}\right\|_{L^{1}(D)}+\|\eta \mid g\|_{L^{2}(D)}\right) \tag{3.7}
\end{align*}
$$

with $D=\Omega^{\prime} \times\left(a^{\prime}, b^{\prime}\right)$ and $\operatorname{supp}\left(S^{\prime}\right) \subset[-M, M]$.
We recall the following important results, see [13].
Proposition 3.5 Let $\left\{\mu_{n}\right\}$ be a bounded in $\mathfrak{M}_{b}(\Omega \times(a, b))$ and $\sigma_{n}$ be a bounded in $L^{1}(\Omega)$. Let $u_{n}$ be a renormalized solution of (2.4) with data $\mu_{n}=\mu_{n, 0}+\mu_{n, s}$ relative to a decomposition $\left(f_{n}, g_{n}, h_{n}\right)$ of $\mu_{n, 0}$ and initial data $\sigma_{n}$. If $\left\{f_{n}\right\}$ is bounded in $L^{1}\left(\Omega_{T}\right),\left\{g_{n}\right\}$ bounded in $L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$ and $\left\{h_{n}\right\}$ convergent in $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$, then, up to a subsequence, $\left\{u_{n}\right\}$ converges to a function $u$ in $L^{1}(\Omega \times(a, b))$. Moreover, if $\left\{\mu_{n}\right\}$ is a bounded in $L^{1}(\Omega \times(a, b))$ then $\left\{u_{n}\right\}$ is convergent in $L^{s}\left(a, b, W_{0}^{1, s}(\Omega)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$.

We say that a sequence of bounded measures $\left\{\mu_{n}\right\}$ in $\Omega \times(a, b)$ converges to a bounded measure $\mu$ in $\Omega \times(a, b)$ in the narrow topology of measures if

$$
\left.\lim _{n \rightarrow \infty} \int_{\Omega \times(a, b)} \varphi d \mu_{n}=\int_{\Omega \times(a, b)} \varphi d \mu \quad \text { for all } \varphi \in C(\Omega \times(a, b)) \cap L^{\infty}(\Omega \times(a, b))\right) .
$$

We recall the following fundamental stability result of [13].
Theorem 3.6 Suppose that $B \equiv 0$. Let $\sigma \in L^{1}(\Omega)$ and

$$
\mu=f-\operatorname{div} g+h_{t}+\mu_{s}^{+}-\mu_{s}^{-} \in \mathfrak{M}_{b}(\Omega \times(a, b)),
$$

with $f \in L^{1}(\Omega \times(a, b)), g \in L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right), h \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ and $\mu_{s}^{+}, \mu_{s}^{-} \in \mathfrak{M}_{s}^{+}(\Omega \times$ $(a, b))$. Let $\sigma_{n} \in L^{1}(\Omega)$ and

$$
\mu_{n}=f_{n}-\operatorname{div} g_{n}+\left(h_{n}\right)_{t}+\rho_{n}-\eta_{n} \in \mathfrak{M}_{b}(\Omega \times(a, b))
$$

with $f_{n} \in L^{1}(\Omega \times(a, b)), g_{n} \in L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right), h_{n} \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$, and $\rho_{n}, \eta_{n} \in$ $\mathfrak{M}_{b}^{+}(\Omega \times(a, b))$, such that

$$
\rho_{n}=\rho_{n}^{1}-\operatorname{div} \rho_{n}^{2}+\rho_{n, s}, \quad \eta_{n}=\eta_{n}^{1}-\operatorname{div} \eta_{n}^{2}+\eta_{n, s}
$$

with $\rho_{n}^{1}, \eta_{n}^{1} \in L^{1}(\Omega \times(a, b)), \rho_{n}^{2}, \eta_{n}^{2} \in L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$ and $\rho_{n, s}, \eta_{n, s} \in \mathfrak{M}_{s}^{+}(\Omega \times(a, b))$. Assume that $\left\{\mu_{n}\right\}$ is a bounded in $\mathfrak{M}_{b}(\Omega \times(a, b)),\left\{\sigma_{n}\right\},\left\{f_{n}\right\},\left\{g_{n}\right\},\left\{h_{n}\right\}$ converge to $\sigma, f, g, h$ in $L^{1}(\Omega)$, weakly in $L^{1}(\Omega \times(a, b))$, in $L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$, in $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ respectively and $\left\{\rho_{n}\right\},\left\{\eta_{n}\right\}$ converge to $\mu_{s}^{+}, \mu_{s}^{-}$in the narrow topology of measures; and $\left\{\rho_{n}^{1}\right\},\left\{\eta_{n}^{1}\right\}$ are bounded in $L^{1}(\Omega \times(a, b))$, and $\left\{\rho_{n}^{2}\right\},\left\{\eta_{n}^{2}\right\}$ bounded in $L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$.
Let $\left\{u_{n}\right\}$ be a sequence of renormalized solutions of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\mu_{n} \text { in } \Omega \times(a, b),  \tag{3.8}\\
u_{n}=0 \quad \text { on } \partial \Omega \times(a, b), \\
u_{n}(a)=\sigma_{n} \text { in } \Omega
\end{array}\right.
$$

relative to the decomposition $\left(f_{n}+\rho_{n}^{1}-\eta_{n}^{1}, g_{n}+\rho_{n}^{2}-\eta_{n}^{2}, h_{n}\right)$ of $\mu_{n, 0}$. Let $v_{n}=u_{n}-h_{n}$. Then up to a subsequence, $\left\{u_{n}\right\}$ converges a.e. in $\Omega \times(a, b)$ to a renormalized solution $u$ of (3.2), and $\left\{v_{n}\right\}$ converges a.e. in $\Omega \times(a, b)$ to $v=u-h$. Moreover, $\left\{\nabla u_{n}\right\},\left\{\nabla v_{n}\right\}$ converge respectively to $\nabla u, \nabla v$ a.e in $\Omega \times(a, b)$, and $\left\{T_{k}\left(v_{n}\right)\right\}$ converges to $T_{k}(v)$ strongly in $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ for any $k>0$.

In order to apply above Theorem, we need some the following properties concerning approximate measures of $\mu \in \mathfrak{M}_{b}^{+}(\Omega \times(a, b))$, see [13].
Proposition 3.7 Let $\mu=\mu_{0}+\mu_{s} \in \mathfrak{M}_{b}^{+}(\Omega \times(a, b))$ with $\mu_{0} \in \mathfrak{M}_{0}(\Omega \times(a, b)) \cap \mathfrak{M}_{b}^{+}(\Omega \times$ $(a, b))$ and $\mu_{s} \in \mathfrak{M}_{s}^{+}(\Omega \times(a, b))$. Let $\left\{\varphi_{n}\right\}$ be sequence of standard mollifiers in $\mathbb{R}^{N+1}$. Then, there exist a decomposition $(f, g, h)$ of $\mu_{0}$ and $f_{n}, g_{n}, h_{n} \in C_{c}^{\infty}(\Omega \times(a, b)), \mu_{n, s} \in$ $C_{c}^{\infty}(\Omega \times(a, b)) \cap \mathfrak{M}_{b}^{+}(\Omega \times(a, b))$ such that $\left\{f_{n}\right\},\left\{g_{n}\right\},\left\{h_{n}\right\}$ strongly converge to $f, g, h$ in $L^{1}(\Omega \times(a, b)), L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$ and $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right), \mu_{n}=f_{n}-\operatorname{div} g_{n}+\left(h_{n}\right)_{t}+\mu_{n, s}, \mu_{n, s}$ converge to $\mu, \mu_{s}$ in the narrow topology respectively, $0 \leq \mu_{n} \leq \varphi_{n} * \mu$ and
$\left\|f_{n}\right\|_{L^{1}(\Omega \times(a, b))}+\left\|g_{n}\right\|_{L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)}+\left\|h_{n}\right\|_{L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)}+\mu_{n, s}(\Omega \times(a, b)) \leq 2 \mu(\Omega \times(a, b))$.
Proposition 3.8 Let $\mu=\mu_{0}+\mu_{s}, \mu_{n}=\mu_{n, 0}+\mu_{n, s} \in \mathfrak{M}_{b}^{+}(\Omega \times(a, b))$ with $\mu_{0}, \mu_{n, 0} \in$ $\mathfrak{M}_{0}(\Omega \times(a, b)) \cap \mathfrak{M}_{b}^{+}(\Omega \times(a, b))$ and $\mu_{n, s}, \mu_{s} \in \mathfrak{M}_{s}^{+}(\Omega \times(a, b))$ such that $\left\{\mu_{n}\right\}$ nondecreasingly converges to $\mu$ in $\mathfrak{M}_{b}(\Omega \times(a, b))$. Then, $\left\{\mu_{n, s}\right\}$ is nondecreasing and converging to $\mu_{s}$ in $\mathfrak{M}_{b}(\Omega \times(a, b))$ and there exist decompositions $(f, g, h)$ of $\mu_{0},\left(f_{n}, g_{n}, h_{n}\right)$ of $\mu_{n, 0}$ such that $\left\{f_{n}\right\},\left\{g_{n}\right\},\left\{h_{n}\right\}$ strongly converge to $f, g, h$ in $L^{1}(\Omega \times(a, b)), L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)$ and $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ respectively satisfying
$\left\|f_{n}\right\|_{L^{1}(\Omega \times(a, b))}+\left\|g_{n}\right\|_{L^{2}\left(\Omega \times(a, b), \mathbb{R}^{N}\right)}+\left\|h_{n}\right\|_{L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)}+\mu_{n, s}(\Omega \times(a, b)) \leq 2 \mu(\Omega \times(a, b))$.

Remark 3.9 For $0<\rho \leq \frac{1}{3} \min \left\{\sup _{x \in \Omega} d(x, \partial \Omega),(b-a)^{1 / 2}\right\}$, set

$$
\Omega_{\rho}^{j}=\{x \in \Omega: d(x, \partial \Omega)>j \rho\} \times\left(a+(j \rho)^{2}, a+\left((b-a)^{1 / 2}-j \rho\right)^{2}\right) \text { for } j=0, \ldots, k_{\rho}
$$

where $k_{\rho}=\left[\frac{\min \left\{\sup _{x \in \Omega} d(x, \partial \Omega),(b-a)^{1 / 2}\right\}}{2 \rho}\right]$.
We can choose $f_{n}, g_{n}, h_{n}$ in above two Propositions such that for any $j=1, \ldots, k_{\rho}$,

$$
\begin{equation*}
\left\|f_{n}\right\|_{L^{1}\left(\Omega_{\rho}^{j}\right)}+\left\|g_{n}\right\|_{L^{2}\left(\Omega_{\rho}^{j}, \mathbb{R}^{N}\right)}+\left\|\left|h_{n}\right|+\left|\nabla h_{n} \|\right|_{L^{2}\left(\Omega_{\rho}^{j}\right)} \leq 2 \mu\left(\Omega_{\rho}^{j-1}\right) \forall n \in \mathbb{N}\right. \tag{3.9}
\end{equation*}
$$

In fact, set $\mu_{j}=\chi_{\Omega_{\rho}^{k_{\rho}-j} \backslash \Omega_{\rho}^{k_{\rho}-j+1}} \mu$ if $j=1, \ldots, k_{\rho}-1, \mu_{j}=\chi_{\Omega \times(a, b) \backslash \Omega_{\rho}^{1}} \mu$ if $j=k_{\rho}$ and $\mu_{j}=\chi_{\Omega_{\rho}^{k_{\rho}}} \mu$ if $j=0$. From the proof of above two Propositions in [13], for any $\varepsilon>0$ we can assume supports of $f_{n}, g_{n}, h_{n}$ containing in $\operatorname{supp}(\mu)+\tilde{Q}_{\varepsilon}(0,0)$. Thus, for any $\mu=\mu_{j}$ we have $f_{n}^{j}, g_{n}^{j}, h_{n}^{j}$ correspondingly such that their supports contain in $\Omega_{\rho, T}^{k_{\rho}-j-1 / 2} \backslash \Omega_{\rho, T}^{k_{\rho}-j+3 / 2}$ if $j=1, \ldots, k_{\rho}-1$ and $\Omega_{T} \backslash \Omega_{\rho, T}^{3 / 2}$ if $j=k_{\rho}$ and $\Omega_{\rho, T}^{k_{\rho}-1 / 2}$ if $j=0$. By $\mu=\sum_{j=0}^{k_{\rho}} \mu_{j}$, thus it is allowed to choose $f_{n}=\sum_{j=0}^{k_{\rho}} f_{n}^{j}, f_{n}=\sum_{j=0}^{k_{\rho}} g_{n}^{j}$ and $h_{n}=\sum_{j=0}^{k_{\rho}} h_{n}^{j}$ and (3.9) satisfies since

$$
\begin{aligned}
& \left\|f_{n}\right\|_{L^{1}\left(\Omega_{\rho}^{j}\right)}+\left\|g_{n}\right\|_{L^{2}\left(\Omega_{\rho}^{j}, \mathbb{R}^{N}\right)}+\left\|\left|h_{n}\right|+\mid \nabla h_{n}\right\| \|_{L^{2}\left(\Omega_{\rho}^{j}\right)} \\
& \quad \leq \sum_{i=0}^{k_{\rho}}\left(\left\|f_{n}^{i}\right\|_{L^{1}\left(\Omega_{\rho}^{j}\right)}+\left\|g_{n}^{i}\right\|_{L^{2}\left(\Omega_{\rho}^{j}, \mathbb{R}^{N}\right)}+\left|\left\|h_{n}^{i}\left|+\left|\nabla h_{n}^{i}\right| \|_{L^{2}\left(\Omega_{\rho}^{j}\right)}\right)\right.\right.\right. \\
& \quad=\sum_{i=0}^{k_{\rho}-j+1}\left(\left\|f_{n}^{i}\right\|_{L^{1}\left(\Omega_{\rho}^{j}\right)}+\left\|g_{n}^{i}\right\|_{L^{2}\left(\Omega_{\rho}^{j}, \mathbb{R}^{N}\right)}+\left\|\left|\left|h_{n}^{i}\right|+\right| \nabla h_{n}^{i}\right\| \|_{L^{2}\left(\Omega_{\rho}^{j}\right)}\right) \\
& \quad \leq \sum_{i=j-1}^{k_{\rho}-j+1} 2 \mu_{j}(\Omega \times(a, b))=2 \mu\left(\Omega_{\rho}^{j-1}\right) .
\end{aligned}
$$

Definition 3.10 Let $\mu \in \mathfrak{M}_{b}(\Omega \times(a, b))$ and $\sigma \in \mathfrak{M}_{b}(\Omega)$. A measurable function $u$ is $a$ distribution solution to problem (3.2) if $u \in L^{s}\left(a, b, W_{0}^{1, s}(\Omega)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$ and $B(u, \nabla u) \in L^{1}(\Omega \times(a, b))$ such that

$$
\begin{aligned}
& -\int_{\Omega \times(a, b)} u \varphi_{t} d x d t+\int_{\Omega \times(a, b)} A(x, t, \nabla u) \nabla \varphi d x d t \\
& \quad=\int_{\Omega \times(a, b)} B(u, \nabla u) \varphi d x d t+\int_{\Omega \times(a, b)} \varphi d \mu+\int_{\Omega} \varphi(a) d \sigma
\end{aligned}
$$

for every $\varphi \in C_{c}^{1}(\Omega \times[a, b))$.
Remark 3.11 Let $\sigma^{\prime} \in \mathfrak{M}_{b}(\Omega)$ and $a^{\prime} \in(a, b)$, set $\omega=\mu+\sigma^{\prime} \otimes \delta_{\left\{t=a^{\prime}\right\}}$. If $u$ is a distribution solution to problem (3.2) with data $\omega$ and $\sigma=0$ such that $\operatorname{supp}(\mu) \subset \bar{\Omega} \times\left[a^{\prime}, b\right]$, and $u=0, B(u, \nabla u)=0$ in $\Omega \times\left(a, a^{\prime}\right)$, then $\tilde{u}:=\left.u\right|_{\Omega \times\left[a^{\prime}, b\right)}$ is a distribution solution to problem (3.2) in $\Omega \times\left(a^{\prime}, b\right)$ with data $\mu$ and $\sigma^{\prime}$. Indeed, for any $\varphi \in C_{c}^{1}\left(\Omega \times\left[a^{\prime}, b\right)\right)$ we defined

$$
\tilde{\varphi}(x, t)=\left\{\begin{array}{l}
\varphi(x, t) \text { if }(x, t) \in \Omega \times\left[a^{\prime}, b\right), \\
\left(1+\varepsilon_{0}\right)\left(t-a^{\prime}\right) \varphi_{t}\left(x, a^{\prime}\right)+\varphi\left(x,\left(1+\varepsilon_{0}\right) a^{\prime}-\varepsilon_{0} t\right) \text { if }(x, t) \in \Omega \times\left[a, a^{\prime}\right)
\end{array}\right.
$$

where $\varepsilon_{0} \in\left(0, \frac{b-a^{\prime}}{a^{\prime}-a}\right)$.
Clearly, $\tilde{\varphi} \in C_{c}^{1}(\Omega \times[a, b))$, thus we have

$$
\begin{aligned}
-\int_{\Omega \times(a, b)} u \tilde{\varphi}_{t} d x d t & +\int_{\Omega \times(a, b)} A(x, t, \nabla u) \nabla \tilde{\varphi} d x d t \\
= & \int_{\Omega \times(a, b)} B(u, \nabla u) \tilde{\varphi} d x d t+\int_{\Omega \times(a, b)} \tilde{\varphi} d \omega
\end{aligned}
$$

which implies

$$
\begin{aligned}
-\int_{\Omega \times\left(a^{\prime}, b\right)} & \tilde{u} \varphi_{t} d x d t+\int_{\Omega \times\left(a^{\prime}, b\right)} A(x, t, \nabla \tilde{u}) \nabla \varphi d x d t \\
& =\int_{\Omega \times\left(a^{\prime}, b\right)} B(\tilde{u}, \nabla \tilde{u}) \varphi d x d t+\int_{\Omega \times\left(a^{\prime}, b\right)} \varphi d \mu+\int_{\Omega} \varphi\left(a^{\prime}\right) d \sigma^{\prime}
\end{aligned}
$$

Definition 3.12 Let $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times[a,+\infty)\right)$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$. A measurable function $u$ is a distribution solution to problem

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=B(u, \nabla u)+\mu \text { in } \mathbb{R}^{N} \times(a,+\infty)  \tag{3.10}\\
u(a)=\sigma \quad \text { in } \mathbb{R}^{N}
\end{array}\right.
$$

if $u \in L_{l o c}^{s}\left(a, \infty, W_{l o c}^{1, s}\left(\mathbb{R}^{N}\right)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$ and $B(u, \nabla u) \in L_{l o c}^{1}\left(\mathbb{R}^{N} \times[a, \infty)\right)$ such that

$$
\begin{aligned}
-\int_{\mathbb{R}^{N} \times(a, \infty)} & u \varphi_{t} d x d t+\int_{\mathbb{R}^{N} \times(a, \infty)} A(x, t, \nabla u) \nabla \varphi d x d t \\
& =\int_{\mathbb{R}^{N} \times(a, \infty)} B(u, \nabla u) \varphi d x d t+\int_{\mathbb{R}^{N} \times(a, \infty)} \varphi d \mu+\int_{\mathbb{R}^{N}} \varphi(a) d \sigma
\end{aligned}
$$

for every $\varphi \in C_{c}^{1}\left(\mathbb{R}^{N} \times[a, \infty)\right)$.
Definition 3.13 Let $\omega \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$. A measurable function $u$ is a distribution solution to problem

$$
\begin{equation*}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=B(u, \nabla u)+\omega \text { in } \mathbb{R}^{N+1} \tag{3.11}
\end{equation*}
$$

if $u \in L_{\text {loc }}^{s}\left(\mathbb{R} ; W_{\text {loc }}^{1, s}\left(\mathbb{R}^{N}\right)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$ and $B(u, \nabla u) \in L_{\text {loc }}^{1}\left(\mathbb{R}^{N+1}\right)$ such that

$$
-\int_{\mathbb{R}^{N+1}} u \varphi_{t} d x d t+\int_{\mathbb{R}^{N+1}} A(x, t, \nabla u) \nabla \varphi d x d t=\int_{\mathbb{R}^{N+1}} B(u, \nabla u) \varphi d x d t+\int_{\mathbb{R}^{N+1}} \varphi d \omega,
$$

for every $\varphi \in C_{c}^{1}\left(\mathbb{R}^{N+1}\right)$.
Remark 3.14 Let $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times[a,+\infty)\right)$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$. If $u$ is a distribution solution to problem (3.11) with data $\omega=\mu+\sigma \otimes \delta_{\{t=a\}}$ such that $u=0, B(u, \nabla u)=0$ in $\mathbb{R}^{N} \times(-\infty, a)$, then $\tilde{u}:=\left.u\right|_{\mathbb{R}^{N} \times[a, \infty)}$ is a distribution solution to problem (3.10) in $\mathbb{R}^{N} \times(a, \infty)$ with data $\mu$ and $\sigma$, see Remark 3.11.

To prove the existence distribution solution of problem (3.10) we need the following results. First, we have local estimates of the renormalized solution which get from [13, Proposition 2.8 ].

Proposition 3.15 Let $u, v$ be in Definition 3.1. There exists $C=C\left(\Lambda_{1}, \Lambda_{2}\right)>0$ such that for $k \geq 1$ and $0 \leq \eta \in C_{c}^{\infty}(\Omega \times(a, b))$

$$
\begin{equation*}
\int_{|v| \leq k} \eta|\nabla u|^{2} d x d t+\int_{|v| \leq k} \eta|\nabla v|^{2} d x d t \leq C k A \tag{3.12}
\end{equation*}
$$

where

$$
\begin{gathered}
A=\left\|v \eta_{t}\right\|_{L^{1}(\Omega \times(a, b))}+\| \| \nabla u\left\|\left.\nabla \eta\left|\left\|_{L^{1}(\Omega \times(a, b))}+\right\| \eta f\left\|_{L^{1}(\Omega \times(a, b))}+\right\| \eta\right| g\right|^{2}\right\|_{L^{1}(\Omega \times(a, b))} \\
+\left|\|\nabla \eta\| g\| \|_{L^{1}(\Omega \times(a, b))}+\left\|\eta|\nabla h|^{2}\right\|_{L^{1}(\Omega \times(a, b))}+\int_{\Omega \times(a, b)} \eta d\right| \mu_{s} \mid .
\end{gathered}
$$

For our purpose, we recall the Landes-time approximation of functions $w$ belonging to $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$, introduced in [45], used in [24, 17, 8]. For $\nu>0$ we define

$$
\langle w\rangle_{\nu}(x, t)=\nu \int_{a}^{\min \{t, b\}} w(x, s) e^{\nu(s-t)} d s \quad \text { for all } \quad(x, t) \in \Omega \times(a, b)
$$

We have that $\langle w\rangle_{\nu}$ converges to $w$ strongly in $L^{2}\left(a, b, H_{0}^{1}(\Omega)\right)$ and $\left\|\langle w\rangle_{\nu^{\prime}}\right\|_{L^{q}(\Omega \times(a, b))} \leq$ $\|w\|_{L^{q}(\Omega \times(a, b))}$ for every $q \in[1, \infty]$. Moreover,

$$
\left(\langle w\rangle_{\nu}\right)_{t}=\nu\left(w-\langle w\rangle_{\nu}\right) \quad \text { in the sense of distributions }
$$

if $w \in L^{\infty}(\Omega \times(a, b))$ then

$$
\int_{\Omega \times(a, b)}\left(\langle w\rangle_{\nu}\right)_{t} \varphi d x d t=\nu \int_{\Omega \times(a, b)}\left(w-\langle w\rangle_{\nu}\right) \varphi d x d t \quad \text { for all } \varphi \in L^{2}\left(a, b, H_{0}^{1}(\Omega)\right) .
$$

Proposition 3.16 Let $q_{0}>1$ and $0<\alpha<1 / 2$ such that $q_{0}>\alpha+1$. Let $L: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and nondecreasing such that $L(0)=0$. If $u$ is a solution of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+L(u)=\mu \text { in } \Omega \times(a, b),  \tag{3.13}\\
u=0 \quad \text { on } \partial \Omega \times(a, b), \\
u(a)=0 \quad \text { in } \Omega
\end{array}\right.
$$

with $\mu \in C_{c}^{\infty}(\Omega \times(a, b))$ there exists $C_{1}>0$ depending on $\Lambda_{1}, \Lambda_{2}, \alpha, q_{0}$ such that for $0 \leq \eta \in$ $C_{c}^{\infty}(D)$ where $D=\Omega^{\prime} \times\left(a^{\prime}, b^{\prime}\right), \Omega^{\prime} \subset \subset \Omega$ and $a<a^{\prime}<b^{\prime}<b$, then

$$
\begin{align*}
& \frac{1}{k} \int_{D}\left|\nabla T_{k}(u)\right|^{2} \eta d x d t \\
& \quad+\int_{D} \frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \eta d x d t+|\|\nabla u\| \nabla \eta|\left\|_{L^{1}(D)}+\right\| L(u) \eta \|_{L^{1}(D)} \leq C_{1} B \tag{3.14}
\end{align*}
$$

where $q_{1}=\frac{q_{0}-\alpha-1}{2 q_{0}}$,

$$
B=\left.\left|\left|\eta_{t}(|u|+1)\right|_{L^{1}(D)}+\int_{D}(|u|+1)^{q_{0}} \eta d x d t+\int_{D}\right| \nabla \eta^{1 / q_{1}}\right|^{q_{1}} d x d t+\int_{D} \eta d|\mu|
$$

Furthermore, for $T_{k}(w) \in L^{2}\left(a^{\prime}, b^{\prime}, H_{0}^{1}\left(\Omega^{\prime}\right)\right)$, the Landes-time approximation $\left\langle T_{k}(w)\right\rangle_{\nu}$ of the truncate function $T_{k}(w)$ in $D$ then for any $\varepsilon \in(0,1)$ and $\nu>0$

$$
\begin{align*}
& \nu \int_{D} \eta\left(T_{k}(w)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) d x d t \\
& \quad+\int_{D} \eta A\left(x, t, \nabla T_{k}(u)\right) \nabla T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) d x d t \leq C_{2} \varepsilon(1+k) B \tag{3.15}
\end{align*}
$$

for some $C_{2}=C_{2}\left(\Lambda_{1}, \Lambda_{2}, \alpha, q_{0}\right)$.
Proposition 3.17 Let $q_{0}>1$, $\mu_{n}=\mu_{n, 0}+\mu_{n, s} \in \mathfrak{M}_{b}\left(B_{n}(0) \times\left(-n^{2}, n^{2}\right)\right)$. Let $u_{n}$ be a renormalized solution of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\mu_{n} \text { in } B_{n}(0) \times\left(-n^{2}, n^{2}\right)  \tag{3.16}\\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right) \\
u_{n}\left(-n^{2}\right)=0 \text { in } B_{n}(0)
\end{array}\right.
$$

relative to the decomposition $\left(f_{n}, g_{n}, h_{n}\right)$ of $\mu_{n, 0}$ satisfying (3.15) in Proposition 3.16 with $L \equiv 0$. Assume that for any $m \in \mathbb{N}$ and $\alpha \in(0,1 / 2), D_{m}:=B_{m}(0) \times\left(-m^{2}, m^{2}\right)$

$$
\begin{aligned}
& \frac{1}{k}\left|\left\|\left.\nabla T_{k}(u)\right|^{2}\right\|_{L^{1}\left(D_{m}\right)}+\left\|\left||\nabla u|^{2}(|u|+1)^{-\alpha-1}\left\|_{L^{1}\left(D_{m}\right)}+\right\|\|\nabla u\| \|_{L^{1}\left(D_{m}\right)}+\left|\mu_{n}\right|\left(D_{m}\right)\right.\right.\right. \\
& \quad+\left\|f_{n}\right\|_{L^{1}\left(D_{m}\right)}+\left\|g_{n}\right\|_{L^{2}\left(D_{m}, \mathbb{R}^{N}\right)}+\left\|\left|\left|h_{n}\right|+\left|\nabla h_{n}\right|\left\|_{L^{2}\left(D_{m}\right)}+\right\| u_{n} \|_{L^{q_{0}\left(D_{m}\right)}} \leq C(m, \alpha)\right.\right.
\end{aligned}
$$

for all $n \geq m$ and $h_{n}$ is convergent in $L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$. Then, there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ such that $u_{n}$ converges to $u$ a.e in $\mathbb{R}^{N+1}$ and in $L_{\text {loc }}^{s}\left(\mathbb{R} ; W_{\text {loc }}^{1, s}\left(\mathbb{R}^{N}\right)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$.

Proofs of above two Propositions are given in the Appendix section. The following result is as a consequence of Proposition 3.17.
Corollary 3.18 Let $\mu_{n} \in L^{1}\left(B_{n}(0) \times\left(-n^{2}, n^{2}\right)\right)$. Let $u_{n}$ be a unique renormalized solution of problem 3.16. Assume that for any $m \in \mathbb{N}$,

$$
\sup _{n \geq m}\left|\mu_{n}\right|\left(B_{m}(0) \times\left(-m^{2}, m^{2}\right)\right)<\infty \quad \text { and } \sup _{n \geq m} \int_{B_{m}(0) \times\left(-m^{2}, m^{2}\right)}\left|u_{n}\right|^{q_{0}} d x d t<\infty .
$$

then there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ such that $u_{n}$ converges to $u$ a.e in $\mathbb{R}^{N+1}$ and in $L_{\text {loc }}^{s}\left(\mathbb{R} ; W_{\text {loc }}^{1, s}\left(\mathbb{R}^{N}\right)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$.

Finally, we would like to present a technical lemma which will be used several times in the paper, specially in the proof of Theorem 2.17, 2.19 and 2.20. It is a consequence of Vitali Covering Lemma, a proof of lemma can be seen in [22, 21, 54].

Lemma 3.19 Let $\Omega$ be a $\left(R_{0}, \delta\right)$ - Reifenberg flat domain with $\delta<1 / 4$ and let $w$ be an $A_{\infty}$ weight. Suppose that the sequence of balls $\left\{B_{r}\left(y_{i}\right)\right\}_{i=1}^{L}$ with centers $y_{i} \in \bar{\Omega}$ and a common radius $r \leq R_{0} / 4$ covers $\Omega$. Set $s_{i}=T-i r^{2} / 2$ for all $i=0,1, \ldots,\left[\frac{2 T}{r^{2}}\right]$. Let $E \subset F \subset \Omega_{T}$ be measurable sets for which there exists $0<\varepsilon<1$ such that $w(E)<\varepsilon w\left(\tilde{Q}_{r}\left(y_{i}, s_{j}\right)\right)$ for all $i=1, \ldots, L, j=0,1, \ldots,\left[\frac{2 T}{r^{2}}\right] ;$ and for all $(x, t) \in \Omega_{T}, \rho \in(0,2 r]$, we have $\tilde{Q}_{\rho}(x, t) \cap \Omega_{T} \subset F$ if $w\left(E \cap \tilde{Q}_{\rho}(x, t)\right) \geq \varepsilon w\left(\tilde{Q}_{\rho}(x, t)\right)$. Then $w(E) \leq B \varepsilon w(F)$ for a constant $B$ depending only on $N$ and $[w]_{A_{\infty}}$.

Clearly, the Lemma contains the following two Lemmas
Lemma 3.20 Let $0<\varepsilon<1, R>0$ and cylinder $\tilde{Q}_{R}:=\tilde{Q}_{R}\left(x_{0}, t_{0}\right)$ for some $\left(x_{0}, t_{0}\right) \in$ $\mathbb{R}^{N+1}$ and $w \in A_{\infty}$. let $E \subset F \subset \tilde{Q}_{R}$ be two measurable sets in $\mathbb{R}^{N+1}$ with $w(E)<$ $\varepsilon w\left(\tilde{Q}_{R}\right)$ and satisfying the following property: for all $(x, t) \in \tilde{Q}_{R}$ and $r \in(0, R]$, we have $\tilde{Q}_{r}(x, t) \cap \tilde{Q}_{R} \subset F$ provided $w\left(E \cap \tilde{Q}_{r}(x, t)\right) \geq \varepsilon w\left(\tilde{Q}_{r}(x, t)\right)$. Then $w(E) \leq B \varepsilon w(F)$ for some $B=B\left(N,[w]_{A_{\infty}}\right)$.

Lemma 3.21 Let $0<\varepsilon<1$ and $R>R^{\prime}>0$ and let $E \subset F \subset Q=B_{R}\left(x_{0}\right) \times(a, b)$ be two measurable sets in $\mathbb{R}^{N+1}$ with $|E|<\varepsilon\left|\tilde{Q}_{R^{\prime}}\right|$ and satisfying the following property: for all $(x, t) \in Q$ and $r \in\left(0, R^{\prime}\right]$, we have $Q_{r}(x, t) \cap Q \subset F$ if $\left|E \cap \tilde{Q}_{r}(x, t)\right| \geq \varepsilon\left|\tilde{Q}_{r}(x, t)\right|$. Then $|E| \leq B \varepsilon|F|$ for a constant $B$ depending only on $N$.

## 4 Estimates on Potential

In this section, we will develop nonlinear potential theory corresponding to quasilinear parabolic equations.

First we introduce the Wolff parabolic potential of $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ by

$$
\mathbb{W}_{\alpha, p}^{R}[\mu](x, t)=\int_{0}^{R}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho} \quad \text { for any } \quad(x, t) \in \mathbb{R}^{N+1}
$$

where $\alpha>0,1<p<\alpha^{-1}(N+2)$ and $0<R \leq \infty$. For convenience, $\mathbb{W}_{\alpha, p}[\mu]:=\mathbb{W}_{\alpha, p}^{\infty}[\mu]$.
The following result is an extension of [36, Theorem 1.1], [16, Proposition 2.2] to Parabolic potential.

Theorem 4.1 Let $\alpha>0,1<p<\alpha^{-1}(N+2)$ and $w \in A_{\infty}, \mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$. There exist constants $C_{1}, C_{2}>0$ and $\varepsilon_{0} \in(0,1)$ depending on $N, \alpha, p,[w]_{A_{\infty}}$ such that for any $\lambda>0$ and $\varepsilon \in\left(0, \varepsilon_{0}\right)$

$$
\begin{equation*}
w\left(\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>a \lambda,\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda\right\}\right) \leq C_{1} \exp \left(-C_{2} \varepsilon^{-1}\right) w\left(\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>\lambda\right\}\right) \tag{4.1}
\end{equation*}
$$

where $a=2+3^{\frac{N+2-\alpha p}{p-1}}$.
Proof of Theorem 4.1. We only consider case $R<\infty$. Let $\left\{\tilde{Q}_{R}\left(x_{j}, t_{j}\right)\right\}$ be a cover of $\mathbb{R}^{N+1}$ such that $\sum_{j} \chi_{\tilde{Q}_{R}\left(x_{j}, t_{j}\right)} \leq M$ in $\mathbb{R}^{N+1}$ for some constant $M=M(N)>0$. It is enough to show that there exist constants $c_{1}, c_{2}>0$ and $\varepsilon_{0} \in(0,1)$ depending on $N, \alpha, p,[w]_{A_{\infty}}$ such that for any $Q \in\left\{\tilde{Q}_{R}\left(x_{j}, t_{j}\right)\right\}, \lambda>0$ and $\varepsilon \in\left(0, \varepsilon_{0}\right)$

$$
\begin{equation*}
w\left(Q \cap\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>a \lambda,\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda\right\}\right) \leq c_{1} \exp \left(-c_{2} \varepsilon^{-1}\right) w\left(Q \cap\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>\lambda\right\}\right) \tag{4.2}
\end{equation*}
$$

Fix $\lambda>0$ and $0<\varepsilon<1 / 10$. We set

$$
E=Q \cap\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>a \lambda,\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda\right\} \quad \text { and } \quad F=Q \cap\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>\lambda\right\}
$$

Thanks to Lemma 3.20 we will get (4.2) if we verify the following two claims:

$$
\begin{equation*}
w(E) \leq c_{3} \exp \left(-c_{4} \varepsilon^{-1}\right) w(Q) \tag{4.3}
\end{equation*}
$$

and for any $(x, t) \in Q, 0<r \leq R$,

$$
\begin{equation*}
w\left(E \cap \tilde{Q}_{r}(x, t)\right)<c_{5} \exp \left(-c_{6} \varepsilon^{-1}\right) w\left(\tilde{Q}_{r}(x, t)\right) \tag{4.4}
\end{equation*}
$$

provided that $\tilde{Q}_{r}(x, t) \cap Q \cap F^{c} \neq \emptyset$ and $E \cap \tilde{Q}_{r}(x, t) \neq \emptyset$, where constants $c_{3}, c_{4}, c_{5}$ and $c_{6}$ depend on $N, \alpha, p$ and $[w]_{A_{\infty}}$.
Claim (4.3): Set

$$
g_{k}(x, t)=\int_{2^{-k} R}^{2^{-k+1} R}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho}
$$

We have for $m \in \mathbb{N}$ and $(x, t) \in E$

$$
\begin{aligned}
\mathbb{W}_{\alpha, p}^{R}[\mu](x, t) & =\sum_{k=m+1}^{\infty} g_{k}(x, t)+\int_{2^{-m} R}^{R}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho} \\
& \leq \sum_{k=m+1}^{\infty} g_{k}(x, t)+m\left(\mathbb{M}_{\alpha p}^{R}[\mu](x, t)\right)^{\frac{1}{p-1}} \\
& \leq \sum_{k=m+1}^{\infty} g_{k}(x, t)+m \varepsilon \lambda .
\end{aligned}
$$

We deduce that for $\beta>0, m \in \mathbb{N}$

$$
\begin{aligned}
|E| & \leq\left|Q \cap\left\{\sum_{k=m+1}^{\infty} g_{k}>(1-m \varepsilon) \lambda\right\}\right| \\
& =\left|Q \cap\left\{\sum_{k=m+1}^{\infty} g_{k}>\sum_{k=m+1}^{\infty} 2^{-\beta(k-m-1)}\left(1-2^{-\beta}\right)(1-m \varepsilon) \lambda\right\}\right| \\
& \leq \sum_{k=m+1}^{\infty}\left|Q \cap\left\{g_{k}>2^{-\beta(k-m-1)}\left(1-2^{-\beta}\right)(1-m \varepsilon) \lambda\right\}\right|
\end{aligned}
$$

We can assume that $\left(x_{0}, t_{0}\right) \in Q,\left(\mathbb{M}_{\alpha p}^{R}[\mu]\left(x_{0}, t_{0}\right)\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda$. Thus, by computing, see $[16$, Proof of Proposition 2.2] we have for any $k \in \mathbb{N}$

$$
\left|Q \cap\left\{g_{k}>s\right\}\right| \leq \frac{c_{7}}{s^{p-1}} 2^{-k \alpha p}|Q|(\varepsilon \lambda)^{p-1}
$$

Consequently,

$$
\begin{aligned}
|E| & \leq \sum_{k=m+1}^{\infty} \frac{c_{7}}{\left(2^{-\beta(k-m-1)}\left(1-2^{-\beta}\right)(1-m \varepsilon) \lambda\right)^{p-1}} 2^{-k \alpha p}|Q|(\varepsilon \lambda)^{p-1} \\
& \leq c_{7} 2^{-(m+1) \alpha p}\left(\frac{\varepsilon}{1-m \varepsilon}\right)^{p-1}|Q|\left(1-2^{-\beta}\right)^{-p+1} \sum_{k=m+1}^{\infty} 2^{(\beta(p-1)-\alpha p)(k-m-1)} .
\end{aligned}
$$

If we choose $\varepsilon^{-1}-2<m \leq \varepsilon^{-1}-1$ and $\beta=\beta(\alpha, p)$ so that $\beta(p-1)-\alpha p<0$, we obtain

$$
|E| \leq c_{8} \exp \left(-\alpha p \ln (2) \varepsilon^{-1}\right)|Q|
$$

Thus, we get (4.3).
Claim (4.4). Take $(x, t) \in Q$ and $0<r \leq R$. Now assume that $\tilde{Q}_{r}(x, t) \cap Q \cap F^{c} \neq \emptyset$ and $E \cap \tilde{Q}_{r}(x, t) \neq \emptyset$ i.e, there exist $\left(x_{1}, t_{1}\right),\left(\bar{x}_{2}, t_{2}\right) \in \tilde{Q}_{r}(x, t) \cap Q$ such that $\mathbb{W}_{\alpha, p}^{R}[\mu]\left(x_{1}, t_{1}\right) \leq \lambda$ and $\left(\mathbb{M}_{\alpha p}^{R}[\mu]\left(x_{2}, t_{2}\right)\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda$. We need to prove that

$$
w\left(E \cap \tilde{Q}_{r}(x, t)\right)<c_{9} \exp \left(-c_{10} \varepsilon^{-1}\right) w\left(\tilde{Q}_{r}(x, t)\right)
$$

To do this, for all $(y, s) \in E \cap \tilde{Q}_{r}(x, t) . \tilde{Q}_{\rho}(y, s) \subset \tilde{Q}_{3 \rho}\left(x_{1}, t_{1}\right)$ if $\rho>r$. If $r \leq R / 3$,

$$
\begin{aligned}
\mathbb{W}_{\alpha, p}^{R}[\mu](y, s) & =\mathbb{W}_{\alpha, p}^{r}[\mu](y, s)+\int_{r}^{R / 3}\left(\frac{\mu\left(\tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho}+\int_{R / 3}^{R}\left(\frac{\mu\left(\tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho} \\
& \leq \mathbb{W}_{\alpha, p}^{r}[\mu](y, s)+\int_{r}^{R / 3}\left(\frac{\mu\left(\tilde{Q}_{3 \rho}\left(x x_{1}, t_{1}\right)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho}+2\left(\mathbb{M}_{\alpha p}^{R}[\mu](y, s)\right)^{\frac{1}{p-1}} \\
& \leq \mathbb{W}_{\alpha, p}^{r}[\mu](y, s)+3^{\frac{N+2-\alpha p}{p-1}} \lambda+2 \varepsilon \lambda .
\end{aligned}
$$

which follows $\mathbb{W}_{\alpha, p}^{r}[\mu](y, s)>\lambda$.
If $r \geq R / 3$

$$
\begin{aligned}
\mathbb{W}_{\alpha, p}^{R}[\mu](y, s) & \leq \mathbb{W}_{\alpha, p}^{r}[\mu](y, s)+\int_{R / 3}^{R}\left(\frac{\mu\left(\tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho} \\
& \leq \mathbb{W}_{\alpha, p}^{r}[\mu](y, s)+2 \varepsilon \lambda
\end{aligned}
$$

which follows $\mathbb{W}_{\alpha, p}^{r}[\mu](y, s)>\lambda$.
Thus,

$$
w\left(E \cap \tilde{Q}_{r}(x, t)\right) \leq w\left(\tilde{Q}_{r}(x, t) \cap\left\{\mathbb{W}_{\alpha, p}^{r}[\mu]>\lambda\right\}\right)
$$

Since $\left(x_{2}, t_{2}\right) \in \tilde{Q}_{r}(x, t),\left(\mathbb{M}_{\alpha p}^{R}[\mu]\left(x_{2}, t_{2}\right)\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda$, so as above we also obtain

$$
w\left(\tilde{Q}_{r}(x, t) \cap\left\{\mathbb{W}_{\alpha, p}^{r}[\mu]>\lambda\right\}\right) \leq c_{9} \exp \left(-c_{10} \varepsilon^{-1}\right) w\left(\tilde{Q}_{r}(x, t)\right)
$$

which implies (4.4). This completes the proof of the Theorem.

Theorem 4.2 Let $\alpha>0,1<p<\alpha^{-1}(N+2)$, $p-1<q<\infty$ and $0<s \leq \infty$ and $w \in A_{\infty}$. There holds

$$
\begin{equation*}
C^{-1}\left\|\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)} \leq\left\|\mathbb{W}_{\alpha, p}^{R}[\mu]\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)} \leq C\left\|\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)} \tag{4.5}
\end{equation*}
$$

for all $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ and $R \in(0, \infty]$ where $C$ is a positive constant only depending on $N, \alpha, p, q, s$ and $[w]_{A_{\infty}}$.

Proof. From (4.1) in Theorem (4.1), we have for $0<s<\infty$

$$
\begin{aligned}
& \left\|\mathbb{W}_{\alpha, p}^{R}[\mu]\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)}^{s}=a^{s} q \int_{0}^{\infty} \lambda^{s} w\left(\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>a \lambda\right\}\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda} \\
& \quad \leq c_{1} \exp \left(-c_{2} \varepsilon^{-1}\right) q \int_{0}^{\infty} \lambda^{s} w\left(\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>\lambda\right\}\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda}+c_{3} s \int_{0}^{\infty} \lambda^{s} w\left(\left\{\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}}>\varepsilon \lambda\right\}\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda} \\
& \quad=c_{1} \exp \left(-c_{2} \varepsilon^{-1}\right)\left\|\mathbb{W}_{\alpha, p}^{R}[\mu]\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)}^{s}+c_{3} \varepsilon^{-s}\left\|\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)}^{s} .
\end{aligned}
$$

Choose $0<\varepsilon<\varepsilon_{0}$ such that $c_{1} \exp \left(-c_{2} \varepsilon^{-1}\right)<1 / 2$ we get

$$
\left\|\mathbb{W}_{\alpha, p}^{R}[\mu]\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)}^{s} \leq c_{4}\left\|\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q, s}\left(\mathbb{R}^{N+1}, d w\right)}^{s}
$$

Similarly, we also get above inequality in case $s=\infty$. So, we proved the right-hand side inequality of (4.5).
To complete the proof, we prove the left-hand side inequality of (4.5). Since for every $(x, t) \in \mathbb{R}^{N+1}$

$$
\begin{gathered}
\left(\mathbb{W}_{\alpha p}^{R}[\mu](x, t)\right)^{\frac{1}{p-1}} \leq c_{5}\left(\mathbb{W}_{\alpha, p}^{R}[\mu](x, t)+\left(\frac{\mu\left(\tilde{Q}_{2 R}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\right) \text { and } \\
\left(\frac{\mu\left(\tilde{Q}_{R / 2}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \leq c_{6} \mathbb{W}_{\alpha, p}^{R}[\mu](x, t)
\end{gathered}
$$

thus it is enough to show that for any $\lambda>0$

$$
\begin{equation*}
w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{2 R}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right) \leq c_{7} w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{R / 2}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>c_{8} \lambda\right\}\right) \tag{4.6}
\end{equation*}
$$

Let $\left\{Q_{j}\right\}=\left\{\tilde{Q}_{R / 4}\left(x_{j}, t_{j}\right)\right\}$ be a cover of $\mathbb{R}^{N+1}$ such that for any $Q_{j} \in\left\{Q_{j}\right\}$, there exist $Q_{j, 1}, \ldots, Q_{j, M_{1}} \in\left\{Q_{j}\right\}$ with $\sum_{j} \sum_{k=1}^{M_{1}} \chi_{Q_{j, k}} \leq M_{2}$ and $Q_{j}+\tilde{Q}_{2 R}(0,0) \subset \bigcup_{k=1}^{M_{1}} Q_{j, k}$ for some integer constants $M_{i}=M_{i}(N), i=1,2$. Then,

$$
\begin{aligned}
& w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{2 R}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right) \leq \sum_{j} w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{2 R}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\} \cap Q_{j}\right) \\
& \quad \leq \sum_{j} w\left(\left\{(x, t): \sum_{k=1}^{M_{1}} \frac{\mu\left(Q_{j, k}\right)}{\left.\left.R^{N+2-\alpha p}>\lambda^{p-1}\right\} \cap Q_{j}\right)}\right.\right. \\
& \quad \leq \sum_{j} \sum_{k=1}^{M_{1}} w\left(\left\{(x, t):\left(\frac{\mu\left(Q_{j, k}\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_{1}^{-1 /(p-1)} \lambda\right\} \cap Q_{j}\right) \\
& \quad=\sum_{j} \sum_{k=1}^{M_{1}} a_{j, k} w\left(Q_{j}\right)
\end{aligned}
$$

where $a_{j, k}=1$ if $\left(\frac{\mu\left(Q_{j, k}\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_{1}^{-1 /(p-1)} \lambda$ and $a_{j, k}=0$ if otherwise.
Using the strong doubling property of $w$, there is $c_{9}=c_{9}\left(N,[w]_{A_{\infty}}\right)$ such that $w\left(Q_{j}\right) \leq$ $c_{9} w\left(Q_{j, k}\right)$. On the other hand, if $a_{j, k}=1$ then $Q_{j, k} \subset\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{R / 2}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_{1}^{-1 /(p-1)} \lambda\right\}$. Therefore,

$$
\begin{aligned}
& w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{2 R}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right) \leq \sum_{j} \sum_{k=1}^{M_{1}} c_{9} a_{j, k} w\left(Q_{j, k}\right) \\
& \quad \leq \sum_{j} \sum_{k=1}^{M_{1}} c_{9} w\left(\left\{(x, t):\left(\frac{\mu\left(\tilde{Q}_{R / 2}(x, t)\right)}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_{1}^{-1 /(p-1)} \lambda\right\} \cap Q_{j, k}\right)
\end{aligned}
$$

which implies (4.6) since $\sum_{j} \sum_{k=1}^{M_{1}} \chi_{Q_{j, k}} \leq M_{2}$ in $\mathbb{R}^{N+1}$.
Theorem 4.3 Let $0<\alpha p<N+2$ and $w \in A_{\infty}$ There exist $C_{1}, C_{2}>0$ depending on $N, \alpha, p$ and $[w]_{A_{\infty}}$ such that for any $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, any cylinder $\tilde{Q}_{\rho} \subset \mathbb{R}^{N+1}$ there holds

$$
\begin{equation*}
\frac{1}{w\left(\tilde{Q}_{2 \rho}\right)} \int_{\tilde{Q}_{2 \rho}} \exp \left(C_{1} \mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right](x, t)\right) d w(x, t) \leq C_{2} \tag{4.7}
\end{equation*}
$$

provided $\left\|\mathbb{M}_{\alpha p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]\right\|_{L^{\infty}\left(\tilde{Q}_{\rho}\right)} \leq 1$, where $\mu_{\tilde{Q}_{\rho}}=\chi_{\tilde{Q}_{\rho}} \mu$.
Proof. Assume that $\left\|\mathbb{M}_{\alpha p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]\right\|_{L^{\infty}\left(\tilde{Q}_{\rho}\right)} \leq 1$. We apply Theorem (4.1) to $\mu_{\tilde{Q}_{\rho}}$. Then, choose $\varepsilon=\lambda^{-1}$ for all $\lambda \geq \lambda_{0}:=\max \left\{\varepsilon_{0}^{-1}, \frac{N+2-\alpha p}{p-1}\right\}$, we obtain

$$
w\left(\left\{\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]>a \lambda\right\} \cap \tilde{Q}_{2 \rho}\right) \leq C_{1} \exp \left(-C_{2} \varepsilon^{-1}\right) w\left(\left\{\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]>\lambda\right\}\right) \quad \forall \lambda \geq \lambda_{0}
$$

On the other hand, if $\rho>R$, clearly we have $\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right] \equiv 0$ in $\mathbb{R}^{N+1} \backslash \tilde{Q}_{2 \rho}$, if $\rho \leq R$, for any $(x, t) \in \mathbb{R}^{N+1} \backslash \tilde{Q}_{2 \rho}$

$$
\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right](x, t)=\int_{\rho}^{R}\left(\frac{\mu_{\tilde{Q}_{\rho}}\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d r}{r} \leq \frac{N+2-\alpha p}{p-1}\left(\frac{\mu\left(\tilde{Q}_{\rho}\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \leq \lambda_{0}
$$

So, we get $\left\{\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]>\lambda\right\} \subset \tilde{Q}_{2 \rho}$ for all $\lambda \geq \lambda_{0}$. This can be written under the form

$$
w\left(\left\{\mathbb{W}_{\alpha, p}^{R}\left[\mu_{\tilde{Q}_{\rho}}\right]>a \lambda\right\} \cap \tilde{Q}_{2 \rho}\right) \leq\left(\chi_{\left(0, t_{0}\right]}+C_{1} \exp \left(-C_{2} \lambda\right)\right) w\left(\tilde{Q}_{2 \rho}\right)
$$

for all $\lambda>0$. Therefore, we get (4.7).
In what follows, we need some estimates on Wolff parabolic potential:
Proposition 4.4 Let $p>1,0<\alpha p<N+2$ and $q>1, \alpha p q<N+2$. There exist $C_{1}, C_{2}$ such that

$$
\begin{gather*}
\left\|\mathbb{W}_{\alpha, p}[\mu]\right\|_{L^{\frac{(N+2)(p-1)}{N+2-\alpha p}, \infty}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\left(\mu\left(\mathbb{R}^{N+1}\right)\right)^{\frac{1}{p-1}} \quad \forall \mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)  \tag{4.8}\\
\left\|\mathbb{W}_{\alpha, p}[\mu]\right\|_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha p q}, \infty}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\|\mu\|_{L^{q, \infty}\left(\mathbb{R}^{N+1}\right)}^{\frac{1}{p-1}} \forall \mu \in L^{q, \infty}\left(\mathbb{R}^{N+1}\right), \mu \geq 0 \tag{4.9}
\end{gather*}
$$

and

$$
\begin{equation*}
\left\|\left.\mathbb{W}_{\alpha, p}[\mu]\right|_{L^{q(N+2)(p-1)}}\left(\mathbb{R}^{N+1}\right) \leq C_{2}\right\| \mu \|_{L^{q}\left(\mathbb{R}^{N+1}\right)}^{\frac{1}{p-1}} \quad \forall \mu \in L^{q}\left(\mathbb{R}^{N+1}\right), \mu \geq 0 \tag{4.10}
\end{equation*}
$$

In particular, for $s>\frac{(p-1)(N+2)}{N+2-\alpha p}$, we define $F(\mu):=\left(\mathbb{W}_{\alpha, p}[\mu]\right)^{s}$ for all $\mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)$. Then,

$$
\begin{aligned}
& \|F(\mu)\|_{L}^{\frac{(N+2)(s-p+1)}{\alpha s p}}\left(\mathbb{R}^{N+1}\right) \\
& \|F(\mu)\|_{L_{3}\|\mu\|_{\left.L^{\frac{(N+2)(s-p+1)}{\left(\frac{s}{p-1}\right.}},_{\left(\mathbb{R}^{N+1}\right)(s-p+1)}^{\alpha s p}\right)} \leq C_{3}\|\mu\|_{\left.\mathbb{R}^{N+1}\right)}^{\frac{s}{p-1}} \quad \text { and }}^{L^{\frac{(N+2)(s-p+1)}{\alpha s p}}, \infty_{\left(\mathbb{R}^{N+1}\right)}},
\end{aligned}
$$

for some constant $C_{i}=C_{i}(N, p, \alpha, s)$ for $i=3,4$.
Proof. Let $s \geq 1$ such that $\alpha s p<N+2$. It is known that if $\mu \in L^{s, \infty}\left(\mathbb{R}^{N+1}\right)$ then

$$
|\mu|\left(\tilde{Q}_{\rho}(x, t)\right) \leq c_{1}\|\mu\|_{L^{s, \infty}\left(\mathbb{R}^{N+1}\right)} \rho^{\frac{N+2}{s^{\prime}}} \forall \rho>0 .
$$

Thus for $\delta=\|\mu\|_{L^{s, \infty}\left(\mathbb{R}^{N+1}\right)}^{\frac{s}{N+2}}(\mathbb{M}(\mu)(x, t))^{-\frac{s}{N+2}}$ we have

$$
\begin{aligned}
\mathbb{W}_{\alpha, p}[\mu](x, t) & =\int_{0}^{\delta}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho}+\int_{\delta}^{\infty}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d \rho}{\rho} \\
& \leq c_{2}(\mathbb{M}(\mu)(x, t))^{\frac{1}{p-1}} \delta^{\frac{\alpha p}{p-1}}+c_{2}\|\mu\|_{L^{s, \infty}\left(\mathbb{R}^{N+1}\right)} \delta^{-\frac{1}{\rho-1}} \frac{{ }^{\frac{N+2-\alpha s p}{s(p-1)}}}{} \\
& =c_{3}(\mathbb{M}(\mu)(x, t))^{\frac{N+2-\alpha s p}{(p-1)(N+2)}}\|\mu\|_{L^{s, \infty}\left(\mathbb{R}^{N+1}\right)}^{(p-\alpha s p} .
\end{aligned}
$$

So, for any $\lambda>0$

$$
\left|\left\{\mathbb{W}_{\alpha, p}[\mu]>\lambda\right\}\right| \leq\left|\left\{\mathbb{M}(\mu)>c_{4}\|\mu\|_{L^{s, 2}\left(\mathbb{R}^{N+1}\right)}^{-\frac{\alpha s p}{N+2-\alpha s p}} \lambda^{\frac{(p-1)(N+2)}{N+2-\alpha s p}}\right\}\right| .
$$

Hence, since $\mathbb{M}$ is bounded from $\mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)$ to $L^{1, \infty}\left(\mathbb{R}^{N+1}\right)$ and $L^{q}\left(\mathbb{R}^{N+1}\right)\left(L^{q, \infty}\left(\mathbb{R}^{N+1}\right)\right.$ resp. $)$ to itself, we get the result.

Remark 4.5 Assume that $\alpha p=N+2$ and $R>0$. As above we also have for any $\varepsilon>0$
$\mathbb{W}_{\alpha, p}^{R}[\mu](x, t) \leq C_{1, \varepsilon} \max \left\{\left(|\mu|\left(\mathbb{R}^{N+1}\right)\right)^{\frac{1}{p-1}},\left((\mathbb{M}(\mu)(x, t))^{\varepsilon}\left(|\mu|\left(\mathbb{R}^{N+1}\right)\right)^{\frac{\alpha p}{p-1}} R^{\varepsilon \alpha p}\right)^{\frac{1}{\alpha p+\varepsilon(p-1)}}\right\}$
where $C_{1, \varepsilon}=C_{1}(N, \alpha, p, \varepsilon)$.
Therefore, for any $\lambda>C_{\varepsilon}\left(|\mu|\left(\mathbb{R}^{N+1}\right)\right)^{\frac{1}{p-1}}$,

$$
\begin{equation*}
\left|\left\{\mathbb{W}_{\alpha, p}^{R}[\mu]>\lambda\right\}\right| \leq C_{2, \varepsilon}\left(\frac{\left(|\mu|\left(\mathbb{R}^{N+1}\right)\right)^{\frac{1}{p^{-1}}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} R^{\alpha p} \tag{4.11}
\end{equation*}
$$

where $C_{2, \varepsilon}=C_{2}(N, \alpha, p, \varepsilon)$. In particular, if $\mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)$ then $\mathbb{W}_{\alpha, p}^{R}[\mu] \in L_{l o c}^{s}\left(\mathbb{R}^{N+1}\right)$ for all $s>0$.

Remark 4.6 Assume that $p, q>1,0<\alpha p q<N+2$. As in [59, Theorem 3], it is easy to prove that if $w \in A_{\frac{q(N+2-\alpha)}{N+2-\alpha p q}}$, i.e, $0<w \in L_{\text {loc }}^{1}\left(\mathbb{R}^{N+1}\right)$ and for any $\tilde{Q}_{\rho}(y, s) \subset \mathbb{R}^{N+1}$

$$
\sup _{\tilde{Q}_{\rho}(y, s) \subset \mathbb{R}^{N+1}}\left(\left(f_{\tilde{Q}_{\rho}(y, s)} w d x d t\right)\left(f_{\tilde{Q}_{\rho}(y, s)} w^{-\frac{N+2-\alpha p q}{(q-1)(N+2)}} d x d t\right)^{\frac{(q-1)(N+2)}{N+2-\alpha p q}}\right)=C_{1}<\infty
$$

then

$$
\left(\int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_{\alpha p}[|f|]\right)^{\frac{(N+2) q}{N+2-\alpha p q}} w d x d t\right)^{\frac{N+2-\alpha p q}{(N+2) q}} \leq C_{2}\left(\int_{\mathbb{R}^{N+1}}|f|^{q} w^{1-\frac{\alpha p q}{N+2}} d x d t\right)^{\frac{1}{q}}
$$

for some a constant $C_{2}=C_{2}\left(N, \alpha p, q, C_{1}\right)$.
Therefore, from (4.5) in Theorem 4.2 we get a weighted version of (4.10)

$$
\left(\int_{\mathbb{R}^{N+1}}\left(\mathbb{W}_{\alpha, p}[|f|]\right)^{\frac{(N+2)(p-1) q}{N+2-\alpha p q}} w d x d t\right)^{\frac{N+2-\alpha p q}{(N+2) q}} \leq C_{2}\left(\int_{\mathbb{R}^{N+1}}|f|^{p} w^{1-\frac{\alpha p}{N+2}} d x d t\right)^{\frac{1}{p}}
$$

The following another version of (4.10) in the Lorentz-Morrey spaces involving calorie.
Proposition 4.7 Let $p, q>1$, and $0<\alpha p q<\theta \leq N+2$. There exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\left(\mathbb{W}_{\alpha, p}[|\mu|]\right)^{p-1}\right\|_{L^{\frac{\theta q}{\theta-\alpha p q} ; \theta}\left(\mathbb{R}^{N+1}\right)} \leq C\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)} \quad \forall \mu \in L^{q ; \theta}\left(\mathbb{R}^{N+1}\right) \tag{4.12}
\end{equation*}
$$

Proof. As the proof of Proposition 4.4 we have

$$
\mathbb{W}_{\alpha, p}[|\mu|] \leq c_{1}\left(\mathbb{M}_{\theta / q}[|\mu|]\right)^{\frac{\alpha p q}{\theta(p-1)}}(\mathbb{M}[|\mu|])^{\frac{\theta-\alpha p q}{\theta(p-1)}}
$$

Since $\mathbb{M}_{\theta / q}[|\mu|] \leq c_{2}\left(\mathbb{M}_{\theta}\left[|\mu|^{q}\right]\right)^{1 / q}$, above inequality becomes

$$
\begin{equation*}
\mathbb{W}_{\alpha, p}[\mu] \leq c_{3}\left(\mathbb{M}_{\theta}\left[|\mu|^{q}\right]\right)^{\frac{\alpha p}{\theta(p-1)}}(\mathbb{M}[\mu])^{\frac{\theta-\alpha p q}{\theta(p-1)}} \tag{4.13}
\end{equation*}
$$

Take $\tilde{Q}_{\rho}(y, s) \subset \mathbb{R}^{N+1}$, we have

$$
\begin{aligned}
& \int_{\tilde{Q}_{\rho}(y, s)}\left(\mathbb{W}_{\alpha, p}[\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t \leq c_{4}\left(\int_{\tilde{Q}_{\rho}(y, s)}\left(\mathbb{W}_{\alpha, p}\left[\chi_{\tilde{Q}_{2 \rho}(y, s)} \mu\right]\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t\right. \\
&\left.+\int_{\tilde{Q}_{\rho}(y, s)}\left(\mathbb{W}_{\alpha, p}\left[\chi_{\left(\tilde{Q}_{2 \rho}(y, s)\right)^{c}} \mu\right]\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t\right) \\
&=A+B .
\end{aligned}
$$

Using inequality (4.13) and boundless $\mathbb{M}$ from $L^{q}\left(\mathbb{R}^{N+1}\right)$ to itself, yield

$$
\begin{aligned}
A & \leq c_{5} \int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_{\theta}\left[|\mu|^{q}\right]\right)^{\frac{\alpha q}{\theta-\alpha p q}}\left(\mathbb{M}\left[\chi_{\tilde{Q}_{2 \rho}(y, s)} \mu\right]\right)^{q} d x d t \\
& \leq c_{6}\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)}^{\frac{\alpha q^{2}}{\theta-p q}} \int_{\chi_{\tilde{Q}_{2 \rho}(y, s)}}|\mu|^{q} d x d t \\
& \leq c_{7}\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)}^{\frac{\theta q}{\theta-q p}} \rho^{N+2-\theta}
\end{aligned}
$$

On the other hand, since $|\mu|\left(\tilde{Q}_{r}(x, t)\right) \leq c_{8}\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)} r^{N+2-\frac{\theta}{q}}$ for all $\tilde{Q}_{r}(x, t) \subset \mathbb{R}^{N+1}$,

$$
\begin{aligned}
B & \leq \int_{\tilde{Q}_{\rho}(y, s)}\left(\int_{\rho}^{\infty}\left(\frac{|\mu|\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d r}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t \\
& \leq c_{9} \int_{\tilde{Q}_{\rho}(y, s)}\left(\int_{\rho}^{\infty}\left(\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)} r^{-\frac{\theta}{q}+\alpha}\right)^{\frac{1}{p-1}} \frac{d r}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t \\
& \leq c_{10}\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)}^{\frac{\theta q}{\theta-\alpha q}} \rho^{N+2-\theta} .
\end{aligned}
$$

Therefore,

$$
\int_{\tilde{Q}_{\rho}(y, s)}\left(\mathbb{W}_{\alpha, p}[\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha p q}} d x d t \leq c_{11}\|\mu\|_{L^{q ; \theta}\left(\mathbb{R}^{N+1}\right)}^{\frac{\theta q}{\theta-\alpha p}} \rho^{N+2-\theta},
$$

which follows (4.12).
In the next result we state a series of equivalent norms concerning potentials $\mathbb{I}_{\alpha}[\mu], \mathbb{I}_{\alpha}^{R}[\mu], \mathcal{H}_{\alpha}[\mu], \mathcal{G}_{\alpha}[\mu]$.

Proposition 4.8 Let $q>1,0<\alpha<N+2$ and $R>0$. There exist constants $C_{1}=$ $C_{1}(N, \alpha, q)$ and $C_{2}=C_{2}(N, \alpha, q, R)$ such that the following statements hold
a. for any $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$

$$
\begin{align*}
& C_{1}^{-1}\left\|\mathbb{I}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\mathcal{H}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\left\|\mathbb{I}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \text { and }  \tag{4.14}\\
& C_{1}^{-1}\left\|\mathbb{I}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\mathcal{H}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C_{1}\left\|\mathbb{I}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \tag{4.15}
\end{align*}
$$

b. for any $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$

$$
\begin{align*}
& C_{2}^{-1}\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\mathcal{G}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \quad \text { and }  \tag{4.16}\\
& C_{2}^{-1}\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\left\|\mathcal{G}_{\alpha}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C_{2}\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \tag{4.17}
\end{align*}
$$

where $\stackrel{\vee}{\mathcal{H}}_{\alpha}[\mu]$ is the backward parabolic Riesz potential, defined by

$$
\stackrel{\mathcal{H}}{\alpha}^{\vee}[\mu](x, t)=\left(\stackrel{\mathcal{H}}{\alpha}^{v} \mu\right)(x, t)=\int_{\mathbb{R}^{N+1}} \mathcal{H}_{\alpha}(x-y, s-t) d \mu(y, s),
$$

and $\stackrel{\vee}{\mathcal{G}}_{\alpha}[\mu]$ is the backward parabolic Bessel potential:

$$
\stackrel{\vee}{\mathcal{G}}_{\alpha}[\mu](x, t)=\left(\stackrel{\vee}{\mathcal{G}}_{\alpha} * \mu\right)(x, t)=\int_{\mathbb{R}^{N+1}} \mathcal{G}_{\alpha}(y-x, s-t) d \mu(y, s)
$$

Proof. a. We have:

$$
\frac{c_{1}^{-1}}{t^{\frac{N+2-\alpha}{2}}} \chi_{t>0} \chi_{|x| \leq 2 \sqrt{t}} \leq \mathcal{H}_{\alpha}(x, t) \leq \frac{c_{1}}{\max \{|x|, \sqrt{2|t|}\}^{N+2-\alpha}}
$$

which implies

$$
c_{2}^{-1} \int_{0}^{\infty} \frac{\chi_{B_{r}(0) \times\left(\frac{r^{2}}{4}, r^{2}\right)}(x, t)}{r^{N+2-\alpha}} \frac{d r}{r} \leq \mathcal{H}_{\alpha}(x, t) \leq c_{2} \int_{0}^{\infty} \frac{\chi_{\tilde{Q}_{r}(0,0)}(x, t)}{r^{N+2-\alpha}} \frac{d r}{r}
$$

Thus,

$$
\begin{equation*}
c_{2}^{-1} \int_{0}^{\infty} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-\frac{r^{2}}{4}\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r} \leq \mathcal{H}_{\alpha}[\mu](x, t) \leq c_{2} \mathbb{I}_{\alpha}[\mu](x, t) \tag{4.18}
\end{equation*}
$$

Thanks to Theorem 4.2 we will finish the proof of (4.14) when we show that

$$
\int_{\mathbb{R}}\left(\int_{0}^{\infty} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-\frac{r^{2}}{4}\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \geq c_{3} \int_{\mathbb{R}} \int_{0}^{+\infty}\left(\frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}}\right)^{q} \frac{d r}{r} d t
$$

Indeed, we have for $r_{k}=\left(\frac{2}{\sqrt{3}}\right)^{-k}$,

$$
\begin{aligned}
& \left(\int_{0}^{\infty} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-r^{2} / 4\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} \\
& \quad \geq c_{4}\left(\sum_{k=-\infty}^{\infty} \frac{\mu\left(B\left(x, r_{k}\right) \times\left(t-r_{k}^{2}, t-\frac{1}{3} r_{k}^{2}\right)\right)}{r_{k}^{N+2-\alpha}}\right)^{q} \\
& \quad \geq c_{4} \sum_{k=-\infty}^{\infty}\left(\frac{\mu\left(B\left(x, r_{k}\right) \times\left(t-r_{k}^{2}, t-\frac{1}{3} r_{k}^{2}\right)\right)}{r_{k}^{N+2-\alpha}}\right)^{q}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \int_{\mathbb{R}}\left(\int_{0}^{\infty} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-\frac{1}{4} r^{2}\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \\
& \quad \geq c_{4} \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}}\left(\frac{\mu\left(B\left(x, r_{k}\right) \times\left(t-r_{k}^{2}, t-\frac{1}{3} r_{k}^{2}\right)\right)}{r_{k}^{N+2-\alpha}}\right)^{q} d t \\
& \quad=c_{4} \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}}\left(\frac{\mu\left(B\left(x, r_{k}\right) \times\left(t-\frac{1}{3} r_{k}^{2}, t+\frac{1}{3} r_{k}^{2}\right)\right)}{r_{k}^{N+2-\alpha}}\right)^{q} d t \\
& \quad \geq c_{5} \int_{\mathbb{R}} \int_{0}^{+\infty}\left(\frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}}\right)^{q} \frac{d r}{r} d t .
\end{aligned}
$$

Similarly, we also can prove (4.15).
b. Obviously

$$
\begin{aligned}
& \frac{c_{6}^{-1} \exp \left(-4 R^{2}\right)}{t^{\frac{N+2-\alpha}{2}}} \chi_{0<t<4 R^{2}} \chi_{|x| \leq 2 \sqrt{t}} \leq \mathcal{G}_{\alpha}(x, t) \\
& \quad \leq \frac{c_{6}}{\max \{|x|, \sqrt{2|t|}\}^{N+2-\alpha}} \chi_{\tilde{Q}_{R / 2}(0,0)}(x, t)+\frac{c_{6}}{R^{N+2-\alpha}} \exp (-\max \{|x|, \sqrt{2|t|}\})
\end{aligned}
$$

Thus, we can assert that

$$
\begin{aligned}
& c_{7}(R) \int_{0}^{2 R} \frac{\chi_{B_{r}(0) \times\left(\frac{r^{2}}{4}, r^{2}\right)}(x, t)}{r^{N+2-\alpha}} \frac{d r}{r} \leq \mathcal{G}_{\alpha}(x, t) \leq c_{8} \int_{0}^{R} \frac{\chi_{\tilde{Q}_{r}(0,0)}(x, t)}{r^{N+2-\alpha}} \frac{d r}{r} \\
& \quad+c_{9}(R) \int_{\mathbb{R}^{N+1}} \exp (-\max \{|y|, \sqrt{2|s|}\}) \chi_{\tilde{Q}_{R / 2}(0,0)}(x-y, t-s) d y d s
\end{aligned}
$$

Immediately, we get

$$
\begin{equation*}
c_{7}(R) \int_{0}^{2 R} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-\frac{r^{2}}{4}\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r} \leq \mathcal{G}_{\alpha}[\mu](x, t) \leq c_{8} \mathbb{I}_{\alpha}^{R}[\mu](x, t)+c_{9}(R) F(x, t) \tag{4.19}
\end{equation*}
$$

where $F(x, t)=\int_{\mathbb{R}^{N+1}} \exp (-\max \{|y|, \sqrt{2|s|}\}) \mu\left(\tilde{Q}_{R / 2}(x-y, t-s)\right) d y d s$.
As above, we can show that

$$
\int_{0}^{\infty}\left(\int_{0}^{2 R} \frac{\mu\left(B(x, r) \times\left(t-r^{2}, t-\frac{r^{2}}{4}\right)\right)}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \geq c_{10} \int_{0}^{\infty} \int_{0}^{R}\left(\frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}}\right)^{q} \frac{d r}{r}
$$

Thus, thanks to Theorem 4.2 we get the left-hand side inequality of (4.16).
To show the right-hand side of (4.16), we use $\mu\left(\tilde{Q}_{R / 2}(x-y, t-s)\right) \leq c_{10} R^{-(N+2-\alpha)} \mathbb{I}_{\alpha}^{R}[\mu](x-$ $y, t-s$ ) and Young inequality

$$
\begin{aligned}
& \| \mathcal{G}_{\alpha} {[\mu]\left\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq c_{8}\right\| \mathbb{I}_{\alpha}^{R}[\mu]\left\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}+c_{9}(R)\right\| F \|_{L^{q}\left(\mathbb{R}^{N+1}\right)} } \\
& \quad \leq c_{8}\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}+c_{11}(R)\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \int_{\mathbb{R}^{N+1}} \exp (-\max \{|x|, \sqrt{2|t|}) d x d t \\
& \quad=c_{12}(R)\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}
\end{aligned}
$$

Similarly, we also can prove (4.17). This completes the proof of the Proposition.

Remark 4.9 Assume that $0<\alpha<N+2$. From (4.8) in Proposition 4.4 and $\left\|\mathcal{G}_{\alpha}[\mu]\right\|_{L^{1}\left(\mathbb{R}^{N+1}\right)} \leq$ $c_{1} \mu\left(\mathbb{R}^{N+1}\right)$ we deduce that for $1 \leq s<\frac{N+2}{N+2-\alpha}$

$$
\left\|\mathcal{G}_{\alpha}[\mu]\right\|_{L^{s}\left(\mathbb{R}^{N+1}\right)} \leq c_{2} \mu\left(\mathbb{R}^{N+1}\right) \quad \forall \mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)
$$

Next, we introduce the following kernel:

$$
E_{\alpha}^{R}(x, t)=\max \{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)} \chi_{\tilde{Q}_{R}(0,0)}(x, t)
$$

where $0<\alpha<N+2$ and $0<R \leq \infty$. We denote $E_{\alpha}^{\infty}$ by $E_{\alpha}$. It is easy to see that $E_{\alpha} * \mu=(N+2-\alpha) \mathbb{I}_{\alpha}[\mu]$ and $\left\|E_{\alpha}^{R} * \mu\right\|_{L^{s}\left(\mathbb{R}^{N+1}\right)}$ is equivalent to $\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{s}\left(\mathbb{R}^{N+1}\right)}$ for every $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ where $1 \leq s<\infty$.
We obtain equivalences of capacities $\operatorname{Cap}_{E_{\alpha}, p}, \operatorname{Cap}_{E_{\alpha}^{R}, p}, \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}$ and $\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}$.
Corollary 4.10 Let $p>1,1<\alpha<N+2$ and $R>0$. There exist constants $C_{1}=$ $C_{1}(N, \alpha, p)$ and $C_{2}=C_{2}(N, \alpha, p, R)$ such that the following statements hold
a. for any compact $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
C_{1}^{-1} \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E) \leq \operatorname{Cap}_{E_{\alpha}, p}(E) \leq C_{1} \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E) \tag{4.20}
\end{equation*}
$$

b. for any compact $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
C_{2}^{-1} \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E) \leq \operatorname{Cap}_{E_{\alpha}^{R}, p}(E) \leq C_{2} \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E) \tag{4.21}
\end{equation*}
$$

c. for any compact $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E) \leq \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E) \leq C_{1}\left(\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)+\left(\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)\right)^{\frac{N+2}{N+2-\alpha p}}\right) \tag{4.22}
\end{equation*}
$$

provided $1<\alpha p<N+2$.
Proof. By [2, Chapter 2], we have

$$
\begin{aligned}
& \operatorname{Cap}_{E_{\alpha}, p}(E)^{1 / p}=\sup \left\{\mu(E): \mu \in \mathfrak{M}^{+}(E),\left\|E_{\alpha} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\} \\
& \operatorname{Cap}_{E_{\alpha}^{R}, p}(E)^{1 / p}=\sup \left\{\mu(E): \mu \in \mathfrak{M}^{+}(E),\left\|E_{\alpha}^{R} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\} \\
& \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)^{1 / p}=\sup \left\{\mu(E): \mu \in \mathfrak{M}^{+}(E),\|\stackrel{\mathcal{H}}{\alpha}[\mu]\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\} \quad \text { and } \\
& \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)^{1 / p}=\sup \left\{\mu(E): \mu \in \mathfrak{M}^{+}(E),\left\|\stackrel{\vee}{\mathcal{G}_{\alpha}}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}
\end{aligned}
$$

Thanks to (4.15), (4.17) in Proposition 4.8 and $\mathbb{I}_{\alpha}[\mu]=E_{\alpha} * \mu$ and $\left\|E_{\alpha}^{R} * \mu\right\|_{L^{s}\left(\mathbb{R}^{N+1}\right)}$ is equivalent to $\left\|\mathbb{I}_{\alpha}^{R}[\mu]\right\|_{L^{s}\left(\mathbb{R}^{N+1}\right)}$, we get (4.20) and (4.21).
Since $\mathcal{G}_{\alpha} \leq \mathcal{H}_{\alpha}$, thus $\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E) \leq \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)$ for any compact $E \subset \mathbb{R}^{N+1}$. Put $\operatorname{Cap}_{E_{\alpha}, p}(E)=a>0$. We need to prove that

$$
\begin{equation*}
\operatorname{Cap}_{E_{\alpha}^{1}, p}(E) \leq c_{1}\left(a+a^{\frac{N+2}{N+2-\alpha p}}\right) \tag{4.23}
\end{equation*}
$$

We will follow a proof of Yu.V. Netrusov in [2, Chapter 5]. First, we can find $f \in L_{+}^{p}\left(\mathbb{R}^{N+1}\right)$ such that $\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)} \leq 2 a$ and $E_{\alpha} * f \geq \chi_{E}$. Set $F_{\alpha}=E_{\alpha}-E_{\alpha}^{1}$, we have $c_{2} F_{\alpha} \leq E_{\alpha}^{1} * F_{\alpha}$ for some $c_{1}>0$. Thus, $E \subset\left\{E_{\alpha}^{1} * f \geq 1 / 2\right\} \cup\left\{E_{\alpha}^{1} *\left(F_{\alpha} * f\right) \geq c_{2} / 2\right\}$.
Since $\left\|E_{\alpha}^{1}\right\|_{L^{1}\left(\mathbb{R}^{N+1}\right)}<\infty$, for $c_{3}=c_{2}\left(4| | E_{\alpha}^{1} \|_{L^{1}\left(\mathbb{R}^{N+1}\right)}\right)^{-1}$

$$
E_{\alpha}^{1} *\left(F_{\alpha} * f\right) \leq c_{2} / 4+E_{\alpha}^{1} * g \text { with } g=\chi_{F_{\alpha} * f \geq c_{3}} F_{\alpha} * f
$$

which follows $E \subset\left\{E_{\alpha}^{1} * f \geq 1 / 2\right\} \cup\left\{E_{\alpha}^{1} * g \geq c_{2} / 4\right\}$.
Using the subadditivity of capacity, we have

$$
\begin{aligned}
\operatorname{Cap}_{E_{\alpha}^{1}, p}(E) & \leq \operatorname{Cap}_{E_{\alpha}^{1}, p}\left(\left\{E_{\alpha}^{1} * f \geq 1 / 2\right\}\right)+\operatorname{Cap}_{E_{\alpha}^{1}, p}\left(\left\{E_{\alpha}^{1} * g \geq c_{1} / 4\right\}\right) \\
& \leq 2^{p}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}^{p}+\left(4 / c_{1}\right)^{p}\|g\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}^{p} \\
& \leq 2^{p}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}^{p}+\left(4 / c_{1}\right)^{p} c_{3}^{p *-p}\left\|E_{\alpha} * f\right\|_{L^{p *}\left(\mathbb{R}^{N+1}\right)}^{p *}, \text { with } p *=\frac{(N+2) p}{N+2-\alpha p} .
\end{aligned}
$$

On the other hand, from (4.10) in Proposition 4.4 we have

$$
\left\|E_{\alpha} * f\right\|_{L^{p *}\left(\mathbb{R}^{N+1}\right)} \leq c_{4}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}
$$

Hence, we get (4.23).

Remark 4.11 Since $\mathcal{G}_{\alpha} \in L^{1}\left(\mathbb{R}^{N+1}\right)$,

$$
\int_{\mathbb{R}^{N+1}}\left(\mathcal{G}_{\alpha} * f\right)^{p} d x d t \leq\left\|\mathcal{G}_{\alpha}\right\|_{L^{1}\left(\mathbb{R}^{N+1}\right)}^{p} \int_{\mathbb{R}^{N+1}} f^{p} d x d t \quad \forall f \in L_{+}^{p}\left(\mathbb{R}^{N+1}\right)
$$

Thus, for any Borel set $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E) \geq C|E| \text { with } C=\left\|\mathcal{G}_{\alpha}\right\|_{L^{1}\left(\mathbb{R}^{N+1}\right)}^{-p} \tag{4.24}
\end{equation*}
$$

Remark 4.12 It is well-known that $\mathcal{H}_{2}$ is the fundamental solution of the heat operator $\frac{\partial}{\partial t}-\Delta$. In [31], R. Gariepy and W. P. Ziemer introduced the following capacity:

$$
C_{\mathcal{H}_{2}}(K)=\sup \left\{\mu(K): \mu \in \mathfrak{M}^{+}(K), \mathcal{H}_{2}[\mu] \leq 1\right\},
$$

whenever $K \subset \mathbb{R}^{N+1}$ is compact. Thanks to [2, Theorem 2.5.5], we obtain

$$
\operatorname{Cap}_{\mathcal{H}_{1}, 2}(K)=C_{\mathcal{H}_{2}}(K) .
$$

Remark 4.13 For any Borel set $E \subset \mathbb{R}^{N}$, then we always have $\operatorname{Cap}_{\mathcal{G}_{1}, 2}(E \times\{t=0\})=0$ In fact,
$\operatorname{Cap}_{E_{1}^{1}, 2}\left(B_{1}(0) \times\{t=0\}\right)=\sup \left\{\omega\left(B_{1}(0)\right): \omega \in \mathfrak{M}^{+}\left(B_{1}(0)\right),\left\|E_{1}^{1} *\left(\omega \otimes \delta_{0}\right)\right\|_{L^{2}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}$.
Since $\left\|E_{1}^{1} *\left(\omega \otimes \delta_{0}\right)\right\|_{L^{2}\left(\mathbb{R}^{N+1}\right)}=\infty$ if $\omega \neq 0$, thus $\operatorname{Cap}_{\mathcal{G}_{1}, 2}\left(B_{1}(0) \times\{t=0\}\right)=\operatorname{Cap}_{E_{1}^{1}, 2}\left(B_{1}(0) \times\right.$ $\{t=0\})=0$. In particular, $C_{a p} p_{\mathcal{G}_{1}, 2}$ is not absolutely continuous with respect to capacity $C_{1,2}(., \Omega \times(a, b))$. This capacity will be defined in next section.
Remark 4.14 Let $p>1$ and $\alpha>0$. Case $\alpha p \geq p+1$, we always have $\left\|\mathcal{H}_{\alpha}[\mu]\right\|_{L^{p^{\prime}\left(\mathbb{R}^{N}\right)}}=\infty$ for any $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ which implies $\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}\left(\tilde{Q}_{1}(0,0)\right)=0$. If $0<\alpha p<N+2$, Cap $_{\mathcal{H}_{\alpha}, p}\left(\tilde{Q}_{\rho}(0,0)\right)=c \rho^{N+2-\alpha p}$ for some constant c. From (4.22) in Corollary 4.10 we get $\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}\left(\tilde{Q}_{\rho}(0,0)\right) \approx \rho^{N+2-\alpha p}$ for $0<\rho<1$ if $\alpha p<N+2$. Since $\left\|\mathcal{G}_{\alpha}\left[\delta_{(0,0)}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}<\infty$ thus Cap $_{\mathcal{G}_{\alpha}, p}((0,0))>0$ if $\alpha p>N+2$.
If $\alpha p=N+2, \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}\left(\tilde{Q}_{\rho}(0,0)\right) \approx(\log (1 / \rho))^{1-p}$ for any $0<\rho<1 / 2$. In fact, we can prove that $\left\|\mathbb{I}_{\alpha}^{1 / 2}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N}\right)} \leq c_{1}$ for any $d \mu(x, t)=(\log (1 / \rho))^{-1 / p^{\prime}} \rho^{-N-2} \chi_{\tilde{Q}_{\rho}(0,0)} d x d t$ it follows $\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}\left(\tilde{Q}_{\rho}(0,0)\right) \geq c_{2}(\log (1 / \rho))^{1-p}$. Moreover, for $\mu \in \mathfrak{M}^{+}\left(\tilde{Q}_{\rho}\right)$, if $\left\|\mathbb{I}_{\alpha}^{3}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}^{p^{\prime}} \leq 1$,

$$
\begin{aligned}
1 & \geq \int_{\tilde{Q}_{1}(0,0) \backslash \tilde{Q}_{\rho}(0,0)}\left(\int_{2 \max \left\{|x|,|2 t|^{1 / 2}\right\}}^{3} \frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{p^{\prime}} d x d t \\
& \geq \int_{\tilde{Q}_{1}(0,0) \backslash \tilde{Q}_{\rho}(0,0)}\left(\int_{2 \max \left\{|x|,|2 t|^{1 / 2}\right\}}^{3} \frac{1}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{p^{\prime}} d x d t \mu\left(\tilde{Q}_{\rho}(0,0)\right)^{p^{\prime}} \\
& \geq c_{3} \log (1 / \rho) \mu\left(\tilde{Q}_{\rho}(0,0)\right)^{p^{\prime}} .
\end{aligned}
$$

So $\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}\left(\tilde{Q}_{\rho}(0,0)\right) \leq c_{4} \mu\left(\tilde{Q}_{\rho}(0,0)\right)^{p} \leq c_{5}(\log (1 / \rho))^{1-p}$.

Definition 4.15 The parabolic Bessel potential $\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right), \alpha>0$ and $p>1$ is defined by

$$
\begin{equation*}
\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right)=\left\{f: f=\mathcal{G}_{\alpha} * g, g \in L^{p}\left(\mathbb{R}^{N+1}\right)\right\} \tag{4.25}
\end{equation*}
$$

with the norm $\|f\|_{\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right)}:=\|g\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}$. We denote its dual space by $\left(\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right)\right)^{*}$.
Definition 4.16 For $k$ a positive integer, the Sobolev space $W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)$ is defined by

$$
W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)=\left\{\varphi: \frac{\partial^{i_{1}+\ldots+i_{N}+i} \varphi}{\partial x_{1}^{i_{1}} \ldots \partial x_{N}^{i_{N}} \partial t^{i}} \in L^{p}\left(\mathbb{R}^{N+1}\right) \text { for any } i_{1}+\ldots+i_{N}+2 i \leq 2 k\right\}
$$

with the norm

We denote its dual space by $\left(W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)\right)^{*}$. We also define a corresponding capacity on compact set $E \subset \mathbb{R}^{N+1}$,

$$
\operatorname{Cap}_{2 k, k, p}(E)=\inf \left\{\|\varphi\|_{W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)}^{p}: \varphi \in S\left(\mathbb{R}^{N+1}\right), \varphi \geq 1 \text { in a neighborhood of } E\right\}
$$

Let us recall Richard J. Bagby's result, proved in [4].
Theorem 4.17 Let $p>1$ and $k$ be a positive integer. Then, there exists a constant $C$ depending on $N, k, p$ such that for any $u \in \mathcal{L}_{2 k}^{p}\left(\mathbb{R}^{N+1}\right)$,

$$
C^{-1}\|u\|_{W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)} \leq\|u\|_{\mathcal{L}_{2 k}^{p}\left(\mathbb{R}^{N+1}\right)} \leq C\|u\|_{W_{p}^{2 k, k}\left(\mathbb{R}^{N+1}\right)}
$$

Above Theorem gives the assertion of equivalence of capacity $\mathrm{Cap}_{2 k, k, p}, \mathrm{Cap}_{\mathcal{G}_{2 k}, p}$.
Corollary 4.18 Let $p>1$ and $k$ be a positive integer. There exists a constant $C$ depending on $N, k, p$ such that for any compact set $E \subset \mathbb{R}^{N+1}$

$$
\begin{equation*}
C^{-1} \operatorname{Cap}_{2 k, k, p}(E) \leq \operatorname{Cap}_{\mathcal{G}_{2 k}, p}(E) \leq C C a p_{2 k, k, p}(E) \tag{4.26}
\end{equation*}
$$

Next result provides some relations of Riesz, Bessel parabolic potential and Riesz, Bessel potential.
Proposition 4.19 Let $q>1$ and $\frac{2}{q^{\prime}}<\alpha<N+\frac{2}{q^{\prime}}$. There exists a constant $C$ depending on $N, q, \alpha$ such that for any $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$

$$
\begin{align*}
& C^{-1}| | \mathbf{I}_{\alpha-\frac{2}{q^{\prime}}}[\omega] \|_{L^{q}\left(\mathbb{R}^{N}\right)} \\
& \quad \leq\left\|\mathcal{H}_{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)},\left\|\mathcal{H}_{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C\left\|\mathbf{I}_{\alpha-\frac{2}{q^{\prime}}}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N}\right)} \tag{4.27}
\end{align*}
$$

and

$$
\begin{align*}
& C^{-1}\left\|\mathbf{G}_{\alpha-\frac{2}{q^{\prime}}}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N}\right)} \\
& \quad \leq\left\|\mathcal{G}_{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)},\left\|\stackrel{\mathcal{G}}{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq C\left\|\mathbf{G}_{\alpha-\frac{2}{q^{\prime}}}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N}\right)} \tag{4.28}
\end{align*}
$$

where $\delta_{\{t=0\}}$ is the Dirac mass in time at 0 .
Proof. We have
$\mathbb{I}_{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right](x, t)=\int_{\sqrt{2|t|}}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}, \mathbb{I}_{\alpha}^{1}\left[\omega \otimes \delta_{\{t=0\}}\right](x, t)=\int_{\min \{1, \sqrt{2|t|}\}}^{1} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}$.

By [16, Theorem 2.3] and Proposition 4.8, thus it is enough to show that
$c_{1}^{-1} \int_{0}^{\infty}\left(\frac{\omega(B(x, r))}{r^{N+2-\alpha-2 / q}}\right)^{q} \frac{d r}{r} \leq \int_{\mathbb{R}}\left(\int_{\sqrt{2|t|}}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \leq c_{1} \int_{0}^{\infty}\left(\frac{\omega(B(x, r))}{r^{N+2-\alpha-2 / q}}\right)^{q} \frac{d r}{r}$,
and

$$
\begin{align*}
c_{1}^{-1} \int_{0}^{1 / 2}( & \left.\frac{\omega(B(x, r))}{r^{N+2-\alpha-2 / q}}\right)^{q} \frac{d r}{r} \\
& \leq \int_{\mathbb{R}}\left(\int_{\min \{1, \sqrt{2|t|}\}}^{1} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \leq c_{1} \int_{0}^{1}\left(\frac{\omega(B(x, r))}{r^{N+2-\alpha-2 / q}}\right)^{q} \frac{d r}{r} \tag{4.30}
\end{align*}
$$

Indeed, by changing of variables

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left(\int_{\sqrt{2|t|}}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t=2 \int_{0}^{\infty} t\left(\int_{t}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \tag{4.31}
\end{equation*}
$$

Using Hardy's inequality, we have

$$
\int_{0}^{\infty} t\left(\int_{t}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \leq c_{2} \int_{0}^{\infty} r\left(\frac{\omega(B(x, r))}{r^{N+2-\alpha}}\right)^{q} d r
$$

and using the fact that

$$
\int_{t}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r} \geq c_{3} \frac{\omega(B(x, r))}{r^{N+2-\alpha}}
$$

we get

$$
\int_{0}^{\infty} t\left(\int_{t}^{\infty} \frac{\omega(B(x, r))}{r^{N+2-\alpha}} \frac{d r}{r}\right)^{q} d t \geq c_{3} \int_{0}^{\infty} r\left(\frac{\omega(B(x, r))}{r^{N+2-\alpha}}\right)^{q} d r
$$

Thus, we get (4.29). Likewise, we also obtain (4.30).
We have comparisons of $\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}, \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}, \operatorname{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}}, p}, \operatorname{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}}, p}$.
Corollary 4.20 Let $p>1$ and $\frac{2}{p}<\alpha<N+\frac{2}{p}$. There exists a constant $C$ depending on $N, q, \alpha$ such that for any compact $K \subset \mathbb{R}^{N}$

$$
\begin{equation*}
C^{-1} \operatorname{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}}, p}(K) \leq \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(K \times\{0\}) \leq C \operatorname{Cap} \mathbf{I}_{\mathbf{I}_{\alpha-\frac{2}{p}}, p}(K) \tag{4.32}
\end{equation*}
$$

and

$$
\begin{equation*}
C^{-1} \operatorname{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}}, p}(K) \leq \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times\{0\}) \leq C \operatorname{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}}, p}(K) \tag{4.33}
\end{equation*}
$$

Proof. By [2, Chapter 2], we have

$$
\begin{aligned}
\operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(K \times\{0\})^{1 / p} & =\sup \left\{\mu(K \times\{0\}): \mu \in \mathfrak{M}^{+}(K \times\{0\}),\left\|\mathcal{H}_{\alpha}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\} \\
& =\sup \left\{\omega(K): \omega \in \mathfrak{M}^{+}(K),\left\|\mathcal{H}_{\alpha}\left[\omega \otimes \delta_{\{t=0\}}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}
\end{aligned}
$$

$\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times\{0\})^{1 / p}=\sup \left\{\omega(K): \omega \in \mathfrak{M}^{+}(K),\left\|\stackrel{\vee}{\mathcal{G}_{\alpha}}\left[\omega \otimes \delta_{0}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}$,
$\operatorname{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}}^{p}}(K)^{1 / p}=\sup \left\{\omega(K): \omega \in \mathfrak{M}^{+}(K),\left\|\mathbf{I}_{\alpha-\frac{2}{p}}[\omega]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}$,
$\operatorname{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}}, p}(K)^{1 / p}=\sup \left\{\omega(K): \omega \in \mathfrak{M}^{+}(K),\left\|\mathbf{G}_{\alpha-\frac{2}{p}}[\omega]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq 1\right\}$.
Therefore, thanks to Proposition (4.19) we get the results.

Corollary 4.21 Let $p>1$ and $k$ be a positive integer such that $2 k<N+2 / p$. There exists $a$ constant $C$ depending on $N, k, p$ such that for any compact set $K \subset \mathbb{R}^{N}$

$$
\begin{equation*}
C^{-1} \operatorname{Cap}_{\mathbf{G}_{2 k-\frac{2}{p}}, p}(K) \leq \operatorname{Cap}_{2 k, k, p}(K \times\{0\}) \leq C C a p_{\mathbf{G}_{2 k-\frac{2}{p}}, p}(K) \tag{4.34}
\end{equation*}
$$

We also have comparisons of $\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}, \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}$.
Proposition 4.22 Let $0<\alpha<N, p>1$. For $a>0$ there exists a constant $C$ depending on $N, \alpha, p$, a such that for any compact $K \subset \mathbb{R}^{N}$,

$$
C^{-1} \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K) \leq \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times[-a, a]) \leq C \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K)
$$

Proof. By [2], we have

$$
\operatorname{Cap}_{\mathbf{I}_{\alpha}^{\sqrt{a}}, p}(K) \leq c_{1} \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K)
$$

for some $c_{1}=c_{1}(N, \alpha, p, a)>0$. So, we can find $f \in L_{+}^{p}\left(\mathbb{R}^{N}\right)$ such that $\mathbf{I}_{\alpha}^{\frac{\sqrt{a}}{2}} * f \geq \chi_{K}$ and

$$
\int_{\mathbb{R}^{N}}|f|^{p} d x \leq 2 c_{1} \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K)
$$

Note that $\left(E_{\alpha}^{\sqrt{a}} * \tilde{f}\right)(x, t) \geq c_{2}\left(\mathbf{I}_{\alpha}^{\frac{\sqrt{a}}{2}} * f\right)(x, t)$ for all $(x, t) \in \mathbb{R}^{N} \times[-a, a]$ where $\tilde{f}(x, t)=$ $f(x) \chi_{[-2 a, 2 a]}(t)$ and constant $c_{2}=c_{2}(N, \alpha, p)$. So,

$$
\begin{aligned}
\operatorname{Cap}_{E_{\alpha}^{\sqrt{a}}, p}(K \times[-a, a]) & \leq c_{2}^{-p} \int_{\mathbb{R}^{N+1}}|\tilde{f}|^{p} d x d t \\
& =4 c_{2}^{-p} a \int_{\mathbb{R}^{N}}|f|^{p} d x
\end{aligned}
$$

By Corollary 4.10, there is $c_{1}=c_{1}(N, \alpha, p, a)>0$ such that

$$
\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times[-a, a]) \leq c_{1} \operatorname{Cap}_{E_{\alpha}^{\sqrt{a}}, p}(K \times[-a, a])
$$

Thus, we get

$$
\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times[-a, a]) \leq c_{3} \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K)
$$

for some $c_{3}=c_{3}(N, \alpha, p, a)$.
Finally, we prove other one. It is easy to see that

$$
\left\|\mathbb{I}_{\alpha}^{\sqrt{\frac{a}{2}}}\left[\omega \otimes \chi_{[-a, a]}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq c_{4}\left\|\mathbf{I}_{\alpha}^{\sqrt{\frac{a}{2}}}[\omega]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N}\right)} \forall \omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)
$$

for some $c_{4}=c_{4}(N, \alpha, p)$, which implies

$$
\left\|\mathcal{G}_{\alpha}\left[\omega \otimes \chi_{[-a, a]}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq c_{5}\left\|\mathbf{G}_{\alpha}[\omega]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N}\right)} \quad \forall \omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)
$$

for some $c_{4}=c_{4}(N, \alpha, p, a)$.
It follows,

$$
\operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K \times[-a, a]) \geq c_{6} \operatorname{Cap}_{\mathbf{G}_{\alpha}, p}(K)
$$

for some $c_{6}=c_{6}(N, \alpha, p, a)$.

The following proposition is useful for proving that many operators of classical analysis are bounded in the space the space of functions $f$ such that

$$
\int_{K}|f|^{p} d x d t \leq C \operatorname{Cap}(K)
$$

for every compact set $K \subset \mathbb{R}^{N+1},(1<p<\infty)$, if they are bounded in $L^{q}\left(\mathbb{R}^{N+1}, d w\right)$ with $w \in A_{\infty}$.

Proposition 4.23 Let $0<R \leq \infty, 1<p \leq \alpha^{-1}(N+2), 0<\delta<\alpha$ and $f, g \in L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$. Suppose that

1. There exists a positive constant $C_{1}$ such that

$$
\begin{equation*}
\int_{K}|f| d x d t \leq C_{1} C a p_{E_{\alpha}^{R, \delta}, p}(K) \text { for any compact sets } K \subset \mathbb{R}^{N+1} \tag{4.35}
\end{equation*}
$$

2. For all weights $w \in A_{1}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{N+1}}|g| w d x d t \leq C_{2} \int_{\mathbb{R}^{N+1}}|f| w d x d t \tag{4.36}
\end{equation*}
$$

where the constant $C_{2}$ depends only on $N$ and $[w]_{A_{1}}$.
Then,

$$
\begin{equation*}
\int_{K}|g| d x d t \leq C_{3} C a p_{E_{\alpha}^{R, \delta}, p}(K) \text { for any compact set } K \subset \mathbb{R}^{N+1} \tag{4.37}
\end{equation*}
$$

where the constant $C_{3}$ depends only on $N, \alpha, p, \delta$ and $C_{1}, C_{2}$.
The capacity is mentioned in the Proposition (4.23), that is $\left(E_{\alpha}^{R, \delta}, p\right)$-capacity defined by

$$
\operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E)=\inf \left\{\int_{\mathbb{R}^{N+1}}|f|^{p} d x d t: f \in L_{+}^{p}\left(\mathbb{R}^{N+1}\right), E_{\alpha}^{R, \delta} * f \geq \chi_{E}\right\}
$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$, where $0<R \leq \infty, 0<\delta<\alpha<N+2$,

$$
E_{\alpha}^{R, \delta}(x, t)=\max \{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)} \min \left\{1,\left(\frac{\max \{|x|, \sqrt{2|t|}\}}{R}\right)^{-\delta}\right\} .
$$

Remark 4.24 For $0<\alpha q<N+2$, the inequality (4.10) in Proposition 4.4 implies

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{N+1}}\left(E_{\alpha}^{R, \delta} * f\right)^{\frac{q(N+2)}{N+2-\alpha q}} d x d t\right)^{1-\frac{\alpha q}{N+2}} \leq C \int_{\mathbb{R}^{N+1}} f^{q} d x d t \quad \forall f \in L^{q}\left(\mathbb{R}^{N+1}\right), f \geq 0 \tag{4.38}
\end{equation*}
$$

Hence, we get the isoperimetric inequality:

$$
\begin{equation*}
|E|^{1-\frac{\alpha p}{N+2}} \leq \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E) \tag{4.39}
\end{equation*}
$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$.

Also, we recall that a positive function $w \in L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$ is called an $A_{1}$ weight, if the quality

$$
[w]_{A_{1}}:=\sup \left(\left(f_{Q} w d y d s\right) \underset{(x, t) \in Q}{\operatorname{ess} \sup } \frac{1}{w(x, t)}\right)<\infty
$$

where the supremum is taken over all cylinder $Q=\tilde{Q}_{R}(x, t) \subset \mathbb{R}^{N+1}$. The constant $[w]_{A_{1}}$ is called the $A_{1}$ constant of $w$.

To prove the Proposition (4.23), we need to introduce the $(R, \delta)$-Wolff parabolic potential,

$$
\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)=\int_{0}^{\infty}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \text { for any }(x, t) \in \mathbb{R}^{N+1}
$$

where $p>1,0<\alpha p \leq N+2,0<\delta<\alpha p^{\prime}$ and $R \in(0, \infty]$ and $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$.
It is easy to see that

$$
\begin{equation*}
\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t) \leq C \sup _{(y, s) \in \operatorname{supp} \mu} \mathbb{W}_{\alpha, p}^{R, \delta}[\mu](y, s) . \tag{4.40}
\end{equation*}
$$

for some a constant $C=C(N, \alpha, p, \delta)>0$.
Remark 4.25 We easily verify that the Theorem 4.1 also holds for $\mathbb{W}_{\alpha, p}^{R, \delta, R_{1}}[\mu]$ and $\mathbb{M}_{\alpha p}^{R, \delta, R_{1}}[\mu]$ :
$\mathbb{W}_{\alpha, p}^{R, \delta, R_{1}}[\mu](x, t)=\int_{0}^{R_{1}}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho}$,
$\mathbb{M}_{\alpha, p}^{R, \delta /(p-1), R_{1}}[\mu](x, t)=\sup _{0<\rho<R_{1}}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha p}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta(p-1)}\right\}\right) \quad$ for any $(x, t) \in \mathbb{R}^{N+1}$,
where $0<\delta<\alpha p^{\prime}, 1<p<\alpha^{-1}(N+2)$ and $R_{1}>R>0$. This means, for $w \in A_{\infty}, \mu \in$ $\mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, there exist constants $C_{1}, C_{2}>0$ and $\varepsilon_{0} \in(0,1)$ depending on $N, \alpha, p, \delta,[w]_{A_{\infty}}$ such that for any $\lambda>0$ and $\varepsilon \in\left(0, \varepsilon_{0}\right)$
$w\left(\left\{\mathbb{W}_{\alpha, p}^{R, \delta, R_{1}}[\mu]>a \lambda,\left(\mathbb{M}_{\alpha p}^{R, \delta(p-1), R_{1}}[\mu]\right)^{\frac{1}{p-1}} \leq \varepsilon \lambda\right\}\right) \leq C_{1} \exp \left(-C_{2} \varepsilon^{-1}\right) w\left(\left\{\mathbb{W}_{\alpha, p}^{R, \delta, R_{1}}[\mu]>\lambda\right\}\right)$,
where $a=2+3^{\frac{N+2-\alpha p+\delta(p-1)}{p-1}}$.
Therefore, for $q>p-1$

$$
\left\|\mathbb{W}_{\alpha, p}^{R, \delta, R_{1}}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}, d w\right)} \leq C_{3}\left\|\left(\mathbb{M}_{\alpha p}^{R, \delta(p-1), R_{1}}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q}\left(\mathbb{R}^{N+1}, d w\right)}
$$

where $C_{3}=C_{3}(N, \alpha, p, \delta, q)$. Letting $R_{1} \rightarrow \infty$, we get

$$
\begin{equation*}
\left\|\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]\right\|_{L^{q}\left(\mathbb{R}^{N+1}, d w\right)} \leq C_{3}\left\|\left(\mathbb{M}_{\alpha p}^{R, \delta(p-1)}[\mu]\right)^{\frac{1}{p-1}}\right\|_{L^{q}\left(\mathbb{R}^{N+1}, d w\right)} \tag{4.42}
\end{equation*}
$$

where $\mathbb{M}_{\alpha p}^{R, \delta(p-1)}[\mu]:=\mathbb{M}_{\alpha p}^{R, \delta(p-1), \infty}[\mu]$.
We will need the following three Lemmas to prove the Proposition (4.23).
Lemma 4.26 Let $0<p \leq \alpha^{-1}(N+2)$ and $0<\beta<\frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$. There exists a constant c depending on $\delta$ such that for each $\tilde{Q}_{r}=\tilde{Q}_{r}(x, t)$

$$
\begin{equation*}
f_{\tilde{Q}_{r}}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](y, s)\right)^{\beta} d y d s \leq c\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)\right)^{\beta} . \tag{4.43}
\end{equation*}
$$

Proof. We set

$$
\begin{aligned}
& U_{\alpha, p}^{r}[\mu](y, s)=\int_{r}^{\infty}\left(\frac{|\mu|\left(\tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \text { and } \\
& L_{\alpha, p}^{r}[\mu](y, s)=\int_{0}^{r}\left(\frac{\mu\left(\tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho}
\end{aligned}
$$

Thus,

$$
f_{\tilde{Q}_{r}}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](y, s)\right)^{\delta} d y d s \leq c_{1} f_{\tilde{Q}_{r}}\left(U_{\alpha, p}^{r}[\mu](y, s)\right)^{\delta} d y d s+c_{1} f_{\tilde{Q}_{r}}\left(L_{\alpha, p}^{r}[\mu](y, s)\right)^{\delta} d y d s
$$

Since for each $(y, s) \in \tilde{Q}_{r}$ and $\rho \geq r$ we have $\tilde{Q}_{\rho}(y, s) \subset \tilde{Q}_{2 \rho}(x, t)$, thus for each $(y, s) \in \tilde{Q}_{r}$,

$$
\begin{aligned}
U_{\alpha, p}^{r}[\mu](y, s) & \leq \int_{r}^{\infty}\left(\frac{\mu\left(\tilde{Q}_{2 \rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\left(\max \left\{1, \frac{\rho}{R}\right\}\right)^{-\delta} \frac{d \rho}{\rho} \\
& \leq c_{2} \mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)
\end{aligned}
$$

which implies

$$
f_{\tilde{Q}_{r}}\left(U_{\alpha, p}^{r}[\mu](y, s)\right)^{\delta} d y d s \leq c_{2}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)\right)^{\delta} .
$$

Since for each $(y, s) \in \tilde{Q}_{r}$ and $\rho \leq r$ we have $\tilde{Q}_{\rho}(y, s) \subset \tilde{Q}_{2 r}(x, t)$ thus, $L_{\alpha, p}^{r}[\mu]=$ $L_{\alpha, p}^{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right] \leq \mathbb{W}_{\alpha, p}^{R, \delta}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right]$ in $\tilde{Q}_{r}(x, t)$. We now consider two cases.
Case 1: $r \leq R$. We have for $a>0$,

$$
\begin{aligned}
f_{\tilde{Q}_{r}}\left(L_{\alpha, p}^{r}[\mu](y, s)\right)^{\beta} d y d s & \leq f_{\tilde{Q}_{r}}\left(\mathbb{W}_{\alpha, p}^{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right](y, s)\right)^{\beta} d y d s \\
& =\frac{1}{\left|\tilde{Q}_{r}\right|} \beta \int_{0}^{\infty} \lambda^{\beta-1}\left|\left\{\mathbb{W}_{\alpha, p}^{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right]>\lambda\right\} \cap \tilde{Q}_{r}\right| d \lambda \\
& \leq a^{\beta}+c_{2} r^{-N-2} \int_{a}^{\infty} \lambda^{\beta-1}\left|\left\{\mathbb{W}_{\alpha, p}^{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right]>\lambda\right\}\right| d \lambda .
\end{aligned}
$$

If $\alpha p=N+2$, we use (4.11) in Remark 4.5 with $\varepsilon=\frac{\alpha p}{\beta}$ and take $a=\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)\right)^{\frac{1}{p-1}}$

$$
\begin{aligned}
f_{\tilde{Q}_{r}}\left(L_{\alpha, p}^{r}[\mu](y, s)\right)^{\beta} d y d s & \leq a^{\beta}+c_{3} r^{-N-2} \int_{a}^{\infty} \lambda^{\beta-1}\left(\frac{\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)\right)^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{e}} r^{\alpha p} d \lambda \\
& \leq c_{4}\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)\right)^{\frac{\beta}{p-1}} \\
& \leq c_{5}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)\right)^{\beta} .
\end{aligned}
$$

If $\alpha p<N+2$, we use (4.8) in Proposition 4.4 and take $a=\mu\left(\tilde{Q}_{2 r}(x, t)\right)^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}$, we get

$$
\begin{aligned}
f_{\tilde{Q}_{r}}\left(L_{\alpha, p}^{r}[\mu](y, s)\right)^{\beta} d y d s & \leq c_{6}\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}\right)^{\beta} \\
& \leq c_{7}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)\right)^{\beta} .
\end{aligned}
$$

Case 2: $r \geq R$. As above case, we have

$$
f_{\tilde{Q}_{r}}\left(\mathbb{W}_{\alpha-\frac{\delta}{p}, p}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right](y, s)\right)^{\beta} d y d s \leq c_{6}\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}}\right)^{\beta} .
$$

Since $\mathbb{W}_{\alpha, p}^{R, \delta}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right] \leq R^{\delta} \mathbb{W}_{\alpha-\frac{\delta}{p}, p}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right]$, thus

$$
\begin{aligned}
f_{\tilde{Q}_{r}}\left(L_{\alpha, p}^{r}[\mu](y, s)\right)^{\beta} d y d s & \leq c_{6}\left(\mu\left(\tilde{Q}_{2 r}(x, t)\right)^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}} R^{\delta}\right)^{\beta} \\
& \leq c_{5}\left(\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t)\right)^{\beta} .
\end{aligned}
$$

Therefore, we get (4.43). The proof completes.
Remark 4.27 It is easy to see that the inequality (4.43) does not true for $\mathbb{W}_{\alpha, p}^{R}\left[\delta_{(0,0)}\right]$ where $\delta_{(0,0)}$ is the Dirac mass at $(x, t)=(0,0)$.

Remark 4.28 From Lemma (4.26), we have, if there exists $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$ such that $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]\left(x_{0}, t_{0}\right)<\infty$ then $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu] \in L_{\text {loc }}^{\beta}\left(\mathbb{R}^{N+1}\right)$ for any $0<\beta<\frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$.

Lemma 4.29 Let $R \in(0, \infty]$, $1<p \leq \alpha^{-1}(N+2)$ and $0<\delta<\alpha p^{\prime}$. Assume that $\alpha p<N+2$ if $R=\infty$. Then, for any compact set $K \subset \mathbb{R}^{N+1}$ there exists a $\mu \in \mathfrak{M}^{+}(K)$, called a capacitary measure of $K$ such that
and $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t) \geq C_{2}$ a.e in $K$ and $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu] \leq C_{3}$ a.e in $\mathbb{R}^{N+1}$ for some constants $C_{i}=$ $C_{i}(N, \alpha, p), i=1,2,3$.

Proof. We consider a measure $\nu$ on $M=\mathbb{R}^{N+1} \times \mathbb{Z}$ as follows

$$
\nu=m \otimes \sum_{n=-\infty}^{\infty} \delta_{n}
$$

where $m$ is Lebesgue measure, and $\delta_{n}$ denotes unit mass at $n$. Thus, $f \in L^{p}(M, d \nu)$, means $f=\left\{f_{n}\right\}_{-\infty}^{\infty}$, with

$$
\|f\|_{L^{p}(M, d \nu)}^{p}=\sum_{n=-\infty}^{\infty}\left\|f_{n}\right\|_{L^{p}\left(\mathbb{R}^{N+1}\right)}^{p}
$$

Let $n_{R} \in \mathbb{Z} \cup\{+\infty\}$ such that $2^{-n_{R}} \leq R<2^{-n_{R}+1}$ if $R<+\infty$ and $n_{R}-\infty$ if $R=+\infty$. We define a kernel $\mathbb{P}_{\alpha}$ in $\mathbb{R}^{N+1} \times M=\mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{Z}$ by

$$
\mathbb{P}_{\alpha}\left(x, t, x^{\prime}, t^{\prime}, n\right)=\min \left\{1,2^{\left(n-n_{R}\right) \delta / p^{\prime}}\right\} 2^{n(N+2-\alpha)} \chi_{\tilde{Q}_{2-n}}\left(x-x^{\prime}, t-t^{\prime}\right)
$$

If $f$ is $\nu$-measurable and nonnegative and $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, the corresponding potentials $\mathcal{P}_{\alpha} f, \stackrel{\mathcal{P}}{\alpha} \mu$ and $V_{\mathbb{P}_{\alpha}, p}^{\mu}$ are everywhere well defined and given by

$$
\begin{aligned}
&\left(\mathcal{P}_{\alpha} f\right)(x, t)= \int_{M} \mathbb{P}_{\alpha}\left(x, t, x^{\prime}, t^{\prime}, n\right) f\left(x^{\prime}, t^{\prime}, n\right) d \nu\left(x^{\prime}, t^{\prime}, n\right) \\
&= \sum_{n=-\infty}^{\infty} \min \left\{1,2^{\left(n-n_{R}\right) \delta / p^{\prime}}\right\} 2^{n(N+2-\alpha)}\left(\chi_{\tilde{Q}_{2-n}} * f_{n}\right)(x, t) \\
&\left(\stackrel{\vee}{\mathcal{P}}_{\alpha} \mu\right)\left(x^{\prime}, t^{\prime}, n\right)=\int_{\mathbb{R}^{N+1}} \mathbb{P}_{\alpha}\left(x, t, x^{\prime}, t^{\prime}, n\right) d \mu(x, t) \\
&=\min \left\{1,2^{\left(n-n_{R}\right) \delta / p^{\prime}}\right\} 2^{n(N+2-\alpha)}\left(\chi_{\tilde{Q}_{2-n}} * \mu\right)\left(x^{\prime}, t^{\prime}\right) \\
& V_{P_{\alpha}, p}^{\mu}(x, t)=\left(\mathcal{P}_{\alpha}\left(\mathcal{P}_{\alpha} \mu\right)^{p^{\prime}-1}\right)(x, t) \\
&= \sum_{n=-\infty}^{\infty} \min \left\{1,2^{\left(n-n_{R}\right) \delta}\right\} 2^{n p^{\prime}(N+2-\alpha)}\left(\chi_{\tilde{Q}_{2-n}} *\left(\chi_{\tilde{Q}_{2-n}} * \mu\right)^{p^{\prime}-1}\right)(x, t)
\end{aligned}
$$

for any $\left(x, t, x^{\prime}, t^{\prime}, n\right) \in \mathbb{R}^{N+1} \times M$.
Since for all $(x, t) \in \mathbb{R}^{N+1}$,

$$
\begin{aligned}
\left|\tilde{Q}_{1}\right| 2^{-(n+1)(N+2)}\left(\mu\left(\tilde{Q}_{2^{-n-1}}(x, t)\right)\right)^{p^{\prime}-1} & \leq\left(\chi_{\tilde{Q}_{2-n}} *\left(\chi_{\tilde{Q}_{2-n}} * \mu\right)^{p^{\prime}-1}\right)(x, t) \\
& \leq\left|\tilde{Q}_{1}\right| 2^{-n(N+2)}\left(\mu\left(\tilde{Q}_{2^{-n+1}}(x, t)\right)\right)^{p^{\prime}-1}
\end{aligned}
$$

thus,

$$
\begin{equation*}
c_{1}^{-1} V_{\mathbb{P}_{\alpha}, p}^{\mu} \leq \mathbb{W}_{\alpha, p}^{R, \delta}[\mu] \leq c_{1} V_{\mathbb{P}_{\alpha}, p}^{\mu} \tag{4.44}
\end{equation*}
$$

for some a positive constant $c_{1}$.
We now define the $L^{p}$-capacity with $1<p<\infty$

$$
\operatorname{Cap}_{\mathbb{P}_{\alpha}, p}(E)=\inf \left\{\|f\|_{L^{p}(M, d \nu)}^{p}: f \in L_{+}^{p}(M, d \nu), \mathcal{P}_{\alpha} f \geq \chi_{E}\right\}
$$

for any Borel set $E \subset \mathbb{R}^{N+1}$. By [2, Theorem 2.5.1], for any compact set $K \subset \mathbb{R}^{N+1}$

$$
\operatorname{Cap}_{\mathbb{P}_{\alpha}, p}(K)^{1 / p}=\sup \left\{\mu(K): \mu \in \mathfrak{M}^{+}(K),\|\stackrel{\vee}{\mathcal{P}} \mu\|_{L^{p^{\prime}}(M, d \nu)} \leq 1\right\}
$$

By [2, Theorem 2.5.6], for any compact set $K$ in $\mathbb{R}^{N+1}$, there exists $\mu \in \mathfrak{M}^{+}(K)$, called a capacitary measure for $K$, such that $V_{\mathbb{P}_{\alpha}, p}^{\mu} \geq 1 \mathrm{Cap}_{\mathbb{P}_{\alpha}, p}$ - q.e. in $K, V_{P_{\alpha}, p}^{\mu} \leq 1$ a.e in $\operatorname{supp}(\mu)$ and $\mu(K)=\operatorname{Cap}_{\mathbb{P}_{\alpha}, p}(K)$. Thanks to (4.44) and (4.40), we have $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu] \geq c_{1}^{-1}$ $\mathrm{Cap}_{\mathbb{P}_{\alpha}, p}$ - q.e. in $K, \mathbb{W}_{\alpha, p}^{R, \delta}[\mu] \leq c_{2}$ a.e in $\mathbb{R}^{N+1}$ and $\mu(K)=\operatorname{Cap}_{\mathbb{P}_{\alpha}, p}(K)$.
On the other hand,

$$
\begin{aligned}
\|\stackrel{\vee}{\mathcal{P}} \mu\| \|_{L^{p^{\prime}}(M, d \nu)}^{p^{\prime}} & =\sum_{n=-\infty}^{\infty}\left\|\min \left\{1,2^{\left(n-n_{R}\right) \delta / p^{\prime}}\right\} 2^{n(N+2-\alpha)} \chi_{\tilde{Q}_{2-n}} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}^{p^{\prime}} \\
& =\sum_{n=-\infty}^{\infty} \min \left\{1,2^{\left(n-n_{R}\right) \delta}\right\} 2^{n p^{\prime}(N+2-\alpha)} \int_{\mathbb{R}^{N+1}}\left(\chi_{\tilde{Q}_{2-n}} * \mu\right)^{p^{\prime}} d x d t
\end{aligned}
$$

this quantity is equivalent to

$$
\int_{\mathbb{R}^{N+1}} \int_{0}^{\infty}\left(\frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha}}\right)^{p^{\prime}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} d x d t
$$

So, thanks to (4.42) in Remark 4.25, we obtain

$$
c_{2}^{-1}\left\|E_{\alpha}^{R, \delta / p^{\prime}} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}^{p^{\prime}} \leq\left\|\stackrel{\rightharpoonup}{\mathcal{P}}_{\alpha} \mu\right\|_{L^{p^{\prime}(M, d \nu)}}^{p^{\prime}} \leq c_{2}\left\|E_{\alpha}^{R, \delta / p^{\prime}} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}^{p^{\prime}}
$$

for $c_{2}=c_{2}(N, p, \alpha, \delta)$. It follows that two capacities $\operatorname{Cap}_{\mathbb{P}_{\alpha}, p}$ and $\operatorname{Cap}_{E_{\alpha}^{R, \delta / p^{\prime}, p}}$ are equivalent. Therefore, we obtain the desired results.

Lemma 4.30 Let $R \in(0, \infty]$, $1<p \leq \alpha^{-1}(N+2)$ and $0<\delta<\alpha p^{\prime}$. Assume that $\alpha p<N+2$ if $R=\infty$. Then there exists $\bar{C}=C(N, \alpha, p, \delta)$ such that for any $\mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)$

$$
\begin{equation*}
C a p_{E_{\alpha}^{R, \delta / p^{\prime}}, p}\left(\left\{\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]>\lambda\right\}\right) \leq C \lambda^{-p+1} \mu\left(\mathbb{R}^{N+1}\right) \quad \forall \lambda>0 \tag{4.45}
\end{equation*}
$$

In particular, $\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]<\infty \quad$ Cap $E_{E_{\alpha}^{R, \delta / p^{\prime}}{ }_{, p}}-$ q.e. in $\mathbb{R}^{N+1}$.
Proof. By Lemma 4.29, there is a capacitary measure $\sigma$ for a compact subset $K$ of $\left\{\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]>\lambda\right\}$ such that $\mathbb{W}_{\alpha, p}^{R, \delta}[\sigma](x, t) \leq c_{1}$ on supp $\sigma$ and $\operatorname{Cap}_{E_{\alpha}^{R, \delta / p^{\prime}}{ }_{, p}}(K) \approx \sigma(K)$ where $c_{1}=c_{1}(N, \alpha, p, \delta)$.
Set $\mathbb{M}[\mu, \sigma](x, t)=\sup _{\rho>0} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\sigma\left(\tilde{Q}_{3 \rho}(x, t)\right)}$ for any $(x, t) \in \operatorname{supp} \sigma$. Then, for any $(x, t) \in \operatorname{supp} \sigma$

$$
\begin{aligned}
\lambda<\mathbb{W}_{\alpha, p}^{R, \delta}[\mu](x, t) & \leq(\mathbb{M}[\mu, \sigma](x, t))^{\frac{1}{p-1}} \int_{0}^{\infty}\left(\frac{\sigma\left(\tilde{Q}_{3 \rho}(x, t)\right)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \\
& \leq c_{2}(\mathbb{M}[\mu, \sigma](x, t))^{\frac{1}{p-1}}
\end{aligned}
$$

Thus, for any $\lambda>0$, $\operatorname{supp} \sigma \subset\left\{c_{2}(\mathbb{M}[\mu, \sigma])^{\frac{1}{p-1}}>\lambda\right\}=\left\{\mathbb{M}[\mu, \sigma]>\left(\frac{\lambda}{c_{2}}\right)^{p-1}\right\}$. By Vitali Covering Lemma one can cover supp $\sigma$ with a union of $\tilde{Q}_{3 \rho_{i}}\left(x_{i}, t_{i}\right)$ for $i=1, \ldots, m(K)$ so that
$\tilde{Q}_{\rho_{i}}\left(x_{i}, t_{i}\right)$ are disjoint and $\sigma\left(\tilde{Q}_{3 \rho_{i}}\left(x_{i}, t_{i}\right)\right)<\left(\lambda / c_{2}\right)^{-p+1} \mu\left(\tilde{Q}_{\rho_{i}}\left(x_{i}, t_{i}\right)\right)$. It follows that

$$
\begin{aligned}
\operatorname{Cap}_{E_{\alpha}^{R}, p}(K) & \leq c_{3} \sum_{i=1}^{m(K)} \sigma\left(\tilde{Q}_{3 \rho_{i}}\left(x_{i}, t_{i}\right)\right) \\
& \leq c_{3} c_{2}^{p-1} \lambda^{-p+1} \sum_{i=1}^{m(K)} \mu\left(\tilde{Q}_{\rho_{i}}\left(x_{i}, t_{i}\right)\right) \\
& \leq c_{3} c_{2}^{p-1} \lambda^{-p+1} \mu\left(\mathbb{R}^{N+1}\right)
\end{aligned}
$$

So, for all compact subset $K$ of $\left\{\mathbb{W}_{\alpha, p}^{R, \delta}[\mu]>\lambda\right\}$,

$$
\operatorname{Cap}_{E_{\alpha}^{R, \delta / p^{\prime}}{ }_{, p}}(K) \leq c_{1} c_{2}^{p-1} \lambda^{-p+1} \mu\left(\mathbb{R}^{N+1}\right)
$$

Therefore we obtain (4.45).
Remark 4.31 Let $0<\delta<\alpha<N+2$ and $\delta \leq 1$. From the following inequality

$$
\begin{aligned}
& \left|\max \left\{\left|x_{1}-z\right|, \sqrt{2\left|t_{1}-s\right|}\right\}^{-N-2+\alpha}-\max \left\{\left|x_{2}-z\right|, \sqrt{2\left|t_{2}-s\right|}\right\}^{-N-2+\alpha}\right| \\
& \leq c_{1}\left(\max \left\{\left|x_{1}-z\right|, \sqrt{2\left|t_{1}-s\right|}\right\}^{-N-2+\alpha-\delta}+\max \left\{\left|x_{2}-z\right|, \sqrt{2\left|t_{2}-s\right|}\right\}^{-N-2+\alpha-\delta}\right) \\
& \quad \times\left(\left|x_{1}-x_{2}\right|+\left|t_{1}-t_{2}\right|^{1 / 2}\right)^{\delta}
\end{aligned}
$$

for all $\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right),(z, s) \in \mathbb{R}^{N+1}$, where $c_{1}$ is a constant depending on $N, \alpha, \delta$. Thus, for $\mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right)$
$\left|\mathbb{I}_{\alpha}[\mu]\left(x_{1}, t_{1}\right)-\mathbb{I}_{\alpha}[\mu]\left(x_{2}, t_{2}\right)\right| \leq c_{2}\left(\mathbb{I}_{\alpha-\delta}[\mu]\left(x_{1}, t_{1}\right)+\mathbb{I}_{\alpha-\delta}[\mu]\left(x_{2}, t_{2}\right)\right)\left(\left|x_{1}-x_{2}\right|+\left|t_{1}-t_{2}\right|^{1 / 2}\right)^{\delta}$, for all $\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right) \in \mathbb{R}^{N+1}$ and $c_{2}=c_{1} \frac{N+2-\alpha+\delta}{N+2-\alpha}$.
Consequently, for any $\mu \in \mathfrak{M}_{b}^{+}\left(\mathbb{R}^{N+1}\right), \mathbb{I}_{\alpha}[\mu]$ is $\delta$-Holder Cap $\operatorname{E}_{\frac{\alpha-\delta}{2}, 2}-q u a s i c o n t i n u o u s ~ t h i s$ means, for any $\varepsilon>0$ there exists a Borel set $O_{\varepsilon} \subset \mathbb{R}^{N+1}$ and $c_{\varepsilon}>0$ such that

$$
\left|\mathbb{I}_{\alpha}[\mu]\left(x_{1}, t_{1}\right)-\mathbb{I}_{\alpha}[\mu]\left(x_{2}, t_{2}\right)\right| \leq c_{\varepsilon}\left(\left|x_{1}-x_{2}\right|+\left|t_{1}-t_{2}\right|^{1 / 2}\right)^{\delta} \quad \forall\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right) \in O_{\varepsilon}
$$

and $\operatorname{Cap}_{E_{\frac{\alpha-\delta}{2}, 2}}\left(\mathbb{R}^{N+1} \backslash O_{\varepsilon}\right)<\varepsilon$.
Now we are ready to prove Proposition 4.23.
Proof of Proposition 4.23. By Lemma 4.26, 4.29 and 4.30, there is the capacitary measure $\mu$ of a compact subset $K \subset \mathbb{R}^{N+1}$ such that $\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu] \geq c_{1}$ a.e in $K, \mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu] \leq$ $c_{2}$ a.e in $\mathbb{R}^{N+1}$ and $\operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}\left(\left\{\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]>\lambda\right\}\right) \leq c_{2} \lambda^{-p+1} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(K)$ for all $\lambda>0$, $\left(\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]\right)^{\beta} \in A_{1}$ for any $0<\beta<\frac{(N+2)(p-1)}{N+2-\alpha p+\delta p}$. From second assumption we have

$$
\int_{\mathbb{R}^{N+1}}|g|\left(\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]\right)^{\beta} d x d t \leq C_{2} \int_{\mathbb{R}^{N+1}}|f|\left(\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]\right)^{\beta} d x d t
$$

Thus

$$
\begin{aligned}
\int_{K}|g| d x d t & \leq c_{1}^{-\delta} \int_{\mathbb{R}^{N+1}}|g|\left(\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]\right)^{\beta} d x d t \\
& \leq c_{3} \int_{\mathbb{R}^{N+1}}|f|\left(\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]\right)^{\beta} d x d t \\
& =c_{3} \beta \int_{0}^{c_{1}} \int_{\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]>\lambda}|f| d x d t \lambda^{\beta-1} d \lambda .
\end{aligned}
$$

By first assumption we get

$$
\int_{\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]>\lambda}|f| d x d t \leq C_{1} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}\left(\left\{\mathbb{W}_{\alpha, p}^{R, \delta p^{\prime}}[\mu]>\lambda\right\}\right) \leq c_{4} \lambda^{-p+1} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(K) .
$$

Therefore,

$$
\int_{K}|g| d x d t \leq c_{5} \delta \int_{0}^{c_{1}} \lambda^{-p+1} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(K) \lambda^{\delta-1} d \lambda=c_{6} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(K)
$$

since one can choose $\delta>p-1$. This completes the proof of the Proposition.
Definition 4.32 Let $s>1$, $\alpha>0$. We define the space $\mathfrak{M}^{\mathcal{H}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)\left(\mathfrak{M}^{\mathcal{G}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)\right.$ resp.) to be the set of all measure $\mu \in \mathfrak{M}\left(\mathbb{R}^{N+1}\right)$ such that

$$
\left.\begin{array}{l}
{[\mu]_{\mathfrak{M}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)} \\
\left([\mu]_{\mathfrak{M}_{\alpha}, s}:=\sup \left\{|\mu|(K) / \mathbb{R}^{N+1}\right)\right.
\end{array}=\sup \left\{|\mu|(K) / \operatorname{Cap}_{\mathcal{H}_{\alpha}, s}(K): \operatorname{Cap}_{\mathcal{G}_{\alpha}, s}(K): \operatorname{Cap}_{\mathcal{G}_{\alpha}, s}(K)>0\right\}<\infty \text { resp. }\right) \text { (K)>0\}< } .
$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^{N+1}$.
For simplicity, we will write $\mathfrak{M}^{\mathcal{H}_{\alpha}, s}, \mathfrak{M}^{\mathcal{G}_{\alpha}, s}$ to denote $\mathfrak{M}^{\mathcal{H}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right), \mathfrak{M}^{\mathcal{G}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)$ resp.
We see that if $\alpha s \geq N+2, \mathfrak{M}^{\mathcal{H}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)=\{0\}$, if $\alpha s<N+2, \mathfrak{M}^{\mathcal{H}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right) \subset$ $\mathfrak{M}^{\mathcal{G}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right)$. On the other hand, $\mathfrak{M}^{\mathcal{G}_{\alpha}, s}\left(\mathbb{R}^{N+1}\right) \supset \mathfrak{M}_{b}\left(\mathbb{R}^{N+1}\right)$ if $\alpha s>N+2$.
We now have the following two remarks:
Remark 4.33 For $s>1$, there is $C=C(N, \alpha, s)>0$ such that

$$
\begin{equation*}
[f]_{\mathfrak{M}^{G_{\alpha}, p}} \leq C\left[|f|^{s}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}}^{1 / s} \quad \text { for all function } f \tag{4.46}
\end{equation*}
$$

Indeed, set $a=\left[|f|^{s}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}}$, so for any compact set $K$ in $\mathbb{R}^{N+1}$,

$$
\int_{K}|f|^{s} d x d t \leq a C a p_{\mathcal{G}_{\alpha}, p}(K)
$$

This gives $2 a \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(K) \geq \int_{K}\left(|f|^{s}+c_{1} a\right) d x d t \geq c_{2} a^{1-1 / s} \int_{K}|f| d x d t$, here we used (4.24) in Remark 4.11 at the first inequality and Holder's inequality at the second one. It follows (4.46).

Remark 4.34 Assume that $p>1$ and $\frac{2}{p}<\alpha<N+\frac{2}{p}$. Clearly, from Corollary 4.20 we assert that for $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$

$$
\begin{aligned}
& C_{1}^{-1}[\omega]_{\mathfrak{M}^{\mathbf{I}} \alpha-2 / p, p} \leq\left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}} \leq C_{1}[\omega]_{\mathfrak{M}^{\mathbf{I}} \alpha-2 / p, p}, \\
& C_{2}^{-1}[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2 / p}, p}} \leq\left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{G_{\alpha}, p}} \leq C_{2}[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2 / p}, p}},
\end{aligned}
$$

for some $C_{i}=C_{i}(N, p, \alpha), i=1,2$. Where $\mathfrak{M}^{\mathbf{I}_{\alpha-2 / p}, p}:=\mathfrak{M}^{\mathbf{I}_{\alpha-2 / p}, p}\left(\mathbb{R}^{N}\right), \mathfrak{M}^{\mathbf{G}_{\alpha-2 / p}, p}:=$ $\mathfrak{M}^{\mathbf{G}_{\alpha-2 / p}, p}\left(\mathbb{R}^{N}\right)$ and

$$
\begin{aligned}
& {[\omega]_{\mathfrak{M}^{\mathbf{I}} \alpha-2 / p, p}\left(\mathbb{R}^{N}\right)} \\
& {[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2 / p}, p}\left(\mathbb{R}^{N}\right)}:=\sup \left\{\omega(K) / \operatorname{Cap}_{\mathbf{I}_{\alpha-2 / p}, p}(K): \operatorname{Cap}_{\mathbf{I}_{\alpha-2 / p}, p}(K)>0\right\},} \\
& \left\{\omega(K) / \operatorname{Cap}_{\mathbf{G}_{\alpha-2 / p}, p}(K): \operatorname{Cap}_{\mathbf{G}_{\alpha-2 / p}, p}(K)>0\right\},
\end{aligned}
$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^{N}$.
Clearly, Theorem 4.2 and Proposition 4.23 lead to the following result.

Proposition 4.35 Let $q>p-1, s>1$ and $0<\alpha p<N+2$. Then the following quantities are equivalent

$$
\left[\left(\mathbb{W}_{\alpha, p}^{R}[\mu]\right)^{q}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, s}}, \quad\left[\left(\mathbb{I}_{\alpha p}^{R}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, s}} \quad \text { and }\left[\left(\mathbb{M}_{\alpha p}^{R}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, s}}
$$

for every $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ and $0<R \leq \infty$.
In the next result, we present a characterization of the following trace inequality:

$$
\begin{equation*}
\left\|E_{\alpha}^{R, \delta} * f\right\|_{L^{p}\left(\mathbb{R}^{N+1}, d \mu\right)} \leq C_{1}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)} \quad \forall f \in L^{p}\left(\mathbb{R}^{N+1}\right) \tag{4.47}
\end{equation*}
$$

Theorem 4.36 Let $0<R \leq \infty, 1<p<\alpha^{-1}(N+2), 0<\delta<\alpha$ and $\mu$ be a nonnegative Radon measure on $\mathbb{R}^{N+1}$. Then the following statements are equivalent.

1. The trace inequality (4.47) holds.
2. There holds

$$
\begin{equation*}
\left\|E_{\alpha}^{R, \delta} * f\right\|_{L^{p}\left(\mathbb{R}^{N+1}, d \omega\right)} \leq C_{2}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)} \forall f \in L^{p}\left(\mathbb{R}^{N+1}\right) \tag{4.48}
\end{equation*}
$$

$$
\text { where d } \omega=\left(\mathbb{I}_{\alpha}^{R, \delta} \mu\right)^{p^{\prime}} d x d t
$$

3. There holds

$$
\begin{equation*}
\left\|E_{\alpha}^{R, \delta} * f\right\|_{L^{p, \infty}\left(\mathbb{R}^{N+1}, d \mu\right)} \leq C_{3}\|f\|_{L^{p}\left(\mathbb{R}^{N+1}\right)} \quad \forall f \in L^{p}\left(\mathbb{R}^{N+1}\right) \tag{4.49}
\end{equation*}
$$

4. For every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\mu(E) \leq C_{4} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E) \tag{4.50}
\end{equation*}
$$

5. $\mathbb{I}_{\alpha}^{R, \delta}[\mu]<\infty$ a.e and

$$
\begin{equation*}
\mathbb{I}_{\alpha}^{R, \delta}\left[\left(\mathbb{I}_{\alpha}^{R, \delta}[\mu]\right)^{p^{\prime}}\right] \leq C_{5} \mathbb{I}_{\alpha}^{R, \delta}[\mu] \text { a.e. } \tag{4.51}
\end{equation*}
$$

6. For every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\int_{E}\left(\mathbb{I}_{\alpha}^{R, \delta}[\mu]\right)^{p^{\prime}} d x d t \leq C_{6} C a p_{E_{\alpha}^{R, \delta}, p}(E) \tag{4.52}
\end{equation*}
$$

7. For every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}^{R, \delta}\left[\mu \chi_{E}\right]\right)^{p^{\prime}} d x d t \leq C_{7} \mu(E) \tag{4.53}
\end{equation*}
$$

8. For every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\int_{E}\left(\mathbb{I}_{\alpha}^{R, \delta}\left[\mu \chi_{E}\right]\right)^{p^{\prime}} d x d t \leq C_{8} \mu(E) \tag{4.54}
\end{equation*}
$$

We can find a simple sufficient condition on $\mu$ so that trace inequality (4.47) is satisfied from the isoperimetric inequality (4.39).
Proof of Theorem 4.36. As in [80] we can show that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6 \Leftrightarrow 7$ and $7 \Rightarrow 8,5 \Rightarrow 2$. Thus, it is enough to show that $8 . \Rightarrow 5$. First, we need to show that

$$
\begin{equation*}
\left(\int_{r}^{\infty} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N+2-\alpha}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho}\right)^{p^{\prime}-1} \leq c_{1} r^{-\alpha}\left(\min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\right)^{-1} \tag{4.55}
\end{equation*}
$$

We have for any $(y, s) \in \tilde{Q}_{r}(x, t)$

$$
\begin{aligned}
\mathbb{I}_{\alpha}^{R, \delta}\left[\mu \chi_{\tilde{Q}_{r}(x, t)}\right](y, s) & =\int_{0}^{\infty} \frac{\mu\left(\tilde{Q}_{r}(x, t) \cap \tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \\
& \geq \int_{2 r}^{4 r} \frac{\mu\left(\tilde{Q}_{r}(x, t) \cap \tilde{Q}_{\rho}(y, s)\right)}{\rho^{N+2-\alpha}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \\
& \geq c_{2} \frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} .
\end{aligned}
$$

In (4.54), we take $E=\tilde{Q}_{r}(x, t)$

$$
\begin{aligned}
c \mu\left(\tilde{Q}_{r}(x, t)\right) & \geq \int_{\tilde{Q}_{r}(x, t)}\left(\mathbb{I}_{\alpha}\left[\mu \chi_{\tilde{Q}_{r}(x, t)}\right]\right)^{p^{\prime}} \\
& \geq c_{2}^{p^{\prime}}\left(\frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\right)^{p^{\prime}}\left|\tilde{Q}_{r}(x, t)\right|
\end{aligned}
$$

So $\mu\left(\tilde{Q}_{r}(x, t)\right) \leq c_{3} r^{N+2-\alpha p}\left(\min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\right)^{-p}$ which implies (4.55).
Next we set

$$
\begin{aligned}
L_{r}[\mu](x, t) & =\int_{r}^{+\infty} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \\
U_{r}[\mu](x, t) & =\int_{0}^{r} \frac{\mu\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho}
\end{aligned}
$$

and

$$
d \omega=\left(I_{\alpha} \mu\right)^{p^{\prime}} d x d t, \quad d \sigma_{1, r}=\left(L_{r}[\mu]\right)^{p^{\prime}} d x d t, \quad d \sigma_{2, r}=\left(U_{r}[\mu]\right)^{p^{\prime}} d x d t .
$$

We have $d \omega \leq 2^{p^{\prime}-1}\left(d \sigma_{1, r}+d \sigma_{2, r}\right)$. To prove (4.51) we need to show that

$$
\begin{align*}
& \int_{0}^{\infty} \frac{\sigma_{1, r}\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} \frac{d r}{r} \leq c_{4} \mathbb{I}_{\alpha}^{R, \delta}[\mu](x, t),  \tag{4.56}\\
& \int_{0}^{\infty} \frac{\sigma_{2, r}\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} \frac{d r}{r} \leq c_{5} \mathbb{I}_{\alpha}^{R, \delta}[\mu](x, t) \tag{4.57}
\end{align*}
$$

Since, for all $r>0,0<\rho<r$ and $(y, s) \in \tilde{Q}_{r}(x, t)$ we have $\tilde{Q}_{\rho}(y, s) \subset \tilde{Q}_{2 r}(x, t)$. So,

$$
\sigma_{2, r}\left(\tilde{Q}_{r}(x, t)\right)=\int_{\tilde{Q}_{r}(x, t)}\left(U_{r}[\mu](y, s)\right)^{p^{\prime}} d y d s=\int_{\tilde{Q}_{r}(x, t)}\left(U_{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right](y, s)\right)^{p^{\prime}} d y d s
$$

Thus, from (4.54) we get

$$
\begin{aligned}
\sigma_{2, r}\left(\tilde{Q}_{r}(x, t)\right) & \leq \int_{\tilde{Q}_{2 r}(x, t)}\left(U_{r}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right](y, s)\right)^{p^{\prime}} d y d s \\
& \leq \int_{\tilde{Q}_{2 r}(x, t)}\left(\mathbb{I}_{\alpha}^{R, \delta}\left[\mu \chi_{\tilde{Q}_{2 r}(x, t)}\right](y, s)\right)^{p^{\prime}} d y d s \\
& \leq c_{6} \mu\left(\tilde{Q}_{2 r}(x, t)\right)
\end{aligned}
$$

Therefore, (4.57) follows.
Since, for all $r>0, \rho \geq r$ and $(y, s) \in \tilde{Q}_{r}(x, t)$ we have $\tilde{Q}_{\rho}(y, s) \subset \tilde{Q}_{2 \rho}(x, t)$. So, for all $(y, s) \in \tilde{Q}_{r}(x, t)$ we have

$$
\begin{aligned}
L_{r}[\mu](y, s) & \leq \int_{r}^{+\infty} \frac{\mu\left(\tilde{Q}_{2 \rho}(x, t)\right)}{\rho^{N+2-\alpha}} \min \left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d \rho}{\rho} \\
& \leq c_{7} L_{r}[\mu](x, t)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_{1, r}\left(\tilde{Q}_{r}(x, t)\right) & =\int_{\tilde{Q}_{r}(x, t)}\left(L_{r}[\mu](y, s)\right)^{p^{\prime}} d y d s \\
& \leq c_{8} r^{N+2}\left(L_{r}[\mu](x, t)\right)^{p^{\prime}}
\end{aligned}
$$

Since $r^{\alpha-1} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} \leq \frac{1}{\alpha-\delta} \frac{d}{d r}\left(r^{\alpha} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\right)$, we deduce that

$$
\begin{aligned}
\int_{0}^{\infty} \frac{\sigma_{1, r}\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min & \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} \frac{d r}{r} \leq c_{7} \int_{0}^{\infty} r^{\alpha-1}\left(L_{r}[\mu](x, t)\right)^{p^{\prime}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\} d r \\
& \leq \frac{c_{7}}{\alpha-\delta} \int_{0}^{\infty} \frac{d}{d r}\left(r^{\alpha} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\right)\left(L_{r}[\mu](x, t)\right)^{p^{\prime}} d r \\
& \leq c_{8} \int_{0}^{\infty} r^{\alpha}\left(L_{r}[\mu](x, t)\right)^{p^{\prime}-1} \frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+2-\alpha}} \min \left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}^{2} \frac{d r}{r}
\end{aligned}
$$

Therefore, we get (4.56) from (4.55). This completes the proof of Theorem.
Remark 4.37 It is easy to assert that if $\mathbf{8}$. holds then for any $0<\beta<N+2$

$$
\begin{equation*}
\mathbb{I}_{\beta}\left[\left(\mathbb{I}_{\alpha}^{R, \delta}[\mu]\right)^{p^{\prime}}\right] \leq C \mathbb{I}_{\beta}[\mu] \tag{4.58}
\end{equation*}
$$

for some $C=C(N, \alpha, \beta, \delta, p)>0$.
Corollary 4.38 Let $p>1, \alpha>0$ such that $0<\alpha p<N+2$. There holds

$$
\begin{equation*}
C_{1}^{-1}[\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}}^{p^{\prime}} \leq\left[\left(\mathbb{I}_{\alpha}[\mu]\right)^{p^{\prime}}\right]_{\mathfrak{M}_{\alpha}, p} \leq C_{1}[\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}}^{p^{\prime}} \tag{4.59}
\end{equation*}
$$

for all $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$. Furthermore,

$$
\begin{equation*}
\left[\varphi_{n} * \mu\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}} \leq C_{2}[\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}} \tag{4.60}
\end{equation*}
$$

for $n \in \mathbb{N}, \mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ where $\left\{\varphi_{n}\right\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$. Here $C_{i}=$ $C_{i}(N, p, \alpha), i=1,2$.

Proof. For $R=\infty$ we have $\mathbb{I}_{\alpha}^{R, \delta}[\mu]=\mathbb{I}_{\alpha}[\mu]$ and $E_{\alpha}^{R, \delta}=E_{\alpha}$. Thus, by (4.20) in Corollary 4.10 and Theorem 4.36 we get for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\mu(E) \leq c_{1} \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)
$$

if and only if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\int_{E}\left(\mathbb{I}_{\alpha}[\mu]\right)^{p^{\prime}} d x d t \leq c_{2} \operatorname{Cap}_{\mathcal{H}_{\alpha}, p}(E)
$$

It follows (4.59).
Since $\mathbb{I}_{\alpha}\left[\varphi_{n} * \mu\right]=\varphi_{n} * \mathbb{I}_{\alpha}[\mu] \leq \mathbb{M}\left(\mathbb{I}_{\alpha}[\mu]\right)$ and $\mathbb{M}$ is bounded in $L^{p^{\prime}}\left(\mathbb{R}^{N+1}, d w\right)$ with $w \in A_{p^{\prime}}$ yield

$$
\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}\left[\varphi_{n} * \mu\right]\right)^{p^{\prime}} d w \leq c_{3}\left([w]_{A_{p^{\prime}}}\right) \int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}[\mu]\right)^{p^{\prime}} d w
$$

Thanks to Proposition 4.23 we have

$$
\left[\left(\mathbb{I}_{\alpha}\left[\varphi_{n} * \mu\right]\right)^{p^{\prime}}\right]_{\mathfrak{M}_{\alpha}, p} \leq c_{4}\left[\left(\mathbb{I}_{\alpha}[\mu]\right)^{p^{\prime}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}}
$$

which implies (4.60).

Corollary 4.39 Let $p>1, \alpha>0$ with $0<\alpha p \leq N+2,0<\delta<\alpha$ and $R, d>0$. There holds

$$
\begin{equation*}
\left[\left(\mathbb{I}_{\alpha}^{R, \delta}[\mu]\right)^{p^{\prime}}\right]_{\mathfrak{M}_{\alpha}, p} \leq C_{1}(d / R, R)[\mu]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}}^{p^{\prime}} \tag{4.61}
\end{equation*}
$$

for all $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ with diam $(\operatorname{supp}(\mu)) \leq d$. Furthermore,

$$
\begin{equation*}
\left[\varphi_{n} * \mu\right]_{\mathfrak{M}^{\boldsymbol{G}_{\alpha}, p}} \leq C_{2}(d)[\mu]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}} \tag{4.62}
\end{equation*}
$$

for $n \in \mathbb{N}, \mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ with $\operatorname{diam}(\operatorname{supp}(\mu)) \leq d$ where $\left\{\varphi_{n}\right\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Proof. It is easy to see that

$$
\left(c_{1}(d / R)\right)^{-1}\left\|E_{\alpha}^{R}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq\left\|E_{\alpha}^{R, \delta} * \mu\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)} \leq c_{1}(d / R)\left\|E_{\alpha}^{R}[\mu]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}
$$

for any $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ with $\operatorname{diam}(\operatorname{supp}(\mu)) \leq d$, thus two quantities $\operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E)$ and $\operatorname{Cap}_{E_{\alpha}^{R}, p}(E)$ are equivalent for every compact set $E \subset \mathbb{R}^{N+1}$, $\operatorname{diam}(E) \leq d$ where equivalent constants depend only on $N, p, \alpha$ and $d / R$. Therefore, by Corollary 4.10 we get $\operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E) \approx \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)$ for every compact set $E \subset \mathbb{R}^{N+1}, \operatorname{diam}(E) \leq d$ where equivalent constants depend on $d / R$ and $R$. Thus, by Theorem 4.36 and $\operatorname{diam}(\operatorname{supp}(\mu)) \leq d$ we get, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\mu(E) \leq c_{2}(d / R, R) \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)
$$

then for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\int_{E}\left(\mathbb{I}_{\alpha}^{R, \delta}[\mu]\right)^{p^{\prime}} d x d t \leq c_{3}(d / R, R) \operatorname{Cap}_{E_{\alpha}^{R, \delta}, p}(E) \leq c_{4}(d / R, R) \operatorname{Cap}_{\mathcal{G}_{\alpha}, p}(E)
$$

It follows (4.61). As in the Proof of Corollary 4.38 we also have for $w \in A_{p^{\prime}}$

$$
\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}^{1, \delta}\left[\varphi_{n} * \mu\right]\right)^{p^{\prime}} d w \leq c_{5}\left([w]_{A_{p^{\prime}}}\right) \int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}^{1, \delta}[\mu]\right)^{p^{\prime}} d w
$$

Thanks to Proposition 4.23 and Theorem 4.36 we obtain (4.62).
Remark 4.40 Likewise (see [71, Lemma 5.7]), we can verify that if $\frac{2}{p}<\alpha<N+\frac{2}{p}$,

$$
\begin{aligned}
& {\left[\varphi_{1, n} * \omega_{1}\right]_{\mathfrak{M}^{\mathbf{I}} \alpha-2 / p, p} \leq C_{1}\left[\omega_{1}\right]_{\mathfrak{M}^{\mathbf{I}}-2 / p, p} \quad \text { and }} \\
& {\left[\varphi_{1, n} * \omega_{2}\right]_{\mathfrak{M}^{\mathbf{1}}}^{\mathbf{G}_{\alpha-2 / p}, p} \leq C_{2}(d)\left[\omega_{2}\right]_{\mathfrak{M}^{\mathbf{G}}}{ }^{\mathbf{G}_{\alpha-2 / p}, p}}
\end{aligned}
$$

for $n \in \mathbb{N}$ and $\omega_{1}, \omega_{2} \in \mathfrak{M}^{+}\left(\mathbb{R}^{N}\right)$ with diam $\left(\operatorname{supp}\left(\omega_{2}\right)\right) \leq d$ where $C_{1}=C_{1}(N, \alpha, p), C_{2}(d)=$ $C_{2}(N, \alpha, p, d),\left\{\varphi_{1, n}\right\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N}$ and $[\cdot]_{\mathfrak{M}^{\mathbf{I}} \alpha-2 / p, p},[\cdot]_{\mathfrak{M}^{\mathbf{G}}} \mathbf{G}_{\alpha-2 / p}, p$ was defined in Remark 4.34. Hence, we obtain

$$
\begin{aligned}
& {\left[\left(\varphi_{1, n} * \omega_{1}\right) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}} \leq C_{3}\left[\omega_{1} \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha}, p}}} \\
& {\left[\left(\varphi_{1, n} * \omega_{2}\right) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}} \leq C_{4}(d)\left[\omega_{2} \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha}, p}}}
\end{aligned}
$$

for $n \in \mathbb{N}$ and $\omega_{1}, \omega_{2} \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, diam $(\operatorname{supp}(\mu)) \leq d$ where $C_{3}=C_{3}(N, \alpha, p), C_{4}(d)=$ $C_{4}(N, \alpha, p, d)$.

Proposition 4.41 Let $q>1,0<\alpha q<N+2,0<R \leq \infty, 0<\delta<\alpha$ and $K>0$. Let $0 \leq f \in L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$. Let $C_{4}, C_{5}$ be constants in inequalities (4.50) and (4.51) in Theorem 4.36 with $p=q^{\prime}$. Suppose that $\left\{u_{n}\right\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^{N+1}$ satisfying

$$
\begin{align*}
& u_{n+1} \leq K \mathbb{I}_{\alpha}^{R, \delta}\left[u_{n}^{q}\right]+f \quad \forall n \in \mathbb{N} \\
& \quad u_{0} \leq f \tag{4.63}
\end{align*}
$$

Then, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\int_{E} f^{q} d x d t \leq C C a p_{E_{\alpha}^{R, \delta}, q^{\prime}}(E) \tag{4.64}
\end{equation*}
$$

with

$$
\begin{equation*}
C \leq C_{4}\left(\frac{2^{-q+1}}{C_{5}(q-1)}\left(\frac{q-1}{q K 2^{q-1}}\right)^{q}\right)^{q-1} \tag{4.65}
\end{equation*}
$$

then

$$
\begin{equation*}
u_{n} \leq \frac{K q 2^{q-1}}{q-1} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]+f \quad \forall n \in \mathbb{N} \tag{4.66}
\end{equation*}
$$

Proof. From (4.50) and (4.51) in Theorem 4.36, we see that (4.64) implies

$$
\begin{equation*}
\mathbb{I}_{\alpha}^{R, \delta}\left[\left(I_{\alpha}^{R, \delta}\left[f^{q}\right]\right)^{q}\right] \leq\left(\frac{C}{C_{4}}\right)^{\frac{1}{q-1}} C_{5} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right] \tag{4.67}
\end{equation*}
$$

Now we prove (4.66) by induction. Clearly, (4.66) holds with $n=0$. Next we assume that (4.66) holds with $n=m$. Then, by (4.65), (4.67) and (4.63) we have

$$
\begin{aligned}
u_{m+1} & \leq K \mathbb{I}_{\alpha}^{R, \delta}\left[u_{n}^{q}\right]+f \\
& \leq K 2^{q-1}\left(\frac{K q 2^{q-1}}{q-1}\right)^{q} \mathbb{I}_{\alpha}^{R, \delta}\left[\left(\mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]\right)^{q}\right]+K 2^{q-1} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]+f \\
& \leq K 2^{q-1}\left(\frac{K q 2^{q-1}}{q-1}\right)^{q}\left(\frac{C}{C_{4}}\right)^{\frac{1}{q-1}} C_{5} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]+K 2^{q-1} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]+f \\
& \leq \frac{K q 2^{q-1}}{q-1} \mathbb{I}_{\alpha}^{R, \delta}\left[f^{q}\right]+f
\end{aligned}
$$

Therefore (4.66) also holds true with $n=m+1$. This completes the proof of the Theorem.

Corollary 4.42 Let $q>\frac{N+2}{N+2-\alpha}, \alpha>0$ and $f \in L_{+}^{q}\left(\mathbb{R}^{N+1}\right)$. There exists a constant $C>0$ depending on $N, \alpha, q$ such that if for every compact set $E \subset \mathbb{R}^{N+1}, \int_{E} f^{q} d x d t \leq$ $C C a p_{\mathcal{H}_{\alpha}, q^{\prime}}(E)$, then $u=\mathcal{H}_{\alpha}\left[u^{q}\right]+f$ admits a positive solution $u \in L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$.
Proof. Consider the sequence $\left\{u_{n}\right\}$ of nonnegative functions defined by $u_{0}=f$ and $u_{n+1}=$ $\mathcal{H}_{\alpha}\left[u_{n}^{q}\right]+f \forall n \geq 0$. It is easy to see that $u_{n+1} \leq c_{1} \mathbb{I}_{2}\left[u_{n}^{q}\right]+f \quad \forall n \geq 0$. By Proposition 4.41 and Corollary 4.38, there exists a constant $c_{2}=c_{2}(N, \alpha, q)>0$ such that if for every compact set $E \subset \mathbb{R}^{N+1}, \int_{E} f^{q} d x d t \leq c_{2} \operatorname{Cap}_{\mathcal{H}_{\alpha}, q^{\prime}}(E)$ then $u_{n}$ is well defined and

$$
u_{n} \leq \frac{c_{1} q 3^{q-1}}{q-1} \mathbb{I}_{\alpha}\left[f^{q}\right]+f \quad \forall n \geq 0
$$

Since $\left\{u_{n}\right\}$ is nondecreasing, thus thanks to the dominated convergence theorem we obtain $u(x, t)=\lim _{n \rightarrow \infty} u_{n}(x, t)$ is a solution of $u=\mathcal{H}_{\alpha}\left[u^{q}\right]+f$ which $u \in L_{\text {loc }}^{q}\left(\mathbb{R}^{N+1}\right)$. This completes the proof of the Corollary.
Corollary 4.43 Let $q>1, \alpha>0,0<R \leq \infty, 0<\delta<\alpha$ and $\mu \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$. The following two statements are equivalent.
a. for every compact set $E \subset \mathbb{R}^{N+1}, \int_{E} f^{q} \leq C \operatorname{Cap}{E_{\alpha}^{R, \delta}, q^{\prime}}(E)$ for some a constant $C>0$
b. There exists a function $u \in L_{\text {loc }}^{q}\left(\mathbb{R}^{N+1}\right)$ such that $u=\mathbb{I}_{\alpha}^{R, \delta}\left[u^{q}\right]+\varepsilon f$ for some $\varepsilon>0$.

Proof. We will prove $b . \Rightarrow a$. Set $d \omega(x, t)=\left(\left(\mathbb{I}_{\alpha}^{R, \delta}\left[u^{q}\right]\right)^{q}+\varepsilon^{q} f^{q}\right) d x d t$, thus we have $d w(x, t) \geq\left(I_{\alpha}^{R, \delta}[\omega]\right)^{q} d x d t$. Let $\mathbb{M}_{\omega}$ denote the centered Hardy-littlewoood maximal function which is defined for $g \in L_{l o c}^{1}\left(\mathbb{R}^{N+1}, d \omega\right)$,

$$
\mathbb{M}_{\omega} g(x, t)=\sup _{\rho>0} \frac{1}{\omega\left(\tilde{Q}_{\rho}(x, t)\right)} \int_{\tilde{Q}_{\rho}(x, t)}|g| d \omega(x, t)
$$

For $E \subset \mathbb{R}^{N+1}$ is a compact set, we have

$$
\int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_{\omega} \chi_{E}\right)^{q}\left(\mathbb{I}_{\alpha}^{R, \delta}[\omega]\right)^{q} d x d t \leq \int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_{\omega} \chi_{E}\right)^{q} d \omega(x, t)
$$

Since $\mathbb{M}_{\omega}$ is bounded on $L^{s}\left(\mathbb{R}^{N+1}, d \omega\right)$ for $s>1$ and $\left(\mathbb{M}_{\omega} \chi_{E}\right)^{q}\left(\mathbb{I}_{\alpha}^{R, \delta}[\omega]\right)^{q} \geq\left(\mathbb{I}_{\alpha}^{R, \delta}\left[\omega \chi_{E}\right]\right)^{q}$, thus

$$
\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_{\alpha}^{R, \delta}\left[\omega \chi_{E}\right]\right)^{q} d x d t \leq c_{1} \omega(E)
$$

By Theorem 4.36, we get for any compact set $E \subset \mathbb{R}^{N+1}$

$$
\omega(E) \leq c_{2} \operatorname{Cap}_{E_{\alpha}^{R, \delta}, q^{\prime}}(E)
$$

It follows the results.

Remark 4.44 In [64], we also use Theorem (4.36) to show existence of mild solutions to the Navier-Stokes Equations

$$
\left\{\begin{array}{l}
\partial_{t} u-\Delta u+\mathbb{P} \operatorname{div}(u \otimes u)=\mathbb{P} F \quad \text { in } \mathbb{R}^{N} \times(0, \infty)  \tag{4.68}\\
u(0)=u_{0} \text { in } \mathbb{R}^{N}
\end{array}\right.
$$

where $u, F \in \mathbb{R}^{N}, \mathbb{P}=i d-\nabla \Delta^{-1} \nabla$. is the Helmholtz Leray projection onto the vector fields of zero divergence, i.e, for $f \in \mathbb{R}^{N}, \mathbb{P} f=f-\nabla u$ and $\Delta u=$ divf. Namely, there exists $C=C(N)>0$ such that if $\operatorname{div}\left(u_{0}\right)=0$ and

$$
\begin{equation*}
\int_{K}|D(x, t)|^{2} d x d t \leq C \operatorname{Cap}_{\mathcal{H}_{1}, 2}(K) \tag{4.69}
\end{equation*}
$$

for any compact set $K \subset \mathbb{R}^{N+1}$, where if $(x, t) \in \mathbb{R}^{N} \times[0,+\infty)$,

$$
D(x, t)=\left(e^{t \Delta} u_{0}\right)(x)+\int_{0}^{t}\left(e^{(t-s) \Delta} \mathbb{P} F\right)(x) d s
$$

and $D(x, t)=0$ otherwise. Then, the equation (4.68) has global solution u satisfying

$$
\begin{equation*}
|u(x, t)| \leq|D(x, t)|+c \mathbb{I}_{1}\left[|D|^{2}\right](x, t) \tag{4.70}
\end{equation*}
$$

for all $(x, t) \in \mathbb{R}^{N} \times(0, \infty)$ for some $c=c(N)$.

## 5 Global point wise estimates of solutions to the parabolic equations

First, we recall Duzzar and Mingione's result [27], also see [42, 43] which involves local pointwise estimates for solutions of equations (2.4).

Theorem 5.1 Then, there exists a constant $C$ depending only $N, \Lambda_{1}, \Lambda_{2}$ such that if $u \in$ $L^{2}\left(0, T, H^{1}(\Omega)\right) \cap C\left(\Omega_{T}\right)$ is a weak solution to (2.4) with $\mu \in L^{2}\left(\Omega_{T}\right)$ and $u(0)=0$

$$
\begin{equation*}
|u(x, t)| \leq C f_{\tilde{Q}_{R}(x, t)}|u| d y d s+C \mathbb{I}_{2}^{2 R}[|\mu|](x, t) \tag{5.1}
\end{equation*}
$$

for all $Q_{2 R}(x, t) \subset \Omega \times(-\infty, T)$.
Furthermore, if $A$ is independent of space variable $x$, (2.27) is satisfied and $\nabla u \in C\left(\Omega_{T}\right)$ then

$$
\begin{equation*}
|\nabla u(x, t)| \leq C f_{\tilde{Q}_{R}(x, t)}|\nabla u| d y d s+C \mathbb{I}_{1}^{2 R}[|\mu|](x, t) \tag{5.2}
\end{equation*}
$$

for all $Q_{2 R}(x, t) \subset \Omega \times(-\infty, T)$.
Proof of Theorem 2.1. Let $\mu=\mu_{0}+\mu_{s} \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$, with $\mu_{0} \in \mathfrak{M}_{0}\left(\Omega_{T}\right), \mu_{s} \in \mathfrak{M}_{s}\left(\Omega_{T}\right)$. By Proposition 3.7, there exist sequences of nonnegative measures $\mu_{n, 0, i}=\left(f_{n, i}, g_{n, i}, h_{n, i}\right)$ and $\mu_{n, s, i}$ such that $f_{n, i}, g_{n, i}, h_{n, i} \in C_{c}^{\infty}\left(\Omega_{T}\right)$ and strongly converge to some $f_{i}, g_{i}, h_{i}$ in $L^{1}\left(\Omega_{T}\right), L^{2}\left(\Omega_{T}, \mathbb{R}^{N}\right)$ and $L^{2}\left(0, T, H_{0}^{1}(\Omega)\right)$ respectively and $\mu_{n, 1}, \mu_{n, 2}, \mu_{n, s, 1}, \mu_{n, s, 2} \in C_{c}^{\infty}\left(\Omega_{T}\right)$ converge to $\mu^{+}, \mu^{-}, \mu_{s}^{+}, \mu_{s}^{-}$resp. in the narrow topology with $\mu_{n, i}=\mu_{n, 0, i}+\mu_{n, s, i}$, for $i=1,2$ and satisfying $\mu_{0}^{+}=\left(f_{1}, g_{1}, h_{1}\right), \mu_{0}^{-}=\left(f_{2}, g_{2}, h_{2}\right)$ and $0 \leq \mu_{n, 1} \leq \varphi_{n} * \mu^{+}, 0 \leq \mu_{n, 2} \leq \varphi_{n} * \mu^{-}$, where $\left\{\varphi_{n}\right\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Let $\sigma_{1, n}, \sigma_{2, n} \in C_{c}^{\infty}(\Omega)$ be convergent to $\sigma^{+}$and $\sigma^{-}$in the narrow topology and in $L^{1}(\Omega)$ if $\sigma \in L^{1}(\Omega)$ resp. such that $0 \leq \sigma_{1, n} \leq \varphi_{1, n} * \sigma^{+}, 0 \leq \sigma_{2, n} \leq \varphi_{1, n} * \sigma^{-}$where $\left\{\varphi_{1, n}\right\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N}$. Set $\mu_{n}=\mu_{n, 1}-\mu_{n, 2}$ and $\sigma_{n}=\sigma_{1, n}-\sigma_{2, n}$. Let $u_{n}, u_{n, 1}, u_{n, 2}$ be solutions of equations

$$
\begin{gather*}
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\mu_{n} \text { in } \Omega_{T}, \\
u_{n}=0 \text { on } \partial \Omega \times(0, T), \\
u_{n}(0)=\sigma_{n} \text { on } \Omega,
\end{array}\right.  \tag{5.3}\\
\left\{\begin{array}{l}
\left(u_{n, 1}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n, 1}\right)\right)=\chi_{\Omega_{T}} \mu_{n, 1} \text { in } B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right), \\
u_{n, 1}=0 \text { on } \partial B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right), \\
u_{n, 1}(0)=\sigma_{1, n} \text { on } B_{2 T_{0}}\left(x_{0}\right),
\end{array}\right.  \tag{5.4}\\
\left\{\begin{array}{l}
\left(u_{n, 2}\right)_{t}+\operatorname{div}\left(A\left(x, t,-\nabla u_{n, 2}\right)\right)=\chi_{\Omega_{T}} \mu_{n, 2} \text { in } B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right), \\
u_{n, 2}=0 \text { on } \partial B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right), \\
u_{n, 2}(0)=\sigma_{2, n} \text { on } B_{2 T_{0}}\left(x_{0}\right),
\end{array}\right. \tag{5.5}
\end{gather*}
$$

where $\Omega \subset B_{T_{0}}\left(x_{0}\right)$ for $x_{0} \in \Omega$.
We see that $u_{n, 1}, u_{n, 2} \geq 0$ in $B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right)$ and $-u_{n, 2} \leq u_{n} \leq u_{n, 1}$ in $\Omega_{T}$.
Now, we estimate $u_{n, 1}$. By Remark 3.3 and Theorem 3.6, a sequence $\left\{u_{n, 1, m}\right\}$ of solutions to equations

$$
\left\{\begin{array}{l}
\left(u_{n, 1, m}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n, 1, m}\right)\right)=\left(g_{n, m}\right)_{t}+\chi_{\Omega_{T}} \mu_{n, 1} \text { in } B_{2 T_{0}}\left(x_{0}\right) \times\left(-2 T_{0}^{2}, 2 T_{0}^{2}\right),  \tag{5.6}\\
u_{n, 1, m}=0 \text { on } \partial B_{2 T_{0}}\left(x_{0}\right) \times\left(-2 T_{0}^{2}, 2 T_{0}^{2}\right), \\
u_{n, 1, m}\left(-2 T_{0}^{2}\right)=0 \text { on } B_{2 T_{0}}\left(x_{0}\right),
\end{array}\right.
$$

converges to $u_{n, 1}$ in $B_{2 T_{0}}\left(x_{0}\right) \times\left(0,2 T_{0}^{2}\right)$, where $g_{n, m}(x, t)=\sigma_{1, n}(x) \int_{-2 T_{0}^{2}}^{t} \varphi_{2, m}(s) d s$ and $\left\{\varphi_{2, m}\right\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Remark 3.2, we have

$$
\begin{equation*}
\left\|u_{n, 1, m}\right\|_{L^{1}\left(\tilde{Q}_{2 T_{0}}\left(x_{0}, 0\right)\right)} \leq c_{1} T_{0}^{2} A_{n, m}, \tag{5.7}
\end{equation*}
$$

where $A_{n, m}=\mu_{n, 1}\left(\Omega_{T}\right)+\int_{\tilde{Q}_{2 T_{0}}\left(x_{0}, 0\right)} \sigma_{1, n}(x) \varphi_{2, m}(t) d x d t$.
Hence, thanks to Theorem 5.1 we have for $(x, t) \in \Omega_{T}$

$$
\begin{aligned}
u_{n, 1, m}(x, t) & \leq c_{8} T_{0}^{-N-2}\left\|u_{n, 1, m}\right\|_{L^{1}\left(\tilde{Q}_{2 T_{0}}\left(x_{0}, 0\right)\right)}+c_{8} \mathbb{I}_{2}\left[\mu_{n, 1}\right](x, t)+c_{8} \mathbb{I}_{2}\left[\sigma_{1, n} \varphi_{m}\right](x, t) \\
& \leq c_{9} \mathbb{I}_{2}\left[\mu_{n, 1}\right](x, t)+c_{9} \mathbb{I}_{2}\left[\sigma_{1, n} \varphi_{m}\right](x, t) .
\end{aligned}
$$

Since $0 \leq \mu_{n, 1} \leq \varphi_{n} * \mu^{+}, \sigma_{1, n} \leq \varphi_{1, n} * \sigma^{+}$,

$$
u_{n, 1, m}(x, t) \leq c_{9} \varphi_{n} * \mathbb{I}_{2}\left[\mu^{+}\right](x, t)+c_{9}\left(\varphi_{1, n} \varphi_{2, m}\right) * \mathbb{I}_{2}\left[\sigma^{+} \otimes \delta_{\{t=0\}}\right](x, t) \forall(x, t) \in \Omega_{T} .
$$

Letting $m \rightarrow \infty$, we get

$$
u_{n, 1}(x, t) \leq c_{9} \varphi_{n} * \mathbb{I}_{2}\left[\mu^{+}\right](x, t)+c_{9} \varphi_{1, n} *\left(\mathbb{I}_{2}\left[\sigma^{+} \otimes \delta_{\{t=0\}}\right](., t)\right)(x) \forall(x, t) \in \Omega_{T}
$$

Similarly, we also get

$$
u_{n, 2}(x, t) \leq c_{9} \varphi_{n} * \mathbb{I}_{2}\left[\mu^{-}\right](x, t)+c_{9} \varphi_{1, n} *\left(\mathbb{I}_{2}\left[\sigma^{-} \otimes \delta_{\{t=0\}}\right](., t)\right)(x) \forall(x, t) \in \Omega_{T}
$$

Consequently, by Proposition 3.5 and Theorem 3.6 , up to a subsequence, $\left\{u_{n}\right\}$ converges to a distribution solution (a renormalized solution if $\left.\sigma \in L^{1}(\Omega)\right) u$ of (2.4) and satisfied (2.7).

Remark 5.2 Obviously, if $\sigma \equiv 0$ and $\operatorname{supp}(\mu) \subset \bar{\Omega} \times[a, T], a>0$ then $u=0$ in $\Omega \times(0, a)$.
Remark 5.3 If $A$ is independent of space variable $x$, (2.27) is satisfied then

$$
\begin{equation*}
|\nabla u(x, t)| \leq C\left(N, \Lambda_{1}, \Lambda_{2}, T_{0} / d\right) \mathbb{I}_{1}^{2 T_{0}}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right](x, t) \tag{5.8}
\end{equation*}
$$

for any $(x, t) \in \Omega^{d} \times(0, T)$ and $0<d \leq \frac{1}{2} \min \left\{\sup _{x \in \Omega} d(x, \partial \Omega), T_{0}^{1 / 2}\right\}$ where $\Omega^{d}=\{x \in \Omega$ : $d(x, \partial \Omega)>d\}$. Indeed, by Remark 3.3 and Theorem 3.6, a sequence $\left\{v_{n, m}\right\}$ of solutions to equations

$$
\left\{\begin{array}{l}
\left(v_{n, m}\right)_{t}-\operatorname{div}\left(A\left(t, \nabla u_{n, m}\right)\right)=\left(g_{n, m}\right)_{t}+\chi_{\Omega_{T}} \mu_{n} \quad \text { in } \Omega \times\left(-2 T_{0}^{2}, T\right)  \tag{5.9}\\
v_{n, m}=0 \text { on } \partial \Omega \times\left(-2 T_{0}^{2}, T\right) \\
v_{n, m}\left(-2 T_{0}^{2}\right)=0 \text { on } \Omega
\end{array}\right.
$$

converges to $u_{n}$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$, where $g_{n, m}(x, t)=\sigma_{n}(x) \int_{-2 T_{0}^{2}}^{t} \varphi_{2, m}(s) d s$ and $\left\{\varphi_{2, m}\right\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Theorem 5.1, we have for any $(x, t) \in \Omega^{d} \times(0, T)$

$$
\left|\nabla v_{n, m}(x, t)\right| \leq c_{1} f_{\tilde{Q}_{d / 2}(x, t)}\left|\nabla v_{n, m}\right| d y d s+c_{1} \mathbb{I}_{1}^{d}\left[\left|\mu_{n}\right|+\left|\sigma_{n}\right| \otimes \varphi_{2, m}\right](x, t)
$$

On the other hand, by remark 3.2,

$$
\left\|\left|\nabla v_{n, m}\right|\right\|_{L^{1}\left(\Omega \times\left(-T_{0}^{2}, T\right)\right)} \leq c_{2} T_{0}\left(\left|\mu_{n}\right|+\left|\sigma_{n}\right| \otimes \varphi_{2, m}\right)\left(\Omega \times\left(-T_{0}^{2}, T\right)\right)
$$

Therefore, for any $(x, t) \in \Omega^{d} \times(0, T)$

$$
\left|\nabla v_{n, m}(x, t)\right| \leq c_{3} \mathbb{I}_{1}\left[\left|\mu_{n}\right|+\left|\sigma_{n}\right| \otimes \varphi_{2, m}\right](x, t)
$$

where $c_{3}$ depends on $T_{0} / d$.
Finally, letting $m \rightarrow \infty$ and $n \rightarrow \infty$ we get for any $(x, t) \in \Omega^{d} \times(0, T)$

$$
|\nabla u(x, t)| \leq c_{3} \mathbb{I}_{1}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right](x, t) .
$$

We conclude (5.8) since $\mathbb{I}_{1}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right] \leq c_{4} \mathbb{I}_{1}^{2 T_{0}}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]$ in $\Omega_{T}$.
Next, we will establish pointwise estimates from below for solutions of equations (2.4).
Theorem 5.4 If $u \in C\left(Q_{r}(y, s)\right) \cap L^{2}\left(s-r^{2}, s, H^{1}\left(B_{r}(y)\right)\right)$ is a nonnegative weak solution of (2.4) with data $\mu \in \mathfrak{M}^{+}\left(Q_{r}(y, s)\right)$ and $u\left(s-r^{2}\right) \geq 0$, then there exists a constant $C$ depending on $N, \Lambda_{1}, \Lambda_{2}$ such that

$$
\begin{equation*}
u(y, s) \geq C \sum_{k=0}^{\infty} \frac{\mu\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}} \tag{5.10}
\end{equation*}
$$

where $r_{k}=4^{-k} r$.

Proof. It is enough to show that for $\rho \in(0, r)$

$$
\begin{equation*}
\frac{\mu\left(Q_{\rho / 8}\left(y, s-\frac{35}{128} \rho^{2}\right)\right)}{\rho^{N}} \leq c_{1}\left(\inf _{Q_{\rho / 4}(y, s)} u-\inf _{Q_{\rho}(y, s)} u\right) \tag{5.11}
\end{equation*}
$$

By [50, Theorem 6.18, p. 122 ], we have for any $\theta \in(0,1+2 / N)$,

$$
\begin{equation*}
\left(f_{Q_{\rho / 4}\left(y, s-\rho^{2} / 4\right)}(u-a)^{\theta}\right)^{1 / \theta} \leq c_{2}(b-a) \tag{5.12}
\end{equation*}
$$

where $b=\inf _{Q_{\rho / 4}(y, s)} u, a=\inf _{Q_{\rho}(y, s)} u$ and a constant $c_{2}$ depends on $N, \Lambda_{1}, \Lambda_{2}, \theta$.
Let $\eta \in C_{c}^{\infty}\left(Q_{\rho}(y, s)\right)$ such that $0 \leq \eta \leq 1, \operatorname{supp} \eta \subset Q_{\rho / 4}\left(y, s-\frac{1}{4} \rho^{2}\right), \eta=1$ in $Q_{\rho / 8}\left(y, s-\frac{35}{128} \rho^{2}\right)$ and $|\nabla \eta| \leq c_{3} / \rho^{2},\left|\eta_{t}\right| \leq c_{3} / \rho^{2}$ where $c_{3}=c_{3}(N)$. We have

$$
\begin{aligned}
\mu\left(Q_{\rho / 8}\left(y, s-\frac{35}{128} \rho^{2}\right)\right) & \leq \int_{Q_{\rho}(y, s)} \eta^{2} d \mu(x, t) \\
& =\int_{Q_{\rho}(y, s)} u_{t} \eta^{2} d x d t+2 \int_{Q_{\rho}(y, s)} \eta A(x, t, \nabla u) \nabla \eta d x d t \\
& =-2 \int_{Q_{\rho}(y, s)}(u-a) \eta_{t} \eta d x d t+2 \int_{Q_{\rho}(y, s)} \eta A(x, t, \nabla u) \nabla \eta d x d t \\
& \leq c_{3} r^{-2} \int_{Q_{\rho / 4}\left(y, s-\frac{1}{4} \rho^{2}\right)}(u-a) d x d t+2 \Lambda_{1} \int_{Q_{\rho}(y, s)} \eta|\nabla u \| \nabla \eta| d x d t \\
& \leq c_{4} r^{N}(b-a)+c_{4} \int_{Q_{\rho}(y, s)} \eta|\nabla u| \| \eta \mid d x d t
\end{aligned}
$$

Here we used (5.12) with $\theta=1$ in the last inequality. It remains to show that

$$
\begin{equation*}
\int_{Q_{r}(y, s)} \eta|\nabla u \| \nabla \eta| d x d t \leq c_{5} r^{N}(b-a) \tag{5.13}
\end{equation*}
$$

First, we verify that for $\varepsilon \in(0,1)$

$$
\begin{equation*}
\int_{Q_{\rho}(y, s)}|\nabla u|^{2}(u-a)^{-\varepsilon-1} \eta^{2} d x d t \leq c_{6} \int_{Q_{\rho}(y, s)}(u-a)^{1-\varepsilon}\left(\eta\left|\eta_{t}\right|+|\nabla \eta|^{2}\right) d x d t \tag{5.14}
\end{equation*}
$$

Indeed, for $\delta \in(0,1)$ we choose $\varphi=(u-a+\delta)^{-\varepsilon} \eta^{2}$ as test function in (2.4),

$$
\begin{aligned}
0 \leq & \int_{Q_{\rho}(y, s)} u_{t}(u-a+\delta)^{-\varepsilon} \eta^{2} d x d t+\int_{Q_{\rho}(y, s)} A(x, t, \nabla u) \nabla\left((u-a+\delta)^{-\varepsilon} \eta^{2}\right) d x d t \\
\leq & 2(1-\varepsilon) \int_{Q_{\rho}(y, s)}(u-a+\delta)^{1-\varepsilon}\left|\eta_{t}\right| \eta d x d t-\varepsilon \Lambda_{2} \int_{Q_{\rho}(y, s)}|\nabla u|^{2}(u-a+\delta)^{-\varepsilon-1} \eta^{2} d x d t \\
& +2 \Lambda_{1} \int_{Q_{\rho}(y, s)} \eta|\nabla u|(u-a+\delta)^{-\varepsilon}|\nabla \eta| d x d t .
\end{aligned}
$$

So, we deduce (5.14) from using the Holder inequality and letting $\delta \rightarrow 0$.
Therefore, for $\varepsilon \in(0,2 / N)$ using the Holder inequality and we get

$$
\begin{aligned}
& \int_{Q_{r}(y, s)} \eta|\nabla u||\nabla \eta| d x d t \\
& \quad \leq\left(\int_{Q_{\rho}(y, s)}|\nabla u|^{2}(u-a)^{-\varepsilon-1} \eta^{2} d x d t\right)^{1 / 2}\left(\int_{Q_{\rho}(y, s)}(u-a)^{\varepsilon+1}|\nabla \eta|^{2} d x d t\right)^{1 / 2} \\
& \quad \leq c_{7}\left(\int_{Q_{\rho}(y, s)}(u-a)^{1-\varepsilon}\left(\eta\left|\eta_{t}\right|+|\nabla \eta|^{2}\right) d x d t\right)^{1 / 2}\left(\int_{Q_{\rho}(y, s)}(u-a)^{\varepsilon+1}|\nabla \eta|^{2} d x d t\right)^{1 / 2} \\
& \quad \leq c_{8} \rho^{-2}\left(\int_{Q_{\rho / 4}\left(y, s-\frac{1}{4} \rho^{2}\right)}(u-a)^{1-\varepsilon} d x d t\right)^{1 / 2}\left(\int_{Q_{\rho / 4}\left(y, s-\frac{1}{4} \rho^{2}\right)}(u-a)^{\varepsilon+1} d x d t\right)^{1 / 2}
\end{aligned}
$$

Consequently, we get (5.11) from (5.12).
Proof of Theorem 2.3. Let $\mu_{n} \in\left(C_{c}^{\infty}\left(\Omega_{T}\right)\right)^{+}, \sigma_{n} \in\left(C_{c}^{\infty}(\Omega)\right)^{+}$be in the proof of Theorem 2.1. Let $u_{n}$ be a weak solution of equation

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\mu_{n} \text { in } \Omega_{T}, \\
u_{n}=0 \text { on } \partial \Omega \times(0, T) \\
u_{n}(0)=\sigma_{n} \text { on } \Omega
\end{array}\right.
$$

As the proof of Theorem 2.1, thanks to Theorem 5.4 we get By Remark for any $Q_{r}(y, s) \subset$ $\Omega \times(-\operatorname{diam}(\Omega), T)$ and $r_{k}=4^{-k} r$

$$
u_{n}(y, s) \geq c_{1} \sum_{k=0}^{\infty} \frac{\mu_{n}\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}+c_{1} \sum_{k=0}^{\infty} \frac{\left(\sigma_{n} \otimes \delta_{\{t=0\}}\right)\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}
$$

Finally, by Proposition 3.5 and Theorem 3.6 we get the results.

Remark 5.5 If $u \in L^{q}\left(\Omega_{T}\right)$ satisfies (2.8) then $\mathcal{G}_{2}\left[\chi_{E} \mu\right] \in L^{q}\left(\mathbb{R}^{N+1}\right)$ and $\mathbf{G}_{\frac{2}{q}}\left[\chi_{F} \sigma\right] \in$ $L^{q}\left(\mathbb{R}^{N}\right)$ for every $E \subset \subset \Omega \times[0, T)$ and $F \subset \subset \Omega$. Indeed, for $E \subset \subset \Omega \times[0, T), \varepsilon=$ $\operatorname{dist}(E,(\Omega \times(0, T)) \cup(\Omega \times\{t=T\}))>0$, we can see that for any $(y, s) \in \Omega_{T}, r_{k}=4^{-k} \varepsilon / 4$

$$
\begin{equation*}
u(y, s) \geq c_{1} \sum_{k=0}^{\infty} \frac{\tilde{\mu}\left(E \cap Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}} \tag{5.15}
\end{equation*}
$$

where $\tilde{\mu}=\mu+\sigma \otimes \delta_{\{t=0\}}$.
Moreover, for any $(y, s) \notin \Omega_{T}$

$$
\sum_{k=0}^{\infty} \frac{\tilde{\mu}\left(E \cap Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}=0
$$

Thus,

$$
\begin{aligned}
\infty & >\int_{\mathbb{R}^{N+1}} \sum_{k=0}^{\infty}\left(\frac{\tilde{\mu}\left(E \cap Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}\right)^{q} d y d s \\
& =\int_{\mathbb{R}^{N}} \sum_{k=0}^{\infty} \int_{\mathbb{R}}\left(\frac{\tilde{\mu}\left(E \cap Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right)\right)}{r_{k}^{N}}\right)^{q} d s d y \\
& \geq \int_{\mathbb{R}^{N}} \sum_{k=0}^{\infty} \int_{\mathbb{R}}\left(\frac{\tilde{\mu}\left(E \cap \tilde{Q}_{r_{k} / 8}(y, s)\right)}{r_{k}^{N}}\right)^{q} d s d y \\
& \geq c_{2} \int_{\mathbb{R}^{N+1}} \int_{0}^{\varepsilon / 64}\left(\frac{\tilde{\mu}\left(E \cap \tilde{Q}_{\rho}(y, s)\right)}{\rho^{N}}\right)^{q} \frac{d \rho}{\rho} d s d y \\
& \geq c_{3}(\varepsilon) \int_{\mathbb{R}^{N+1}}\left(\mathcal{G}_{2}\left[\tilde{\mu} \chi_{E}\right]\right)^{q} d s d y .
\end{aligned}
$$

Thus, from Proposition 4.19, we get the results.
Proof of Theorem 2.5. Set $D_{n}=B_{n}(0) \times\left(-n^{2}, n^{2}\right)$. For $n \geq 4$, by Theorem 2.1, there exists a renormalized solution $u_{n}$ to problem

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\chi_{D_{n-1}} \omega \text { in } D_{n} \\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right) \\
u_{n}\left(-n^{2}\right)=0 \text { on } B_{n}(0)
\end{array}\right.
$$

relative to a decomposition $\left(f_{n}, g_{n}, h_{n}\right)$ of $\chi_{D_{n-1}} \omega_{0}$ satisfying

$$
\begin{equation*}
-K \mathbb{I}_{2}\left[\omega^{-}\right](x, t) \leq u_{n}(x, t) \leq K \mathbb{I}_{2}\left[\omega^{+}\right](x, t) \quad \forall(x, t) \in D_{n} \tag{5.16}
\end{equation*}
$$

From the proof of Theorem 2.1 and Remark 3.9, we can assume that $u_{n}$ satisfies (3.14) and (3.15) in Proposition 3.16 with $1<q_{0}<\frac{N+2}{N}, L \equiv 0$ and

$$
\begin{equation*}
\left\|f_{n}\right\|_{L^{1}\left(D_{i}\right)}+\left\|g_{n}\right\|_{L^{2}\left(D_{i}\right)}+\left\|\left|\left|h_{n}\right|+\left|\nabla h_{n}\right| \|_{L^{2}\left(D_{i}\right)} \leq 2\right| \omega \mid\left(D_{i+1}\right)\right. \tag{5.17}
\end{equation*}
$$

for any $i=1, \ldots, n-1$ and $h_{n}$ is convergent in $L_{\text {loc }}^{1}\left(\mathbb{R}^{N+1}\right)$.
On the other hand, by Lemma 4.26 we have for any $s \in\left(1, \frac{N+2}{N}\right)$

$$
\begin{align*}
\int_{D_{m}}\left|u_{n}\right|^{s} d x d t & \leq K^{s} \int_{D_{m}}\left(I_{2}[|\omega|]\right)^{s} d x d t \\
& \leq K^{s} \int_{\tilde{Q}_{4 m}\left(x_{0}, t_{0}\right)}\left(I_{2}[|\omega|]\right)^{s} d x d t \\
& \leq c_{1} M m^{N+2}, \tag{5.18}
\end{align*}
$$

for $n \geq m \geq\left|x_{0}\right|+\left|t_{0}\right|^{1 / 2}$. Consequently, we can apply Proposition 3.17 and obtain that $u_{n}$ converges to some $u$ in $L_{l o c}^{1}\left(\mathbb{R} ; W_{l o c}^{1,1}\left(\mathbb{R}^{N}\right)\right.$ ).
Since for any $\alpha \in(0,1 / 2)$

$$
\int_{D_{m}} \frac{\left|\nabla u_{n}\right|^{2}}{\left(\left|u_{n}\right|+1\right)^{\alpha+1}} d x d t \leq C_{m}(\alpha) \quad \forall n \geq m
$$

thus using (5.18) and Holder inequality, we get for any $1 \leq s_{1}<\frac{N+2}{N+1}$

$$
\int_{D_{m}}\left|\nabla u_{n}\right|^{s_{1}} d x d t \leq C_{m}\left(s_{1}\right) \quad \text { for all } n \geq m \geq\left|x_{0}\right|+\left|t_{0}\right|^{1 / 2}
$$

This yields $u_{n} \rightarrow u$ in $L_{\text {loc }}^{s_{1}}\left(\mathbb{R} ; W_{\text {loc }}^{1, s_{1}}\left(\mathbb{R}^{N}\right)\right)$.
Take $\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{N+1}\right)$ and $m_{0} \in \mathbb{N}$ with $\operatorname{supp}(\varphi) \subset D_{m_{0}}$, we have for $n \geq m_{0}+1$

$$
-\int_{\mathbb{R}^{N+1}} u_{n} \varphi_{t} d x d t+\int_{\mathbb{R}^{N+1}} A\left(x, t, \nabla u_{n}\right) \nabla \varphi d x d t=\int_{\mathbb{R}^{N+1}} \varphi d \omega
$$

Letting $n \rightarrow \infty$, we conclude that $u$ is a distribution solution to problem (2.6) with data $\mu=\omega$ which satisfies (2.9).
Claim 1. If $\omega \geq 0$. By Theorem 2.3, we have for $n \geq 4^{k_{0}+1},(y, s) \in B_{4^{k_{0}}} \times\left(0, n^{2}\right)$

$$
u_{n}(y, s) \geq c_{2} \sum_{k=0}^{\infty} \frac{\omega\left(Q_{r_{k} / 8}\left(y, s-\frac{35}{128} r_{k}^{2}\right) \cap D_{n-1}\right)}{r_{k}^{N}}
$$

where $r_{k}=4^{-k+k_{0}}$. This gives

$$
u_{n}(y, s) \geq c_{2} \sum_{k=-k_{0}}^{\infty} \frac{\omega\left(Q_{2^{-2 k-3}}\left(y, s-35 \times 2^{-4 k-7}\right) \cap B_{n-1}(0) \times\left(0,(n-1)^{2}\right)\right)}{2^{-2 N k}}
$$

Letting $n \rightarrow \infty$ and $k_{0} \rightarrow \infty$ we have (2.10). Finally, thanks to Proposition 4.8 and Theorem 4.2, we will assert (2.11) if we show that for $q>\frac{N+2}{N}$

$$
\int_{\mathbb{R}}\left(\sum_{k=-\infty}^{\infty} \frac{\omega\left(Q_{2-2 k-3}\left(x, t-35 \times 2^{-4 k-7}\right)\right)}{2^{-2 N k}}\right)^{q} d x d t \geq c_{3} \int_{\mathbb{R}} \int_{0}^{+\infty}\left(\frac{\omega\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N}}\right)^{q} \frac{d \rho}{\rho} d x d t
$$

Indeed,

$$
\begin{aligned}
& \int_{\mathbb{R}}\left(\sum_{k=-\infty}^{\infty} \frac{\omega\left(Q_{\left.2^{-2 k-3}\left(x, t-35 \times 2^{-4 k-7}\right)\right)}^{2^{-2 N k}}\right)^{q} d x d t}{}\right. \\
& \quad \geq \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}}\left(\frac{\omega\left(Q_{\left.2^{-2 k-3}\left(x, t-35 \times 2^{-4 k-7}\right)\right)}^{2^{-2 N k}}\right)^{q} d t d x}{} \quad=\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}}\left(\frac{\omega\left(\tilde{Q}_{2^{-2 k-3}}(x, t)\right)}{2^{-2 N k}}\right)^{q} d t\right. \\
& \quad \geq c_{4} \int_{\mathbb{R}^{N+1}} \int_{0}^{+\infty}\left(\frac{\omega\left(\tilde{Q}_{\rho}(x, t)\right)}{\rho^{N}}\right)^{q} \frac{d \rho}{\rho} d x d t
\end{aligned}
$$

Claim 2. If $A$ is independent of space variable $x$ and (2.27) is satisfied. By Remark 5.3 we get for any $(x, t) \in D_{n / 4}$

$$
\left|\nabla u_{n}(x, t)\right| \leq c_{5} \mathbb{I}_{1}[|\omega|](x, t)
$$

Letting $n \rightarrow \infty$, we get (2.12).
Claim 3. If $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then by Remark (5.2) we can assume that $u_{n}=0$ in $B_{n}(0) \times\left(-n^{2}, 0\right)$. So, $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$. Therefore, clearly $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.5). The proof is complete.
Remark 5.6 If $\omega \in \mathfrak{M}_{b}\left(\mathbb{R}^{N+1}\right)$ then $u$ satisfies

$$
\left.\|\mid \nabla u\| \|_{L^{N+2}, \infty} \mathbb{R}^{N+1}\right)<C\left(N, \Lambda_{1}, \Lambda_{2}\right)|\omega|\left(\mathbb{R}^{N+1}\right)
$$

Moreover, $I_{2}[|\omega|] \in L^{\frac{N+2}{N}, \infty}\left(\mathbb{R}^{N+1}\right)$ and $I_{2}[|\omega|]<\infty$ a.e in $\mathbb{R}^{N+1}$.

## 6 Quasilinear Lane-Emden Type Parabolic Equations

### 6.1 Quasilinear Lane-Emden Parabolic Equations in $\Omega_{T}$

To prove Theorem 2.8 we need the following proposition which was proved in [6].
Proposition 6.1 Assume $O$ is an open subset of $\mathbb{R}^{N+1}$. Let $p>1$ and $\mu \in \mathfrak{M}^{+}(O)$. If $\mu$ is absolutely continuous with respect to Cap ${ }_{2,1, p}$ in $O$, there exists a nondecreasing sequence $\left\{\mu_{n}\right\} \subset \mathfrak{M}_{b}^{+}(O) \cap\left(W_{p}^{2,1}\left(\mathbb{R}^{N+1}\right)\right)^{*}$, with compact support in $O$ which converges to $\mu$ weakly in $\mathfrak{M}(O)$. Moreover, if $\mu \in \mathfrak{M}_{b}^{+}(O)$ then $\left\|\mu_{n}-\mu\right\|_{\mathfrak{M}_{b}(O)} \rightarrow 0$ as $n \rightarrow \infty$.

Remark 6.2 By Theorem 4.17, $W_{p}^{2,1}\left(\mathbb{R}^{N+1}\right)=\mathcal{L}_{2}^{p}\left(\mathbb{R}^{N+1}\right)$, it follows $\left\{\mu_{n}\right\} \subset \mathfrak{M}_{b}^{+}(O) \cap$ $\left(\mathcal{L}_{2}^{p}\left(\mathbb{R}^{N+1}\right)\right)^{*}$. Note that $\left\|\mu_{n}\right\|_{\left(\mathcal{L}_{2}^{p}\left(\mathbb{R}^{N+1}\right)\right)^{*}}=\left\|\vee_{\mathcal{G}_{2}}\left[\mu_{n}\right]\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)}$. So $\stackrel{\vee}{\mathcal{G}}_{2}\left[\mu_{n}\right] \in L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)$. Consequently, from (4.17) in Proposition 4.8, we obtain $\mathbb{I}_{2}^{R}\left[\mu_{n}\right] \in L^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)$ for any $n \in \mathbb{N}$ and $R>0$. In particular, $\mathbb{I}_{2}\left[\mu_{n}\right] \in L_{\text {loc }}^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)$ for any $n \in \mathbb{N}$.

Remark 6.3 As in the proof of Theorem 2.5 in [16], we can prove a general version of Proposition 6.1, that is: for $p>1$, if $\mu$ is absolutely continuous with respect to Cap ${\mathcal{\mathcal { G } _ { \alpha } , p}}$ in $O$, there exists a nondecreasing sequence $\left\{\mu_{n}\right\} \subset \mathfrak{M}_{b}^{+}(O) \cap\left(\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right)\right)^{*}$, with compact support in $O$ which converges to $\mu$ weakly in $\mathfrak{M}(O)$. Furthermore, $\mathbb{I}_{\alpha}\left[\mu_{n}\right] \in L_{\text {loc }}^{p^{\prime}}\left(\mathbb{R}^{N+1}\right)$ for all $n \in \mathbb{N}$. Besides, we also obtain that for $\mu \in \mathfrak{M}_{b}(O)$ is absolutely continuous with respect to $C p_{\mathcal{G}_{\alpha}, p}$ in $O$ if and only if $\mu=f+\nu$ where $f \in L^{1}(O)$ and $\nu \in\left(\mathcal{L}_{\alpha}^{p}\left(\mathbb{R}^{N+1}\right)\right)^{*}$.

Proof of Theorem 2.8. First, assume that $\sigma \in L^{1}(\Omega)$. Because $\mu$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1, q^{\prime}}$, so are $\mu^{+}$and $\mu^{-}$. Applying Proposition 6.1 there exist two nondecreasing sequences $\left\{\mu_{1, n}\right\}$ and $\left\{\mu_{2, n}\right\}$ of positive bounded measures with compact support in $\Omega_{T}$ which converge to $\mu^{+}$and $\mu^{-}$in $\mathfrak{M}_{b}\left(\Omega_{T}\right)$ respectively and such that $\mathbb{I}_{2}\left[\mu_{1, n}\right], \mathbb{I}_{2}\left[\mu_{2, n}\right] \in L^{q}\left(\Omega_{T}\right)$.

For $i=1,2$, set $\tilde{\mu}_{i, 1}=\mu_{i, 1}$ and $\tilde{\mu}_{i, j}=\mu_{i, j}-\mu_{i, j-1} \geq 0$, so $\mu_{i, n}=\sum_{j=1}^{n} \tilde{\mu}_{i, j}$. We write $\mu_{i, n}=\mu_{i, n, 0}+\mu_{i, n, s}, \tilde{\mu}_{i, j}=\tilde{\mu}_{i, j, 0}+\tilde{\mu}_{i, j, s}$ with $\mu_{i, n, 0}, \tilde{\mu}_{i, n, 0} \in \mathfrak{M}_{0}\left(\Omega_{T}\right), \mu_{i, n, s}, \tilde{\mu}_{i, n, s} \in \mathfrak{M}_{s}\left(\Omega_{T}\right)$. As in the proof of Theorem 2.1, for any $j \in \mathbb{N}$ and $i=1,2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m, i, j, 0}=\left(f_{m, i, j}, g_{m, i, j}, h_{m, i, j}\right)$ and $\tilde{\mu}_{m, i, j, s}$ such that $f_{m, i, j}, g_{m, i, j}, h_{m, i, j} \in$ $C_{c}^{\infty}\left(\Omega_{T}\right)$ and strongly converge to some $f_{i, j}, g_{i, j}, h_{i, j}$ in $L^{1}\left(\Omega_{T}\right), L^{2}\left(\Omega_{T}, \mathbb{R}^{N}\right)$ and $L^{2}\left(0, T, H_{0}^{1}(\Omega)\right)$ respectively and $\tilde{\mu}_{m, i, j}, \tilde{\mu}_{m, i, j, s} \in C_{c}^{\infty}\left(\Omega_{T}\right)$ converge to $\tilde{\mu}_{i, j}, \tilde{\mu}_{i, j, s}$ resp. in the narrow topology with $\tilde{\mu}_{m, i, j}=\tilde{\mu}_{m, i, j, 0}+\tilde{\mu}_{m, i, j, s}$ which satisfy $\tilde{\mu}_{i, j, 0}=\left(f_{i, j}, g_{i, j}, h_{i, j}\right)$ and $0 \leq \tilde{\mu}_{m, i, j} \leq$ $\varphi_{m} * \tilde{\mu}_{i, j}$ and

$$
\begin{equation*}
\left\|f_{m, i, j}\right\|_{L^{1}\left(\Omega_{T}\right)}+\left\|g_{m, i, j}\right\|_{L^{2}\left(\Omega_{T}, \mathbb{R}^{N}\right)}+\left\|h_{m, i, j}\right\|_{L^{2}\left(0, T, H_{0}^{1}(\Omega)\right)}+\mu_{m, i, j, s}\left(\Omega_{T}\right) \leq 2 \tilde{\mu}_{i, j}\left(\Omega_{T}\right) \tag{6.1}
\end{equation*}
$$

Here $\left\{\varphi_{m}\right\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$.
For any $n, k, m \in \mathbb{N}$, let $u_{n, k, m}, u_{1, n, k, m}, u_{2, n, k, m} \in W$ with $W=\left\{z: z \in L^{2}\left(0, T, H_{0}^{1}(\Omega)\right), z_{t} \in\right.$ $\left.L^{2}\left(0, T, H^{-1}(\Omega)\right)\right\}$ be solutions of problems

$$
\begin{align*}
& \left\{\begin{array}{l}
\left(u_{n, k, m}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n, k, m}\right)\right)+T_{k}\left(\left|u_{n, k, m}\right|^{q-1} u_{n, k, m}\right)=\sum_{j=1}^{n}\left(\tilde{\mu}_{m, 1, j}-\tilde{\mu}_{m, 2, j}\right) \text { in } \Omega_{T}, \\
u_{n, k, m}=0 \quad \text { on } \partial \Omega \times(0, T), \\
u_{n, k, m}(0)=T_{n}\left(\sigma^{+}\right)-T_{n}\left(\sigma^{-}\right) \quad \text { on } \Omega,
\end{array}\right. \\
& \left\{\begin{array}{l}
\left(u_{1, n, k, m}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{1, n, k, m}\right)\right)+T_{k}\left(u_{1, n, k, m}^{q}\right)=\sum_{j=1}^{n} \tilde{\mu}_{m, 1, j} \text { in } \Omega_{T}, \\
u_{1, n, k, m}=0 \quad \text { on } \partial \Omega \times(0, T), \\
u_{1, n, k, m}(0)=T_{n}\left(\sigma^{+}\right) \quad \text { in } \Omega, \\
\left\{\begin{array}{l}
\left(u_{2, n, k, m}\right)_{t}-\operatorname{div}\left(\tilde{A}\left(x, t, \nabla u_{2, n, k, m}\right)\right)+T_{k}\left(u_{2, n, k, m}^{q}\right)=\sum_{j=1}^{n} \tilde{\mu}_{m, 2, j} \text { in } \Omega_{T}, \\
u_{2, n, k, m}=0 \\
u_{2, n, k, m}(0)=T_{n}\left(\sigma^{-}\right) \quad \text { in } \Omega,
\end{array}\right.
\end{array}, \begin{array}{l}
\text { on } \partial \Omega \times(0, T),
\end{array}\right. \tag{6.2}
\end{align*}
$$

where $\tilde{A}(x, t, \xi)=-A(x, t,-\xi)$.
By Comparison Principle Theorem and Theorem 2.1, there holds, for any $m, k$ the sequences $\left\{u_{1, n, k, m}\right\}_{n}$ and $\left\{u_{2, n, k, m}\right\}_{n}$ are increasing and

$$
\begin{aligned}
-K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{-}\right) \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu_{2, n} * \varphi_{m}\right] & \leq-u_{2, n, k, m} \leq u_{n, k, m} \leq u_{1, n, k, m} \\
& \leq K \mathbb{I}_{2}\left[\mu_{1, n} * \varphi_{m}\right]+K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{+}\right) \otimes \delta_{\{t=0\}}\right]
\end{aligned}
$$

where a constant $K$ is in Theorem 2.1. Thus,

$$
\begin{aligned}
-K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{-}\right) \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu_{2, n}\right] * \varphi_{m} & \leq-u_{2, n, k, m} \leq u_{n, k, m} \leq u_{1, n, k, m} \\
& \leq K \mathbb{I}_{2}\left[\mu_{1, n}\right] * \varphi_{m}+K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{+}\right) \otimes \delta_{\{t=0\}}\right]
\end{aligned}
$$

Moreover,

$$
\int_{\Omega_{T}} T_{k}\left(u_{i, n, k, m}^{q}\right) d x d t \leq \int_{\Omega_{T}} \varphi_{m} * \mu_{i, n} d x d t+|\sigma|(\Omega) \leq|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)
$$

As in [14, Proof of Lemma 5.3], thanks to Proposition 3.5 and Theorem 3.6, there exist subsequences of $\left\{u_{n, k, m}\right\}_{m}\left\{u_{1, n, k, m}\right\}_{m},\left\{u_{2, n, k, m}\right\}_{m}$, still denoted them, converging to renormalized solutions $u_{n, k} u_{1, n, k}, u_{2, n, k}$ of equations (6.2) with data $\mu_{1, n}-\mu_{2, n}, u_{n, k}(0)=T_{n}\left(\sigma^{+}\right)-$ $T_{n}\left(\sigma^{-}\right)$and the decomposition ( $\sum_{j=1}^{n} f_{1, j}-\sum_{j=1}^{n} f_{2, j}, \sum_{j=1}^{n} g_{1, j}-\sum_{j=1}^{n} g_{2, j}, \sum_{j=1}^{n} h_{1, j}-$ $\left.\sum_{j=1}^{n} h_{2, j}\right)$ of $\mu_{1, n, 0}-\mu_{2, n, 0},(6.3)$ with data $\mu_{1, n}, u_{1, n, k}(0)=T_{n}\left(\sigma^{+}\right)$and the decomposition $\left(\sum_{j=1}^{n} f_{1, j}, \sum_{j=1}^{n} g_{1, j}, \sum_{j=1}^{n} h_{1, j}\right)$ of $\mu_{1, n, 0}$, (6.4) with data $\mu_{2, n}, u_{2, n, k}(0)=T_{n}\left(\sigma^{-}\right)$and the decomposition ( $\sum_{j=1}^{n} f_{2, j}, \sum_{j=1}^{n} g_{2, j}, \sum_{j=1}^{n} h_{2, j}$ ) of $\mu_{2, n, 0}$ respectively, which satisfy

$$
\begin{aligned}
-K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{-}\right) \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu_{2, n}\right] & \leq-u_{2, n, k} \leq u_{n, k} \leq u_{1, n, k} \\
& \leq K \mathbb{I}_{2}\left[\mu_{1, n}\right]+K \mathbb{I}_{2}\left[T_{n}\left(\sigma^{+}\right) \otimes \delta_{\{t=0\}}\right]
\end{aligned}
$$

Next, as in [14, Proof of Lemma 5.4] since $I_{2}\left[\mu_{i, n}\right] \in L^{q}\left(\Omega_{T}\right)$ for any $n$, thanks to Proposition 3.5 and Theorem 3.6, there exist subsequences of $\left\{u_{n, k}\right\}_{k}\left\{u_{1, n, k}\right\}_{k},\left\{u_{2, n, k}\right\}_{k}$, still denoted them, converging to renormalized solutions $u_{n} u_{1, n}, u_{2, n}$ of equations

$$
\begin{align*}
& \left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)+\left|u_{n}\right|^{q-1} u_{n}=\mu_{1, n}-\mu_{2, n} \text { in } \Omega_{T}, \\
u_{n}=0 \quad \text { on } \partial \Omega \times(0, T), \\
u_{n}(0)=T_{n}\left(\sigma^{+}\right)-T_{n}\left(\sigma^{-}\right) \quad \text { in } \Omega,
\end{array}\right.  \tag{6.5}\\
& \left\{\begin{array}{l}
\left(u_{1, n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{1, n}\right)\right)+u_{1, n}^{q}=\mu_{1, n} \text { in } \Omega_{T}, \\
u_{1, n}=0 \quad \text { on } \partial \Omega \times(0, T), \\
u_{1, n}(0)=T_{n}\left(\sigma^{+}\right) \quad \text { in } \Omega,
\end{array}\right.  \tag{6.6}\\
& \left\{\begin{array}{l}
\left(u_{2, n}\right)_{t}-\operatorname{div}\left(\tilde{A}\left(x, t, \nabla u_{2, n}\right)\right)+u_{2, n}^{q}=\mu_{2, n} \text { in } \Omega_{T}, \\
u_{2, n}=0 \quad \text { on } \partial \Omega \times(0, T), \\
u_{2, n}(0)=T_{n}\left(\sigma^{-}\right) \quad \text { in } \Omega,
\end{array}\right. \tag{6.7}
\end{align*}
$$

relative to the decomposition $\left(\sum_{j=1}^{n} f_{1, j}-\sum_{j=1}^{n} f_{2, j}, \sum_{j=1}^{n} g_{1, j}-\sum_{j=1}^{n} g_{2, j}, \sum_{j=1}^{n} h_{1, j}-\right.$ $\left.\sum_{j=1}^{n} h_{2, j}\right)$ of $\mu_{1, n, 0}-\mu_{2, n, 0},\left(\sum_{j=1}^{n} f_{1, j}, \sum_{j=1}^{n} g_{1, j}, \sum_{j=1}^{n} h_{1, j}\right)$ of $\mu_{1, n, 0}$ and $\left(\sum_{j=1}^{n} f_{2, j}\right.$, $\left.\sum_{j=1}^{n} g_{2, j}, \sum_{j=1}^{n} h_{2, j}\right)$ of $\mu_{2, n, 0}$ respectively, which satisfy

$$
\begin{aligned}
-K \mathbb{I}_{2}\left[T_{n}\left(u_{0}^{-}\right) \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu_{2, n}\right] & \leq-u_{2, n} \leq u_{n} \leq u_{1, n} \\
& \leq K \mathbb{I}_{2}\left[\mu_{1, n}\right]+K \mathbb{I}_{2}\left[T_{n}\left(u_{0}^{+}\right) \otimes \delta_{\{t=0\}}\right]
\end{aligned}
$$

and the sequences $\left\{u_{1, n}\right\}_{n}$ and $\left\{u_{2, n}\right\}_{n}$ are increasing and

$$
\int_{\Omega_{T}} u_{i, n}^{q} d x d t \leq|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)
$$

Note that from (6.1) we have

$$
\left\|f_{i, j}\right\|_{L^{1}\left(\Omega_{T}\right)}+\left\|g_{i, j}\right\|_{L^{2}\left(\Omega_{T}, \mathbb{R}^{N}\right)}+\left\|h_{i, j}\right\|_{L^{2}\left(0, T, H_{0}^{1}(\Omega)\right)} \leq 2 \tilde{\mu}_{i, j}\left(\Omega_{T}\right)
$$

which implies

$$
\sum_{j=1}^{n}\left\|f_{i, j}\right\|_{L^{1}\left(\Omega_{T}\right)}+\sum_{j=1}^{n}\left\|g_{i, j}\right\|_{L^{2}\left(\Omega_{T}, \mathbb{R}^{N}\right)}+\sum_{j=1}^{n}\left\|h_{i, j}\right\|_{L^{2}\left(0, T, H_{0}^{1}(\Omega)\right)} \leq 2 \mu_{i, n}\left(\Omega_{T}\right) \leq 2|\mu|\left(\Omega_{T}\right)
$$

Finally, as in [14, Proof of Theorem 5.2] thanks to Proposition 3.5, Theorem 3.6 and Monotone Convergence Theorem there exist subsequences of $\left\{u_{n}\right\}_{n},\left\{u_{1, n}\right\}_{n},\left\{u_{2, n}\right\}_{n}$, still denoted them, converging to renormalized solutions $u, u_{1}, u_{2}$ of equations (6.5) with data $\mu, u(0)=\sigma$ and the decomposition $\left(\sum_{j=1}^{\infty} f_{1, j}-\sum_{j=1}^{\infty} f_{2, j}, \sum_{j=1}^{\infty} g_{1, j}-\sum_{j=1}^{\infty} g_{2, j}, \sum_{j=1}^{\infty} h_{1, j}-\sum_{j=1}^{\infty} h_{2, j}\right)$ of $\mu_{0}$, (6.6) with data $\mu^{+}, u_{1}(0)=\sigma^{+}$and the decomposition $\left(\sum_{j=1}^{\infty} f_{1, j}, \sum_{j=1}^{\infty} g_{1, j}\right.$, $\left.\sum_{j=1}^{\infty} h_{1, j}\right)$ of $\mu_{0}^{+},(6.7)$ with data $\mu^{-}, u_{2}(0)=\sigma^{-}$and the decomposition $\left(\sum_{j=1}^{\infty} f_{2, j}, \sum_{j=1}^{\infty} g_{2, j}\right.$, $\left.\sum_{j=1}^{\infty} h_{2, j}\right)$ of $\mu_{0}^{-}$, respectively and

$$
-K \mathbb{I}_{2}\left[\sigma^{-} \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu^{-}\right] \leq-u_{2} \leq u \leq u_{1} \leq K \mathbb{I}_{2}\left[\mu^{+}\right]+K \mathbb{I}_{2}\left[\sigma^{+} \otimes \delta_{\{t=0\}}\right]
$$

We now have remark: if $\sigma \equiv 0$ and $\operatorname{supp}(\mu) \subset \bar{\Omega} \times[a, T], a>0$, then $u=u_{1}=u_{2}=0$ in $\Omega \times(0, a)$ since $u_{n, k}=u_{1, n, k}=u_{2, n, k}=0$ in $\Omega \times(0, a)$.
Next, we will consider $\sigma \in \mathfrak{M}_{b}(\Omega)$ such that $\sigma$ is absolutely continuous with respect to the capacity $\operatorname{Cap}_{\mathbf{G}_{\frac{2}{q}, q^{\prime}}}$ in $\Omega$. So, $\chi_{\Omega_{T}} \mu+\sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\operatorname{Cap}_{2,1, q^{\prime}}$ in $\Omega \times(-T, T)$. As above, we verify that there exists a renormalized solution $u$ of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))+|u|^{q-1} u=\chi_{\Omega_{T}} \mu+\sigma \otimes \delta_{\{t=0\}} \text { in } \Omega \times(-T, T),  \tag{6.8}\\
u=0 \quad \text { on } \partial \Omega \times(-T, T), \\
u(-T)=0 \quad \text { on } \Omega
\end{array}\right.
$$

satisfying $u=0$ in $\Omega \times(-T, 0)$ and

$$
-K \mathbb{I}_{2}\left[\sigma^{-} \otimes \delta_{\{t=0\}}\right]-K \mathbb{I}_{2}\left[\mu^{-}\right] \leq u \leq K \mathbb{I}_{2}\left[\mu^{+}\right]+K \mathbb{I}_{2}\left[\sigma^{+} \otimes \delta_{\{t=0\}}\right]
$$

Finally, from remark 3.11 we get the result. This completes the proof of the theorem.
Proof of Theorem 2.9. Let $\left\{\mu_{n, i}\right\} \subset C_{c}^{\infty}\left(\Omega_{T}\right), \sigma_{i, n} \in C_{c}^{\infty}(\Omega)$ for $i=1,2$ be as in the proof of Theorem 2.1. We have $0 \leq \mu_{n, 1} \leq \varphi_{n} * \mu^{+}, 0 \leq \mu_{n, 2} \leq \varphi_{n} * \mu^{-}, 0 \leq \sigma_{1, n} \leq$ $\varphi_{1, n} * \sigma^{+}, 0 \leq \sigma_{2, n} \leq \varphi_{1, n} * \sigma^{-}$for any $n \in \mathbb{N}$ where $\left\{\varphi_{n}\right\}$ and $\left\{\varphi_{1, n}\right\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^{N}$ respectively.
We prove that the problem (2.2) has a solution with data $\mu=\mu_{n_{0}}=\mu_{n_{0}, 1}-\mu_{n_{0}, 2}, \sigma=$ $\sigma_{n_{0}}=\sigma_{1, n_{0}}-\sigma_{2, n_{0}}$ for $n_{0} \in \mathbb{N}$. Put

$$
\begin{aligned}
J=\left\{u \in L^{q}\left(\Omega_{T}\right): u^{+}\right. & \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{0}} \otimes \delta_{\{t=0\}}\right] \\
\text { and } \quad u^{-} & \left.\leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right]\right\}
\end{aligned}
$$

where $\max \left\{-\frac{N+2}{q^{\prime}}+2,0\right\}<\delta<2$.
Clearly, $J$ is closed under the strong topology of $L^{q}\left(\Omega_{T}\right)$ and convex.
We consider a map $S: J \rightarrow J$ defined for each $v \in J$ by $S(v)=u$, where $u \in L^{1}\left(\Omega_{T}\right)$ is the unique renormalized solution of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|v|^{q-1} v+\mu_{n_{0}, 1}-\mu_{n_{0}, 2} \text { in } \Omega_{T},  \tag{6.9}\\
u=0 \quad \text { on } \partial \Omega \times(0, T) \\
u(0)=\sigma_{1, n_{0}}-\sigma_{2, n_{0}} \quad \text { in } \Omega
\end{array}\right.
$$

By Theorem 2.1, we have

$$
\begin{aligned}
u^{+} & \leq K \mathbb{I}_{2}^{2 T_{0}}\left[\left(v^{+}\right)^{q}\right]+K \mathbb{I}_{2}^{2 T_{0}}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{0}} \otimes \delta_{\{t=0\}}\right] \\
u^{-} & \leq K \mathbb{I}_{2}^{2 T_{0}}\left[\left(v^{-}\right)^{q}\right]+K \mathbb{I}_{2}^{2 T_{0}}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right],
\end{aligned}
$$

where $K$ is the constant in Theorem 2.1. Thus,
$u^{+} \leq K\left(\frac{q K}{q-1}\right)^{q} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{0}} \otimes \delta_{\{t=0\}}\right]\right)^{q}\right]+K \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{0}} \otimes \delta_{\{t=0\}}\right]$,
$u^{-} \leq K\left(\frac{q K}{q-1}\right)^{q} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right]\right)^{q}\right]+K \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right]$.
Thus, thanks to Theorem 4.36 there exists $c_{1}=c_{1}(N, K, \delta, q)$ such that if for every compact sets $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
\left|\mu_{n_{0}, i}\right|(E)+\left(\left|\sigma_{i, n_{0}}\right| \otimes \delta_{\{t=0\}}\right)(E) \leq c_{1} \operatorname{Cap}_{E_{2}^{2 T_{0}, \delta}, q^{\prime}}(E) \tag{6.10}
\end{equation*}
$$

then $\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, i}+\sigma_{i, n_{0}} \otimes \delta_{\{t=0\}}\right] \in L^{q}\left(\mathbb{R}^{N+1}\right)$ and

$$
\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, i}+\sigma_{i, n_{0}} \otimes \delta_{\{t=0\}}\right]\right)^{q}\right] \leq \frac{(q-1)^{q-1}}{(K q)^{q}} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, i}+\sigma_{i, n_{0}} \otimes \delta_{\{t=0\}}\right] i=1,2
$$

which implies $u \in L^{q}\left(\Omega_{T}\right)$ and

$$
\begin{aligned}
& u^{+} \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{n}} \otimes \delta_{\{t=0\}}\right] \quad \text { and } \\
& u^{-} \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right] .
\end{aligned}
$$

Now we assume that (6.10) is satisfied, so $S$ is well defined. Therefore, if we can show that the map $S: J \rightarrow J$ is continuous and $S(J)$ is pre-compact under the strong topology of $L^{q}\left(\Omega_{T}\right)$ then by Schauder Fixed Point Theorem, $S$ has a fixed point on $J$. Hence the problem (2.2) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$.
Now we show that $\mathbf{S}$ is continuous. Let $\left\{v_{n}\right\}$ be a sequence in $J$ such that $v_{n}$ converges strongly in $L^{q}\left(\Omega_{T}\right)$ to a function $v \in J$. Set $u_{n}=S\left(v_{n}\right)$. We need to show that $u_{n} \rightarrow S(v)$ in $L^{q}\left(\Omega_{T}\right)$.

By Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by it, converging to $u$ a.e in $\Omega_{T}$. Since

$$
\left|u_{n}\right| \leq \sum_{i=1,2} \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, i}+\sigma_{i, n_{0}} \otimes \delta_{\{t=0\}}\right] \in L^{q}\left(\Omega_{T}\right) \quad \forall n \in \mathbb{N}
$$

Applying Dominated Convergence Theorem, we have $u_{n} \rightarrow u$ in $L^{q}\left(\Omega_{T}\right)$. Hence, thanks to Theorem 3.6 we get $u=S(v)$.
Next we show that $\mathbf{S}$ is pre-compact. Indeed if $\left\{u_{n}\right\}=\left\{S\left(v_{n}\right)\right\}$ is a sequence in $S(J)$. By Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by it, converging to $u$ a.e in $\Omega_{T}$. Again, using get Dominated Convergence Theorem we get $u_{n} \rightarrow u$ in $L^{q}\left(\Omega_{T}\right)$. So $\mathbf{S}$ is pre-compact.
Next, thanks to Corollary 4.39 and Remark 4.40 we have

$$
\left[\mu_{n, i}+\sigma_{i, n} \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{2}, q^{\prime}}} \leq c_{2}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{2}, q^{\prime}}} \quad \forall n \in \mathbb{N}, i=1,2
$$

for some $c_{2}=c_{2}(N, q)$.
In addition, by the proof of Corollary 4.39 we get

$$
\left(c_{3}\left(T_{0}\right)\right)^{-1} \operatorname{Cap}_{\mathcal{G}_{2}, q^{\prime}}(E) \leq \operatorname{Cap}_{E_{2}^{2 T_{0}, \delta}, q^{\prime}}(E) \leq c_{3}\left(T_{0}\right) \operatorname{Cap}_{\mathcal{G}_{2}, q^{\prime}}(E)
$$

for every compact set $E$ with $\operatorname{diam}(E) \leq 2 T_{0}$. Thus, there is $c_{4}=c_{4}\left(N, K, \delta, q, T_{0}\right)$ such that if

$$
\begin{equation*}
\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]_{\mathfrak{N}^{\mathcal{G}_{2}, q^{\prime}}} \leq c_{4} \tag{6.11}
\end{equation*}
$$

then (6.10) holds for any $n_{0} \in \mathbb{N}$.
Now we suppose that (6.11) holds, it is equivalent to (2.13) holding for some constant $C_{1}=C_{1}\left(T_{0}\right)$ by Remark 4.34. Therefore, for any $n \in \mathbb{N}$ there exists a renormalized solution $u_{n}$ of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\left|u_{n}\right|^{q-1} u_{n}+\mu_{n, 1}-\mu_{n, 2} \text { in } \Omega_{T},  \tag{6.12}\\
u_{n}=0 \text { on } \partial \Omega \times(0, T), \\
u_{n}(0)=\sigma_{1, n}-\sigma_{2, n} \quad \text { in } \Omega,
\end{array}\right.
$$

which satisfies

$$
-\frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n, 2}+\sigma_{2, n} \otimes \delta_{\{t=0\}}\right] \leq u_{n} \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n, 1}+\sigma_{1, n} \otimes \delta_{\{t=0\}}\right]
$$

Thus, for every $(x, t) \in \Omega_{T}$,

$$
\begin{aligned}
-\frac{q K}{q-1} \varphi_{n} * & \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu^{-}\right](x, t)-\frac{q K}{q-1} \varphi_{1, n} *\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\sigma^{-} \otimes \delta_{\{t=0\}}\right](., t)\right)(x) \leq u_{n}(x, t) \\
& \leq \frac{q K}{q-1} \varphi_{n} *\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu^{-}\right]\right)(x, t)+\frac{q K}{q-1} \varphi_{1, n} *\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\sigma^{-} \otimes \delta_{\{t=0\}}\right](., t)\right)(x)
\end{aligned}
$$

Since $\varphi_{n} * \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu^{ \pm}\right](x, t), \varphi_{1, n} *\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\sigma^{ \pm} \otimes \delta_{\{t=0\}}\right](., t)\right)(x)$ converge to $\mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu^{ \pm}\right](x, t), \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\sigma^{ \pm} \otimes\right.$ $\left.\delta_{\{t=0\}}\right](x, t)$ in $L^{q}\left(\mathbb{R}^{N+1}\right)$ as $n \rightarrow \infty$, respectively, so $\left|u_{n}\right|^{q}$ is equi-integrable.
By Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by its, converging to $u$ a.e in $\Omega_{T}$. It follows $\left|u_{n}\right|^{q-1} u_{n} \rightarrow|u|^{q-1} u$ in $L^{1}\left(\Omega_{T}\right)$.

Consequently, by Proposition 3.5 and Theorem 3.6, we obtain that $u$ is a distribution (a renormalized solution if $\sigma \in L^{1}(\Omega)$ ) of (2.2) with data $\mu, \sigma$, and satisfies (2.14). Furthermore, by Corollary 4.39 we have

$$
\begin{aligned}
\left(c_{5}\left(T_{0}\right)\right)^{-1} & {\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{2}, q^{\prime}}}^{q} } \\
& \leq\left[\left(\mathbb{I}_{2}^{2 T_{0}, \delta}\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]\right)^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{2}, q^{\prime}}} \leq c_{5}\left(T_{0}\right)\left[|\mu|+|\sigma| \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{2}, q^{\prime}}}^{q}
\end{aligned}
$$

which implies $\left[|u|^{q}\right]_{\mathfrak{M}^{G_{2}, q^{\prime}}} \leq c_{4}\left(T_{0}\right)$ and we get (2.15). This completes the proof of the Theorem.

Remark 6.4 In view of above proof, we can see that
i. The Theorem 2.9 also holds when we replace assumption (2.13) by

$$
|\mu|(E) \leq C C^{-1} p_{\mathcal{H}_{2}, q^{\prime}}(E) \quad \text { and } \quad|\sigma|(F) \leq C C a p_{\mathbf{I}_{\frac{2}{q}}, q^{\prime}}(F)
$$

for every compact sets $E \subset \mathbb{R}^{N+1}, F \subset \mathbb{R}^{N}$ where $C=C\left(N \Lambda_{1}, \Lambda_{2}, q\right)$ is some $a$ constant.
ii. If $\sigma \equiv 0$ and $\operatorname{supp}(\mu) \subset \bar{\Omega} \times[a, T], a>0$, then we can show that a solution $u$ in Theorem 2.9 satisfies $u=0$ in $\Omega \times(0, a)$ since we can replace the set $E$ by $E^{\prime}$ :

$$
\begin{gathered}
E^{\prime}=\left\{u \in L^{q}\left(\Omega_{T}\right): u=0 \text { in } \Omega \times(0, a) \text { and } u^{+} \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 1}+\sigma_{1, n_{0}} \otimes \delta_{\{t=0\}}\right]\right. \\
\text { and } \left.u^{-} \leq \frac{q K}{q-1} \mathbb{I}_{2}^{2 T_{0}, \delta}\left[\mu_{n_{0}, 2}+\sigma_{2, n_{0}} \otimes \delta_{\{t=0\}}\right]\right\} .
\end{gathered}
$$

### 6.2 Quasilinear Lane-Emden Parabolic Equations in $\mathbb{R}^{N} \times(0, \infty)$ and $\mathbb{R}^{N+1}$

This section is devoted to proofs of Theorem 2.12 and 2.14.
Proof of the Theorem 2.12. Since $\omega$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1, q^{\prime}}$ in $\mathbb{R}^{N+1},|\omega|$ is too. Set $D_{n}=B_{n}(0) \times\left(-n^{2}, n^{2}\right)$. From the proof of Theorem 2.8, there exist renormalized solutions $u_{n}, v_{n}$ of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)+\left|u_{n}\right|^{q-1} u_{n}=\chi_{D_{n}} \omega \text { in } D_{n}, \\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right), \\
u_{n}\left(-n^{2}\right)=0 \quad \text { in } B_{n}(0),
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\left(v_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla v_{n}\right)\right)+v_{n}^{q}=\chi_{D_{n}}|\omega| \text { in } D_{n}, \\
v_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right), \\
v_{n}\left(-n^{2}\right)=0 \quad \text { in } B_{n}(0),
\end{array}\right.
$$

relative to decompositions $\left(f_{n}, g_{n}, h_{n}\right)$ of $\chi_{D_{n}} \omega_{0}$ and $\left(\bar{f}_{n}, \bar{g}_{n}, \bar{h}_{n}\right)$ of $\chi_{B_{n}(0) \times\left(0, n^{2}\right)}\left|\omega_{0}\right|$, satisfied (3.14), (3.15) in Proposition 3.16 with $1<q_{0}<q, L\left(u_{n}\right)=\left|u_{n}\right|^{q-1} u_{n}, L\left(v_{n}\right)=v_{n}^{q}$ and $\mu$ is replaced by $\chi_{D_{n}} \omega$ and $\chi_{D_{n}}|\omega|$ respectively. Moreover, there hold

$$
\begin{equation*}
-K I_{2}\left[\omega^{-}\right] \leq u_{n} \leq K I_{2}\left[\omega^{+}\right], 0 \leq v_{n} \leq K I_{2}[|\omega|] \text { in } D_{n}, \tag{6.13}
\end{equation*}
$$

and $v_{n+1} \geq v_{n},\left|u_{n}\right| \leq v_{n}$ in $D_{n}$.
By Remark 3.9, we can assume that

$$
\begin{aligned}
& \left\|f_{n}\right\|_{L^{1}\left(D_{i}\right)}+\left\|g_{n}\right\|_{L^{2}\left(D_{i}, \mathbb{R}^{N}\right)}+\left|\left|\left|h_{n}\right|+\left|\nabla h_{n}\right| \|_{L^{2}\left(D_{i}\right)} \leq 2\right| \omega\right|\left(D_{i+1}\right) \text { and } \\
& \left\|\bar{f}_{n}\right\|_{L^{1}\left(D_{i}\right)}+\left\|\bar{g}_{n}\right\|_{L^{2}\left(D_{i}, \mathbb{R}^{N}\right)}+\left|\left|\left|\bar{h}_{n}\right|+\left|\nabla \bar{h}_{n}\right| \|_{L^{2}\left(D_{i}\right)} \leq 2\right| \omega\right|\left(D_{i+1}\right),
\end{aligned}
$$

for any $i=1, \ldots, n-1$ and $h_{n}, \bar{h}_{n}$ are convergent in $L_{\text {loc }}^{1}\left(\mathbb{R}^{N+1}\right)$. On the other hand, since $u_{n}, v_{n}$ satisfy (3.14) in Proposition 3.16 with $1<q_{0}<q, L\left(u_{n}\right)=\left|u_{n}\right|^{q-1} u_{n}, L\left(v_{n}\right)=v_{n}^{q}$ and thanks to Holder inequality: for any $\varepsilon \in(0,1)$

$$
\left(\left|u_{n}\right|+1\right)^{q_{0}} \leq \varepsilon\left|u_{n}\right|^{q}+c_{1}(\varepsilon) \quad \text { and } \quad\left(\left|v_{n}\right|+1\right)^{q_{0}} \leq \varepsilon\left|v_{n}\right|^{q}+c_{1}(\varepsilon) .
$$

Thus we get

$$
\begin{equation*}
\int_{D_{i}}\left|u_{n}\right|^{q} d x d t+\int_{D_{i}}\left|u_{n}\right|^{q_{0}} d x d t+\int_{D_{i}} v_{n}^{q} d x d t+\int_{D_{i}} v_{n}^{q_{0}} d x d t \leq C(i)+c_{2}|\omega|\left(D_{i+1}\right) . \tag{6.14}
\end{equation*}
$$

for $i=1, \ldots, n-1$, where the constant $C(i)$ depends on $N, \Lambda_{1}, \Lambda_{2}, q_{0}, q$ and $i$.
Consequently, we can apply Proposition 3.17 with $\mu_{n}=-\left|u_{n}\right|^{q-1} u_{n}+\chi_{D_{n}} \omega,-v_{n}^{q}+\chi_{D_{n}}|\omega|$ and obtain that there are subsequences of $u_{n}, v_{n}$, still denoted by them, converging to some $u, v$ in $L_{l o c}^{1}\left(\mathbb{R} ; W_{l o c}^{1,1}\left(\mathbb{R}^{N}\right)\right)$. So, $\frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \in L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$ for all $\alpha>0$ and $u \in L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$ satisfies (2.17). In addition, using Holder inequality we get $u \in L_{\text {loc }}^{\gamma}\left(\mathbb{R} ; W_{\mathrm{loc}}^{1, \gamma}\left(\mathbb{R}^{N}\right)\right)$ for any $1 \leq \gamma<\frac{2 q}{q+1}$.
Thanks to (6.14) and Monotone Convergence Theorem we get $v_{n} \rightarrow v$ in $L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$. After, we also have $u_{n} \rightarrow u$ in $L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$ by $\left|u_{n}\right| \leq v_{n}$ and Dominated Convergence Theorem. Consequently, $u$ is a distribution solution of problem (2.16) which satisfies (2.17).
If $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then by the proof of Theorem 2.8 we can assume that $u_{n}=0$ in $B_{n}(0) \times\left(-n^{2}, 0\right)$. So, $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$. Therefore, clearly $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.18).
This completes the proof of the theorem.
Proof of the Theorem 2.14. By the proof of Theorem 2.9 and Remark 6.4, 4.34, there exists a constant $c_{1}=c_{1}\left(N, q, \Lambda_{1}, \Lambda_{2}\right)$ such that if $\omega$ satisfy for every compact set $E \subset \mathbb{R}^{N+1}$,

$$
\begin{equation*}
|\omega|(E) \leq c_{1} \operatorname{Cap}_{\mathcal{H}_{2}, q^{\prime}}(E) \tag{6.15}
\end{equation*}
$$

then there is a renormalized solution $u_{n}$ of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\left|u_{n}\right|^{q-1} u_{n}+\chi_{D_{n}} \omega \text { in } D_{n} \\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right), \\
u_{n}\left(-n^{2}\right)=0 \quad \text { in } B_{n}(0),
\end{array}\right.
$$

relative to a decomposition $\left(f_{n}, g_{n}, h_{n}\right)$ of $\chi_{D_{n}} \omega_{0}$, satisfying (3.14), (3.15) in Proposition 3.16 with $q_{0}=q, L \equiv 0$ and $\mu$ is replaced by $\left|u_{n}\right|^{q-1} u_{n}+\chi_{D_{n}} \omega$ and

$$
\begin{equation*}
-\frac{q K}{q-1} \mathbb{I}_{2}\left[\omega^{-}\right](x, t) \leq u_{n} \leq \frac{q K}{q-1} \mathbb{I}_{2}\left[\omega^{+}\right](x, t) \tag{6.16}
\end{equation*}
$$

for a.e $(x, t)$ in $D_{n}$ and $I_{2}\left[\omega^{ \pm}\right] \in L_{l o c}^{q}\left(\mathbb{R}^{N+1}\right)$.
Besides, thanks to Remark 3.9, we can assume that $f_{n}, g_{n}, h_{n}$ satisfies (5.17) in proof of Theorem (2.5) and $h_{n}$ is convergent in $L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{N+1}\right)$.
Consequently, we can apply Proposition 3.17 and obtain that there exist a subsequence of $u_{n}$, still denoted by it, converging to some $u$ a.e in $\mathbb{R}^{N+1}$ and in $L_{l o c}^{1}\left(\mathbb{R} ; W_{l o c}^{1,1}\left(\mathbb{R}^{N}\right)\right)$. Also, $u_{n} \rightarrow u$ in $L_{\text {loc }}^{q}\left(\mathbb{R}^{N+1}\right)$ by Dominated Convergence Theorem, $\frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \in L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$ for all $\alpha>0$. Using Holder inequality we get $u \in L_{\text {loc }}^{\gamma}\left(\mathbb{R} ; W_{\text {loc }}^{1, \gamma}\left(\mathbb{R}^{N}\right)\right)$ for any $1 \leq \gamma<\frac{2 q}{q+1}$.
Thus we obtain that $u$ is a distribution solution of (2.20) which satisfies (2.21). Since (6.15) holds, thus by Theorem 4.36 we get

$$
c_{2}^{-1}[|\omega|]_{\mathfrak{M}^{\mathcal{H}_{2}, q^{\prime}}}^{q} \leq\left[\left(\mathbb{I}_{2}[|\omega|]\right)^{q}\right]_{\mathfrak{M}^{\mathcal{H}_{2}, q^{\prime}}} \leq c_{2}[|\omega|]_{\mathfrak{M}^{\mathcal{H}_{2}, q^{\prime}}}^{q}
$$

so we have $\left[|u|^{q}\right]_{\mathfrak{M}^{\mathcal{H}_{2}, q^{\prime}}} \leq c_{3}$. It follows (2.23).
If $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then by Remark 6.4 we can assume that $u_{n}=0$ in $B_{n}(0) \times\left(-n^{2}, 0\right)$. So, $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$. Therefore, clearly $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.22).
This completes the proof of the theorem.

## $7 \quad$ Interior Estimates and Boundary Estimates for Parabolic Equations

In this section we always assume that $u \in C\left(-T, T, L^{2}(\Omega)\right) \cap L^{2}\left(-T, T, H_{0}^{1}(\Omega)\right)$ is a solution to equation (2.4) in $\Omega \times(-T, T)$ with $\mu \in L^{2}(\Omega \times(-T, T))$ and $u(-T)=0$. We extend $u$ by zero to $\Omega \times(-\infty,-T)$, clearly $u$ is a solution to equation

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\chi_{(-T, T)}(t) \mu \quad \text { in } \Omega \times(-\infty, T),  \tag{7.1}\\
u=0 \quad \text { on } \quad \partial \Omega \times(-\infty, T)
\end{array}\right.
$$

### 7.1 Interior Estimates

For each ball $B_{2 R}=B_{2 R}\left(x_{0}\right) \subset \subset \Omega$ and $t_{0} \in(-T, T)$, one considers the unique solution

$$
\begin{equation*}
w \in C\left(t_{0}-4 R^{2}, t_{0} ; L^{2}\left(B_{2 R}\right)\right) \cap L^{2}\left(t_{0}-4 R^{2}, t_{0} ; H^{1}\left(B_{2 R}\right)\right) \tag{7.2}
\end{equation*}
$$

to the following equation

$$
\left\{\begin{array}{l}
w_{t}-\operatorname{div}(A(x, t, \nabla w))=0 \text { in } Q_{2 R}  \tag{7.3}\\
w=u \quad \text { on } \quad \partial_{p} Q_{2 R}
\end{array}\right.
$$

where $Q_{2 R}=B_{2 R} \times\left(t_{0}-4 R^{2}, t_{0}\right)$ and $\partial_{p} Q_{2 R}=\left(\partial B_{2 R} \times\left(t_{0}-4 R^{2}, t_{0}\right)\right) \cup\left(B_{2 R} \times\left\{t=t_{0}-4 R^{2}\right\}\right)$.

Theorem 7.1 There exist constants $\theta_{1}>2, \beta_{1} \in\left(0, \frac{1}{2}\right]$ and $C_{1}, C_{2}, C_{3}$ depending on $N, \Lambda_{1}, \Lambda_{2}$ such that the following estimates are true

$$
\begin{gather*}
f_{Q_{2 R}}|\nabla u-\nabla w| d x d t \leq C_{1} \frac{|\mu|\left(Q_{2 R}\right)}{R^{N+1}}  \tag{7.4}\\
\left(f_{Q_{\rho / 2}(y, s)}|\nabla w|^{\theta_{1}} d x d t\right)^{\frac{1}{\theta_{1}}} \leq C_{2} f_{Q_{\rho}(y, s)}|\nabla w| d x d t  \tag{7.5}\\
\left(f_{Q_{\rho_{1}(y, s)}} \left\lvert\, w-\bar{w}_{\left.\left.Q_{\rho_{1}(y, s)}\right|^{2} d x d t\right)^{1 / 2} \leq C_{3}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{1}}\left(f_{Q_{\rho_{2}(y, s)}} \mid w-\bar{w}_{\left.\left.Q_{\rho_{2}}(y, s)\right|^{2} d x d t\right)^{1 / 2}}\right.}\right.,\right. \tag{7.6}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(f_{Q_{\rho_{1}(y, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \leq C_{3}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{1}-1}\left(f_{Q_{\rho_{2}(y, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \tag{7.7}
\end{equation*}
$$

for any $Q_{\rho}(y, s) \subset Q_{2 R}$, and $Q_{\rho_{1}}(y, s) \subset Q_{\rho_{2}}(y, s) \subset Q_{2 R}$.
Proof. Inequalities (7.4), (7.5) and (7.6) were proved by Duzaar and Mingione in [27]. So, it remains to prove (7.7) in case $\rho_{1} \leq \rho_{2} / 2$. By interior Caccioppoli inequality we have

$$
\left(f_{Q_{\rho_{1}(y, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \leq \frac{c_{1}}{\rho_{1}}\left(f_{Q_{2 \rho_{1}}(y, s)}\left|w-\bar{w}_{Q_{2 \rho_{1}}(y, s)}\right|^{2} d x d t\right)^{1 / 2}
$$

On the other hand, by a Sobolev inequality there holds

$$
\left(f_{Q_{\rho_{2}}(y, s)}\left|w-\bar{w}_{Q_{\rho_{2}}(y, s)}\right|^{2} d x d t\right)^{1 / 2} \leq c_{2} \rho_{2}\left(f_{Q_{\rho_{2}}(y, s)}|\nabla w|^{2} d x d t\right)^{1 / 2}
$$

Therefore, (7.7) follows from (7.6).
Corollary 7.2 Let $\beta_{1}$ be the constant in Theorem 7.1. For $2-\beta_{1}<\theta<N+2$, there exists a constant $C=C\left(N, \Lambda_{1}, \Lambda_{2}, \theta\right)>0$ such that for any $B_{\rho}(y) \subset B_{\rho_{0}}(y) \subset \subset \Omega, s \in(-T, T)$

$$
\begin{equation*}
\int_{Q_{\rho}(y, s)}|\nabla u| d x d t \leq C \rho^{N+3-\theta}\left(\left(\frac{T_{0}}{\rho_{0}}\right)^{N+3-\theta}+1\right)\left\|\mathbb{M}_{\theta}[\mu]\right\|_{L^{\infty}(\Omega \times(-T, T))} \tag{7.8}
\end{equation*}
$$

Proof. Take $B_{\rho_{2}}(y) \subset \subset \Omega$ and $s \in(-T, T)$. For any $Q_{\rho_{1}}(y, s) \subset Q_{\rho_{2}}(y, s)$ with $\rho_{1} \leq \rho_{2} / 2$, we take $w$ as in Theorem 7.1 with $Q_{2 R}=Q_{\rho_{2}}(y, s)$. Thus,

$$
\begin{gathered}
\int_{Q_{\rho_{1}}(y, s)}|\nabla w| d x d t \leq c_{1}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{N+\beta_{1}+1} \int_{Q_{\rho_{2}}(y, s)}|\nabla w| d x d t \\
\int_{Q_{\rho_{2}(y, s)}}|\nabla u-\nabla w| d x d t \leq c_{2} \rho_{2}|\mu|\left(Q_{\rho_{2}}(y, s)\right)
\end{gathered}
$$

and we also have

$$
c_{3}^{-1} \int_{Q_{\rho_{2}}(y, s)}|\nabla u| d x d t \leq \int_{Q_{\rho_{2}}(y, s)}|\nabla w| d x d t \leq c_{3} \int_{Q_{\rho_{2}}(y, s)}|\nabla u| d x d t
$$

It follows that

$$
\begin{aligned}
\int_{Q_{\rho_{1}(y, s)}}|\nabla u| d x d t & \leq \int_{Q_{\rho_{1}(y, s)}}|\nabla w| d x d t+\int_{Q_{\rho_{1}}(y, s)}|\nabla u-\nabla w| d x d t \\
& \leq c_{4}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{N+\beta_{1}+1} \int_{Q_{\rho_{2}(y, s)}}|\nabla w| d x d t+\int_{Q_{\rho_{2}}(y, s)}|\nabla u-\nabla w| d x d t \\
& \leq c_{5}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{N+\beta_{1}+1} \int_{Q_{\rho_{2}(y, s)}}|\nabla u| d x d t+c_{5} \rho_{2}|\mu|\left(Q_{\rho_{2}}(y, s)\right) .
\end{aligned}
$$

This implies

$$
\int_{Q_{\rho_{1}}(y, s)}|\nabla u| d x d t \leq c_{5}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{N+\beta_{1}+1} \int_{Q_{\rho_{2}}(y, s)}|\nabla u| d x d t+c_{5} \rho_{2}^{N+3-\theta}\left\|\mathbb{M}_{\theta}[\mu]\right\|_{L^{\infty}(\Omega \times(-T, T))}
$$

Since $N+3-\beta<N+\beta_{1}+1$, applying [50, Lemma 4.6, page 54] we obtain

$$
\int_{Q_{\rho}(y, s)}|\nabla u| d x d t \leq c_{6}\left(\frac{\rho}{\rho_{0}}\right)^{N+3-\theta}\|\nabla u\|_{L^{1}\left(\Omega_{\times}(-T, T)\right)}+c_{6} \rho^{N+3-\theta}\left\|\mathbb{M}_{\theta}[\mu]\right\|_{L^{\infty}(\Omega \times(-T, T))}
$$

for any $B_{\rho}(y) \subset B_{\rho_{0}}(y) \subset \subset \Omega, s \in(-T, T)$. On the other hand, by Remark 3.2

$$
\|\nabla u\|_{L^{1}(\Omega \times(-T, T))} \leq c_{7} T_{0}|\mu|(\Omega \times(-T, T)) \leq c_{8} T_{0}^{N+3-\theta}\left\|\mathbb{M}_{\theta}[\mu]\right\|_{L^{\infty}(\Omega \times(-T, T))}
$$

Hence, we get the desired result.
To continue, we consider the unique solution

$$
\begin{equation*}
v \in C\left(t_{0}-R^{2}, t_{0} ; L^{2}\left(B_{R}\right)\right) \cap L^{2}\left(t_{0}-R^{2}, t_{0} ; H^{1}\left(B_{R}\right)\right) \tag{7.9}
\end{equation*}
$$

to the following equation

$$
\left\{\begin{array}{l}
v_{t}-\operatorname{div}\left(\bar{A}_{B_{R}\left(x_{0}\right)}(t, \nabla v)\right)=0 \text { in } Q_{R},  \tag{7.10}\\
v=w \quad \text { on } \partial_{p} Q_{R},
\end{array}\right.
$$

where $Q_{R}=B_{R}\left(x_{0}\right) \times\left(t_{0}-R^{2}, t_{0}\right)$ and $\partial_{p} Q_{R}=\left(\partial B_{R} \times\left(t_{0}-R^{2}, t_{0}\right)\right) \cup\left(B_{R} \times\left\{t=t_{0}-R^{2}\right\}\right)$.
Lemma 7.3 Let $\theta_{1}$ be the constant in Theorem 7.1. There exist constants $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}\right)$ and $C_{2}=C_{2}\left(\Lambda_{1}, \Lambda_{2}\right)$ such that

$$
\begin{equation*}
\left(f_{Q_{R}}|\nabla w-\nabla v|^{2} d x d t\right)^{1 / 2} \leq C_{1}[A]_{s_{1}}^{R} f_{Q_{2 R}}|\nabla w| d x d t \tag{7.11}
\end{equation*}
$$

with $s_{1}=\frac{2 \theta_{1}}{\theta_{1}-2}$ and

$$
\begin{equation*}
C_{2}^{-1} \int_{Q_{R}}|\nabla v|^{2} d x d t \leq \int_{Q_{R}}|\nabla w|^{2} d x d t \leq C_{2} \int_{Q_{R}}|\nabla v|^{2} d x d t \tag{7.12}
\end{equation*}
$$

Proof. We can choose $\varphi=w-v$ as a test function for equations (7.3), (7.10) and since

$$
\int_{Q_{R}} w_{t}(w-v) d x d t-\int_{Q_{R}} v_{t}(w-v) d x d t=\frac{1}{2} \int_{B_{R}}(w-v)^{2}\left(t_{0}\right) d x \geq 0
$$

we find

$$
-\int_{Q_{R}} \bar{A}_{B_{R}\left(x_{0}\right)}(t, \nabla v) \nabla(w-v) d x d t \leq-\int_{Q_{R}} A(x, t, \nabla w) \nabla(w-v) d x d t
$$

By using inequalities (1.2) and (1.3) together with Holder's inequality we get

$$
\begin{equation*}
c_{1}^{-1} \int_{Q_{R}}|\nabla v|^{2} d x d t \leq \int_{Q_{R}}|\nabla w|^{2} d x d t \leq c_{1} \int_{Q_{R}}|\nabla v|^{2} d x d t \tag{7.13}
\end{equation*}
$$

and we also have

$$
\begin{aligned}
\Lambda_{2} \int_{Q_{R}}|\nabla w-\nabla v|^{2} d x d t & \leq \int_{Q_{R}}\left(\bar{A}_{B_{R}\left(x_{0}\right)}(t, \nabla w)-\bar{A}_{B_{R}\left(x_{0}\right)}(t, \nabla v)\right)(\nabla w-\nabla v) d x d t \\
& \leq \int_{Q_{R}}\left(\bar{A}_{B_{R}\left(x_{0}\right)}(t, \nabla w)-A(x, t, \nabla w)\right)(\nabla w-\nabla v) d x d t \\
& \leq \int_{Q_{R}} \Theta\left(A, B_{R}\left(x_{0}\right)\right)(x, t)|\nabla w||\nabla w-\nabla v| d x d t
\end{aligned}
$$

Here we used the definition of $\Theta\left(A, B_{R}\left(x_{0}\right)\right)$ in the last inequality. Using Holder's inequality with exponents $s_{1}=\frac{2 \theta_{1}}{\theta_{1}-2}, \theta_{1}$ and 2 gives

$$
\begin{aligned}
\Lambda_{2} f_{Q_{R}}|\nabla w-\nabla v|^{2} \leq & \left(f_{Q_{R}} \Theta\left(A, B_{R}\left(x_{0}\right)\right)(x, t)^{s_{1}} d x d t\right)^{1 / s_{1}}\left(f_{Q_{R}}|\nabla w|^{\theta_{1}} d x d t\right)^{1 / \theta_{1}} \\
& \times\left(f_{Q_{R}}|\nabla w-\nabla v|^{2} d x d t\right)^{1 / 2}
\end{aligned}
$$

In other words,

$$
\left(f_{Q_{R}}|\nabla w-\nabla v|^{2} d x d t\right)^{1 / 2} \leq \Lambda_{2}^{-1}[A]_{s_{1}}^{R}\left(f_{Q_{R}}|\nabla w|^{\theta_{1}} d x d t\right)^{1 / \theta_{1}}
$$

After using the inequality (7.5) in Theorem 7.1 we get (7.11).
Lemma 7.4 Let $\theta_{1}$ be the constant in Theorem 7.1. There exists a functions $v \in C\left(t_{0}-\right.$ $\left.R^{2}, t_{0} ; L^{2}\left(B_{R}\right)\right) \cap L^{2}\left(t_{0}-R^{2}, t_{0} ; H^{1}\left(B_{R}\right)\right) \cap L^{\infty}\left(t_{0}-\frac{1}{4} R^{2}, t_{0} ; W^{1, \infty}\left(B_{R / 2}\right)\right)$ such that

$$
\begin{equation*}
\|\nabla v\|_{L^{\infty}\left(Q_{R / 2}\right)} \leq C f_{Q_{2 R}}|\nabla u| d x d t+C \frac{|\mu|\left(Q_{2 R}\right)}{R^{N+1}} \tag{7.14}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{Q_{R}}|\nabla u-\nabla v| d x d t \leq C \frac{|\mu|\left(Q_{2 R}\right)}{R^{N+1}}+C[A]_{s_{1}}^{R}\left(f_{Q_{2 R}}|\nabla u| d x d t+\frac{|\mu|\left(Q_{2 R}\right)}{R^{N+1}}\right) \tag{7.15}
\end{equation*}
$$

where $s_{1}=\frac{2 \theta_{1}}{\theta_{1}-2}$ and $C=C\left(N, \Lambda_{1}, \Lambda_{2}\right)$.
Proof. Let $w$ and $v$ be in equations (7.3) and (7.10). By standard interior regularity and inequality (7.5) in Theorem 7.1 and (7.12) in Lemma 7.3 we have

$$
\begin{aligned}
\|\nabla v\|_{L^{\infty}\left(Q_{R / 2}\right)} & \leq c_{1}\left(f_{Q_{R}}|\nabla v|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{2}\left(f_{Q_{R}}|\nabla w|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{3} f_{Q_{2 R}}|\nabla w| d x d t .
\end{aligned}
$$

Thus, we get (7.14) from (7.4) in Theorem 7.1.
On the other hand, (7.11) in Lemma 7.3 and Holder's inequality yield

$$
f_{Q_{R}}|\nabla w-\nabla v| d x d t \leq c_{4}[A]_{s_{1}}^{R} f_{Q_{2 R}}|\nabla w| d x d t
$$

It leads

$$
f_{Q_{R}}|\nabla u-\nabla v| d x d t \leq f_{Q_{R}}|\nabla u-\nabla w| d x d t+c_{4}[A]_{s_{1}}^{R} f_{Q_{2 R}}|\nabla w| d x d t
$$

Consequently, we get (7.15) from (7.4) in Theorem 7.1. The proof is complete.

### 7.2 Boundary Estimates

In this subsection, we focus on the corresponding estimates near the boundary.
Let $x_{0} \in \partial \Omega$ be a boundary point and for $R>0$ and $t_{0} \in(-T, T)$, we set $\tilde{\Omega}_{6 R}=$ $\tilde{\Omega}_{6 R}\left(x_{0}, t_{0}\right)=\left(\Omega \cap B_{6 R}\left(x_{0}\right)\right) \times\left(t_{0}-(6 R)^{2}, t_{0}\right)$ and $Q_{6 R}=Q_{6 R}\left(x_{0}, t_{0}\right)$.
We consider the unique solution $w$ to the equation

$$
\left\{\begin{array}{l}
w_{t}-\operatorname{div}(A(x, t, \nabla w))=0 \text { in } \tilde{\Omega}_{6 R}  \tag{7.16}\\
w=u \quad \text { on } \partial_{p} \tilde{\Omega}_{6 R}
\end{array}\right.
$$

In what follows we extend $\mu$ and $u$ by zero to $(\Omega \times(-\infty, T))^{c}$ and then extend $w$ by $u$ to $\mathbb{R}^{N+1} \backslash \tilde{\Omega}_{6 R}$.

In order to obtain estimates for $w$ as in Theorem 7.1 we require the domain $\Omega$ to be satisfied $2-$ Capacity uniform thickness condition.

### 7.2.1 2-Capacity uniform thickness domain

It is well known that if $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}, r_{0}>0$, there exist $p_{0} \in\left(\frac{2 N}{N+2}, 2\right)$ and $C=C\left(N, c_{0}\right)>0$ such that

$$
\begin{equation*}
\operatorname{Cap}_{p_{0}}\left(\overline{B_{r}(x)} \cap\left(\mathbb{R}^{N} \backslash \Omega\right), B_{2 r}(x)\right) \geq C r^{N-p_{0}} \tag{7.17}
\end{equation*}
$$

for all $0<r \leq r_{0}$ and all $x \in \mathbb{R}^{N} \backslash \Omega$, see [47, 57].
Theorem 7.5 Suppose that $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}$, $r_{0}$. Let $w$ be in (7.16) with $0<6 R \leq r_{0}$. There exist constants $\theta_{2}>2, \beta_{2} \in\left(0, \frac{1}{2}\right], C_{2}, C_{3}$ depending on $N, \Lambda_{1}, \Lambda_{2}, c_{0}$ and $C_{1}$ depending on $N, \Lambda_{1}, \Lambda_{2}$ such that

$$
\begin{gather*}
f_{Q_{6 R}}|\nabla u-\nabla w| d x d t \leq C_{1} \frac{|\mu|\left(\tilde{\Omega}_{6 R}\right)}{R^{N+1}}  \tag{7.18}\\
\left(f_{Q_{\rho / 2}(z, s)}|\nabla w|^{\theta_{2}} d x d t\right)^{\frac{1}{\theta_{2}}} \leq C_{2} f_{Q_{3 \rho}(z, s)}|\nabla w| d x d t  \tag{7.19}\\
\left(f_{Q_{\rho_{1}(y, s)}}|w|^{2} d x d t\right)^{1 / 2} \leq C_{3}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{2}}\left(f_{Q_{\rho_{2}(y, s)}}|w|^{2} d x d t\right)^{1 / 2} \tag{7.20}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(f_{Q_{\rho_{1}(z, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \leq C_{3}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{2}-1}\left(f_{Q_{\rho_{2}(z, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \tag{7.21}
\end{equation*}
$$

for any $Q_{3 \rho}(z, s) \subset Q_{6 R}, y \in \partial \Omega, Q_{\rho_{1}}(y, s) \subset Q_{\rho_{2}}(y, s) \subset Q_{6 R}$ and $Q_{\rho_{1}}(z, s) \subset Q_{\rho_{2}}(z, s) \subset$ $Q_{6 R}$

Proof. 1. For $\eta \in C_{c}^{\infty}\left(\left[t_{0}-(6 R)^{2}, t_{0}\right)\right), 0 \leq \eta \leq 1, \eta_{t} \leq 0$ and $\eta\left(t_{0}-(6 R)^{2}\right)=1$. Using $\varphi=T_{k}(u-w) \eta$, for any $k>0$, as a test function for (7.1) and (7.16), we get

$$
\begin{aligned}
\int_{\tilde{\Omega}_{6 R}}(u- & w)_{t} T_{k}(u-w) \eta d x d t \\
& +\int_{\tilde{\Omega}_{6 R}}(A(x, t, \nabla u)-A(x, t, \nabla w)) \nabla T_{k}(u-w) \eta d x d t=\int_{\tilde{\Omega}_{6 R}} T_{k}(u-w) \eta d \mu
\end{aligned}
$$

Thanks to (1.3), we obtain

$$
-\int_{\tilde{\Omega}_{6 R}} \bar{T}_{k}(u-w) \eta_{t} d x d t+\Lambda_{2} \int_{\tilde{\Omega}_{6 R}}\left|\nabla T_{k}(u-w)\right|^{2} \eta d x d t \leq k|\mu|\left(\tilde{\Omega}_{6 R}\right)
$$

where $\bar{T}_{k}(s)=\int_{0}^{s} T_{k}(\tau) d \tau$. As in [13, Proposition 2.8], we also verify that

$$
\||\nabla(u-w)|\|_{L^{\frac{N+2}{N+1}, \infty}\left(\tilde{\Omega}_{6 R}\right)} \leq c_{1}|\mu|\left(\tilde{\Omega}_{6 R}\right)
$$

Hence we get (7.18).
2. We need to prove that

$$
\begin{equation*}
f_{Q_{r / 4}(z, s)}|\nabla w|^{2} d x d t \leq \frac{1}{2} f_{Q_{\frac{26}{10} r}(z, s)}|\nabla w|^{2} d x d t+c_{7}\left(f_{Q_{\frac{26}{10} r}(z, s)}|\nabla w|^{p_{0}} d x d t\right)^{\frac{2}{p_{0}}} \tag{7.22}
\end{equation*}
$$

for all $Q_{\frac{26}{10} r}(z, s) \subset Q_{6 R}=Q_{6 R}\left(x_{0}, t_{0}\right)$. Here the constant $p_{0}$ is in inequality (7.17).
Suppose that $B_{r}(z) \subset \Omega$. Take $\rho \in(0, r]$. Let $\varphi \in C_{c}^{\infty}\left(B_{\rho}(z)\right), \eta \in C_{c}^{\infty}\left(\left(s-\rho^{2}, s\right]\right)$ be such that $0 \leq \varphi, \eta \leq 1, \varphi=1$ in $B_{\rho / 2}(z), \eta=1$ in $\left[s-\rho^{2} / 4, s\right]$ and $|\nabla \varphi| \leq c_{1} / \rho,\left|\eta_{t}\right| \leq c_{1} / \rho^{2}$. We denote

$$
\tilde{w}_{B_{\rho}(z)}(t)=\left(\int_{B_{\rho}(z)} \varphi(x)^{2} d x\right)^{-1} \int_{B_{\rho}(z)} w(x, t) \varphi(x)^{2} d x
$$

Using $\varphi=\left(w-\tilde{w}_{B_{\rho}(z)}\right) \varphi^{2} \eta^{2}$ as a test function for the equation (7.16) we have for all $s^{\prime} \in\left[s-\rho^{2} / 4, s\right]$

$$
\begin{aligned}
\int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)} & \left(w-\tilde{w}_{B_{\rho}(z)}\right)_{t}\left(w-\tilde{w}_{B_{\rho}(z)}\right) \varphi^{2} \eta^{2} d x d t \\
& +\int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)} A(x, t, \nabla w) \nabla\left(\left(w-\tilde{w}_{B_{\rho}(z)}\right) \varphi^{2} \eta^{2}\right) d x d t=0 .
\end{aligned}
$$

Here we used the equality $\int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)}\left(\tilde{w}_{B_{\rho}(z)}\right)_{t}\left(w-\tilde{w}_{B_{\rho}(z)}\right) \varphi^{2} \eta^{2} d x d t=0$.
Thus, we can write

$$
\begin{aligned}
& \frac{1}{2} \int_{B_{\rho}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi^{2} d x+\int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)} A(x, t, \nabla w) \nabla w \varphi^{2} \eta^{2} d x d t \\
&=-2 \int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)} A(x, t, \nabla w) \nabla \varphi \varphi \eta^{2}\left(w-\tilde{w}_{B_{\rho}(z)}\right) d x d t \\
&+\int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)}\left(w-\tilde{w}_{B_{\rho}(z)}\right)^{2} \varphi^{2} \eta \eta_{t} d x d t .
\end{aligned}
$$

From conditions (1.2) and (1.3), we get

$$
\begin{aligned}
& \frac{1}{2} \int_{B_{\rho}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi^{2} d x+\Lambda_{2} \int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)}|\nabla w|^{2} \varphi^{2} \eta^{2} d x d t \\
& \quad \leq 2 \Lambda_{1} \int_{B_{\rho}(z) \times\left(s-\rho^{2}, s^{\prime}\right)}|\nabla w||\nabla \varphi| \varphi \eta^{2}\left|w-\tilde{w}_{B_{\rho}(z)}\right| d x d t+\frac{c_{8}}{\rho^{2}} \int_{Q_{\rho}(z, s)}\left(w-\tilde{w}_{B_{\rho}(z)}\right)^{2} d x d t .
\end{aligned}
$$

Using Holder inequality we can verify that

$$
\begin{align*}
\sup _{s^{\prime} \in\left[s-\rho^{2} / 4, s\right]} \int_{B_{\rho}(z)} & \left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi^{2} d x \\
& +\int_{Q_{\rho / 2}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{9}}{\rho^{2}} \int_{Q_{\rho}(z, s)}\left|w-\tilde{w}_{B_{\rho}(z)}\right|^{2} d x d t . \tag{7.23}
\end{align*}
$$

On the other hand, for any $s^{\prime} \in\left[s-\rho^{2} / 4, s\right]$

$$
\begin{equation*}
\int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)\right)^{2} d x \leq 2\left(1+2^{N+2}\right) \int_{B_{\rho}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi^{2} d x \tag{7.24}
\end{equation*}
$$

where $\varphi_{1}(x)=\varphi(z+2(x-z))$ for all $x \in B_{\rho / 2}(z)$ and

$$
\tilde{w}_{B_{\rho / 2}(z)}=\left(\int_{B_{\rho / 2}(z)} \varphi_{1}(x)^{2} d x\right)^{-1} \int_{B_{\rho / 2}(z)} w(x, t) \varphi_{1}(x)^{2} d x
$$

In fact, since $0 \leq \varphi \leq 1$ and $\varphi=1$ in $B_{\rho / 2}(z)$ thus

$$
\begin{aligned}
& \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)\right)^{2} d x \\
& \quad \leq 2 \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} d x+2^{N+1}\left(\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}}(z)\left(s^{\prime}\right)\right)^{2}\left|B_{\rho / 4}(z)\right| \\
& \quad \leq 2 \int_{B_{\rho}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi^{2} d x+2^{N+2} \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)\right)^{2} \varphi_{1}^{2} d x \\
& \quad+2^{N+2} \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho}(z)}\left(s^{\prime}\right)\right)^{2} \varphi_{1}^{2} d x .
\end{aligned}
$$

which yields (7.24) due to the following inequality

$$
\int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)\right)^{2} \varphi_{1}^{2} d x \leq \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-l\right)^{2} \varphi_{1}^{2} d x \quad \forall l \in \mathbb{R}
$$

Therefore,

$$
\begin{align*}
& \sup _{s^{\prime} \in\left[s-\rho^{2} / 4, s\right]} \int_{B_{\rho / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{\rho / 2}(z)}\left(s^{\prime}\right)\right)^{2} d x \\
& \quad+\int_{Q_{\rho / 2}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{10}}{\rho^{2}} \int_{Q_{\rho}(z, s)}\left|w-\tilde{w}_{B_{\rho}(z)}\right|^{2} d x d t \tag{7.25}
\end{align*}
$$

Now we use estimate (7.25) for $\rho=r / 2$, we have

$$
\begin{aligned}
\int_{Q_{r / 4}(z, s)}|\nabla w|^{2} d x d t \leq & \frac{c_{10}}{r^{2}} \int_{Q_{r / 2}(z, s)}\left(w-\tilde{w}_{B_{r / 2}(z)}\right)^{2} d x d t \\
\leq & \frac{c_{10}}{r^{2}}\left(\sup _{s^{\prime} \in\left[s-r^{2} / 4, s\right]} \int_{B_{r / 2}(z)}\left(w\left(s^{\prime}\right)-\tilde{w}_{B_{r / 2}(z)}\left(s^{\prime}\right)\right)^{2} d x\right)^{\frac{2}{N+2}} \\
& \times \int_{s-r^{2} / 4}^{s}\left(\int_{B_{r / 2}(z)}\left(w-\tilde{w}_{B_{r / 2}(z)}\right)^{2} d x\right)^{\frac{N}{N+2}} d t
\end{aligned}
$$

After we again use estimate (7.25) for $\rho=r$ we get

$$
\begin{aligned}
\int_{Q_{r / 4}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{11}}{r^{2}} & \left(\frac{1}{r^{2}} \int_{Q_{r}(z, s)}\left|w-\tilde{w}_{B_{r}(z)}\right|^{2} d x d t\right)^{\frac{2}{N+2}} \\
& \times \int_{s-r^{2} / 4}^{s}\left(\int_{B_{\rho / 2}(z)}\left(w-\tilde{w}_{B_{r / 2}(z)}\right)^{2} d x\right)^{\frac{N}{N+2}} d t
\end{aligned}
$$

Thanks to a Sobolev-Poincare inequality, we obtain

$$
\int_{Q_{r / 4}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{12}}{r^{2}}\left(\int_{Q_{r}(z, s)}|\nabla w|^{2} d x d t\right)^{\frac{2}{N+2}} \int_{Q_{r / 2}(z, s)}|\nabla w|^{\frac{2 N}{N+2}} d x d t
$$

Since $p_{0} \in\left(\frac{2 N}{N+2}, 2\right)$, thanks to Holder inequality we get (7.22).
Finally, we consider the case $B_{r}(z) \cap \Omega \neq \emptyset$. In this case we choose $z_{0} \in \partial \Omega$ such that $\left|z-z_{0}\right|=\operatorname{dist}(z, \partial \Omega)$. Then $\left|z_{0}-z\right|<r$ and thus $\frac{1}{4} r \leq \rho_{1} \leq \frac{1}{2} r$,

$$
\begin{equation*}
B_{\frac{1}{4} r}(z) \subset B_{\frac{5}{4} r}\left(z_{0}\right) \subset B_{\rho_{1}+r}\left(z_{0}\right) \subset B_{\rho_{1}+\frac{11}{10} r}\left(z_{0}\right) \subset B_{\frac{16}{10} r}\left(z_{0}\right) \subset B_{\frac{26}{10} r}(z) \subset B_{6 R}\left(x_{0}\right) \tag{7.26}
\end{equation*}
$$

Let $\varphi \in C_{c}^{\infty}\left(B_{\rho_{1}+\frac{11}{10} r}\left(z_{0}\right)\right)$ be such that $0 \leq \varphi \leq 1, \varphi=1$ in $B_{\rho_{1}+r}\left(z_{0}\right)$ and $|\nabla \varphi| \leq C / r$. For $\frac{1}{2} r \leq \rho_{2} \leq r$, let $\eta \in C_{c}^{\infty}\left(\left(s-\rho_{2}^{2}, s\right]\right)$ be such that $0 \leq \eta \leq 1, \eta=1$ in $\left[s-\rho_{2}^{2} / 4, s\right]$ and $\left|\eta_{t}\right| \leq c / r^{2}$. Using $\phi=w \varphi^{2} \eta^{2}$ as a test function for (7.16) we have for any $s^{\prime} \in\left(s-\rho_{2}^{2}, s\right)$

$$
\begin{aligned}
& \int_{\left(B_{\rho_{1}+\frac{11}{10} r}\left(z_{0}\right) \cap \Omega\right) \times\left(s-\rho_{2}^{2}, s^{\prime}\right)} w_{t} w \varphi^{2} \eta^{2} d x d t \\
&+\int_{\left(B_{\rho_{1}+\frac{11}{10} r}\left(z_{0}\right) \cap \Omega\right) \times\left(s-\rho_{2}^{2}, s^{\prime}\right)} A(x, t, \nabla w) \nabla\left(w \varphi^{2} \eta^{2}\right) d x d t=0
\end{aligned}
$$

As above we also get

$$
\begin{aligned}
\sup _{s^{\prime} \in\left[s-\rho_{2}^{2} / 4, s\right]} & \int_{B_{\rho_{1}+r}\left(z_{0}\right)} w^{2}\left(s^{\prime}\right) d x \\
& +\int_{B_{\rho_{1}+r}\left(z_{0}\right) \times\left(s-\rho_{2}^{2} / 4, s\right)}|\nabla w|^{2} d x d t \leq \frac{c_{13}}{r^{2}} \int_{B_{\rho_{1}+\frac{11}{10} r}\left(z_{0}\right) \times\left(s-\rho_{2}^{2}, s\right)} w^{2} d x d t .
\end{aligned}
$$

In particular, for $\rho_{1}=\frac{1}{4} r, \rho_{2}=\frac{1}{2} r$ and using (7.26) yield

$$
\begin{equation*}
\int_{Q_{\frac{1}{4} r}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{14}}{r^{2}} \int_{B_{\frac{29}{20} r}\left(z_{0}\right) \times\left(s-r^{2} / 4, s\right)} w^{2} d x d t \tag{7.27}
\end{equation*}
$$

and $\rho_{1}=\left(\frac{1}{4}+\frac{1}{10}\right) r, \rho_{2}=r$,

$$
\sup _{s^{\prime} \in\left[s-r^{2} / 4, s\right]} \int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)} w^{2}\left(s^{\prime}\right) d x \leq \frac{c_{15}}{r^{2}} \int_{B_{\frac{29}{20} r}\left(z_{0}\right) \times\left(s-r^{2}, s\right)} w^{2} d x d t .
$$

Set $K_{1}=\{w=0\} \cap \bar{B}_{\frac{29}{20} r}\left(z_{0}\right)$ and $K_{2}=\{w=0\} \cap \bar{B}_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)$, Since $\mathbb{R}^{N} \backslash \Omega$ satisfies an uniformly $2-$ thick, we have the following estimates

$$
\operatorname{Cap}_{2}\left(K_{1}, B_{\frac{29}{10} r}\left(z_{0}\right)\right) \geq c_{16} r^{N-2} \text { and } \operatorname{Cap}_{p_{0}}\left(K_{2}, B_{\frac{1}{2} r+\frac{11}{5} r}\left(z_{0}\right)\right) \geq c_{16} r^{N-p_{0}}
$$

So, by Sobolev-Poincare's inequality we get

$$
\begin{equation*}
f_{B_{\frac{29}{20} r}\left(z_{0}\right)} w^{2} d x \leq c_{17} r^{2} f_{B_{\frac{5}{2} r}(z)}|\nabla w|^{2} d x \tag{7.28}
\end{equation*}
$$

and

$$
f_{B_{\frac{1}{4} r+\frac{11}{10} r} r\left(z_{0}\right)} w^{2} d x d t \leq c_{18} r^{2}\left(f_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)}|\nabla w|^{p_{0}} d x\right)^{\frac{2}{p_{0}}} \leq c_{19} r^{2}\left(f_{B_{\frac{5}{2} r}\left(z_{0}\right)}|\nabla w|^{p_{0}} d x\right)^{\frac{2}{p_{0}}}
$$

Leads to

$$
\begin{equation*}
\sup _{s^{\prime} \in\left[s-r^{2} / 4, s\right]} \int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)} w^{2}\left(s^{\prime}\right) d x \leq c_{20} \int_{Q_{\frac{5}{2} r}(z, s)}|\nabla w|^{2} d x d t \tag{7.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)} w^{2}(t) d x \leq c_{21} r^{N+2}\left(f_{B_{\frac{5}{2} r}\left(z_{0}\right)}|\nabla w|^{p_{0}}(t) d x\right)^{\frac{2}{p_{0}}} \tag{7.30}
\end{equation*}
$$

From (7.27), we have

$$
\begin{aligned}
& f_{Q_{\frac{1}{4} r}(z, s)}|\nabla w|^{2} d x d t \leq \frac{c_{22}}{r^{N+4}} \int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right) \times\left(s-r^{2} / 4, s\right)} w^{2} d x d t \\
& \leq \frac{c_{22}}{r^{N+4}}\left(\sup _{s^{\prime} \in\left[s-r^{2} / 4, s\right]} \int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)} w^{2}\left(s^{\prime}\right) d x\right)^{1-\frac{p_{0}}{2}} \int_{s-r^{2} / 4}^{s}\left(\int_{B_{\frac{1}{4} r+\frac{11}{10} r}\left(z_{0}\right)} w^{2}(t) d x\right)^{\frac{p_{0}}{2}} d t .
\end{aligned}
$$

Using (7.30), (7.29) and Holder's inequality we get

$$
\begin{aligned}
f_{Q_{\frac{1}{4} r}(z, s)}|\nabla w|^{2} d x d t & \leq \frac{c_{23}}{r^{N+4}}\left(\int_{Q_{\frac{5}{2} r}(z, s)}|\nabla w|^{2} d x d t\right)^{1-\frac{p_{0}}{2}} r^{\frac{N+2}{2} p_{0}-N} \int_{Q_{\frac{5}{2} r}(z, s)}|\nabla w|^{p_{0}} d x d t \\
& =c_{24}\left(f_{Q_{\frac{5}{2} r}(z, s)}|\nabla w|^{2} d x d t\right)^{1-\frac{p_{0}}{2}} f_{Q_{\frac{5}{2} r}(z, s)}|\nabla w|^{p_{0}} d x d t \\
& \leq \frac{1}{2} f_{Q_{\frac{26}{10} r}(z, s)}|\nabla w|^{2} d x d t+c_{25}\left(f_{Q_{\frac{26}{10} r}(z, s)}|\nabla w|^{p_{0}} d x d t\right)^{\frac{2}{p_{0}}} .
\end{aligned}
$$

So we proved (7.22).
Therefore, By Gehring's Lemma (see [60]) we get (7.19).
3. Now we prove (7.20). Let $y \in \partial \Omega, Q_{\rho_{1}}(y, s) \subset Q_{\rho_{2}}(y, s) \subset Q_{6 R}$ with $\rho_{1} \leq \rho_{2} / 4$. First, we will show that there exists a constant $\beta_{2}=\beta_{2}\left(N, \Lambda_{1}, \Lambda_{2}, c_{0}\right) \in(0,1 / 2]$ such that

$$
\begin{equation*}
\operatorname{osc}\left(w, Q_{\rho_{1}}(y, s)\right) \leq c_{26}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{2}} \operatorname{osc}\left(w, Q_{\rho_{2} / 2}(y, s)\right) \tag{7.31}
\end{equation*}
$$

where $\operatorname{osc}(w, A)=\sup _{A} w-\inf _{A} w$.
Indeed, since

$$
\int_{0}^{1} \frac{\operatorname{Cap}_{1,2}\left(\Omega^{c} \cap B_{r}(z), B_{2 r}(z)\right)}{r^{N-2}} \frac{d r}{r}=+\infty \quad \forall z \in \partial \Omega
$$

thus by the Wiener criterion (see [83]), we have $w$ is continuous up to $\partial_{p} \tilde{\Omega}_{6 R}$. So, we can choose $\varphi=\left(V-M_{4 \rho_{1}}\right) \eta^{2} \in L^{2}\left(-\infty, T ; H_{0}^{1}\left(\Omega \cap B_{6 R}\left(x_{0}\right)\right)\right)$ as test function in (7.16), where a. $\eta \in C^{\infty}\left(Q_{4 \rho_{1}}(y, s)\right), 0 \leq \eta \leq 1$ such that $\eta=1$ in $Q_{\rho_{1} / 2}\left(y, s-\frac{17}{4} \rho_{1}^{2}\right), \operatorname{supp}(\eta) \subset \subset$ $Q_{\rho_{1}}\left(y, s-4 \rho_{1}^{2}\right)$ and $|\nabla \eta| \leq c_{27} / \rho_{1},\left|\eta_{t}\right| \leq c_{28} / \rho_{1}^{2}$.
b. $M_{4 \rho_{1}}=\sup _{Q_{4 \rho_{1}(y, s)}} w$ and $V=\inf \left\{M_{4 \rho_{1}}-w, M_{4 \rho_{1}}\right\}$ in $\tilde{\Omega}_{6 R}, V=M_{4 \rho_{1}}$ outside $\tilde{\Omega}_{6 R}$. We have

$$
\begin{aligned}
& \int_{\tilde{\Omega}_{6 R}} w_{t}\left(V-M_{4 \rho_{1}}\right) \eta^{2} d x d t \\
& \quad+\int_{\tilde{\Omega}_{6 R}} 2 \eta A(x, t, \nabla w) \nabla \eta\left(V-M_{4 \rho_{1}}\right) d x d t+\int_{\tilde{\Omega}_{6 R}} \eta^{2} A(x, t, \nabla w) \nabla V d x d t=0,
\end{aligned}
$$

which implies

$$
\begin{gathered}
\int_{\tilde{\Omega}_{6 R}} \eta^{2} A(x, t,-\nabla V)(-\nabla V) d x d t=\int_{\tilde{\Omega}_{6 R}} 2 \eta A(x, t,-\nabla V) \nabla \eta\left(V-M_{4 \rho_{1}}\right) d x d t \\
\quad-\int_{\tilde{\Omega}_{6 R}}\left(V-M_{4 \rho_{1}}\right)_{t}\left(V-M_{4 \rho_{1}}\right) \eta^{2} d x d t .
\end{gathered}
$$

Using (1.2) and (1.3) we get

$$
\begin{aligned}
& \Lambda_{2} \int_{\tilde{\Omega}_{6 R}} \eta^{2}|\nabla V|^{2} d x d t \\
& \quad \leq 2 \Lambda_{1} \int_{\tilde{\Omega}_{6 R}} \eta|\nabla V||\nabla \eta|\left|V-M_{4 \rho_{1} \mid}\right| d x d t-1 / 2 \int_{\tilde{\Omega}_{6 R}}\left(\left(V-M_{4 \rho_{1}}\right)^{2}-M_{4 \rho_{1}}^{2}\right)\left(\eta^{2}\right)_{t} d x d t \\
& \quad \leq 2 \Lambda_{1} M_{4 \rho_{1}} \int_{\tilde{\Omega}_{6 R}} \eta|\nabla V||\nabla \eta| d x d t+2 M_{4 \rho_{1}} \int_{\tilde{\Omega}_{6 R}} \eta V\left|\eta_{t}\right| d x d t .
\end{aligned}
$$

Since $\operatorname{supp}(|\nabla V|) \cap \operatorname{supp}(\eta) \subset \tilde{\Omega}_{6 R}$, thus

$$
\begin{align*}
\int_{\mathbb{R}^{N+1}}|\nabla(\eta V)|^{2} d x d t & \leq c_{29} M_{4 \rho_{1}}\left(\int_{\mathbb{R}^{N+1}} \eta|\nabla V||\nabla \eta| d x d t+\int_{\mathbb{R}^{N+1}} V\left(\eta\left|\eta_{t}\right|+|\nabla \eta|^{2}\right) d x d t\right) \\
& \leq c_{30} M_{4 \rho_{1}}\left(\int_{\mathbb{R}^{N+1}} \eta|\nabla V||\nabla \eta| d x d t+\frac{1}{\rho_{1}^{2}} \int_{Q_{\rho_{1}}\left(y, s-4 \rho_{1}^{2}\right)} V d x d t\right) \cdot(7.3 \tag{7.32}
\end{align*}
$$

By [50, Theorem 6.31 , p. 132], for any $\sigma \in(0,1+2 / N)$ there holds

$$
\begin{equation*}
\left(f_{Q_{\rho_{1}}\left(y, s-4 \rho_{1}^{2}\right)} V^{\sigma} d x d t\right)^{1 / \sigma} \leq c_{31} \inf _{Q_{\rho_{1}}(y, s)} V=c_{31}\left(M_{4 \rho_{1}}-\sup _{Q_{\rho_{1}}(y, s)} w\right)=c_{31}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right) . \tag{7.33}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\frac{1}{\rho_{1}^{2}} \int_{Q_{\rho_{1}\left(y, s-4 \rho_{1}^{2}\right)}} V d x d t \leq c_{32} \rho_{1}^{N}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right) . \tag{7.34}
\end{equation*}
$$

We need to estimate $\int_{\tilde{\Omega}_{6 R}} \eta|\nabla V||\nabla \eta| d x d t$. Using Holder inequality and (7.33), for $\varepsilon \in$ $(0, \min \{2 / N, 1\})$ we have

$$
\begin{aligned}
\int_{\tilde{\Omega}_{6 R}} \eta|\nabla V||\nabla \eta| d x d t & \leq\left(\int_{\tilde{\Omega}_{6 R}} \eta^{2} V^{-(1+\varepsilon)}|\nabla V|^{2} d x d t\right)^{1 / 2}\left(\int_{\tilde{\Omega}_{6 R}} V^{1+\varepsilon}|\nabla \eta|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{28}\left(\int_{\tilde{\Omega}_{6 R}} \eta^{2} V^{-(1+\varepsilon)}|\nabla V|^{2} d x d t\right)^{1 / 2}\left(\int_{Q_{\rho_{1}\left(y, s-4 \rho_{1}^{2}\right)}} V^{1+\varepsilon} d x d t\right)^{1 / 2} \\
& \leq c_{33}\left(\int_{\tilde{\Omega}_{6 R}} \eta^{2} V^{-(1+\varepsilon)}|\nabla V|^{2} d x d t\right)^{1 / 2} \rho_{1}^{N / 2}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right)^{(1+\varepsilon) / 2} .
\end{aligned}
$$

To estimate $\left(\int_{\tilde{\Omega}_{6 R}} \eta^{2} V^{-(1+\varepsilon)}|\nabla V|^{2} d x d t\right)^{1 / 2}$, we can choose $\varphi=\left((V+\delta)^{-\varepsilon}-\left(M_{4 \rho_{1}}+\delta\right)^{-\varepsilon}\right) \eta^{2}$, for $\delta>0$, as test function in (7.16), we will get

$$
\begin{aligned}
& \int_{\tilde{\Omega}_{6 R}} \eta^{2}(V+\delta)^{-(1+\varepsilon)}|\nabla V|^{2} d x d t \\
& \quad \leq c_{34} \int_{\tilde{\Omega}_{6 R}} \eta(V+\delta)^{-\varepsilon}|\nabla V||\nabla \eta| d x d t+c_{34} \int_{\tilde{\Omega}_{6 R}} \eta(V+\delta)^{1-\varepsilon}\left|\eta_{t}\right| d x d t
\end{aligned}
$$

Thanks to Holder's inequality, we obtain

$$
\begin{aligned}
\int_{\tilde{\Omega}_{6 R}} \eta^{2}(V+\delta)^{-(1+\varepsilon)}|\nabla V|^{2} d x d t & \leq c_{35} \int_{\tilde{\Omega}_{6 R}}(V+\delta)^{1-\varepsilon}\left(\eta\left|\eta_{t}\right|+|\nabla \eta|^{2}\right) d x d t \\
& \leq c_{36} \rho_{1}^{2} \int_{Q_{\rho_{1}\left(y, s-4 \rho_{1}^{2}\right)}}(V+\delta)^{1-\varepsilon} d x d t
\end{aligned}
$$

Letting $\delta \rightarrow 0$ and using (7.33), we get

$$
\begin{aligned}
\int_{\tilde{\Omega}_{6 R}} \eta^{2} V^{-(1+\varepsilon)}|\nabla V|^{2} d x d t & \leq c_{36} \rho_{1}^{2} \int_{Q_{\rho_{1}}\left(y, s-4 \rho_{1}^{2}\right)} V^{1-\varepsilon} d x d t \\
& \leq c_{37} \rho_{1}^{N}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right)^{1-\varepsilon}
\end{aligned}
$$

Thus,

$$
\int_{\tilde{\Omega}_{6 R}} \eta|\nabla V||\nabla \eta| d x d t \leq c_{38} \rho_{1}^{N}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right)
$$

Combining this with (7.32) and (7.34),

$$
\int_{\mathbb{R}^{N+1}}|\nabla(\eta V)|^{2} d x d t \leq c_{39} \rho_{1}^{N} M_{4 \rho_{1}}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right)
$$

Note that $\eta V=M_{4 \rho_{1}}$ in $\left(\Omega^{c} \cap B_{\rho_{1} / 2}(y)\right) \times\left(s-\frac{9}{2} \rho_{1}^{2}, s-\frac{17}{4} \rho_{1}^{2}\right)$ thus

$$
\begin{aligned}
\int_{\mathbb{R}^{N+1}}|\nabla(\eta V)|^{2} d x d t & \geq \int_{s-\frac{9}{2} \rho_{1}^{2}}^{s-\frac{17}{4} \rho_{1}^{2}} \int_{\mathbb{R}^{\mathbb{N}}}|\nabla(\eta V)|^{2} d x d t \\
& \geq \int_{s-\frac{9}{2} \rho_{1}^{2}}^{s-\frac{17}{4} \rho_{1}^{2}} M_{4 \rho_{1}}^{2} \operatorname{Cap}_{1,2}\left(\Omega^{c} \cap B_{\rho_{1} / 2}(y), B_{\rho_{1}}(y)\right) d t \\
& \geq c_{40} M_{4 \rho_{1}}^{2} \rho_{1}^{N}
\end{aligned}
$$

Here we used $\operatorname{Cap}_{1,2}\left(\Omega^{c} \cap B_{\rho_{1} / 2}(y), B_{\rho_{1}}(y)\right) \geq c \rho_{1}^{N-2}$ in the last inequality. It follows

$$
M_{4 \rho_{1}} \leq c_{41}\left(M_{4 \rho_{1}}-M_{\rho_{1}}\right)
$$

So

$$
\sup _{Q_{\rho_{1}}(y, s)} w \leq \gamma \sup _{Q_{4 \rho_{1}}(y, s)} w \quad \text { where } \gamma=\frac{c_{41}}{c_{41}+1}<1 .
$$

Of course, above estimate is also true when we replace $w$ by $-w$. These give,

$$
\operatorname{osc}\left(w, Q_{\rho_{1}}(y, s)\right) \leq \gamma \operatorname{osc}\left(w, Q_{4 \rho_{1}}(y, s)\right)
$$

It follows (7.31).
We come back the proof of (7.20).
Since $w=0$ outside $\Omega_{T}$ this leads to

$$
\left(f_{Q_{\rho_{1}(y, s)}}|w|^{2} d x d t\right)^{1 / 2} \leq c_{42} \operatorname{Osc}\left(w, Q_{\rho_{2} / 2}(y, s)\right)
$$

On the other hand, By [50, Theorem 6.30, p. 132] we have

$$
\begin{aligned}
& \sup _{Q_{\rho_{2} / 2}(y, s)} w \leq c_{43}\left(f_{Q_{\rho_{2}}(y, s)}\left(w^{+}\right)^{2} d x d t\right)^{1 / 2} \text { and } \\
& \sup _{Q_{\rho_{2} / 2}(y, s)}(-w) \leq c_{44}\left(f_{Q_{\rho_{2}}(y, s)}\left(w^{-}\right)^{2} d x d t\right)^{1 / 2}
\end{aligned}
$$

Thus, we get (7.20).
Next, we have (7.21) for case $z=y \in \partial \Omega$ since from Caccippoli's inequality,

$$
\int_{Q_{\rho_{1}(z, s)}}|\nabla w|^{2} d x d t \leq \frac{c_{45}}{\rho_{1}^{2}} \int_{Q_{2 \rho_{1}(z, s)}}|w|^{2} d x d t
$$

and using Sobolev-Poincare's inequality as in (7.28),

$$
\int_{Q_{\rho_{2}}(z, s)}|w|^{2} d x d t \leq c_{46} \rho_{2}^{2} \int_{Q_{\rho_{2}(z, s)}}|\nabla w|^{2} d x d t
$$

We now prove (7.21). Take $Q_{\rho_{1}}(z, s) \subset Q_{\rho_{2}}(z, s) \subset Q_{6 R}$, it is enough to consider the case $\rho_{1} \leq \rho_{2} / 20$. Clearly, if $B_{\rho_{2} / 4}(z) \subset \Omega$ then (7.21) follows from (7.7) in Theorem 7.1. We consider $B_{\rho_{2} / 4}(z) \cap \partial \Omega \neq \emptyset$, let $z_{0} \in B_{\rho_{2} / 4}(z) \cap \partial \Omega$ such that $\left|z-z_{0}\right|=\operatorname{dist}(z, \partial \Omega) \leq \rho_{2} / 4$. Obviously, if $\rho_{1}<\left|z-z_{0}\right| / 4$ and $z \notin \Omega$, then (7.21) is trivial. If $\rho_{1}<\left|z-z_{0}\right| / 4$ and $z \in \Omega$, then (7.21) follows from (7.7) in Theorem 7.1.
Now assume $\rho_{1} \geq\left|z-z_{0}\right| / 4$ then since $Q_{\rho_{1}}(z, s) \subset Q_{5 \rho_{1}}\left(z_{0}, s\right)$

$$
\begin{aligned}
\left(f_{Q_{\rho_{1}(z, s)}}|\nabla w|^{2} d x d t\right)^{1 / 2} & \leq c_{47}\left(f_{Q_{5 \rho_{1}\left(z_{0}, s\right)}}|\nabla w|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{48}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{2}-1}\left(f_{Q_{\rho_{2} / 4}\left(z_{0}, s\right)}|\nabla w|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{49}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{\beta_{2}-1}\left(f_{Q_{\rho_{2} / 2}(z, s)}|\nabla w|^{2} d x d t\right)^{1 / 2}
\end{aligned}
$$

which implies (7.21).
Corollary 7.6 Suppose that $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}, r_{0}$. Let $\beta_{2}$ be the constant in Theorem 7.5. For $2-\beta_{2}<\theta<N+2$, there exists a constant $C=C\left(N, \Lambda_{1}, \Lambda_{2}, \theta\right)>0$ such that for any $B_{\rho}(y) \cap \partial \Omega \neq \emptyset, s \in(-T, T), 0<\rho \leq r_{0}$

$$
\begin{equation*}
\int_{Q_{\rho}(y, s)}|\nabla u| d x d t \leq C \rho^{N+3-\theta}\left(\left(\frac{T_{0}}{r_{0}}\right)^{N+3-\theta}+1\right)\left\|\mathbb{M}_{\theta}[\mu]\right\|_{L^{\infty}(\Omega \times(-T, T))} \tag{7.35}
\end{equation*}
$$

where $T_{0}=\operatorname{diam}(\Omega)+T^{1 / 2}$.
Proof. Take $B_{\rho_{2} / 4}(y) \cap \partial \Omega \neq \emptyset$ and $s \in(-T, T), \rho_{2} \leq 2 r_{0}$. Let $y_{0} \in B_{\rho_{2} / 4}(y) \cap \partial \Omega$ such that $\left|y-y_{0}\right|=\operatorname{dist}(y, \partial \Omega) \leq \rho_{2} / 4$, thus $Q_{\rho_{2} / 4}(y, s) \subset Q_{\rho_{2} / 2}\left(y_{0}, s\right)$ For any $Q_{\rho_{1}}(y, s) \subset Q_{\rho_{2}}(y, s)$ with $\rho_{1} \leq \rho_{2} / 4$, we take $w$ as in Theorem 7.5 with $Q_{6 R}=Q_{\rho_{2} / 2}\left(y_{0}, s\right)$. Thus,

$$
\begin{aligned}
& \int_{Q_{\rho_{1}(y, s)}}|\nabla w| d x d t \leq c_{1}\left(\frac{\rho_{1}}{\rho_{2}}\right)^{N+\beta_{1}+1} \int_{Q_{\rho_{2} / 4}(y, s)}|\nabla w| d x d t, \\
& \int_{Q_{\rho_{2} / 2}\left(y_{0}, s\right)}|\nabla u-\nabla w| d x d t \leq c_{2} \rho_{2}|\mu|\left(Q_{\rho_{2} / 2}\left(y_{0}, s\right)\right) .
\end{aligned}
$$

As in the proof of Corollary 7.2, we get the result.

### 7.2.2 Reifenberg flat domain

In this subsection, we always assume that $A$ satisfies (2.27). Also, we assume that $\Omega$ is a $\left(\delta, R_{0}\right)$ - Reifenberg flat domain with $0<\delta<1 / 2$. Fix $x_{0} \in \partial \Omega$ and $0<R<R_{0} / 6$. We have a density estimate

$$
\begin{equation*}
\left|B_{t}(x) \cap\left(\mathbb{R}^{N} \backslash \Omega\right)\right| \geq c\left|B_{t}(x)\right| \forall x \in \partial \Omega, 0<t<R_{0} \tag{7.36}
\end{equation*}
$$

with $c=((1-\delta) / 2)^{N} \geq 4^{-N}$.
In particular, $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c, r_{0}=R_{0}$.
Next we set $\rho=R(1-\delta)$ so that $0<\rho /(1-\delta)<R_{0} / 6$. By the definition of Reifenberg flat domains, there exists a coordinate system $\left\{y_{1}, y_{2}, \ldots, y_{N}\right\}$ with the origin $0 \in \Omega$ such that in this coordinate system $x_{0}=(0, \ldots, 0,-\rho \delta /(1-\delta))$ and

$$
B_{\rho}^{+}(0) \subset \Omega \cap B_{\rho}(0) \subset B_{\rho}(0) \cap\left\{y=\left(y_{1}, y_{2}, \ldots ., y_{N}\right): y_{N}>-2 \rho \delta /(1-\delta)\right\}
$$

Since $\delta<1 / 2$ we have

$$
B_{\rho}^{+}(0) \subset \Omega \cap B_{\rho}(0) \subset B_{\rho}(0) \cap\left\{y=\left(y_{1}, y_{2}, \ldots, y_{N}\right): y_{N}>-4 \rho \delta\right\},
$$

where $B_{\rho}^{+}(0):=B_{\rho}(0) \cap\left\{y=\left(y_{1}, y_{2}, \ldots, y_{N}\right): y_{N}>0\right\}$.
Furthermore we consider the unique solution

$$
\begin{equation*}
v \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(\Omega \cap B_{\rho}(0)\right)\right) \cap L^{2}\left(t_{0}-\rho^{2}, t_{0} ; H^{1}\left(\Omega \cap B_{\rho}(0)\right)\right) \tag{7.37}
\end{equation*}
$$

to the following equation

$$
\left\{\begin{array}{l}
v_{t}-\operatorname{div}\left(\bar{A}_{B_{\rho}(0)}(t, \nabla v)\right)=0 \text { in } \tilde{\Omega}_{\rho}(0),  \tag{7.38}\\
v=w \quad \text { on } \partial_{p} \tilde{\Omega}_{\rho}(0),
\end{array}\right.
$$

where $\tilde{\Omega}_{\rho}(0)=\left(\Omega \cap B_{\rho}(0)\right) \times\left(t_{0}-\rho^{2}, t_{0}\right)\left(-T<t_{0}<T\right)$.
We put $v=w$ outside $\tilde{\Omega}_{\rho}(0)$. As Lemma 7.3 we have the following Lemma.
Lemma 7.7 Let $\theta_{2}$ be the constant in Theorem 7.5. There exists constants $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}\right)$, $C_{2}=C_{2}\left(\Lambda_{1}, \Lambda_{2}\right)$ such that

$$
\begin{equation*}
\left(f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla w-\nabla v|^{2}\right)^{1 / 2} \leq[A]_{s_{2}}^{R} f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla w| d x d t \tag{7.39}
\end{equation*}
$$

with $s_{2}=\frac{2 \theta_{2}}{\theta_{2}-2}$ and

$$
\begin{equation*}
C_{2}^{-1} \int_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \leq \int_{Q_{\rho}\left(0, t_{0}\right)}|\nabla w|^{2} d x d t \leq C_{2} \int_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \tag{7.40}
\end{equation*}
$$

We can see that if the boundary of $\Omega$ is bad enough, then the $L^{\infty}$-norm of $\nabla v$ up to $\partial \Omega \cap B_{\rho}(0) \times\left(t_{0}-\rho^{2}, t_{0}\right)$ could be unbounded. For our purpose, we will consider another equation:

$$
\left\{\begin{array}{l}
V_{t}-\operatorname{div}\left(\bar{A}_{B_{\rho}(0)}(t, \nabla V)\right)=0 \text { in } Q_{\rho}^{+}\left(0, t_{0}\right),  \tag{7.41}\\
V=0 \quad \text { on } T_{\rho}\left(0, t_{0}\right)
\end{array}\right.
$$

where $Q_{\rho}^{+}\left(0, t_{0}\right)=B_{\rho}^{+}(0) \times\left(t_{0}-\rho^{2}, t_{0}\right)$ and $T_{\rho}\left(0, t_{0}\right)=Q_{\rho}\left(0, t_{0}\right) \cap\left\{x_{N}=0\right\}$.
A weak solution $V$ of above problem is understood in the following sense: the zero extension of $V$ to $Q_{\rho}\left(0, t_{0}\right)$ is in $V \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(B_{\rho}(0)\right)\right) \cap L_{\text {loc }}^{2}\left(t_{0}-\rho^{2}, t_{0} ; H^{1}\left(B_{\rho}(0)\right)\right)$ and for every $\varphi \in C_{c}^{1}\left(Q_{\rho}^{+}\left(0, t_{0}\right)\right)$ there holds

$$
-\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} V \varphi_{t} d x d t+\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} \bar{A}_{B_{\rho}(0)}(t, \nabla V) \nabla \varphi d x d t=0
$$

We have the following gradient $L^{\infty}$ estimate up to the boundary for $V$.

Lemma 7.8 (see [48, 49]) For any weak solution $V \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(B_{\rho}^{+}(0)\right)\right) \cap L_{l o c}^{2}\left(t_{0}-\right.$ $\left.\rho^{2}, t_{0} ; H^{1}\left(B_{\rho}^{+}(0)\right)\right)$ of (7.41), we have

$$
\begin{equation*}
\|\nabla V\|_{L^{\infty}\left(Q_{\rho^{\prime} / 2}^{+}\left(0, t_{0}\right)\right)} \leq C f_{Q_{\rho^{\prime}}^{+}\left(0, t_{0}\right)}|\nabla V|^{2} d x d t \quad \forall 0<\rho^{\prime} \leq \rho \tag{7.42}
\end{equation*}
$$

for some constant $C=C\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$. Moreover, $\nabla V$ is continuous up to $T_{\rho}\left(0, t_{0}\right)$.
Lemma 7.9 If $V \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(B_{\rho}^{+}(0)\right)\right) \cap L^{2}\left(t_{0}-\rho^{2}, t_{0} ; H^{1}\left(B_{\rho}^{+}(0)\right)\right)$ is a weak solution of (7.41), then its zero extension from $Q_{\rho}^{+}\left(0, t_{0}\right)$ to $Q_{\rho}\left(0, t_{0}\right)$ solves

$$
\begin{equation*}
V_{t}-\operatorname{div}\left(\bar{A}_{B_{\rho}(0)}(t, \nabla V)\right)=\frac{\partial F}{\partial x_{N}} \tag{7.43}
\end{equation*}
$$

weakly in $Q_{\rho}\left(0, t_{0}\right)$, for $(x, t)=\left(x^{\prime}, x_{N}, t\right) \in Q_{\rho}\left(0, t_{0}\right)$, $\bar{A}_{B_{\rho}(0)}=\left(\bar{A}_{B_{\rho}(0)}^{1}, \bar{A}_{B_{\rho}(0)}^{2}, \ldots, \bar{A}_{B_{\rho}(0)}^{N}\right)$, and $F(x, t)=\chi_{x_{N}<0} \bar{A}_{B_{\rho}(0)}^{N}\left(t, \nabla V\left(x^{\prime}, 0, t\right)\right)$.
Proof. Let $g \in C^{\infty}(\mathbb{R})$ with $g=0$ on $(-\infty, 1 / 2)$ and $g=1$ on $(1, \infty)$. Then, for any $\varphi \in C_{c}^{\infty}\left(Q_{\rho}\left(0, t_{0}\right)\right)$ and $n \in \mathbb{N}$. We have $\varphi_{n}(x, t)=\varphi_{n}\left(x^{\prime}, x_{N}, t\right)=g\left(n x_{N}\right) \varphi(x, t) \in$ $C_{c}^{\infty}\left(Q_{\rho}^{+}\left(0, t_{0}\right)\right.$. Thus, we get

$$
\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} V_{t} \varphi_{n} d x d t+\int_{Q_{\rho}^{+}\left(0, t_{0}\right.} \bar{A}_{B_{\rho}(0)}(t, \nabla V) \nabla\left(g\left(n x_{N}\right) \varphi(x, t)\right) d x d t=0
$$

which implies

$$
\begin{aligned}
\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} V_{t} \varphi_{n} d x d t & +\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} \bar{A}_{B_{\rho}(0)}(t, \nabla V) \nabla \varphi(x, t) g\left(n x_{N}\right) d x d t \\
& =-\int_{0}^{\rho} G\left(x_{N}\right) g^{\prime}\left(n x_{N}\right) n d x_{N}
\end{aligned}
$$

where

$$
G\left(x_{N}\right)=\int_{t_{0}-\rho^{2}}^{t_{0}} \int_{\left|x^{\prime}\right|<\sqrt{\rho^{2}-x_{N}^{2}}} \bar{A}_{B_{\rho}(0)}^{N}(t, \nabla V) \varphi\left(x^{\prime}, x_{N}, t\right) d x^{\prime} d t \in C([0, \infty))
$$

Letting $n \rightarrow \infty$ we get

$$
\begin{gathered}
\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} V_{t} \varphi d x d t+\int_{Q_{\rho}^{+}\left(0, t_{0}\right)} \bar{A}_{B_{\rho}(0)}(t, \nabla V) \nabla \varphi(x, t) d x d t=-G(0) \\
=-\int_{Q_{\rho}\left(0, t_{0}\right)} F \frac{\partial \varphi}{\partial x_{N}} d x d t
\end{gathered}
$$

Since $\nabla V=0, V=0$ outside $Q_{\rho}^{+}$, therefore we get the result.
We now consider a scaled version of equation (7.38)

$$
\left\{\begin{array}{l}
v_{t}-\operatorname{div}\left(\bar{A}_{B_{1}(0)}(t, \nabla v)\right)=0 \quad \text { in } \tilde{\Omega}_{1}(0),  \tag{7.44}\\
v=0 \quad \text { on } \partial_{p} \tilde{\Omega}_{1}(0) \backslash(\Omega \times(-T, T)),
\end{array}\right.
$$

under assumption

$$
\begin{equation*}
B_{1}^{+}(0) \subset \Omega \cap B_{1}(0) \subset B_{1}(0) \cap\left\{x_{N}>-4 \delta\right\} \tag{7.45}
\end{equation*}
$$

Lemma 7.10 For any $\varepsilon>0$ there exists a small $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon\right)>0$ such that if $v \in C\left(t_{0}-1, t_{0} ; L^{2}\left(\Omega \cap B_{1}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(\Omega \cap B_{1}(0)\right)\right)$ is a solution of (7.44) and (7.45) is satisfied and the bounded

$$
\begin{equation*}
f_{Q_{1}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \leq 1 \tag{7.46}
\end{equation*}
$$

then there exists a weak solution $V \in C\left(t_{0}-1, t_{0} ; L^{2}\left(B_{1}^{+}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(B_{1}^{+}(0)\right)\right)$ of (7.41) with $\rho=1$, whose zero extension to $Q_{1}\left(0, t_{0}\right)$ satisfies

$$
\begin{equation*}
f_{Q_{1}\left(0, t_{0}\right)}|v-V|^{2} d x d t \leq \varepsilon^{2} \tag{7.47}
\end{equation*}
$$

Proof. We argue by contradiction. Suppose that the conclusion were false. Then, there exist a constant $\varepsilon_{0}>0, t_{0} \in \mathbb{R}$ and a sequence of nonlinearities $\left\{A_{k}\right\}$ satisfying (1.2) and (2.27), a sequence of domains $\left\{\Omega^{k}\right\}$, and a sequence of functions $\left\{v_{k}\right\} \subset C\left(t_{0}-1, t_{0} ; L^{2}\left(\Omega^{k} \cap\right.\right.$ $\left.\left.B_{1}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(\Omega^{k} \cap B_{1}(0)\right)\right)$ such that

$$
\begin{gather*}
B_{1}^{+}(0) \subset \Omega^{k} \cap B_{1}(0) \subset B_{1}(0) \cap\left\{x_{N}>-1 / 2 k\right\},  \tag{7.48}\\
\left\{\begin{array}{l}
\left(v_{k}\right)_{t}-\operatorname{div}\left(\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right)\right)=0 \text { in } \tilde{\Omega}_{1}^{k}(0), \\
v_{k}=0 \quad \text { on }\left(\partial_{p} \tilde{\Omega}_{1}^{k}(0)\right) \backslash\left(\Omega^{k} \times(-T, T)\right),
\end{array}\right. \tag{7.49}
\end{gather*}
$$

and the zero extension of each $v_{k}$ to $Q_{1}\left(0, t_{0}\right)$ satisfies

$$
\begin{align*}
& f_{Q_{1}\left(0, t_{0}\right)}\left|\nabla v_{k}\right|^{2} d x d t \leq 1 \text { but }  \tag{7.50}\\
& f_{Q_{1}\left(0, t_{0}\right)}\left|v_{k}-V_{k}\right|^{2} d x d t \geq \varepsilon_{0}^{2} \tag{7.51}
\end{align*}
$$

for any weak solution $V_{k}$ of

$$
\left\{\begin{array}{l}
\left(V_{k}\right)_{t}-\operatorname{div}\left(\bar{A}_{k, B_{1}(0)}\left(t, \nabla V_{k}\right)\right)=0, \text { in } Q_{1}^{+}\left(0, t_{0}\right)  \tag{7.52}\\
V_{k}=0 \text { on } T_{1}\left(0, t_{0}\right)
\end{array}\right.
$$

By (7.48) and (7.50) and Poincare's inequality it following that

$$
\left\|v_{k}\right\|_{L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(B_{1}(0)\right)\right)} \leq c_{1}\left\|\nabla v_{k}\right\|_{L^{2}\left(Q_{1}\left(0, t_{0}\right)\right.} \leq c_{2}
$$

and

$$
\begin{aligned}
\left\|\left(v_{k}\right)_{t}\right\|_{L^{2}\left(t_{0}-1, t_{0} ; H^{-1}\left(B_{1}(0)\right)\right)} & =\left\|\bar{A}_{k, Q_{1}\left(0, t_{0}\right)}\left(\nabla v_{k}\right)\right\|_{L^{2}\left(t_{0}-1, t_{0} ; H^{-1}\left(B_{1}(0)\right)\right)} \\
& \leq \int_{Q_{1}\left(0, t_{0}\right)}\left|\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right)\right|^{2} d x d t \\
& \leq c_{3} \int_{Q_{1}\left(0, t_{0}\right)}\left|\nabla v_{k}\right|^{2} d x d t \\
& \leq c_{4} .
\end{aligned}
$$

Therefore, using Aubin-Lions Lemma, one can find $v_{0}$ and a subsequence, still denoted by $\left\{v_{k}\right\}$ such that

$$
v_{k} \rightarrow v_{0} \text { weakly in } L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right) \text { and strongly in } L^{2}\left(t_{0}-1, t_{0}, L^{2}\left(B_{1}(0)\right)\right)
$$

and

$$
\left(v_{k}\right)_{t} \rightarrow\left(v_{0}\right)_{t} \quad \text { weakly in } L^{2}\left(t_{0}-1, t_{0}, H^{-1}\left(B_{1}(0)\right)\right)
$$

Moreover, $v_{0}=0$ in $Q_{1}^{-}\left(0, t_{0}\right):=\left(B_{1}(0) \cap\left\{x_{N}<0\right\}\right) \times\left(1-t_{0}, 1\right)$ since $v_{k}=0$ on outside $\Omega^{k} \cap Q_{1}\left(0, t_{0}\right)$ for all $k$.
To get a contradiction we take $V_{k}$ to be the unique solution of $\left(V_{k}\right)_{t}-\operatorname{div}\left(\bar{A}_{k, B_{1}(0)}\left(t, \nabla V_{k}\right)\right)=$ 0 in $Q_{1}^{+}\left(0, t_{0}\right)$ and $V_{k}-v_{0} \in L^{2}\left(t_{0}-1, t_{0}, H_{0}^{1}\left(B_{1}^{+}(0)\right)\right)$ and $V_{k}\left(t_{0}-1\right)=v_{0}\left(t_{0}-1\right)$. As above, one can find $V_{0}$ and a subsequence, still denoted by $\left\{V_{k}\right\}$ such that
$V_{k} \rightarrow V_{0}$ weakly in $L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)$ and strongly in $L^{2}\left(t_{0}-1, t_{0}, L^{2}\left(B_{1}(0)\right)\right)$,
and

$$
\left(V_{k}\right)_{t} \rightarrow\left(V_{0}\right)_{t} \quad \text { weakly in } L^{2}\left(t_{0}-1, t_{0}, H^{-1}\left(B_{1}\right)\right)
$$

for some $V_{0} \in v_{0}+L^{2}\left(t_{0}-1, t_{0}, H_{0}^{1}\left(B_{1}^{+}(0)\right)\right.$ and $V_{0}\left(t_{0}-1\right)=v_{0}\left(t_{0}-1\right)$.
Thanks to (7.51), the proof would be complete if we could show that $v_{0}=V_{0}$. In fact, Let $\mathcal{J}_{k}: X \rightarrow L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)$ determined by

$$
\mathcal{J}_{k}(\phi(x, t))=\bar{A}_{k, B_{1}(0)}(t, \nabla \phi(x, t)) \text { for any } \phi \in X
$$

where $X \subset L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)$ is closures (in the strong topology of $L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)$ ) of convex combinations of $\left\{v_{k}\right\}_{k \geq 1} \cup\left\{V_{k}\right\}_{k \geq 1} \cup\{0\}$.
Since $v_{k}, V_{k}$ converge weakly to $v_{0}, V_{0}$ in $L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)$ resp., thus by Mazur Theorem, $X$ is compact subset of $L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)$ and $v_{0}, V_{0} \in X$.
Thanks to (1.2) and (2.27), we get $\mathcal{J}_{k}(0)=0$ and

$$
\left\|\mathcal{J}_{k}\left(\phi_{1}\right)-\mathcal{J}_{k}\left(\phi_{2}\right)\right\|_{L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)} \leq \Lambda_{1}\left\|\phi_{1}-\phi_{2}\right\|_{L^{2}\left(t_{0}-1, t_{0}, H^{1}\left(B_{1}(0)\right)\right)}
$$

for every $\phi_{1}, \phi_{2} \in X$ and $k \in \mathbb{N}$. Thus, by Ascoli Theorem, there exist $\mathcal{J} \in C\left(X, L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)\right)$ and a subsequence of $\left\{\mathcal{J}_{k}\right\}$, still denote by it, such that

$$
\begin{equation*}
\sup _{\phi \in X}\left\|\mathcal{J}_{k}(\phi)-\mathcal{J}(\phi)\right\|_{L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)} \rightarrow 0 \quad \text { as } \quad k \rightarrow \infty \tag{7.53}
\end{equation*}
$$

and also for any $\phi_{1}, \phi_{2} \in X$,

$$
\begin{equation*}
\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(\mathcal{J}\left(\phi_{1}\right)-\mathcal{J}\left(\phi_{2}\right)\right) \cdot\left(\nabla \phi_{1}-\nabla \phi_{2}\right) d x d t \geq \Lambda_{2}| |\left|\nabla \phi_{1}-\nabla \phi_{2}\right| \|_{L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right)\right)} \tag{7.54}
\end{equation*}
$$

From (7.48), we deduce

$$
\begin{aligned}
& \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(v_{k}-V_{k}\right)_{t}\left(v_{0}-V_{0}\right) d x d t \\
& \quad+\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right)-\bar{A}_{k, B_{1}(0)}\left(t, \nabla V_{k}\right)\right) \cdot \nabla\left(v_{0}-V_{0}\right) d x d t=0
\end{aligned}
$$

We have

$$
\begin{aligned}
& \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left|\bar{A}_{k, B_{1}(0)}\left(\nabla v_{k}\right)\right|^{2} d x d t \leq c_{9} \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left|\nabla v_{k}\right|^{2} d x d t \leq c_{10} \text { and } \\
& \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left|\bar{A}_{k, B_{1}(0)}\left(\nabla V_{k}\right)\right|^{2} d x d t \leq c_{9} \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left|\nabla V_{k}\right|^{2} d x d t \leq c_{11}
\end{aligned}
$$

for every $k$.
Thus there exists a subsequence, still denoted by $\left\{\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right), \bar{A}_{k, B_{1}(0)}\left(t, \nabla V_{k}\right)\right\}$ and a vector field $A_{1}, A_{2}$ belonging to $L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)$ such that

$$
\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right) \rightarrow A_{1} \text { and } \bar{A}_{k, B_{1}(0)}\left(t, \nabla V_{k}\right) \rightarrow A_{2}
$$

weakly in $L^{2}\left(Q_{1}^{+}\left(0, t_{0}\right), \mathbb{R}^{N}\right)$. It follows

$$
\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(v_{0}-V_{0}\right)_{t}\left(v_{0}-V_{0}\right) d x d t+\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(A_{1}-A_{2}\right) \cdot \nabla\left(v_{0}-V_{0}\right) d x d t=0
$$

Since

$$
\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(v_{0}-V_{0}\right)_{t}\left(v_{0}-V_{0}\right) d x d t=\int_{B_{1}^{+}(0)}\left(v_{0}-V_{0}\right)^{2}\left(t_{0}\right) d x \geq 0
$$

we get

$$
\begin{equation*}
\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(A_{1}-A_{2}\right) \cdot \nabla\left(v_{0}-V_{0}\right) d x d t \leq 0 \tag{7.55}
\end{equation*}
$$

For our purpose, we need to show that

$$
\begin{align*}
& \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(A_{1}-\mathcal{J}\left(v_{0}\right)\right) \cdot \nabla\left(v_{0}-V_{0}\right) d x d t \geq 0 \text { and }  \tag{7.56}\\
& \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(A_{2}-\mathcal{J}\left(V_{0}\right)\right) \cdot \nabla\left(V_{0}-v_{0}\right) d x d t \geq 0 \tag{7.57}
\end{align*}
$$

To do this, we fix a function $g \in X$ and any $\varphi \in C_{c}^{1}\left(Q_{1}^{+}\left(0, t_{0}\right)\right)$ such that $\varphi \geq 0$. We have

$$
\begin{aligned}
0 & \leq \int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi\left(\bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right)-\bar{A}_{k, B_{1}(0)}(t, \nabla g)\right)\left(\nabla v_{k}-\nabla g\right) d x d t \\
& =\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi \bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right) \nabla v_{k} d x d t-\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi \bar{A}_{k, B_{1}(0)}\left(t, \nabla v_{k}\right) \nabla g d x d t \\
& -\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi \bar{A}_{k, B_{1}(0)}(t, \nabla g)\left(\nabla v_{k}-\nabla g\right) d x d t \\
& :=B_{1}+B_{2}+B_{3}
\end{aligned}
$$

It is easy to see that

$$
\lim _{k \rightarrow \infty} B_{2}=-\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi A_{1} \nabla g d x d t \quad \text { and } \quad \lim _{k \rightarrow \infty} B_{3}=-\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi \mathcal{J}(g)\left(\nabla v_{0}-\nabla g\right) d x d t
$$

Moreover, we have

$$
\begin{aligned}
B_{1} & =-\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(v_{k}\right)_{t} \varphi v_{k} d x d t-\int_{Q_{1}^{+}\left(0, t_{0}\right)} \bar{A}_{k, Q_{1}\left(0, t_{0}\right)}\left(\nabla v_{k}\right) \nabla \varphi v_{k} d x d t \\
& =\frac{1}{2} \int_{Q_{1}^{+}\left(0, t_{0}\right)} v_{k}^{2} \varphi_{t} d x d t-\int_{Q_{1}^{+}\left(0, t_{0}\right)} \bar{A}_{k, Q_{1}\left(0, t_{0}\right)}\left(\nabla v_{k}\right) \nabla \varphi v_{k} d x d t
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim _{k \rightarrow \infty} B_{1} & =\frac{1}{2} \int_{Q_{1}^{+}\left(0, t_{0}\right)} v_{0}^{2} \varphi_{t} d x d t-\int_{Q_{1}^{+}\left(0, t_{0}\right)} A_{1} \nabla \varphi v_{0} d x d t \\
& =-\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(v_{0}\right)_{t} \varphi v_{0} d x d t-\int_{Q_{1}^{+}\left(0, t_{0}\right)} A_{1} \nabla\left(\varphi v_{0}\right) d x d t+\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi A_{1} \nabla v_{0} d x d t \\
& =\int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi A_{1} \nabla v_{0} d x d t
\end{aligned}
$$

Hence,

$$
0 \leq \int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi\left(A_{1}-\mathcal{J}(g)\right)\left(\nabla v_{0}-\nabla g\right) d x d t
$$

holds for all $\varphi \in C_{c}^{1}\left(Q_{1}^{+}\left(0, t_{0}\right)\right), \varphi \geq 0$ and $g \in X$. Now we choose $g=v_{0}-\xi\left(v_{0}-V_{0}\right)=$ $(1-\xi) v_{0}+\xi V_{0} \in X$ for $\xi \in(0,1)$, so

$$
0 \leq \int_{Q_{1}^{+}\left(0, t_{0}\right)} \varphi\left(A-\mathcal{J}\left(v_{0}-\xi\left(v_{0}-V_{0}\right)\right)\right)\left(\nabla v_{0}-\nabla V_{0}\right) d x d t
$$

Letting $\xi \rightarrow 0^{+}$and $\varphi \rightarrow \chi_{Q_{1}^{+}\left(0, t_{0}\right)}$, we get (7.56). Similarly, we also obtain (7.57).
Thus,

$$
\int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(A_{1}-A_{2}\right) \nabla\left(v_{0}-V_{0}\right) d x d t \geq \int_{Q_{1}^{+}\left(0, t_{0}\right)}\left(\mathcal{J}\left(v_{0}\right)-\mathcal{J}\left(V_{0}\right)\right) \nabla\left(v_{0}-V_{0}\right) d x d t
$$

Combining this with (7.54), (7.55) and $v_{0}-V_{0} \in L^{2}\left(t_{0}-1, t_{0}, H_{0}^{1}\left(B_{1}^{+}(0)\right)\right)$, yields $v_{0}=V_{0}$. This completes the proof of Lemma.
Lemma 7.11 For any $\varepsilon>0$ there exists a small $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon\right)>0$ such that if $v \in C\left(t_{0}-1, t_{0} ; L^{2}\left(\Omega \cap B_{1}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(\Omega \cap B_{1}(0)\right)\right)$ is a solution of (7.44) and (7.45) is satisfied and the bounded

$$
\begin{equation*}
f_{Q_{1}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \leq 1 \tag{7.58}
\end{equation*}
$$

then there exists a weak solution $V \in C\left(t_{0}-1, t_{0} ; L^{2}\left(B_{1}^{+}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(B_{1}^{+}(0)\right)\right)$ of (7.41) with $\rho=1$, whose zero extension to $Q_{1}\left(0, t_{0}\right)$ satisfies

$$
\begin{align*}
& \|\nabla V\|_{L^{\infty}\left(Q_{1 / 4}\left(0, t_{0}\right)\right)} \leq C \quad \text { and }  \tag{7.59}\\
& f_{Q_{1 / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V|^{2} d x d t \leq \varepsilon^{2} \tag{7.60}
\end{align*}
$$

for some $C=C\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$.
Proof. Given $\varepsilon_{1} \in(0,1)$ by applying Lemma 7.10 one finds a small $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon_{1}\right)>0$ and a weak solution $V \in C\left(t_{0}-1, t_{0} ; L^{2}\left(B_{1}^{+}(0)\right)\right) \cap L^{2}\left(t_{0}-1, t_{0} ; H^{1}\left(B_{1}^{+}(0)\right)\right)$ of (7.41) with $\rho=1$ such that

$$
\begin{equation*}
f_{Q_{1}\left(0, t_{0}\right)}|v-V|^{2} d x d t \leq \varepsilon_{1}^{2} \tag{7.61}
\end{equation*}
$$

Using $\phi^{2} V$ with $\phi \in C_{c}^{\infty}\left(B_{1} \times\left(t_{0}-1, t_{0}\right]\right), 0 \leq \phi \leq 1$ and $\phi=1$ in $Q_{1 / 2}\left(0, t_{0}\right)$ as test function in (7.41), we can obtain

$$
\int_{Q_{1 / 2}\left(0, t_{0}\right)}|\nabla V|^{2} d x d t \leq c_{1} \int_{Q_{1}\left(0, t_{0}\right)}|V|^{2} d x d t
$$

This implies

$$
\begin{aligned}
\int_{Q_{1 / 2}\left(0, t_{0}\right)}|\nabla V|^{2} d x d t & \leq c_{2} \int_{Q_{1}\left(0, t_{0}\right)}\left(|v-V|^{2}+|v|^{2}\right) d x d t \\
& \leq c_{3} \int_{Q_{1}\left(0, t_{0}\right)}\left(|v-V|^{2}+|\nabla v|^{2}\right) d x d t \\
& \leq c_{4}
\end{aligned}
$$

since (7.58), (7.61) and Poincare's inequality. Thus, using Lemma 7.8 we get (7.59).
Next, we will prove (7.60). By Lemma 7.9, the zero extension of $V$ to $Q_{1}\left(0, t_{0}\right)$ satisfies

$$
V_{t}-\operatorname{div}\left(\bar{A}_{B_{1}(0)}(t, \nabla V)\right)=\frac{\partial F}{\partial x_{N}} \quad \text { in weakly } Q_{1}\left(0, t_{0}\right)
$$

where $F(x, t)=\chi_{x_{N}<0} \bar{A}_{B_{\rho}(0)}^{N}\left(t, \nabla V\left(x^{\prime}, 0, t\right)\right)$. Thus, we can write

$$
\begin{aligned}
& \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}(V-v)_{t} \varphi d x d t \\
& \quad+\int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}\left(\bar{A}_{B_{1}(0)}(t, \nabla V)-\bar{A}_{B_{1}(0)}(t, \nabla v)\right) \nabla \varphi d x d t=-\int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} F \frac{\partial \varphi}{\partial x_{N}} d x d t,
\end{aligned}
$$

for any $\varphi \in L^{2}\left(t_{0}-1, t_{0}, H_{0}^{1}\left(\Omega \cap B_{1}(0)\right)\right)$.
We take $\varphi=\phi^{2}(V-v)$ where $\varphi \in C_{c}^{\infty}\left(B_{1 / 4} \times\left(t_{0}-(1 / 4)^{2}, t_{0}\right]\right), 0 \leq \phi \leq 1$ and $\phi=1$ on $\bar{Q}_{1 / 8}\left(0, t_{0}\right)$, so

$$
\begin{aligned}
& \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi^{2}\left(\bar{A}_{B_{1}(0)}(t, \nabla V)-\bar{A}_{B_{1}(0)}(t, \nabla v)\right)(\nabla V-\nabla v) d x d t \\
& \quad=-2 \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi(V-v)\left(\bar{A}_{B_{1}(0)}(t, \nabla V)-\bar{A}_{B_{1}(0)}(t, \nabla v)\right) \nabla \phi d x d t \\
& \quad-\int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi^{2}(V-v)_{t}(V-v) d x d t \\
& \quad-\int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}\left(\phi^{2} F \frac{\partial(V-v)}{\partial x_{N}}+2 \phi F(V-v) \frac{\partial \phi}{\partial x_{N}}\right) d x d t .
\end{aligned}
$$

We can rewrite $I_{1}=I_{2}+I_{3}+I_{4}$.
We see that

$$
I_{1} \geq c_{5} \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi^{2}|\nabla V-\nabla v|^{2} d x d t
$$

and using Holder's inequality

$$
\begin{aligned}
\left|I_{2}\right| & \leq c_{6} \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi|V-v|(|\nabla V|+|\nabla v|)|\nabla \phi| d x d t \\
& \leq \varepsilon_{2} \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi^{2}\left(|\nabla V|^{2}+|\nabla v|^{2}\right) d x d t+c_{7}\left(\varepsilon_{2}\right) \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}|V-v|^{2}|\nabla \phi|^{2} d x d t
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
& \left|I_{4}\right| \leq \varepsilon_{2} \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi^{2}\left(|\nabla V|^{2}+|\nabla v|^{2}\right) d x d t+c_{8}\left(\varepsilon_{2}\right) \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}|V-v|^{2}|\nabla \phi|^{2} d x d t \\
& \quad+c_{8}\left(\varepsilon_{2}\right) \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)}|F|^{2} \phi^{2} d x d t
\end{aligned}
$$

and

$$
I_{3} \leq \int_{\tilde{\Omega}_{1}\left(0, t_{0}\right)} \phi_{t} \phi(V-v)^{2} d x d t \leq c_{9} \int_{\tilde{\Omega}_{1 / 4}\left(0, t_{0}\right)}|V-v|^{2} d x d t
$$

Hence,

$$
\begin{aligned}
& \int_{\tilde{\Omega}_{1 / 8}\left(0, t_{0}\right)}|\nabla V-\nabla v|^{2} \\
& \leq c_{10} \varepsilon_{2} \int_{\tilde{\Omega}_{1 / 4}\left(0, t_{0}\right)}\left(|\nabla V|^{2}+|\nabla v|^{2}\right)+c_{11}\left(\varepsilon_{2}\right) \int_{\tilde{\Omega}_{1 / 4}\left(0, t_{0}\right)}\left(|V-v|^{2}+|F|^{2}\right) \\
& \leq c_{12} \varepsilon_{2}+c_{13}\left(\varepsilon_{2}\right)\left(\varepsilon_{1}^{2}+\int_{\tilde{\Omega}_{1 / 4}\left(0, t_{0}\right) \cap\left\{-4 \delta<x_{N}<0\right\}}\left|\nabla V\left(x^{\prime}, 0, t\right)\right|^{2} d x d t\right) \\
& \leq c_{12} \varepsilon_{2}+c_{14}\left(\varepsilon_{2}\right)\left(\varepsilon_{1}^{2}+\delta\right)
\end{aligned}
$$

Finally, for any $\varepsilon>0$ by choosing $\varepsilon_{2}, \varepsilon_{1}$ and $\delta$ appropriately we get (7.60). This completes the proof of Lemma.

Lemma 7.12 For any $\varepsilon>0$ there exists a small $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon\right)>0$ such that if $v \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(\Omega \cap B_{\rho}(0)\right)\right) \cap L^{2}\left(t_{0}-\rho^{2}, t_{0} ; H^{1}\left(\Omega \cap B_{\rho}(0)\right)\right)$ is a solution of

$$
\left\{\begin{array}{l}
v_{t}-\operatorname{div}\left(\bar{A}_{B_{\rho}(0)}(t, \nabla v)\right)=0 \text { in } \tilde{\Omega}_{\rho}(0)  \tag{7.62}\\
v=0 \quad \text { on } \partial_{p} \tilde{\Omega}_{\rho}(0) \backslash(\Omega \times(-T, T))
\end{array}\right.
$$

and

$$
\begin{equation*}
B_{\rho}^{+}(0) \subset \Omega \cap B_{\rho}(0) \subset B_{\rho}(0) \cap\left\{x_{N}>-4 \rho \delta\right\} . \tag{7.63}
\end{equation*}
$$

then there exists a weak solution $V \in C\left(t_{0}-\rho^{2}, t_{0} ; L^{2}\left(B_{\rho}^{+}(0)\right)\right) \cap L^{2}\left(t_{0}-\rho^{2}, t_{0} ; H^{1}\left(B_{\rho}^{+}(0)\right)\right)$ of (7.41), whose zero extension to $Q_{1}\left(0, t_{0}\right)$ satisfies

$$
\begin{align*}
& \|\nabla V\|_{L^{\infty}\left(Q_{\rho / 4}\left(0, t_{0}\right)\right)}^{2} \leq C f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \text { and }  \tag{7.64}\\
& f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V|^{2} d x d t \leq \varepsilon^{2} f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \tag{7.65}
\end{align*}
$$

for some $C=C\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$.
Proof. We set

$$
\mathcal{A}(x, t, \xi)=A\left(\rho x, t_{0}+\rho^{2}\left(t-t_{0}\right), \kappa \xi\right) / \kappa \text { and } \tilde{v}(x, t)=v\left(\rho x, t_{0}+\rho^{2}\left(t-t_{0}\right)\right) /(\rho \kappa)
$$

where $\kappa=\left(\frac{1}{\left|Q_{\rho}\left(0, t_{0}\right)\right|} \int_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t\right)^{1 / 2}$. Then $\mathcal{A}$ satisfies conditions (1.2) and (2.27) with the same constants $\Lambda_{1}$ and $\Lambda_{2}$. We can see that $\tilde{v}$ is a solution of

$$
\left\{\begin{array}{l}
\tilde{v}_{t}-\operatorname{div}\left(\overline{\mathcal{A}}_{B_{1}(0)}(t, \nabla \tilde{v})\right)=0 \quad \text { in } \tilde{\Omega}_{1}^{\rho}(0)  \tag{7.66}\\
\tilde{v}=0 \quad \text { on } \quad\left(\left(\partial \Omega^{\rho} \cap B_{1}(0)\right) \times\left(t_{0}-1, t_{0}\right)\right) \cup\left(\left(\Omega^{\rho} \cap B_{1}(0)\right) \times\left\{t=t_{0}-1\right\}\right)
\end{array}\right.
$$

where $\Omega^{\rho}=\{z=x / \rho: x \in \Omega\}$ and satisfies $f_{Q_{1}\left(0, t_{0}\right)}|\nabla \tilde{v}|^{2} d x d t=1$. We also have

$$
B_{1}^{+}(0) \subset \Omega^{\rho} \cap B_{1}(0) \subset B_{1}(0) \cap\left\{x_{N}>-4 \delta\right\}
$$

Therefore, applying Lemma 7.11 for any $\varepsilon>0$, there exist a constant $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon\right)>0$ and $\tilde{V}$ satisfies

$$
\|\nabla \tilde{V}\|_{L^{\infty}\left(Q_{1 / 4}\left(0, t_{0}\right)\right)} \leq c_{1} \text { and } f_{Q_{1 / 8}\left(0, t_{0}\right)}|\nabla \tilde{v}-\nabla \tilde{V}|^{2} d x d t \leq \varepsilon^{2}
$$

We complete the proof by choosing $V(x, t)=k \rho \tilde{V}\left(x / \rho, t_{0}+\left(t-t_{0}\right) / \rho^{2}\right)$.
Lemma 7.13 Let $s_{2}$ be as in Lemma 7.7. For any $\varepsilon>0$ there exists a small $\delta=$ $\delta\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon\right)>0$ such that the following holds. If $\Omega$ is a $\left(\delta, R_{0}\right)$-Reifenberg flat domain and $u \in C\left(0, T ; L^{2}(\Omega)\right) \cap L^{2}\left(0, T ; H^{1}(\Omega)\right)$ is a solution to equation (2.4) with $\mu \in L^{2}(\Omega \times(-T, T))$ and $u(-T)=0$, for $x_{0} \in \partial \Omega,-T<t_{0}<T$ and $0<R<R_{0} / 6$ then there is a function $V \in L^{2}\left(t_{0}-(R / 9)^{2}, t_{0} ; H^{1}\left(B_{R / 9}\left(x_{0}\right)\right)\right) \cap L^{\infty}\left(t_{0}-(R / 9)^{2}, t_{0} ; W^{1, \infty}\left(B_{R / 9}\left(x_{0}\right)\right)\right)$ such that

$$
\begin{equation*}
\|\nabla V\|_{L^{\infty}\left(Q_{R / 9}\left(x_{0}, t_{0}\right)\right)} \leq c f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla u| d x d t+c \frac{|\mu|\left(Q_{6 R}\left(x_{0}, t_{0}\right)\right)}{R^{N+1}} \tag{7.67}
\end{equation*}
$$

and

$$
\begin{align*}
& f_{Q_{R / 9}\left(x_{0}, t_{0}\right)}|\nabla u-\nabla V| d x d t \\
& \quad \leq c\left(\varepsilon+[A]_{s_{2}}^{R_{0}}\right) f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla u| d x d t+c\left(\varepsilon+1+[A]_{s_{2}}^{R_{0}}\right) \frac{|\mu|\left(Q_{6 R}\left(x_{0}, t_{0}\right)\right)}{R^{N+1}}, \tag{7.68}
\end{align*}
$$

for some $c=c\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$.
Proof. Let $x_{0} \in \partial \Omega,-T<t_{0}<T$ and $\rho=R(1-\delta)$, we may assume that $0 \in \Omega$, $x_{0}=(0, \ldots,-\delta \rho /(1-\delta))$ and

$$
\begin{equation*}
B_{\rho}^{+}(0) \subset \Omega \cap B_{\rho}(0) \subset B_{\rho}(0) \cap\left\{x_{N}>-4 \rho \delta\right\} . \tag{7.69}
\end{equation*}
$$

We also have

$$
\begin{equation*}
Q_{R / 9}\left(x_{0}, t_{0}\right) \subset Q_{\rho / 8}\left(0, t_{0}\right) \subset Q_{\rho / 4}\left(0, t_{0}\right) \subset Q_{\rho}\left(0, t_{0}\right) \subset Q_{6 \rho}\left(0, t_{0}\right) \subset Q_{6 R}\left(x_{0}, t_{0}\right) \tag{7.70}
\end{equation*}
$$

provided that $0<\delta<1 / 625$.
Let $w$ and $v$ be in Theorem 7.5 and Lemma 7.7. By Lemma 7.12 for any $\varepsilon>0$ we can find a small positive $\delta=\delta(N, \alpha, \beta, \varepsilon)<1 / 625$ such that there is a function $V \in L^{2}\left(t_{0}-\right.$ $\left.\rho^{2}, t_{0} ; H^{1}\left(B_{\rho}(0)\right)\right) \cap L^{\infty}\left(t_{0}-\rho^{2}, t_{0} ; W^{1, \infty}\left(B_{\rho}(0)\right)\right)$ satisfying

$$
\begin{aligned}
& \|\nabla V\|_{L^{\infty}\left(Q_{\rho / 4}\left(0, t_{0}\right)\right)}^{2} \leq c_{1} f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t \text { and } \\
& f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V|^{2} \leq \varepsilon^{2} f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla v|^{2} d x d t
\end{aligned}
$$

Then, by (7.40) in Lemma 7.7 and (7.19) in Theorem 7.5 and (7.70) we get

$$
\begin{align*}
\|\nabla V\|_{L^{\infty}\left(Q_{R / 9}\left(x_{0}, t_{0}\right)\right)} & \leq c_{2}\left(f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla w|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{3} f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla w| d x d t \tag{7.71}
\end{align*}
$$

and

$$
\begin{align*}
f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V| d x d t & \leq c_{4} \varepsilon\left(f_{Q_{\rho}\left(0, t_{0}\right)}|\nabla w|^{2} d x d t\right)^{1 / 2} \\
& \leq c_{5} \varepsilon f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla w| d x d t \tag{7.72}
\end{align*}
$$

Therefore, from (7.18) in Theorem 7.5 and (7.71) we get (7.67).
Now we prove (7.68), we have

$$
\begin{aligned}
& f_{Q_{R / 9}\left(x_{0}, t_{0}\right)}|\nabla u-\nabla V| d x d t \leq c_{6} f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla u-\nabla V| d x d t \\
& \quad \leq c_{6} f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla u-\nabla w| d x d t+c_{6} f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla w-\nabla v| d x d t \\
& \quad+c_{8} f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V| d x d t .
\end{aligned}
$$

From Lemma 7.7 and Theorem 7.5 and (7.72) it follows that

$$
\begin{aligned}
f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla u-\nabla w| d x d t & \leq c_{7} \frac{|\mu|\left(Q_{6 R}\left(x_{0}, t_{0}\right)\right)}{R^{N+1}} \\
f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla w| d x d t & \leq c_{8}[A]_{s_{2}}^{R_{0}} f_{Q_{6 \rho}\left(0, t_{0}\right)}|\nabla w| d x d t \\
& \leq c_{9}[A]_{s_{2}}^{R_{0}} f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla w| d x d t \\
& \leq c_{10}[A]_{s_{2}}^{R_{0}}\left(f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla u| d x d t+\frac{|\mu|\left(Q_{6 R}\left(x_{0}, t_{0}\right)\right)}{R^{N+1}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
f_{Q_{\rho / 8}\left(0, t_{0}\right)}|\nabla v-\nabla V| d x d t & \leq c_{11} \varepsilon f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla w| d x d t \\
& \leq c_{12} \varepsilon\left(f_{Q_{6 R}\left(x_{0}, t_{0}\right)}|\nabla u| d x d t+\frac{|\mu|\left(Q_{6 R}\left(x_{0}, t_{0}\right)\right)}{R^{N+1}}\right)
\end{aligned}
$$

Hence we get (7.68).

## 8 Global Integral Gradient Bounds for Parabolic equations

### 8.1 Global estimates on 2-Capacity uniform thickness domains

We use the Theorem 7.1 and 7.5 to prove the following theorem.
Theorem 8.1 Suppose that $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}, r_{0}$. Let $\theta_{1}, \theta_{2}$ be in Theorem 7.1 and 7.5. Set $\theta=\min \left\{\theta_{1}, \theta_{2}\right\}$ and $T_{0}=\operatorname{diam}(\Omega)+T^{1 / 2}$. Let $Q=$ $B_{\text {diam }(\Omega)}\left(x_{0}\right) \times(0, T)$ that contains $\Omega_{T}$. Let $B_{1}=\tilde{Q}_{R_{1}}\left(y_{0}, s_{0}\right), B_{2}=4 B_{1}:=\tilde{Q}_{4 R_{1}}\left(y_{0}, s_{0}\right)$ for $R_{1}>0$. For $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$, there exist a distribution solution $u$ of equation (2.4) with data $\mu, u_{0}=\sigma$ and constants $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, c_{0}, T_{0} / r_{0}\right), c_{2}>$ $0, \varepsilon_{1}=\varepsilon_{1}\left(N, \Lambda_{1}, \Lambda_{2}, c_{0}, T_{0} / r_{0}\right), \varepsilon_{2}=\varepsilon_{1}\left(N, \Lambda_{1}, \Lambda_{2}, c_{0}\right)>0$ such that

$$
\begin{equation*}
\left|\left\{\mathbb{M}(|\nabla u|)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}[\omega] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap Q\right| \leq C_{1} \varepsilon|\{\mathbb{M}(|\nabla u|)>\lambda\} \cap Q|, \tag{8.1}
\end{equation*}
$$

for all $\lambda>0, \varepsilon \in\left(0, \varepsilon_{1}\right)$ and

$$
\begin{equation*}
\left|\left\{\mathbb{M}\left(\chi_{B_{2}}|\nabla u|\right)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}\left[\chi_{B_{2}} \omega\right] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap B_{1}\right| \leq C_{1} \varepsilon\left|\left\{\mathbb{M}\left(\chi_{B_{2}}|\nabla u|\right)>\lambda\right\} \cap B_{1}\right| \tag{8.2}
\end{equation*}
$$

for all $\lambda>\varepsilon^{-1+\frac{1}{\theta}}\|\nabla u\|_{L^{1}\left(\Omega_{T} \cap B_{2}\right)} R_{2}^{-N-2}, \varepsilon \in\left(0, \varepsilon_{2}\right)$ with $R_{2}=\inf \left\{r_{0}, R_{1}\right\} / 16$.
Moreover, if $\sigma \in L^{1}(\Omega)$ then $u$ is a renormalized solution.
Proof of Theorem 8.1. Let $\left\{\mu_{n}\right\} \subset C_{c}^{\infty}\left(\Omega_{T}\right),\left\{\sigma_{n}\right\} \subset C_{c}^{\infty}(\Omega)$ be as in the proof of Theorem 2.1. We have $\left|\mu_{n}\right| \leq \varphi_{n} *|\mu|$ and $\left|\sigma_{n}\right| \leq \varphi_{1, n} *|\sigma|$ for any $n \in \mathbb{N},\left\{\varphi_{n}\right\},\left\{\varphi_{1, n}\right\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^{N}$, respectively.
Let $u_{n}$ be solution of equation

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\mu_{n} \text { in } \Omega_{T},  \tag{8.3}\\
u_{n}=0 \text { on } \partial \Omega \times(0, T), \\
u_{n}(0)=\sigma_{n} \text { in } \Omega
\end{array}\right.
$$

By Proposition 3.5 and Theorem 3.6, there exists a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ converging to a distribution solution $u$ of (2.4) with data $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right)$ and $u_{0}=\sigma$ such that $u_{n} \rightarrow u$ in $L^{s}\left(0, T, W_{0}^{1, s}(\Omega)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$ and if $\sigma \in L^{1}(\Omega)$ then $u$ is a renormalized solution.
By Remark 3.3 and Theorem 3.6, a sequence $\left\{u_{n, m}\right\}_{m}$ of solutions to equations

$$
\left\{\begin{array}{l}
\left(u_{n, m}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n, m}\right)\right)=\mu_{n, m} \text { in } \Omega \times(-T, T), \\
u_{n, m}=0 \text { on } \partial \Omega \times(-T, T), \\
u_{n, m}(-T)=0 \text { on } \Omega
\end{array}\right.
$$

converges to $\chi_{\Omega_{T}} u_{n}$ in $L^{s}\left(-T, T, W_{0}^{1, s}(\Omega)\right)$ for any $s \in\left[1, \frac{N+2}{N+1}\right)$, where $\mu_{n, m}=\left(g_{n, m}\right)_{t}+$ $\chi_{\Omega_{T}} \mu_{n}, g_{n, m}(x, t)=\sigma_{n}(x) \int_{-T}^{t} \varphi_{2, m}(s) d s$ and $\left\{\varphi_{2, m}\right\}$ is a sequence of mollifiers in $\mathbb{R}$.
Set

$$
\begin{aligned}
& E_{\lambda, \varepsilon}^{1}=\left\{\mathbb{M}(|\nabla u|)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}[\omega] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap Q, \quad F_{\lambda}^{1}=\{\mathbb{M}(|\nabla u|)>\lambda\} \cap Q, \\
& E_{\lambda, \varepsilon}^{2}=\left\{\mathbb{M}\left(\chi_{B_{2}}|\nabla u|\right)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}\left[\chi_{B_{2}} \omega\right] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap B_{1}, \quad F_{\lambda}^{2}=\left\{\mathbb{M}\left(\chi_{B_{2}}|\nabla u|\right)>\lambda\right\} \cap B_{1},
\end{aligned}
$$

for $\varepsilon \in(0,1)$ and $\lambda>0$.
We verify that

$$
\begin{equation*}
\left|E_{\lambda, \varepsilon}^{1}\right| \leq c_{1} \varepsilon\left|\tilde{Q}_{R_{3}}\right| \quad \forall \lambda>0, \varepsilon \in(0,1) \text { and } \tag{8.4}
\end{equation*}
$$

$$
\begin{equation*}
\left|E_{\lambda, \varepsilon}^{2}\right| \leq c_{2} \varepsilon\left|\tilde{Q}_{R_{2}}\right| \forall \lambda>\varepsilon^{-1+\frac{1}{\theta}}\|\nabla u\|_{L^{1}\left(\Omega_{T} \cap A\right)} R_{2}^{-N-2}, \varepsilon \in(0,1) \tag{8.5}
\end{equation*}
$$

for some $c_{1}=c_{1}\left(T_{0} / r_{0}\right), c_{2}>0$ and $R_{3}=\inf \left\{r_{0}, T_{0}\right\} / 16$.
In fact, we can assume that $E_{\lambda, \varepsilon}^{1} \neq \emptyset$ so $\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right) \leq T_{0}^{N+1} \varepsilon^{1-\frac{1}{\theta}} \lambda$. We have

$$
\left|E_{\lambda, \varepsilon}^{1}\right| \leq \frac{c_{3}}{\varepsilon^{-1 / \theta} \lambda} \int_{\Omega_{T}}|\nabla u| d x d t
$$

By Remark 3.2, $\int_{\Omega_{T}}\left|\nabla u_{n}\right| d x d t \leq c_{4} T_{0}\left(\left|\mu_{n}\right|\left(\Omega_{T}\right)+\left|\sigma_{n}\right|(\Omega)\right)$ for all $n$. Letting $n \rightarrow \infty$ we get $\int_{\Omega_{T}}|\nabla u| d x d t \leq c_{4} T_{0}\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right)$. Thus,

$$
\left|E_{\lambda, \varepsilon}^{1}\right| \leq \frac{c_{3} c_{4}}{\varepsilon^{-1 / \theta} \lambda} T_{0}\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right) \leq \frac{c_{3} c_{4}}{\varepsilon^{-1 / \theta} \lambda} T_{0}^{N+2} \varepsilon^{1-\frac{1}{\theta}} \lambda=c_{5} \varepsilon\left|\tilde{Q}_{R_{3}}\right|
$$

Hence, (8.4) holds with $c_{1}=c_{5}\left(T_{0} / r_{0}\right)$.
For any $\lambda>\varepsilon^{-1+\frac{1}{\theta}}\|\nabla u\|_{L^{1}\left(\Omega_{T} \cap B_{2}\right)} R_{2}^{-N-2}$ we have

$$
\left|E_{\lambda, \varepsilon}^{2}\right| \leq \frac{c_{3}}{\varepsilon^{-1 / \theta} \lambda} \int_{\Omega_{T}} \chi_{B_{2}}|\nabla u| d x d t<c_{2} \varepsilon\left|\tilde{Q}_{R_{2}}\right|
$$

Hence, (8.5) holds.
Next we verify that for all $(x, t) \in Q$ and $r \in\left(0, R_{3}\right]$ and $\lambda>0, \varepsilon \in(0,1)$ we have $\tilde{Q}_{r}(x, t) \cap$ $Q \subset F_{\lambda}^{1}$ if $\left|E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)\right| \geq c_{6} \varepsilon\left|\tilde{Q}_{r}(x, t)\right|$ where the constant $c_{6}$ does not depend on $\lambda$ and $\varepsilon$. Indeed, take $(x, t) \in Q$ and $0<r \leq R_{3}$. Now assume that $\tilde{Q}_{r}(x, t) \cap Q \cap\left(F_{\lambda}^{1}\right)^{c} \neq \emptyset$ and $E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t) \neq \emptyset$ i.e, there exist $\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right) \in \tilde{Q}_{r}(x, t) \cap Q$ such that $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq$ $\lambda$ and $\mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \varepsilon^{1-\frac{1}{\theta}} \lambda$. We need to prove that

$$
\begin{equation*}
\left.\mid E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)\right)\left|<c_{6} \varepsilon\right| \tilde{Q}_{r}(x, t) \mid \tag{8.6}
\end{equation*}
$$

Obviously, we have for all $(y, s) \in \tilde{Q}_{r}(x, t)$ there holds

$$
\mathbb{M}(|\nabla u|)(y, s) \leq \max \left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}|\nabla u|\right)(y, s), 3^{N+2} \lambda\right\}
$$

Leads to, for all $\lambda>0$ and $\varepsilon \in\left(0, \varepsilon_{0}\right)$ with $\varepsilon_{0} \leq 3^{-(N+2) \theta}$,

$$
\begin{equation*}
E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)=\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}|\nabla u|\right)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}[\omega] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap Q \cap \tilde{Q}_{r}(x, t) \tag{8.7}
\end{equation*}
$$

In particular, $E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)=\emptyset$ if $\bar{B}_{4 r}(x) \subset \subset \mathbb{R}^{N} \backslash \Omega$. Thus, it is enough to consider the case $B_{4 r}(x) \subset \subset \Omega$ and $B_{4 r}(x) \cap \Omega \neq \emptyset$.
We consider the case $B_{4 r}(x) \subset \subset \Omega$. Let $w_{n, m}$ be as in Theorem 7.1 with $Q_{2 R}=Q_{4 r}\left(x, t_{0}\right)$ and $u=u_{n, m}$ where $t_{0}=\min \left\{t+2 r^{2}, T\right\}$. We have

$$
\begin{align*}
& f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla w_{n, m}\right| d x d t \leq c_{7} \frac{\left|\mu_{n, m}\right|\left(Q_{4 r}\left(x, t_{0}\right)\right)}{r^{N+1}} \text { and }  \tag{8.8}\\
& f_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla w_{n, m}\right|^{\theta} d x d t \leq c_{8}\left(f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla w_{n, m}\right| d x d t\right)^{\theta} \tag{8.9}
\end{align*}
$$

From (8.7), we have

$$
\begin{aligned}
\mid E_{\lambda, \varepsilon}^{1} \cap & \tilde{Q}_{r}(x, t)\left|\leq\left|\left\{\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla w_{n, m}\right|\right)>\varepsilon^{-1 / \theta} \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right\}\right|\right. \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla w_{n, m}\right|\right)>\varepsilon^{-1 / \theta} \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla u_{n}\right|\right)>\varepsilon^{-1 / \theta} \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u\right|\right)>\varepsilon^{-1 / \theta} \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
\leq & c_{9} \varepsilon \lambda^{-\theta} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla w_{n, m}\right|^{\theta} d x d t+c_{9} \varepsilon^{1 / \theta} \lambda^{-1} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla w_{n, m}\right| d x d t \\
& +c_{9} \varepsilon^{1 / \theta} \lambda^{-1} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla u_{n}\right| d x d t+c_{9} \varepsilon^{1 / \theta} \lambda^{-1} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u\right| d x d t .
\end{aligned}
$$

Thanks to (8.8) and (8.9) we can continue

$$
\begin{aligned}
& \left|E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{10} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}\right| d x d t\right)^{\theta} \\
& \quad+c_{10} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(\frac{\left|\mu_{n, m}\right|\left(Q_{4 r}\left(x, t_{0}\right)\right)}{r^{N+1}}\right)^{\theta}+c_{10} \varepsilon^{1 / \theta} \lambda^{-1}\left|\tilde{Q}_{r}(x, t)\right| \frac{\left|\mu_{n, m}\right|\left(Q_{4 r}\left(x, t_{0}\right)\right)}{r^{N+1}} \\
& \quad+c_{10} \varepsilon^{1 / \theta} \lambda^{-1} \int_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla u_{n}\right| d x d t+c_{10} \varepsilon^{1 / \theta} \lambda^{-1} \int_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla u_{n}-\nabla u\right| d x d t .
\end{aligned}
$$

Letting $m \rightarrow \infty$ and $n \rightarrow \infty$, we get

$$
\begin{aligned}
& \left|E_{\lambda, \varepsilon} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{10} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(f_{Q_{4 r}\left(x, t_{0}\right)}|\nabla u| d x d t\right)^{\theta} \\
& \quad+c_{10} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(\frac{\omega\left(\overline{Q_{4 r}\left(x, t_{0}\right)}\right)}{r^{N+1}}\right)^{\theta}+c_{10} \varepsilon^{1 / \theta} \lambda^{-1}\left|\tilde{Q}_{r}(x, t)\right| \frac{\omega\left(\overline{Q_{4 r}\left(x, t_{0}\right)}\right)}{r^{N+1}}
\end{aligned}
$$

Since, $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq \lambda$ and $\mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \varepsilon^{1-\frac{1}{\theta}} \lambda$ we have

$$
\int_{Q_{4 r}\left(x, t_{0}\right)}|\nabla u| d x d t \leq \int_{\tilde{Q}_{8 r}(x, t)}|\nabla u| d x d t \leq \int_{\tilde{Q}_{9 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t \leq\left|\tilde{Q}_{9 r}\left(x_{1}, t_{1}\right)\right| \lambda
$$

and

$$
\omega\left(\overline{Q_{4 r}\left(x, t_{0}\right)}\right) \leq \omega\left(\tilde{Q}_{8 r}(x, t)\right) \leq \omega\left(\tilde{Q}_{9 r}\left(x_{2}, t_{2}\right) \leq \varepsilon^{1-\frac{1}{\theta}} \lambda(9 r)^{N+1}\right.
$$

Thus

$$
\left|E_{\lambda, \varepsilon} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{11} \varepsilon\left|\tilde{Q}_{r}(x, t)\right|
$$

Next, we consider the case $B_{4 r}(x) \cap \Omega \neq \emptyset$. Let $x_{3} \in \partial \Omega$ such that $\left|x_{3}-x\right|=\operatorname{dist}(x, \partial \Omega)$. Let $w_{n}$ be as in Theorem 7.5 with $\tilde{\Omega}_{6 R}=\tilde{\Omega}_{16 r}\left(x_{3}, t_{0}\right)$ and $u=u_{n, m}$ where $t_{0}=\min \left\{t+2 r^{2}, T\right\}$. We have $Q_{12 r}\left(x, t_{0}\right) \subset Q_{16 r}\left(x_{3}, t_{0}\right)$,

$$
\begin{aligned}
& f_{Q_{12 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla w_{n, m}\right| d x d t \leq c_{12} \frac{\left|\mu_{n, m}\right|\left(\tilde{\Omega}_{16 r}\left(x_{3}, t_{0}\right)\right)}{r^{N+1}} \text { and } \\
& \left(f_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla w_{n, m}\right|^{\theta} d x d t\right)^{\frac{1}{\theta}} \leq c_{13} f_{Q_{12 r}\left(x, t_{0}\right)}\left|\nabla w_{n, m}\right| d x d t
\end{aligned}
$$

As above we also obtain

$$
\begin{aligned}
& \left|E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{14} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(f_{Q_{12 r}\left(x, t_{0}\right)}|\nabla u| d x d t\right)^{\theta} \\
& \quad+c_{14} \varepsilon \lambda^{-\theta}\left|\tilde{Q}_{r}(x, t)\right|\left(\frac{\omega\left(\overline{Q_{16 r}\left(x_{3}, t_{0}\right)}\right)}{r^{N+1}}\right)^{\theta}+c_{14} \varepsilon^{1 / \theta} \lambda^{-1}\left|\tilde{Q}_{r}(x, t)\right| \frac{\omega\left(\overline{Q_{16 r}\left(x_{3}, t_{0}\right)}\right)}{r^{N+1}}
\end{aligned}
$$

Since, $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq \lambda$ and $\mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \varepsilon^{1-\frac{1}{\theta}} \lambda$ we have

$$
\int_{Q_{12 r}\left(x, t_{0}\right)}|\nabla u| d x d t \leq \int_{\tilde{Q}_{24 r}(x, t)}|\nabla u| d x d t \leq \int_{\tilde{Q}_{25 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t \leq\left|\tilde{Q}_{25 r}\left(x_{1}, t_{1}\right)\right| \lambda
$$

and

$$
\omega\left(\overline{Q_{16 r}\left(x_{3}, t_{0}\right)}\right) \leq \omega\left(\tilde{Q}_{32 r}\left(x_{3}, t\right)\right) \leq \omega\left(\tilde{Q}_{36 r}(x, t)\right) \leq \omega\left(\tilde{Q}_{37 r}\left(x_{2}, t_{2}\right)\right) \leq \varepsilon^{1-\frac{1}{\theta}} \lambda(37 r)^{N+1}
$$

Thus

$$
\left|E_{\lambda, \varepsilon}^{1} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{15} \varepsilon\left|\tilde{Q}_{r}(x, t)\right|
$$

Hence, (8.6) holds with $c_{6}=2 \max \left\{c_{11}, c_{15}\right\}$.
Similarly, we also prove that for all $(x, t) \in B_{\tilde{1}}$ and $r \in\left(0, R_{2}\right]$ and $\lambda>0, \varepsilon \in(0,1)$ we have $\tilde{Q}_{r}(x, t) \cap B_{1} \subset F_{\lambda}^{2}$ if $\left|E_{\lambda, \varepsilon}^{2} \cap \tilde{Q}_{r}(x, t)\right| \geq c_{16} \varepsilon\left|\tilde{Q}_{r}(x, t)\right|$ where a constant $c_{26}$ does not depend on $\lambda$ and $\varepsilon$. Now, choose $\varepsilon_{1}=\left(2 \max \left\{1, c_{1}, c_{6}\right\}\right)^{-1}$ and $\varepsilon_{2}=\left(2 \max \left\{1, c_{2}, c_{16}\right\}^{-1}\right.$. We apply Lemma 3.21 with $E=E_{\lambda, \varepsilon}^{1}, F=F_{\lambda}^{1}$ and $\varepsilon$ is replaced by $\max \left\{c_{1}, c_{6}\right\} \varepsilon$ for any $0<\varepsilon<\varepsilon_{1}$ and $\lambda>0$ we get (8.1), for $E=E_{\lambda, \varepsilon}^{2}, F=F_{\lambda}^{2}$ and $\varepsilon$ is replaced by $\max \left\{c_{1}, c_{17}\right\} \varepsilon$ for any $0<\varepsilon<\varepsilon_{2}$ and $\lambda>\varepsilon^{-1+\frac{1}{\theta}}\|\nabla u\|_{L^{1}\left(\Omega_{T} \cap B_{2}\right)} R_{2}^{-N-2}$ we get (8.2).
This completes the proof of the Theorem.
Proof of Theorem 2.17. By theorem 8.1, there exist constants $c_{1}>0,0<\varepsilon_{0}<1$ and a renormalized solution $u$ of equation (2.4) with data $\mu, u_{0}=\sigma$ such that for any $\varepsilon \in(0,1)$, $\lambda>0$

$$
\left|\left\{\mathbb{M}(|\nabla u|)>\varepsilon^{-1 / \theta} \lambda, \mathbb{M}_{1}[\omega] \leq \varepsilon^{1-\frac{1}{\theta}} \lambda\right\} \cap Q\right| \leq c_{1} \varepsilon|\{\mathbb{M}(|\nabla u|)>\lambda\} \cap Q|
$$

Therefore, if $0<s<\infty$

$$
\begin{aligned}
\|\mathbb{M}(|\nabla u|)\|_{L^{p, s}(Q)}^{s} & =\varepsilon^{-s / \theta} p \int_{0}^{\infty} \lambda^{s}\left|\left\{(x, t) \in Q: \mathbb{M}(|\nabla u|)>\varepsilon^{-1 / \theta} \lambda\right\}\right|^{\frac{s}{p}} \frac{d \lambda}{\lambda} \\
& \leq c_{1}^{s / p} \varepsilon^{\frac{s(\theta-p)}{\theta_{p}}} p \int_{0}^{\infty} \lambda^{s}|\{(x, t) \in Q: \mathbb{M}(|\nabla u|)>\lambda\}|^{\frac{s}{p}} \frac{d \lambda}{\lambda} \\
& +\varepsilon^{-s / \theta} p \int_{0}^{\infty} \lambda^{s}\left|\left\{(x, t) \in Q: \mathbb{M}_{1}[\omega]>\varepsilon^{1-\frac{1}{\theta}} \lambda\right\}\right|^{\frac{s}{p}} \frac{d \lambda}{\lambda} \\
& =c_{1}^{s / p} \varepsilon^{\frac{s(\theta-p)}{\theta_{p}}}\|\mathbb{M}(|\nabla u|)\|_{L^{p, s}(Q)}^{s}+\varepsilon^{-s}| | \mathbb{M}_{1}[\omega] \|_{L^{p, s}(Q)}^{s}
\end{aligned}
$$

Since $p<\theta$, we can choose $0<\varepsilon<\varepsilon_{0}$ such that $c_{1}^{s / p} \varepsilon^{\frac{s(\theta-p)}{\theta p}} \leq 1 / 2$ we get the result for case $0<s<\infty$. Similarly, we also get the result for case $s=\infty$.
Also, we get (2.29) by using (4.16) in Proposition 4.8, (4.28) in Proposition 4.19. This completes the proof.
Remark 8.2 Thanks to Proposition 4.4 we have for any $s \in\left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ if $\mu \in L^{\frac{(s-1)(N+2)}{s}, \infty}\left(\Omega_{T}\right)$ and $\sigma \equiv 0$ then

$$
\left\||\nabla u|^{s}\right\|_{L} \frac{(s-1)(N+2)}{s}, \infty_{\left(\Omega_{T}\right)} \leq c_{2}\|\mu\|_{L}^{s}{ }_{L}^{s-1)(N+2)}, \infty_{\left(\Omega_{T}\right)},
$$

where constant $c_{2}$ depends on $N, \Lambda_{1}, \Lambda_{2}, s, c_{0}, T_{0} / r_{0}$.
As the proof of Theorem 8.1, we also get

Theorem 8.3 Suppose that $\mathbb{R}^{N} \backslash \Omega$ satisfies uniformly 2 -thick with constants $c_{0}, r_{0}$. Let $\theta$ be as in Theorem 8.1. Let $1 \leq p<\theta, 0<s \leq \infty$ and $\mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. There exist $C_{1}=C_{1}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, c_{0}\right)>0$ and a distribution solution $u$ of equation (2.4) with data $\mu$ and $u_{0}=\sigma$ such that

$$
\begin{align*}
\left\|\mathbb{M}\left(\chi_{\tilde{Q}_{4 R}\left(y_{0}, s_{0}\right)}|\nabla u|\right)\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)} \leq & C_{1} R^{\frac{N+2}{p}} \inf \left\{r_{0}, R\right\}^{-N-2}\|\nabla u\|_{L^{1}\left(\tilde{Q}_{4 R}\left(y_{0}, s_{0}\right)\right)} \\
& +C_{1}\left\|\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{4 R}\left(y_{0}, s_{0}\right)} \omega\right]\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)} \tag{8.10}
\end{align*}
$$

for any $\tilde{Q}_{R}\left(y_{0}, s_{0}\right) \subset \mathbb{R}^{N+1}$ and if $\sigma \in L^{1}(\Omega)$ then $u$ is a renormalized solution.
Proof of Theorem 2.19. Let $\left\{u_{n, m}\right\}$ and $\mu_{n, m}$ be in the proof of Theorem 8.1. From Corollary 7.2 and 7.6 we assert: for $2-\inf \left\{\beta_{1}, \beta_{2}\right\}<\gamma<N+2$, there exists a constant $C=C\left(N, \Lambda_{1}, \Lambda_{2}, c_{0}, \gamma\right)>0$ such that for any $0<\rho \leq T_{0}$

$$
\int_{Q_{\rho}(y, s)}\left|\nabla u_{n, m}\right| d x d t \leq C\left(N, \Lambda_{1}, \Lambda_{2}, \gamma, c_{0}, T_{0} / r_{0}\right) \rho^{N+3-\gamma}| | \mathbb{M}_{\gamma}\left[\left|\mu_{n, m}\right|\right]| |_{L^{\infty}(\Omega \times(-T, T))}
$$

where $\beta_{1}, \beta_{2}$ are constants in Theorem 7.1 and Theorem 7.5. It is easy to see that

$$
\left\|\mathbb{M}_{\gamma}\left[\left|\mu_{n, m}\right|\right]\right\|_{L^{\infty}(\Omega \times(-T, T))} \leq\left\|\mathbb{M}_{\gamma}[\omega]\right\|_{L^{\infty}(\Omega \times(-T, T))}=\left\|\mathbb{M}_{\gamma}[\omega]\right\|_{L^{\infty}\left(\Omega_{T}\right)}
$$

for any $n, m$ large enough.
Letting $m \rightarrow \infty, n \rightarrow \infty$, yield

$$
\int_{Q_{\rho}(y, s)}|\nabla u| d x d t \leq\left. C\left(N, \Lambda_{1}, \Lambda_{2}, \gamma, c_{0}, T_{0} / r_{0}\right) \rho^{N+3-\gamma}| | \mathbb{M}_{\gamma}[\omega]\right|_{L^{\infty}\left(\Omega_{T}\right)}
$$

By Theorem 8.3 we get

$$
\begin{aligned}
\|\|\nabla\|\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right) \cap \Omega_{T}\right)} \leq & c_{1}\left(T_{0} / r_{0}\right) R^{\frac{N+2}{p}+1-\gamma}\left\|\mathbb{M}_{\gamma}[\omega]\right\|_{L^{\infty}\left(\Omega_{T}\right)} \\
& +c_{2}\left\|\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)} \omega\right]\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)}
\end{aligned}
$$

for any $\tilde{Q}_{R}\left(y_{0}, s_{0}\right) \subset \mathbb{R}^{N+1}$ and $0<R \leq T_{0}$. It follows (2.30).
Finally, if $\mu \in L_{*}^{\frac{(\gamma-1) p}{\gamma}}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\Omega_{T}\right)$ and $\sigma \equiv 0$, then clearly $u$ is a unique renormalized solution. It suffices to show that

$$
\begin{gather*}
\left\|\mathbb{M}_{\gamma}[|\mu|]\right\|_{L^{\infty}\left(\Omega_{T}\right)} \leq c_{3}\|\mu\|_{L_{*}}^{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p}\left(\Omega_{T}\right)
\end{gather*} \text { and } \quad \begin{aligned}
& R^{\frac{p(\gamma-1)-N-2}{p}}\left\|\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y, s_{0}\right)}|\mu|\right]\right\|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)} \leq c_{3}\|\mu\|_{L_{*}^{*}}^{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p}{ }_{\left(\Omega_{T}\right)} \tag{8.11}
\end{aligned}
$$

for any $\tilde{Q}_{R}\left(y_{0}, s_{0}\right) \subset \mathbb{R}^{N+1}$ and $0<R \leq T_{0}$, where $c_{3}=c_{3}\left(N, \Lambda_{1}, \Lambda_{2}, p, s, \gamma, c_{0}, T_{0} / r_{0}\right)$. In fact, for $0<\rho<T_{0}$ and $(x, t) \in \Omega_{T}$ we have

$$
\begin{aligned}
&\|\mu\|_{L_{*}} \frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p \\
&\left(\Omega_{T}\right) \geq\|\mu\|_{L_{*}}^{\frac{(\gamma-1) p}{\gamma}, \infty ;(\gamma-1) p}\left(\Omega_{T}\right) \\
& \geq \rho^{\frac{(\gamma-1) p-N-2}{\frac{(\gamma-1) p}{\gamma}}}\|\mu\|_{L^{\frac{(\gamma-1) p}{\gamma}, \infty}\left(\tilde{Q}_{\rho}(x, t) \cap \Omega_{T}\right)} \\
& \geq c_{4} \rho^{\frac{(\gamma-1) p-N-2}{\frac{(\gamma-1) p}{\gamma}}}\left|\tilde{Q}_{\rho}(x, t)\right|^{-1+\frac{\gamma}{(\gamma-1) p}}|\mu|\left(\tilde{Q}_{\rho}(x, t) \cap \Omega_{T}\right) \\
&=c_{5} \frac{|\mu|\left(\tilde{Q}_{\rho}(x, t) \cap \Omega_{T}\right)}{\rho^{N+2-\gamma}},
\end{aligned}
$$

which obviously implies (8.11).
Next, we note that

$$
\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)}|\mu|\right](x, t) \leq c_{6}\left(\mathbb{M}\left(\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)}|\mu|\right)(x, t)\right)^{1-\frac{1}{\gamma}}\|\mu\|_{L_{*}}^{\frac{1}{\gamma}} \frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\Omega_{T}\right)
$$

We derive

$$
\begin{aligned}
& R^{\frac{p(\gamma-1)-N-2}{p}} \|\left.\mathbb{M}_{1}\left[\chi_{\tilde{Q}_{R}\left(y, s_{0}\right)}|\mu|\right]\right|_{L^{p, s}\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)} \\
& \left.\leq c_{6} R^{\frac{p(\gamma-1)-N-2}{p}}\left\|\mathbb{M}\left(\chi_{\tilde{Q}_{R}\left(y_{0}, s_{0}\right)}|\mu|\right)\right\|_{L^{\frac{(\gamma-1) p}{\gamma}}, \frac{(\gamma-1) s}{\gamma}}^{1-\frac{1}{\gamma}} \tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right) \quad\|\mu\|_{L_{*}}^{\frac{1}{\gamma}}{ }^{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p}\left(\Omega_{T}\right) \\
& \leq c_{7} R^{\frac{p(\gamma-1)-N-2}{p}}\| \| \mu\| \|_{L^{\frac{(\gamma-1) p}{\gamma}}}^{1-\frac{1}{\gamma}}, \frac{(\gamma-1) s}{\gamma}{ }_{\left(\tilde{Q}_{R}\left(y_{0}, s_{0}\right)\right)}\|\mu\|_{L_{*}}^{\frac{1}{\gamma}}{ }^{\frac{(\gamma-1) p}{\gamma}}, \frac{(\gamma-1) s}{\gamma} ;(\gamma-1) p\left(\Omega_{T}\right) .
\end{aligned}
$$

Here we used the boundedness property of $\mathbb{M}$ in $L^{\frac{(\gamma-1) p}{\gamma}, \frac{(\gamma-1) s}{\gamma}}\left(\mathbb{R}^{N+1}\right)$ for $\frac{(\gamma-1) p}{\gamma}>1$. Therefore, immediately we get (8.12). This completes the proof of theorem.

### 8.2 Global estimates on Reifenberg flat domains

Now we prove results for Reifenberg flat domain. First, we will use Lemma 7.4, 7.13 and Lemma 3.19 to get the following result.

Theorem 8.4 Suppose that A satisfies (2.27). Let $s_{1}, s_{2}$ be in Lemma 7.3 and 7.7, set $s_{0}=\max \left\{s_{1}, s_{2}\right\}$. Let $w \in A_{\infty}, \mu \in \mathfrak{M}_{b}\left(\Omega_{T}\right), \sigma \in \mathfrak{M}_{b}(\Omega)$, set $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (2.4) with data $\mu$ and $u_{0}=\sigma$ such that following holds. For any $\varepsilon>0, R_{0}>0$ one finds $\delta_{1}=\delta_{1}\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon,[w]_{A_{\infty}}\right) \in(0,1)$ and $\delta_{2}=$ $\delta_{2}\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon,[w]_{A_{\infty}}, T_{0} / R_{0}\right) \in(0,1)$ and $\Lambda=\Lambda\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$ such that if $\Omega$ is $\left(\delta_{1}, R_{0}\right)$ Reifenberg flat domain and $[\mathcal{A}]_{s_{0}}^{R_{0}} \leq \delta_{1}$ then

$$
\begin{equation*}
w\left(\left\{\mathbb{M}(|\nabla u|)>\Lambda \lambda, \mathbb{M}_{1}[\omega] \leq \delta_{2} \lambda\right\} \cap \Omega_{T}\right) \leq B \varepsilon w\left(\{\mathbb{M}(|\nabla u|)>\lambda\} \cap \Omega_{T}\right) \tag{8.13}
\end{equation*}
$$

for all $\lambda>0$, where the constant $B$ depends only on $N, \Lambda_{1}, \Lambda_{2}, T_{0} / R_{0},[w]_{A_{\infty}}$. Furthermore, if $\sigma \in L^{1}(\Omega)$ then $u$ is a renormalized solution.

Proof. Let $\left\{\mu_{n}\right\},\left\{\sigma_{n}\right\},\left\{\mu_{n, m}\right\},\left\{u_{n}\right\},\left\{u_{n, m}\right\}, u$ be as in the proof of Theorem 8.1. Let $\varepsilon$ be in $(0,1)$. Set $E_{\lambda, \delta_{2}}=\left\{\mathbb{M}(|\nabla u|)>\Lambda \lambda, \mathbb{M}_{1}[\omega] \leq \delta_{2} \lambda\right\} \cap \Omega_{T}$ and $F_{\lambda}=\{\mathbb{M}(|\nabla u|)>\lambda\} \cap \Omega_{T}$ for $\varepsilon \in(0,1)$ and $\lambda>0$. Let $\left\{y_{i}\right\}_{i=1}^{L} \subset \Omega$ and a ball $B_{0}$ with radius $2 T_{0}$ such that

$$
\Omega \subset \bigcup_{i=1}^{L} B_{r_{0}}\left(y_{i}\right) \subset B_{0}
$$

where $r_{0}=\min \left\{R_{0} / 1080, T_{0}\right\}$. Let $s_{j}=T-j r_{0}^{2} / 2$ for all $j=0,1, \ldots,\left[\frac{2 T}{r_{0}^{2}}\right]$ and $Q_{2 T_{0}}=$ $B_{0} \times\left(T-4 T_{0}^{2}, T\right)$. So,

$$
\Omega_{T} \subset \bigcup_{i, j} Q_{r_{0}}\left(y_{i}, s_{j}\right) \subset Q_{2 T_{0}}
$$

We verify that

$$
\begin{equation*}
w\left(E_{\lambda, \delta_{2}}\right) \leq \varepsilon w\left(\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right) \quad \forall \lambda>0 \tag{8.14}
\end{equation*}
$$

for some $\delta_{2}$ small enough, depended on $n, p, \alpha, \beta, \epsilon,[w]_{A_{\infty}}, T_{0} / R_{0}$.
In fact, we can assume that $E_{\lambda, \delta_{2}} \neq \emptyset$ so $|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega) \leq T_{0}^{N+1} \delta_{2} \lambda$. We have

$$
\left|E_{\lambda, \delta_{2}}\right| \leq \frac{c_{1}}{\Lambda \lambda} \int_{\Omega_{T}}|\nabla u| d x d t
$$

We also have

$$
\int_{\Omega_{T}}|\nabla u| d x d t \leq c_{2} T_{0}\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right)
$$

Thus,

$$
\left|E_{\lambda, \varepsilon}\right| \leq \frac{c_{3}}{\Lambda \lambda} T_{0}\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right) \leq \frac{c_{3}}{\Lambda \lambda} T_{0}^{N+2} \delta_{2} \lambda=c_{4} \delta_{2}\left|Q_{2 T_{0}}\right| .
$$

which implies

$$
w\left(E_{\lambda, \delta_{2}}\right) \leq A\left(\frac{\left|E_{\lambda, \delta_{2}}\right|}{\left|Q_{2 T_{0}}\right|}\right)^{\nu} w\left(Q_{2 T_{0}}\right) \leq A\left(c_{4} \delta_{2}\right)^{\nu} w\left(Q_{2 T_{0}}\right)
$$

where $(A, \nu)$ is a pair of $A_{\infty}$ constants of $w$. It is known that (see, e.g [33]) there exist $A_{1}=A_{1}(N, A, \nu)$ and $\nu_{1}=\nu_{1}(N, A, \nu)$ such that

$$
\frac{w\left(\tilde{Q}_{2 T_{0}}\right)}{w\left(\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right)} \leq A_{1}\left(\frac{\left|\tilde{Q}_{2 T_{0}}\right|}{\left|\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right|}\right)^{\nu_{1}} \forall i, j .
$$

So,

$$
w\left(E_{\lambda, \delta_{2}}\right) \leq A\left(c_{4} \delta_{2}\right)^{\nu} A_{1}\left(\frac{\left|\tilde{Q}_{T_{0}}\right|}{\left|\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right|}\right)^{\nu_{1}} w\left(\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right)<\varepsilon w\left(\tilde{Q}_{r_{0}}\left(y_{i}, s_{j}\right)\right) \forall i, j
$$

where $\delta_{2} \leq\left(\frac{\varepsilon}{2 c_{5}\left(T_{0} r_{0}^{-1}\right)^{(N+2) \nu_{1}}}\right)^{1 / \nu}$. It follows (8.14).
Next we verify that for all $(x, t) \in \Omega_{T}$ and $r \in\left(0,2 r_{0}\right]$ and $\lambda>0$ we have $\tilde{Q}_{r}(x, t) \cap \Omega_{T} \subset F_{\lambda}$ if $w\left(E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right) \geq \varepsilon w\left(Q_{r}(x, t)\right)$ for some $\delta_{2} \leq\left(\frac{\varepsilon}{2 c_{5}\left(T_{0} r_{0}^{-1}\right)^{(N+2) \nu_{1}}}\right)^{1 / \nu}$.
Indeed, take $(x, t) \in \Omega_{T}$ and $0<r \leq 2 r_{0}$. Now assume that $\tilde{Q}_{r}(x, t) \cap \Omega_{T} \cap F_{\lambda}^{c} \neq \emptyset$ and $E_{\lambda, \delta_{2}} \cap$ $\tilde{Q}_{r}(x, t) \neq \emptyset$ i.e, there exist $\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right) \in \tilde{Q}_{r}(x, t) \cap \Omega_{T}$ such that $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq \lambda$ and $\mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \delta_{2} \lambda$. We need to prove that

$$
\begin{equation*}
\left.w\left(E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right)\right)<\varepsilon w\left(\tilde{Q}_{r}(x, t)\right) . \tag{8.15}
\end{equation*}
$$

Clearly,

$$
\mathbb{M}(|\nabla u|)(y, s) \leq \max \left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}|\nabla u|\right)(y, s), 3^{N+2} \lambda\right\} \quad \forall(y, s) \in \tilde{Q}_{r}(x, t)
$$

Therefore, for all $\lambda>0$ and $\Lambda \geq 3^{N+2}$,

$$
\begin{equation*}
E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)=\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}|\nabla u|\right)>\Lambda \lambda, \mathbb{M}_{1}[\omega] \leq \delta_{2} \lambda\right\} \cap \Omega_{T} \cap \tilde{Q}_{r}(x, t) \tag{8.16}
\end{equation*}
$$

In particular, $E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)=\emptyset$ if $\bar{B}_{8 r}(x) \subset \subset \mathbb{R}^{N} \backslash \Omega$. Thus, it is enough to consider the case $B_{8 r}(x) \subset \subset \Omega$ and $B_{8 r}(x) \cap \Omega \neq \emptyset$.
We consider the case $B_{8 r}(x) \subset \subset \Omega$. Let $v_{n, m}$ be as in Lemma 7.4 with $Q_{2 R}=Q_{8 r}\left(x, t_{0}\right)$ and $u=u_{n, m}$ where $t_{0}=\min \left\{t+2 r^{2}, T\right\}$. We have

$$
\begin{equation*}
\left\|\nabla v_{n, m}\right\|_{L^{\infty}\left(Q_{2 r}\left(x, t_{0}\right)\right)} \leq c_{6} f_{Q_{8 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}\right| d x d t+c_{6} \frac{\left|\mu_{n, m}\right|\left(Q_{8 r}\left(x, t_{0}\right)\right)}{r^{N+1}} \tag{8.17}
\end{equation*}
$$

and

$$
\begin{gathered}
f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla v_{n, m}\right| d x d t \leq c_{8} \frac{\left|\mu_{n, m}\right|\left(Q_{8 r}\left(x, t_{0}\right)\right)}{r^{N+1}}+c_{8}[A]_{s_{0}}^{R_{0}}\left(f_{Q_{8 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}\right| d x d t\right. \\
\left.+\frac{\left|\mu_{n, m}\right|\left(Q_{8 r}\left(x, t_{0}\right)\right)}{r^{N+1}}\right)
\end{gathered}
$$

Thanks to $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq \lambda$ and $\mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \delta_{2} \lambda$ with $\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right) \in Q_{r}(x, t)$, we get

$$
\begin{aligned}
\limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty}\left\|\nabla v_{n, m}\right\|_{L^{\infty}\left(Q_{2 r}(x, t)\right)} & \leq c_{9} f_{\tilde{Q}_{17 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t+c_{9} \frac{\omega\left(\tilde{Q}_{17 r}\left(x_{2}, t_{2}\right)\right)}{r^{N+1}} \\
& \leq c_{9} \lambda+c_{9} \delta_{2} \lambda \\
& \leq c_{10} \lambda
\end{aligned}
$$

and

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty} f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla u_{n}-\nabla v_{n}\right| d x d t \\
& \quad \leq c_{11} \frac{\omega\left(\tilde{Q}_{17 r}\left(x_{2}, t_{2}\right)\right)}{r^{N+1}}+c_{11}[A]_{s_{0}}^{R_{0}}\left(f_{\tilde{Q}_{17 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t+\frac{\omega\left(\overline{\tilde{Q}_{17 r}\left(x_{2}, t_{2}\right)}\right)}{r^{N+1}}\right) \\
& \quad \leq c_{11} \delta_{2} \lambda+c_{11}[A]_{s_{0}}^{R_{0}}\left(\lambda+\delta_{2} \lambda\right) \\
& \quad \leq c_{11}\left(\delta_{2}+\delta_{1}\left(1+\delta_{2}\right)\right) \lambda .
\end{aligned}
$$

Here we used $[A]_{s_{0}}^{R_{0}} \leq \delta_{1}$ in the last inequality.
So, we can find $n_{0}$ large enough and a sequence $\left\{k_{n}\right\}$ such that

$$
\begin{align*}
& \left\|\nabla v_{n, m}\right\|_{L^{\infty}\left(\tilde{Q}_{2 r}(x, t)\right)}=\left\|\nabla v_{n, m}\right\|_{L^{\infty}\left(Q_{2 r}\left(x, t_{0}\right)\right)} \leq 2 c_{10} \lambda \text { and }  \tag{8.18}\\
& f_{Q_{4 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla v_{n, m}\right| d x d t \leq 2 c_{11}\left(\delta_{2}+\delta_{1}\left(1+\delta_{2}\right)\right) \lambda \tag{8.19}
\end{align*}
$$

for all $n \geq n_{0}$ and $m \geq k_{n}$.
In view of (8.18) we see that for $\Lambda \geq \max \left\{3^{N+2}, 8 c_{10}\right\}$ and $n \geq n_{0}, m \geq k_{n}$,

$$
\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla v_{n, m}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right|=0
$$

Leads to

$$
\begin{aligned}
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| & \leq\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla v_{n, m}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right|
\end{aligned}
$$

Therefore, by (8.19) and $\tilde{Q}_{2 r}(x, t) \subset Q_{4 r}\left(x, t_{0}\right)$ we obtain for any $n \geq n_{0}$ and $m \geq k_{n}$

$$
\begin{aligned}
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| & \leq \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla v_{n, m}\right| d x d t \\
& +\frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right| d x d t+\frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right| d x d t \\
& \leq c_{13}\left(\delta_{2}+\delta_{1}\left(1+\delta_{2}\right)\right)\left|Q_{r}(x, t)\right| \\
& +\frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right| d x d t+\frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right| d x d t .
\end{aligned}
$$

Letting $m \rightarrow \infty$ and $n \rightarrow \infty$ we get

$$
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{13}\left(\delta_{2}+\delta_{1}\left(1+\delta_{2}\right)\right)\left|\tilde{Q}_{r}(x, t)\right| .
$$

Thus,

$$
\begin{aligned}
w\left(E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right) & \leq C\left(\frac{\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right|}{\left|\tilde{Q}_{r}(x, t)\right|}\right)^{\nu} w\left(\tilde{Q}_{r}(x, t)\right) \\
& \leq C\left(c_{13}\left(\delta_{2}+\delta_{1}\left(1+\delta_{2}\right)\right)\right)^{\nu} w\left(\tilde{Q}_{r}(x, t)\right) \\
& <\varepsilon w\left(\tilde{Q}_{r}(x, t)\right)
\end{aligned}
$$

where $\delta_{2}, \delta_{1}$ are appropriately chosen, $(C, \nu)$ is a pair of $A_{\infty}$ constants of $w$.
Next we consider the case $B_{8 r}(x) \cap \Omega \neq \emptyset$. Let $x_{3} \in \partial \Omega$ such that $\left|x_{3}-x\right|=\operatorname{dist}(x, \partial \Omega)$. Set $t_{0}=\min \left\{t+2 r^{2}, T\right\}$. We have

$$
\begin{equation*}
Q_{2 r}\left(x, t_{0}\right) \subset Q_{10 r}\left(x_{3}, t_{0}\right) \subset Q_{540 r}\left(x_{3}, t_{0}\right) \subset \tilde{Q}_{1080 r}\left(x_{3}, t\right) \subset \tilde{Q}_{1088 r}(x, t) \subset \tilde{Q}_{1089 r}\left(x_{1}, t_{1}\right) \tag{8.20}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{540 r}\left(x_{3}, t_{0}\right) \subset \tilde{Q}_{1080 r}\left(x_{3}, t\right) \subset \tilde{Q}_{1088 r}(x, t) \subset \tilde{Q}_{1089 r}\left(x_{2}, t_{2}\right) \tag{8.21}
\end{equation*}
$$

Let $V_{n, m}$ be as in Lemma 7.13 with $Q_{6 R}=Q_{540 r}\left(x_{3}, t_{0}\right), u=u_{n, m}$ and $\varepsilon=\delta_{3} \in(0,1)$. We have

$$
\left\|\nabla V_{n, m}\right\|_{L^{\infty}\left(Q_{10 r}\left(x_{3}, t_{0}\right)\right)} \leq c_{14} f_{Q_{540 r}\left(x_{3}, t_{0}\right)}\left|\nabla u_{n, m}\right| d x d t+c_{14} \frac{\left|\mu_{n, m}\right|\left(Q_{540 r}\left(x_{3}, t_{0}\right)\right)}{R^{N+1}}
$$

and

$$
\begin{aligned}
& f_{Q_{10 r}\left(x_{3}, t_{0}\right)}\left|\nabla u_{n, m}-\nabla V_{n, m}\right| d x d t \\
& \quad \leq c_{15}\left(\delta_{3}+[A]_{s_{0}}^{R_{0}}\right) f_{Q_{540 r}\left(x_{3}, t_{0}\right)}\left|\nabla u_{n, m}\right| d x d t+c_{15}\left(\delta_{3}+1+[A]_{s_{0}}^{R_{0}}\right) \frac{\left|\mu_{n, m}\right|\left(Q_{540 r}\left(x_{3}, t_{0}\right)\right)}{R^{N+1}}
\end{aligned}
$$

Since $\mathbb{M}(|\nabla u|)\left(x_{1}, t_{1}\right) \leq \lambda, \mathbb{M}_{1}[\omega]\left(x_{2}, t_{2}\right) \leq \delta_{2} \lambda$ and (8.20), (8.21) we get

$$
\begin{aligned}
\limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty}\left\|\nabla V_{n, m}\right\|_{L^{\infty}\left(Q_{2 r}\left(x, t_{0}\right)\right)} & \leq \limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty}\left\|\nabla V_{n, m}\right\|_{L^{\infty}\left(Q_{10 r}\left(x_{3}, t_{0}\right)\right)} \\
& \leq c_{14} f_{Q_{540 r}\left(x_{3}, t_{0}\right)}|\nabla u| d x d t+c_{14} \frac{\omega\left(\overline{Q_{540 r}\left(x_{3}, t_{0}\right)}\right)}{R^{N+1}} \\
& \leq c_{15} f_{\tilde{Q}_{1089 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t+c_{15} \frac{\omega\left(\tilde{Q}_{1089 r}\left(x_{2}, t_{2}\right)\right)}{R^{N+1}} \\
& \leq c_{16} \lambda+c_{16} \delta_{2} \lambda \\
& \leq c_{17} \lambda
\end{aligned}
$$

and

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty} \limsup _{m \rightarrow \infty} f_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla V_{n, m}\right| d x d t \\
& \quad \leq c_{18}\left(\delta_{3}+[A]_{s_{0}}^{R_{0}}\right) f_{Q_{540 r}\left(x_{3}, t_{0}\right)}|\nabla u| d x d t+c_{18}\left(\delta_{3}+1+[A]_{s_{0}}^{R_{0}}\right) \frac{\omega\left(\overline{Q_{540 r}\left(x_{3}, t_{0}\right)}\right)}{r^{N+1}} \\
& \quad \leq c_{19}\left(\delta_{3}+[A]_{s_{0}}^{R_{0}}\right) f_{\tilde{Q}_{1089 r}\left(x_{1}, t_{1}\right)}|\nabla u| d x d t+c_{19}\left(\delta_{3}+1+[A]_{s_{0}}^{R_{0}}\right) \frac{\omega\left(\tilde{Q}_{1089}\left(x_{2}, t_{2}\right)\right)}{r^{N+1}} \\
& \quad \leq c_{20}\left(\delta_{3}+[A]_{s_{0}}^{R_{0}}\right) \lambda+c_{21}\left(\delta_{3}+1+[A]_{s_{0}}^{R_{0}}\right) \delta_{2} \lambda \\
& \quad \leq c_{20}\left(\left(\delta_{3}+\delta_{1}\right)+\left(\delta_{3}+1+\delta_{1}\right) \delta_{2}\right) \lambda .
\end{aligned}
$$

Here we used $[A]_{s}^{R_{0}} \leq \delta_{1}$ in the last inequality.
So, we can find $n_{0}$ large enough and a sequence $\left\{k_{n}\right\}$ such that

$$
\begin{equation*}
\left\|\nabla V_{n, m}\right\|_{L^{\infty}\left(\tilde{Q}_{2 r}(x, t)\right)}=\left\|\nabla V_{n, m}\right\|_{L^{\infty}\left(Q_{2 r}\left(x, t_{0}\right)\right)} \leq 2 c_{17} \lambda \quad \text { and } \tag{8.22}
\end{equation*}
$$

$$
\begin{equation*}
f_{Q_{2 r}\left(x, t_{0}\right)}\left|\nabla u_{n, m}-\nabla V_{n, m}\right| d x d t \leq 2 c_{21}\left(\left(\delta_{3}+\delta_{1}\right)+\left(\delta_{3}+1+\delta_{1}\right) \delta_{2}\right) \lambda \tag{8.23}
\end{equation*}
$$

for all $n \geq n_{0}$ and $m \geq k_{n}$.
Now set $\Lambda=\max \left\{3^{N+2}, 8 c_{10}, 8 c_{17}\right\}$. As above we also have for $n \geq n_{0}, m \geq k_{n}$

$$
\begin{aligned}
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| & \leq\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla V_{n, m}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| \\
& +\left|\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right|\right)>\Lambda \lambda / 4\right\} \cap \tilde{Q}_{r}(x, t)\right| .
\end{aligned}
$$

Therefore from (8.23) we obtain

$$
\begin{aligned}
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| & \leq \frac{c_{22}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n, m}-\nabla V_{n, m}\right| d x d t \\
& +\frac{c_{22}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right| d x d t+\frac{c_{22}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right| d x d t \\
& \leq c_{23}\left(\left(\delta_{3}+\delta_{1}\right)+\left(\delta_{3}+1+\delta_{1}\right) \delta_{2}\right)\left|\tilde{Q}_{r}(x, t)\right| \\
& +\frac{c_{22}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u_{n}-\nabla u_{n, m}\right| d x d t+\frac{c_{22}}{\lambda} \int_{\tilde{Q}_{2 r}(x, t)}\left|\nabla u-\nabla u_{n}\right| d x d t .
\end{aligned}
$$

Letting $m \rightarrow \infty$ and $n \rightarrow \infty$ we get

$$
\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right| \leq c_{22}\left(\left(\delta_{3}+\delta_{1}\right)+\left(\delta_{3}+1+\delta_{1}\right) \delta_{2}\right)\left|\tilde{Q}_{r}(x, t)\right| .
$$

Thus

$$
\begin{aligned}
w\left(E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right) & \leq C\left(\frac{\left|E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right|}{\left|\tilde{Q}_{r}(x, t)\right|}\right)^{\nu} w\left(\tilde{Q}_{r}(x, t)\right) \\
& \leq C\left(c_{22}\left(\left(\delta_{3}+\delta_{1}\right)+\left(\delta_{3}+1+\delta_{1}\right) \delta_{2}\right)\right)^{\nu} w\left(\tilde{Q}_{r}(x, t)\right) \\
& <\varepsilon w\left(\tilde{Q}_{r}(x, t)\right)
\end{aligned}
$$

where $\delta_{3}, \delta_{1}, \delta_{2}$ are appropriately chosen, $(C, \nu)$ is a pair of $A_{\infty}$ constants of $w$.
Therefore, for all $(x, t) \in \Omega_{T}$ and $r \in\left(0,2 r_{0}\right]$ and $\lambda>0$ if $w\left(E_{\lambda, \delta_{2}} \cap \tilde{Q}_{r}(x, t)\right) \geq \varepsilon w\left(\tilde{Q}_{r}(x, t)\right)$
then $\tilde{Q}_{r}(x, t) \cap \Omega_{T} \subset F_{\lambda}$ where $\delta_{1}=\delta_{1}\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon,[w]_{A_{\infty}}\right) \in(0,1)$ and $\delta_{2}=\delta_{2}\left(N, \Lambda_{1}, \Lambda_{2}, \varepsilon,[w]_{A_{\infty}}, T_{0} / R_{0}\right) \in$ $(0,1)$. Applying Lemma 3.19 we get the result.

Proof of Theorem 2.20. As in the proof of Theorem 2.17, we can prove (2.32) by using estimate (8.13) in Theorem 8.4. In particular, thanks to Proposition 4.4 for $q>\frac{N+2}{N+1}$, $\mu \in L^{\frac{(N+2)(q-1)}{q}, \infty}\left(\Omega_{T}\right)$ and $\sigma \equiv 0$,

$$
\begin{equation*}
\left\||\nabla u|^{q}\right\|_{L}^{\frac{(N+2)(q-1)}{q}, \infty}\left(\Omega_{T}\right)<c\|\mu\|_{L}^{q}{ }_{L}^{\frac{(N+2)(q-1)}{q}, \infty}\left(\Omega_{T}\right) \tag{8.24}
\end{equation*}
$$

where the constant $c$ depends only on $N, \Lambda_{1}, \Lambda_{2}, q$ and $T_{0} / R_{0}$.
Proof of Theorem 2.22. By Theorem 2.20, there exists a renormalized solution of (2.4) with data $\mu, u(0)=\sigma$ satisfied

$$
\begin{equation*}
\int_{\Omega_{T}}|\nabla u|^{q} d w \leq c_{1} \int_{\Omega_{T}}\left(\mathbb{M}_{1}[\omega]\right)^{q} d w \tag{8.25}
\end{equation*}
$$

for any $w \in A_{\infty}$, where $c_{1}=c_{1}\left(N, \Lambda_{1}, \Lambda_{2}, q, T_{0} / R_{0},[w]_{A_{\infty}}\right)$.
For $0<\delta<1$ we have $\mathbb{M}_{1}[\omega] \leq c_{2} \mathbb{I}_{1}^{2 T_{0}, \delta}[\omega]$ in $\Omega_{T}$. Thus, (8.25) can be rewritten

$$
\begin{equation*}
\int_{\Omega_{T}}|\nabla u|^{q} d w \leq c_{1} c_{2}^{q} \int_{\Omega_{T}}\left(\mathbb{I}_{1}^{2 T_{0}, \delta}[\omega]\right)^{q} d w . \tag{8.26}
\end{equation*}
$$

Thanks to Proposition 4.23 and Corollary 4.39 and 4.38 we obtain the result.
In follow that we usually employ the the Minkowski inequality, for convenience we recall it, for any $0<q_{1} \leq q_{2}<\infty$ there holds

$$
\left(\int_{X}\left(\int_{Y}|f(x, y)|^{q_{1}} d \mu_{2}(y)\right)^{\frac{q_{2}}{q_{1}}} d \mu_{1}(x)\right)^{\frac{1}{q_{2}}} \leq\left(\int_{Y}\left(\int_{X}|f(x, y)|^{q_{2}} d \mu_{1}(x)\right)^{\frac{q_{1}}{q_{2}}} d \mu_{2}(y)\right)^{\frac{1}{q_{1}}}
$$

for any measure function $f$ in $X \times Y$, where $\mu_{1}, \mu_{2}$ are nonnegative measure in $X$ and $Y$ respectively.
Proof of Theorem 2.21. We will consider only the case $s \neq \infty$ and leave the case $s=\infty$ to the readers. Take $\kappa_{1} \in(0, \kappa)$. It is easy to see that for $\left(x_{0}, t_{0}\right) \in \Omega_{T}$ and $0<\rho<\operatorname{diam}(\Omega)+T^{1 / 2}$

$$
w(x, t)=\min \left\{\rho^{-N-2+\kappa-\kappa_{1}}, \max \left\{\left|x-x_{0}\right|, \sqrt{2\left|t-t_{0}\right|}\right\}^{-N-2+\kappa-\kappa_{1}}\right\} \in A_{\infty}
$$

where $[w]_{A_{\infty}}$ is independent of $\left(x_{0}, t_{0}\right)$ and $\rho$. Thus, from (2.32) in Theorem 2.20 we have

$$
\begin{align*}
\|\mathbb{M}(|\nabla u|)\|_{L^{q, s}\left(\tilde{Q}_{\rho}\left(x_{0}, t_{0}\right) \cap \Omega_{T}\right)}^{s} & =\rho^{\frac{\left(N+2-\kappa+\kappa_{1}\right) s}{q}}\|\mathbb{M}(|\nabla u|)\|_{L^{q, s}\left(\tilde{Q}_{\rho}\left(x_{0}, t_{0}\right) \cap \Omega_{T}, d w\right)}^{s} \\
& \leq c_{1} \rho^{\frac{\left(N+2-\kappa+\kappa_{1}\right) s}{q}}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s}\left(\Omega_{T}, d w\right)}^{s} \\
& =q c_{1} \rho^{\frac{\left(N+2-\kappa+\kappa_{1}\right) s}{q}} \int_{0}^{\infty}\left(\lambda^{q} w\left(\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \Omega_{T}\right)\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda} \\
& =q c_{1} \rho^{\frac{\left(N+2-\kappa+\kappa_{1}\right) s}{q}} \int_{0}^{\infty}\left(\lambda^{q} \int_{0}^{\infty}\left|\left\{\mathbb{M}_{1}[\omega]>\lambda, w>\tau\right\} \cap \Omega_{T}\right| d \tau\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda} \\
& =: c_{1} \rho^{\frac{\left(N+2-\kappa+\kappa_{1}\right) s}{q}} A . \tag{8.27}
\end{align*}
$$

Since $w \leq \rho^{-N-2+\kappa-\kappa_{1}}$ and $\left\{\mathbb{M}_{1}[\omega]>\lambda, w>\tau\right\} \subset\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \tilde{Q} \underset{\tau-N-2+\kappa-\kappa_{1}}{ }\left(x_{0}, t_{0}\right)$,

$$
A \leq q \int_{0}^{\infty}\left(\lambda^{q} \int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}}\left|\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \tilde{Q}_{\tau=N-2+\kappa-\kappa_{1}}\left(x_{0}, t_{0}\right) \cap \Omega_{T}\right| d \tau\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda}
$$

We divide to two cases.
Case 1: $0<s \leq q$. We can verify that for any nonincreasing function $F$ in $(0, \infty)$ and $0<a \leq 1$ we have

$$
\left(\int_{0}^{\infty} F(\tau) d \tau\right)^{a} \leq 4 \int_{0}^{\infty}(\tau F(\tau))^{a} \frac{d \tau}{\tau} .
$$

Hence,

$$
\begin{aligned}
A & \leq 4 q \int_{0}^{\infty} \int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}}\left(\lambda^{q} \tau\left|\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \tilde{Q}_{\tau=N-2+\kappa-\kappa_{1}}\left(x_{0}, t_{0}\right) \cap \Omega_{T}\right|\right)^{\frac{s}{q}} \frac{d \tau}{\tau} \frac{d \lambda}{\lambda} \\
& =4 q \int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}} \int_{0}^{\infty}\left(\lambda^{q} \mid\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \tilde{Q}_{\tau-N-2+\kappa-\kappa_{1}}\right. \\
& \left.\left.=4 \int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}} \| x_{0}\right) \cap \Omega_{T} \mid\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda} \tau^{\frac{s}{q}} \frac{d \tau}{\tau} \\
& \left.\leq 4 \int_{0}[\omega] \|_{L^{q, s}(\tilde{Q}}^{s} \frac{1}{-N-2+\kappa-\kappa_{1}}\left(x_{0}, t_{0}\right) \cap \Omega_{T}\right)^{\frac{s}{q}} \frac{d \tau}{\tau} \\
& =c_{2}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s ; \kappa}\left(\Omega_{T}\right)}^{s} \rho^{-\frac{s \kappa_{1}}{q}} .
\end{aligned}
$$

Case 2: $s>q$. Using the Minkowski inequality, yields

$$
\begin{aligned}
A & \leq c_{3}\left(\int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}}\left(\int_{0}^{\infty}\left(\lambda^{q}\left|\left\{\mathbb{M}_{1}[\omega]>\lambda\right\} \cap \tilde{Q}_{\tau-N-2+\kappa-\kappa_{1}}\left(x_{0}, t_{0}\right) \cap \Omega_{T}\right|\right)^{\frac{s}{q}} \frac{d \lambda}{\lambda}\right)^{\frac{q}{s}} d \tau\right)^{\frac{s}{q}} \\
& \left.\leq c_{4}\left(\int_{0}^{\rho^{-N-2+\kappa-\kappa_{1}}}\left(\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s ; k}\left(\Omega_{T}\right)}^{s}\right)^{\frac{(N+2-\kappa) s}{\left(-N-2+\kappa-\kappa_{1}\right) q}}\right)^{\frac{q}{s}} d \tau\right)^{\frac{s}{q}} \\
& =c_{5}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s, k}\left(\Omega_{T}\right)}^{s} \rho^{-\frac{s \kappa_{1}}{q}} .
\end{aligned}
$$

Therefore, we always have

$$
A \leq c_{6}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q, s ; \kappa}\left(\Omega_{T}\right)}^{s} \rho^{-\frac{s \kappa_{1}}{q}}
$$

which implies (2.33) from (8.27).
Similarly, we obtain estimate (2.46) by adapting

$$
w(x, t)=\min \left\{\rho^{-N+\vartheta-\vartheta_{1}},\left|x-x_{0}\right|^{-N+\vartheta-\vartheta_{1}}\right\} \in A_{\infty}
$$

in above argument, where $0<\vartheta_{1}<\vartheta, x_{0} \in \Omega$ and $0<\rho<\operatorname{diam}(\Omega)$ and $[w]_{A_{\infty}}$ is independent of $x_{0}$ and $\rho$.
Next, to archive (2.35) we need to show that for any ball $B_{\rho} \subset \mathbb{R}^{N}$

$$
\begin{equation*}
\left(\int_{0}^{T}\left|\operatorname{osc}_{B_{\rho} \cap \bar{\Omega}} u(t)\right|^{q} d t\right)^{\frac{1}{q}} \leq c_{7} \rho^{1-\frac{\vartheta}{q}}\||\nabla u|\|_{L_{* *}^{q ; \vartheta}\left(\Omega_{T}\right)} \tag{8.28}
\end{equation*}
$$

Since the extension of $u$ over $\left(\Omega_{T}\right)^{c}$ is zero and $u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ thus we have for a.e $t \in(0, T), u(., t) \in W^{1,1}\left(\mathbb{R}^{N}\right)$. Applying [32, Lemma 7.16] to a ball $B_{\rho} \subset \mathbb{R}^{N}$, we get for a.e $t \in(0, T)$ and $x \in B_{\rho}$

$$
\begin{aligned}
\left|u(x, t)-u_{B \rho}(t)\right| & \leq \frac{2^{N}}{N\left|B_{1}(0)\right|} \int_{B_{\rho}} \frac{|\nabla u(y, t)|}{|x-y|^{N-1}} d y \\
& \leq \frac{2^{N}}{N\left|B_{1}(0)\right|} \int_{B_{2 \rho}(x)} \frac{|\nabla u(y, t)|}{|x-y|^{N-1}} d y \\
& \leq c_{8} \int_{0}^{3 \rho} \frac{\int_{B_{r}(x)}|\nabla u(y, t)| d y}{r^{N-1}} \frac{d r}{r}
\end{aligned}
$$

here $u_{B \rho}(t)$ is the average of $u(., t)$ over $B_{\rho}$, i.e $u_{B \rho}(t)=\frac{1}{\left|B_{\rho}\right|} \int_{B_{\rho}} u(x, t) d x$.
Using the Minkowski and the Holder inequality, we discover that for a.e $x \in B_{\rho}$

$$
\begin{aligned}
\left(\int_{0}^{T}\left|u(x, t)-u_{B \rho}(t)\right|^{q} d t\right)^{\frac{1}{q}} & \leq c_{8}\left(\int_{0}^{T}\left(\int_{0}^{3 \rho} \frac{\int_{B_{r}(x)}|\nabla u(y, t)| d y}{r^{N-1}} \frac{d r}{r}\right)^{q} d t\right)^{\frac{1}{q}} \\
& \leq c_{8} \int_{0}^{3 \rho} \int_{B_{r}(x)}\left(\int_{0}^{T}|\nabla u(y, t)|^{q} d t\right)^{\frac{1}{q}} d y \frac{d r}{r^{N}} \\
& \leq c_{8} \int_{0}^{3 \rho}\left(\int_{B_{r}(x)} \int_{0}^{T}|\nabla u(y, t)|^{q} d t d y\right)^{\frac{1}{q}}\left|B_{r}(x)\right|^{\frac{q-1}{q}} \frac{d r}{r^{N}} \\
& \leq c_{8}\left|B_{1}(x)\right|^{\frac{q-1}{q}} \int_{0}^{3 \rho} r^{\frac{N-\vartheta}{q}} r^{\frac{N(q-1)}{q}} \frac{d r}{r^{N}}\left|\|\nabla u \mid\|_{L_{* *}^{q ; \vartheta}\left(\Omega_{T}\right)}\right. \\
& =c_{9} \rho^{1-\frac{\vartheta}{q}}\||\nabla u|\|_{L_{* *}^{q ; \vartheta}\left(\Omega_{T}\right)}
\end{aligned}
$$

Therefore, we find (8.28) with $c_{7}=2 c_{9}$.
Proof of Proposition 2.28. Clearly, estimate (2.46) is followed by (4.12) in Proposition 4.7. We want to emphasize that almost every estimates in this proof will be used the Minkowski inequality. For a ball $B_{\rho} \subset \mathbb{R}^{N}$, we have for a.e $x \in \mathbb{R}^{N}$

$$
\begin{align*}
\left\|\mathbb{I}_{1}[\mu](x, .)\right\|_{L^{q}(\mathbb{R})} & =\left(\int_{-\infty}^{+\infty}\left(\int_{0}^{\infty} \frac{\mu\left(\tilde{Q}_{r}(x, t)\right)}{r^{N+1}} \frac{d r}{r}\right)^{q} d t\right)^{\frac{1}{q}} \\
& \leq \int_{0}^{\infty}\left(\int_{-\infty}^{+\infty}\left(\mu\left(\tilde{Q}_{r}(x, t)\right)\right)^{q} d t\right)^{\frac{1}{q}} \frac{d r}{r^{N+2}} . \tag{8.29}
\end{align*}
$$

Now, we need to estimate $\left(\int_{-\infty}^{+\infty}\left(\mu\left(\tilde{Q}_{r}(x, t)\right)\right)^{q} d t\right)^{\frac{1}{q}}$.
b. We have

$$
\begin{aligned}
\left(\int_{-\infty}^{+\infty}\left(\mu\left(\tilde{Q}_{r}(x, t)\right)\right)^{q} d t\right)^{\frac{1}{q}} & =\left(\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}^{N+1}} \chi_{\tilde{Q}_{r}(x, t)}\left(x_{1}, t_{1}\right) d \mu\left(x_{1}, t_{1}\right)\right)^{q} d t\right)^{\frac{1}{q}} \\
& \leq \int_{\mathbb{R}^{N+1}}\left(\int_{-\infty}^{+\infty} \chi_{\tilde{Q}_{r}(x, t)}\left(x_{1}, t_{1}\right) d t\right)^{\frac{1}{q}} d \mu\left(x_{1}, t_{1}\right) \\
& =r^{\frac{2}{q}} \mu_{1}\left(B_{r}(x)\right)
\end{aligned}
$$

Combining this with (8.29) we obtain (2.47) and (2.49).
Thus, we also assert (2.49) from [1, Theorem 3.1].
c. Set $d \mu_{2}(x)=\|\mu(x, .)\|_{L^{q_{1}}(\mathbb{R})} d x$. Using Holder's inequality, yields

$$
\mu\left(\tilde{Q}_{r}(x, t)\right) \leq r^{\frac{2\left(q_{1}-1\right)}{q_{1}}} \int_{B_{r}(x)}\left(\int_{t-\frac{\rho^{2}}{2}}^{t+\frac{\rho^{2}}{2}}\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1}\right)^{\frac{1}{q_{1}}} d x_{1}
$$

Leads to

$$
\left(\int_{-\infty}^{+\infty}\left(\mu\left(\tilde{Q}_{r}(x, t)\right)\right)^{q} d t\right)^{\frac{1}{q}} \leq r^{\frac{2\left(q_{1}-1\right)}{q_{1}}} \int_{B_{r}(x)}\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^{2}}{2}}^{t+\frac{\rho^{2}}{2}}\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1}\right)^{\frac{q}{q_{1}}} d t\right)^{\frac{1}{q}} d x_{1}
$$

Note that

$$
\begin{aligned}
& \left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^{2}}{2}}^{t+\frac{\rho^{2}}{2}}\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1}\right)^{\frac{q}{q_{1}}} d t\right)^{\frac{q_{1}}{q}} \\
& \quad=\left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \chi_{\left(t-\frac{\rho^{2}}{2}, t+\frac{\rho^{2}}{2}\right)}\left(t_{1}\right)\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1}\right)^{\frac{q}{q_{1}}} d t\right)^{\frac{q_{1}}{q}} \\
& \quad \leq \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \chi_{\left(t-\frac{\rho^{2}}{2}, t+\frac{\rho^{2}}{2}\right)}\left(t_{1}\right) d t\right)^{\frac{q_{1}}{q}}\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1} \\
& \quad=\rho^{\frac{2 q_{1}}{q}} \int_{-\infty}^{+\infty}\left(w\left(x_{1}, t_{1}\right)\right)^{q_{1}} d t_{1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(\int_{-\infty}^{+\infty}\left(\mu\left(\tilde{Q}_{r}(x, t)\right)\right)^{q} d t\right)^{\frac{1}{q}} & \leq r^{\frac{2\left(q_{1}-1\right)}{q_{1}}+\frac{2}{q}} \int_{B_{r}(x)}\left\|\mu\left(x_{1}, .\right)\right\|_{L^{q_{1}(\mathbb{R})}} d x_{1} \\
& =r^{\frac{2\left(q_{1}-1\right)}{q_{1}}+\frac{2}{q}} \mu_{2}\left(B_{r}(x)\right) .
\end{aligned}
$$

Consequently, since (8.29) we derive (2.50) and (2.51).
We also obtain (2.52) from [1, Theorem 3.1].

### 8.3 Global estimates in $\mathbb{R}^{N} \times(0, \infty)$ and $\mathbb{R}^{N+1}$

Now, we present the proofs of Theorem 2.25 and 2.27.
Proof of Theorem 2.25 and Theorem 2.27. For any $n \geq 1$, it is easy to see that
i. $\mathbb{R}^{N} \backslash B_{n}(0)$ satisfies uniformly $2-$ thick with constants $c_{0}=\frac{\operatorname{Cap}_{p}\left(B_{1 / 4}\left(z_{0}\right), B_{2}(0)\right)}{\operatorname{Cap}_{p}\left(B_{1}(0), B_{2}(0)\right)}, z_{0}=$ $(1 / 2,0, \ldots, 0) \in \mathbb{R}^{N}$ and $r_{0}=n$.
ii. for any $\delta \in(0,1), B_{n}(0)$ is a $(\delta, 2 n \delta)$ - Reifenberg flat domain.
iii. $[\mathcal{A}]_{s_{0}}^{n} \leq[\mathcal{A}]_{s_{0}}^{\infty}$.

Assume that $\| \mathbb{M}_{1}[|\omega|]| |_{L^{p, s}\left(\mathbb{R}^{N+1}\right)}<\infty$. Thus by Remark 2.26 we have

$$
\begin{equation*}
\mathbb{I}_{2}[|\omega|](x, t)<\infty \text { for a.e }(x, t) \in \mathbb{R}^{N+1} \tag{8.30}
\end{equation*}
$$

In view of the proof of the Theorem 2.5 and applying Theorem 2.17 to $B_{n}(0) \times\left(-n^{2}, n^{2}\right)$ and with data $\chi_{B_{n-1}(0) \times\left(-(n-1)^{2},(n-1)^{2}\right)} \omega$ for any $n \geq 2$, there exists a sequence renormalized solution $\left\{u_{n}\right\}$ ( we will take its subsequence if need) of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\chi_{B_{n-1}(0) \times\left(-(n-1)^{2},(n-1)^{2}\right)} \omega \text { in } B_{n}(0) \times\left(-n^{2}, n^{2}\right) \\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right) \\
u_{n}\left(-n^{2}\right)=0 \text { in } B_{n}(0)
\end{array}\right.
$$

converging to a distribution solution $u$ in $L_{\text {loc }}^{1}\left(\mathbb{R} ; W_{\mathrm{loc}}^{1,1}\left(\mathbb{R}^{N}\right)\right)$ of 2.6 with data $\mu=\omega$ such that

$$
\begin{aligned}
\left\|\nabla u_{n} \mid\right\|_{L^{p, s}\left(B_{n}(0) \times\left(-n^{2}, n^{2}\right)\right)} & \leq c_{1}\left\|\mathbb{M}_{1}\left[\chi_{B_{n-1}(0) \times\left(-(n-1)^{2},(n-1)^{2}\right)}|\omega|\right]\right\|_{L^{p, s}\left(B_{2 n}(0) \times\left(-n^{2}, n^{2}\right)\right)} \\
& \leq c_{1}\left\|\mathbb{M}_{1}[|\omega|]\right\|_{L^{p, s}\left(\mathbb{R}^{N+1}\right)} .
\end{aligned}
$$

Here $c_{1}=c_{1}\left(N, \Lambda_{1}, \Lambda_{2}, p, s\right)$ is not depending on $n$ since $\frac{T_{0}}{r_{0}}=\frac{2 n+\left(1+n^{2}\right)^{1 / 2}}{n} \approx 1$.
Using Fatou Lemma, we get estimate (2.38).
As above, we also obtain (2.39).
And similarly, we can prove Theorem 2.27 by this way.
This completes the proof of Theorem.
Remark 8.5 (sharpness) The inequality (2.41) is in a sense optimal as follows:

$$
\begin{equation*}
C^{-1}| | \mathbb{M}_{1}[|\omega|]\left\|_{L^{q}\left(\mathbb{R}^{N+1}\right)} \leq\right\|| | \nabla \mathcal{H}_{2}|*| \omega\left|\left\|_{L^{q}\left(\mathbb{R}^{N} \times(0, \infty)\right)} \leq C| | \mathbb{M}_{1}[|\omega|]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}\right. \tag{8.31}
\end{equation*}
$$

for every $q>1$ where $C=C(N, q)$. Indeed, we have

$$
\nabla \mathcal{H}_{2}(x, t)=-\frac{C_{\alpha}}{2} \frac{\chi_{(0, \infty)}(t)}{t^{(N+1) / 2}} \exp \left(-\frac{|x|^{2}}{4 t}\right) \frac{x}{\sqrt{t}}
$$

leads to

$$
\frac{c_{1}^{-1}}{t^{\frac{N+1}{2}}} \chi_{t>0} \chi_{\frac{1}{2} \sqrt{t} \leq|x| \leq 2 \sqrt{t}} \leq\left|\nabla \mathcal{H}_{\alpha}(x, t)\right| \leq \frac{c_{1}}{\max \{|x|, \sqrt{2|t|}\}^{N+1}}
$$

Immediately, we get

$$
c_{2}^{-1} \int_{0}^{\infty} \frac{\omega\left(\left(B_{r}(x) \backslash B_{r / 2}(x)\right) \times\left(t-r^{2}, t-r^{2} / 4\right)\right)}{r^{N+1}} \frac{d r}{r} \leq\left|\nabla \mathcal{H}_{2}\right| *|\omega|(x, t) \leq c_{2} \mathbb{I}_{1}[\omega](x, t) .
$$

By Theorem 4.2, gives the right-hand side inequality of (8.31). So, it is enough to show that $A:=\int_{\mathbb{R}^{N+1}}\left(\int_{0}^{\infty} \frac{\omega\left(\left(B_{r}(x) \backslash B_{r / 2}(x)\right) \times\left(t-r^{2}, t-r^{2} / 4\right)\right)}{r^{N+1}} \frac{d r}{r}\right)^{q} d x d t \geq c_{3}\left\|\mathbb{M}_{1}[\omega]\right\|_{L^{q}\left(\mathbb{R}^{N+1}\right)}^{q}$

To do this, we take $r_{k}=(3 / 2)^{k}$ for $k \in \mathbb{Z}$,

$$
\begin{aligned}
& \left(\int_{0}^{\infty} \frac{\omega\left(\left(B_{r}(x) \backslash B_{r / 2}(x)\right) \times\left(t-r^{2}, t-r^{2} / 4\right)\right)}{r^{N+1}} \frac{d r}{r}\right)^{q} \\
& \quad \geq c_{4} \sum_{k=-\infty}^{\infty}\left(\frac{\omega\left(\left(B_{r_{k}}(x) \backslash B_{3 r_{k} / 4}(x)\right) \times\left(t-r_{k}^{2}, t-9 r_{k}^{2} / 16\right)\right)}{r_{k}^{N+1}}\right)^{q} .
\end{aligned}
$$

We deduce that

$$
A \geq c_{4} \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^{N+1}}\left(\frac{\omega\left(\left(B_{r_{k}}(x) \backslash B_{3 r_{k} / 4}(x)\right) \times\left(t-r_{k}^{2}, t-9 r_{k}^{2} / 16\right)\right)}{r_{k}^{N+1}}\right)^{q} d x d t
$$

For any $k$, put $y=x+\frac{7}{8} r_{k}$ and $s=t-\frac{25}{32} r_{k}^{2}$, so $B_{r_{k}}(x) \backslash B_{3 r_{k} / 4}(x) \supset B_{r_{k} / 8}(y)$ and

$$
\begin{aligned}
& \int_{\mathbb{R}^{N+1}}\left(\frac{\omega\left(\left(B_{r_{k}}(x) \backslash B_{3 r_{k} / 4}(x)\right) \times\left(t-r_{k}^{2}, t-9 r_{k}^{2} / 16\right)\right)}{r_{k}^{N+1}}\right)^{q} d x d t \\
& \quad \geq \int_{\mathbb{R}^{N+1}}\left(\frac{\omega\left(B_{r_{k} / 8}(y) \times\left(s-7 r_{k}^{2} / 32, t+7 r_{k}^{2} / 32\right)\right)}{r_{k}^{N+1}}\right)^{q} d y d s
\end{aligned}
$$

Consequently,

$$
A \geq c_{4} \int_{\mathbb{R}^{N+1}} \sum_{k=-\infty}^{\infty}\left(\frac{\omega\left(B_{r_{k} / 8}(y) \times\left(s-7 r_{k}^{2} / 32, t+7 r_{k}^{2} / 32\right)\right)}{r_{k}^{N+1}}\right)^{q} d y d s
$$

It follows (8.32).

## 9 Quasilinear Riccati Type Parabolic Equations

### 9.1 Quasilinear Riccati Type Parabolic Equation in $\Omega_{T}$

We provide below only the proof of Theorem 2.30, 2.32 and 2.33. The proof of Theorem 2.31 can be proceeded by a similar argument.

Proof of Theorem 2.30. Let $\left\{\mu_{n}\right\} \subset C_{c}^{\infty}\left(\Omega_{T}\right)$ be as in the proof of Theorem 2.1. We have $\left|\mu_{n}\right|\left(\Omega_{T}\right) \leq|\mu|\left(\Omega_{T}\right)$ for any $n \in \mathbb{N}$. Let $\sigma_{n} \in C_{c}^{\infty}(\Omega)$ be converging to $\sigma$ in the narrow topology of measures and in $L^{1}(\Omega)$ if $\sigma \in L^{1}(\Omega)$ such that $\left\|\sigma_{n}\right\|_{L^{1}(\Omega)} \leq|\sigma|(\Omega)$. For $n_{0} \in \mathbb{N}$, we prove that the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}$ and $\sigma=\sigma_{n_{0}}$. Now we put

$$
\mathbf{E}_{\Lambda}=\left\{u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right):\| \| \nabla \|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)} \leq \Lambda\right\},
$$

where $L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)$ is Lorent space with norm

$$
\|\left. f\right|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)}:=\sup _{0<|D|<\infty}\left(|D|^{-\frac{1}{N+2}} \int_{D \cap \Omega_{T}}|f|\right) .
$$

By Fatou's lemma, $\mathbf{E}_{\Lambda}$ is closed under the strong topology of $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ and convex. We consider a map $S: \mathbf{E}_{\Lambda} \rightarrow \mathbf{E}_{\Lambda}$ defined for each $v \in \mathbf{E}_{\Lambda}$ by $S(v)=u$, where $u \in$ $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ is the unique solution of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=|\nabla v|^{q}+\mu_{n_{0}} \quad \text { in } \Omega_{T},  \tag{9.1}\\
u=0 \text { on } \quad \partial \Omega \times(0, T) \\
u(0)=\sigma_{n_{0}}
\end{array}\right.
$$

By Remark 3.2, we have

$$
\||\nabla u|\|_{L^{N+1}, \Omega_{\left(\Omega_{T}\right)}^{N+1}} \leq c_{1}\left(\left\|\left.| | \nabla v\right|^{q}\right\|_{L^{1}\left(\Omega_{T}\right)}+\left|\mu_{n_{0}}\right|\left(\Omega_{T}\right)+\left\|\sigma_{n_{0}}\right\|_{L^{1}(\Omega)}\right),
$$

for some $c_{1}=c_{1}\left(N, \Lambda_{1}, \Lambda_{2}\right)$. It leads to

$$
\begin{aligned}
\||\nabla u|\|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)} & \leq c_{1}\left(c_{2}\left|\Omega_{T}\right|^{1-\frac{q(N+1)}{N+2}}| | \nabla v\left|\|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)}^{q}+|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right)\right. \\
& \leq c_{1}\left(c_{2}\left|\Omega_{T}\right|^{1-\frac{q(N+1)}{N+2}} \Lambda^{q}+|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right)
\end{aligned}
$$

for some $c_{2}=c_{2}(N, q)>0$. Thus, we now suppose that

$$
\left|\Omega_{T}\right|^{-1+\frac{q^{\prime}}{N+2}}\left(|\mu|\left(\Omega_{T}\right)+|\sigma|(\Omega)\right) \leq\left(2 c_{1}\right)^{-q^{\prime}} c_{2}^{-\frac{1}{q-1}}
$$

then

$$
\||\nabla u|\|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)} \leq \Lambda:=2 c_{1}(|\mu|(\Omega)+|\sigma|(\Omega))
$$

which implies that $S$ is well defined.
Now we show that $S$ is continuous. Let $\left\{v_{n}\right\}$ be a sequence in $\mathbf{E}_{\Lambda}$ such that $v_{n}$ converges strongly in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ to a function $v \in \mathbf{E}_{\Lambda}$. Set $u_{n}=S\left(v_{n}\right)$. We need to show that $u_{n} \rightarrow S(v)$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. We have

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\left|\nabla v_{n}\right|^{q}+\mu_{n_{0}} \text { in } \Omega_{T}  \tag{9.2}\\
u_{n}=0 \text { on } \partial \Omega \times(0, T), \\
u_{n}(0)=\sigma_{n_{0}} \quad \text { in } \Omega
\end{array}\right.
$$

satisfied

$$
\left\|\left|\nabla u_{n}\right|\right\|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)} \leq \Lambda, \quad\left\|\nabla v_{n}\right\| \|_{L^{\frac{N+2}{N+1}, \infty}\left(\Omega_{T}\right)} \leq \Lambda .
$$

Thus, $\left|\nabla v_{n}\right|^{q} \rightarrow|\nabla v|^{q}$ in $L^{1}\left(\Omega_{T}\right)$. Therefore, it is easy to see that we get $u_{n} \rightarrow S(v)$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ by Theorem 3.6.
Next we show that $S$ is pre-compact. Indeed if $\left\{u_{n}\right\}=\left\{S\left(v_{n}\right)\right\}$ is a sequence in $S\left(\mathbf{E}_{\Lambda}\right)$. By Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$ converging to some $u$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. Consequently, by Schauder Fixed Point Theorem, $S$ has a fixed point on $\mathbf{E}_{\Lambda}$ this means: the problem (2.53) has a solution with data $\mu_{n_{0}}, \sigma_{n_{0}}$.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution $u_{n}$ of

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n}\right)\right)=\left|\nabla u_{n}\right|^{q}+\mu_{n} \quad \text { in } \Omega_{T}  \tag{9.3}\\
u=0 \text { on } \partial \Omega \times(0, T) \\
u_{n}(0)=\sigma_{n}
\end{array}\right.
$$

which satisfies

$$
\left\|\left|\nabla u_{n}\right|\right\|_{L^{N+2} N+1}{ }_{\left(\Omega_{T}\right)} \leq 2 c_{1}(|\mu|(\Omega)+|\sigma|(\Omega)) .
$$

Thanks to Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$ converging to $u$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. So, $|||\nabla u|||_{L^{N+2}, \infty\left(\Omega_{T}\right)} \leq 2 c_{1}(|\mu|(\Omega)+|\sigma|(\Omega))$ and $\left|\nabla u_{n}\right|^{q} \rightarrow|\nabla u|^{q}$ in $L^{1}(\Omega)$ since $\left\{\left|\nabla u_{n}\right|^{q}\right\}$ is equi-integrable. It follows the results by Proposition 3.5 and Theorem 3.6.

Proof of Theorem 2.32. Case a. $A$ is linear operator. By Theorem 2.22, there exist $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q\right) \in(0,1)$ and $s_{0}=s_{0}\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$ such that $\Omega$ is $\left(\delta, R_{0}\right)$ - Reifenberg flat domain and $[\mathcal{A}]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ and a sequence $\left\{u_{n}\right\}_{n}$ as distribution solutions of

$$
\left\{\begin{array}{l}
\left(u_{1}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{1}\right)\right)=\mu \text { in } \Omega_{T} \\
u_{1}=0 \text { on } \partial \Omega \times(0, T) \\
u_{1}(0)=\sigma \text { in } \Omega
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\left(u_{n+1}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n+1}\right)\right)=\left|\nabla u_{n}\right|^{q}+\mu \text { in } \Omega_{T}, \\
u_{n+1}=0 \text { on } \partial \Omega \times(0, T), \\
u_{n+1}(0)=\sigma \quad \text { in } \Omega
\end{array}\right.
$$

which satisfy

$$
\begin{equation*}
\left[\left|\nabla u_{n+1}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq c_{1}\left[\left|\nabla u_{n}\right|^{q}+\omega\right]_{\mathfrak{M}^{G_{1}, q^{\prime}}}^{q} \quad \forall n \geq 0 \tag{9.4}
\end{equation*}
$$

where $u_{0} \equiv 0$ and constant $c_{1}$ depends only on $N, \Lambda_{1}, \Lambda_{2}, q$ and $T_{0} / R_{0}, T_{0}$. Moreover, if $\sigma \in L^{1}(\Omega)$ then $\left\{u_{n}\right\}$ is the sequence of renormalized solutions.
i. Suppose

$$
\begin{equation*}
[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq(q-1)^{\frac{1}{q}}\left(q c_{1} 2^{q-1}\right)^{-\frac{1}{q-1}}, \tag{9.5}
\end{equation*}
$$

we prove that

$$
\begin{equation*}
\left[\left|\nabla u_{n}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq \frac{q c_{1} 2^{q-1}}{q-1}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q} \quad \forall n \geq 1 \tag{9.6}
\end{equation*}
$$

Indeed, clearly, we have (9.6) when $n=1$. Now assume that (9.6) is true with $n=m$, that is,

$$
\left[\left|\nabla u_{m}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq \frac{q c_{1} 2^{q-1}}{q-1}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q}
$$

From (9.4) we obtain

$$
\begin{aligned}
{\left[\left|\nabla u_{m+1}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} } & \leq c_{1}\left[\left|\nabla u_{m}\right|^{q}+\omega\right]_{\mathfrak{M}}^{q} \mathcal{G}_{1}, q^{\prime} \\
& \leq c_{1} 2^{q-1}\left(\left[\left|\nabla u_{m}\right|^{q}\right]_{\mathfrak{M}}^{q} \mathcal{G}_{1}, q^{\prime}\right. \\
& \left.+[\omega]_{\mathfrak{M} \mathcal{M}_{1}, q^{\prime}}^{q}\right) \\
& \leq c_{1} 2^{q-1}\left(\left(\frac{q c_{1} 2^{q-1}}{q-1}\right)^{q}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q(q-1)}+1\right)[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q} \\
& \leq \frac{q c_{1} 2^{q-1}}{q-1}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q}
\end{aligned}
$$

Here, the last inequality is obtained by using (9.5). So, (9.6) is also true with $n=m+1$. Thus, (9.6) is true for all $n \geq 1$.
ii. Clearly, $u_{n+1}-u_{n}$ is the unique renormalized solution of

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(x, t, \nabla u))=\left|\nabla u_{n}\right|^{q}-\left|\nabla u_{n-1}\right|^{q} \text { in } \Omega_{T},  \tag{9.7}\\
u=0 \quad \text { on } \partial \Omega \times(0, T), \\
u(0)=0 \quad \text { in } \Omega
\end{array}\right.
$$

So, we have

$$
\left[\left|\nabla u_{n+1}-\nabla u_{n}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq c_{1}\left[\left|\nabla u_{n}\right|^{q}-\left|\nabla u_{n-1}\right|^{q}\right]_{\mathfrak{M}^{\mathfrak{G}_{1}, q^{\prime}}}^{q} \quad \forall n \geq 1
$$

Since, $\left|s_{1}^{q}-s_{2}^{q}\right| \leq q\left|s_{1}-s_{2}\right|\left(\max \left\{s_{1}, s_{2}\right\}\right)^{q-1}$ for any $s_{1}, s_{2} \geq 0$ and using Holder inequality, we get

$$
\begin{aligned}
{\left[\left|\nabla u_{n+1}-\nabla u_{n}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} } & \leq c_{1} q^{q}\left[\left|\nabla u_{n}-\nabla u_{n-1}\right|^{q}\right]_{\mathfrak{M}_{\mathfrak{G}_{1}, q^{\prime}}}\left[\left(\max \left\{\left|\nabla u_{n}\right|,\left|\nabla u_{n-1}\right|\right\}\right)^{q}\right]_{\mathfrak{M} \mathcal{G}_{1}, q^{\prime}}^{q-1} \\
& \leq c_{1} q^{q}\left[\left|\nabla u_{n}-\nabla u_{n-1}\right|^{q}\right]_{\mathfrak{M}_{1}, q^{\prime}}\left(\left[\left|\nabla u_{n}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}+\left[\left|\nabla u_{n-1}\right|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}\right)^{q-1}
\end{aligned}
$$

which follows from (9.6),

$$
\left[\left|\nabla u_{n+1}-\nabla u_{n}\right|^{q}\right]_{\mathfrak{M}^{\mathfrak{c}_{1}}, q^{\prime}} \leq C\left[\left|\nabla u_{n}-\nabla u_{n-1}\right|^{q}\right]_{\mathfrak{M}^{\mathfrak{c}_{1}}, q^{\prime}} \forall n \geq 1
$$

where

$$
C=c_{1} q^{q}\left(\frac{q c_{1} 2^{q}}{q-1}\right)^{q-1}[\omega]_{\mathfrak{M}^{\mathfrak{G}_{1}, q^{\prime}}}^{q(q-1)}
$$

Hence, if $C<1$ then $u_{n}$ converges to $u=u_{1}+\sum_{n=1}^{\infty}\left(u_{n+1}-u_{n}\right)$ in $L^{q}\left(0, T, W_{0}^{1, q}(\Omega)\right)$ and satisfied

$$
\left[|\nabla u|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq \frac{q c_{1} 2^{q-1}}{q-1}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}}^{q}
$$

Note that $C<1$ is equivalent to

$$
[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq\left(c_{1} q^{q}\right)^{-\frac{1}{q(q-1)}}\left(\frac{q c_{1} 2^{q}}{q-1}\right)^{-\frac{1}{q}}
$$

Combining this with (9.5) and using Theorem 3.6, we conclude that the problem (2.53) has a distribution solution $u$ (a renormalized if $\sigma \in L^{1}(\Omega)$ ), if

$$
[\omega]_{\mathfrak{M}^{G_{1}, q^{\prime}}} \leq \min \left\{(q-1)^{\frac{1}{q}}\left(q c_{1} 2^{q-1}\right)^{-\frac{1}{q-1}},\left(c_{1} q^{q}\right)^{-\frac{1}{q(q-1)}}\left(\frac{q c_{1} 2^{q}}{q-1}\right)^{-\frac{1}{q}}\right\}
$$

Next, we will prove Case b. and Case c..
Let $\left\{\mu_{n}\right\} \subset C_{c}^{\infty}\left(\Omega_{T}\right), \sigma_{n} \in C_{c}^{\infty}(\Omega)$ be as in the proof of Theorem 2.1. We have $\left|\mu_{n}\right| \leq$ $\varphi_{n} *|\mu|,\left|\sigma_{n}\right| \leq \varphi_{1, n} *|\sigma|$ for any $n \in \mathbb{N},\left\{\varphi_{n}\right\},\left\{\varphi_{1, n}\right\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^{N}$ respectively. Set $\omega_{n}=\left|\mu_{n}\right|+\left|\sigma_{n}\right| \otimes \delta_{\{t=0\}}$ and $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$.
Case b. For $n_{0} \in \mathbb{N}, \varepsilon>0$, we prove that the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$. Now we put

$$
\mathbf{E}_{\Lambda}=\left\{u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right):\left[|\nabla u|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}\left(\Omega_{T}\right)} \leq \Lambda\right\} .
$$

By Fatou's lemma, $\mathbf{E}_{\Lambda}$ is closed under the strong topology of $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ and convex. We consider a map $S: \mathbf{E}_{\Lambda} \rightarrow \mathbf{E}_{\Lambda}$ defined for each $v \in \mathbf{E}_{\Lambda}$ by $S(v)=u$, where $u \in$ $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ is the unique solution of problem (9.1). By Theorem 2.22, there exist $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q+\varepsilon\right) \in(0,1)$ and $s_{0}=s_{0}\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$ such that $\Omega$ is $\left(\delta, R_{0}\right)$ - Reifenberg flat domain and $[\mathcal{A}]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ we have

$$
\left[|\nabla u|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathfrak{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{2}\left[|\nabla v|^{q}+\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathfrak{G}_{1},(q+\varepsilon)^{\prime}}}^{q+\varepsilon},
$$

where $c_{2}=c_{2}\left(N, \Lambda_{1}, \Lambda_{2}, q+\varepsilon, T_{0} / R_{0}, T_{0}\right)$. By Remark 4.33, we deduce that

$$
\left[|\nabla v|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{3}\left[|\nabla v|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}^{\frac{q}{q+\varepsilon}}
$$

where a constant $c_{3}$ depends on $N, q+\varepsilon$.
Thus,

$$
\begin{aligned}
{\left[|\nabla u|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} } & \leq c_{2}\left(\left[|\nabla v|^{q}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}+\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}\right)^{q+\varepsilon} \\
& \leq c_{2}\left(c_{3}\left[|\nabla u|^{q+\varepsilon}\right]_{\mathfrak{M}^{\underline{G}},(q+\varepsilon)^{\prime}}^{\frac{q}{q+\varepsilon}}+\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}\right)^{q+\varepsilon} \\
& \leq c_{2}\left(c_{3} \Lambda^{\frac{q}{q+\varepsilon}}+\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}\right)^{q+\varepsilon} \\
& \leq \Lambda,
\end{aligned}
$$

provided that $\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{4}:=2^{-q^{\prime}} c_{2}^{-\frac{q^{\prime}}{q+\varepsilon}} c_{3}^{-\frac{1}{q-1}}$ and $\Lambda:=2^{q+\varepsilon} c_{2}\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\prime},(q+\varepsilon)^{\prime}}^{q+\varepsilon}$. which implies that $S$ is well defined with $\left[\omega_{n_{0}}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{4}$.
Now we show that $S$ is continuous. Let $\left\{v_{n}\right\}$ be a sequence in $\mathbf{E}_{\Lambda}$ such that $v_{n}$ converges strongly in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ to a function $v \in \mathbf{E}_{\Lambda}$. Set $u_{n}=S\left(v_{n}\right)$. We need to show that $u_{n} \rightarrow S(v)$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. We have $u_{n}$ satisfied (9.2) and

$$
\left[\left|\nabla u_{n}\right|^{q+\varepsilon}\right]_{\mathfrak{M}_{1},(q+\varepsilon)^{\prime}} \leq \Lambda, \quad\left[\left|\nabla v_{n}\right|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq \Lambda
$$

In particular, $\left\|\nabla v_{n}\right\|_{L^{q+\varepsilon}\left(\Omega_{T}\right)} \leq \Lambda \operatorname{Cap}_{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}\left(\bar{\Omega}_{T}\right)$ for all $n$. Thus, $\left|\nabla v_{n}\right|^{q} \rightarrow|\nabla v|^{q}$ in $L^{1}\left(\Omega_{T}\right)$. Therefore, it is easy to see that we get $u_{n} \rightarrow S(v)$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ by Theorem 3.6. On the other hand, $S$ is pre-compact. Therefore, by Schauder Fixed Point Theorem, $S$ has a fixed point on $\mathbf{E}_{\Lambda}$. Hence the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$. Thanks to Corollary 4.39 and Remark 4.40 we get

$$
\begin{equation*}
\left[\omega_{n}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{5}[\omega]_{\mathfrak{M}_{1},(q+\varepsilon)^{\prime}} \quad \forall n \in \mathbb{N}, \tag{9.8}
\end{equation*}
$$

where $c_{5}=c_{5}\left(N, q+\varepsilon, T_{0}\right)$.
Assume that $[\omega]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq c_{4} c_{5}^{-1}$. So $\left[\omega_{n}\right]_{\mathfrak{M}^{\mathcal{G}_{1}},(q+\varepsilon)^{\prime}} \leq c_{4}$ for all $n$.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution $u_{n}$ of problem (9.3) which satisfies

$$
\left[\left|\nabla u_{n}\right|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}} \leq 2^{q+\varepsilon} c_{2}\left[\omega_{n}\right]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}^{q+\varepsilon} \leq 2^{q+\varepsilon} c_{2} c_{5}^{q+\varepsilon}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}}^{q+\varepsilon} .
$$

By Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$ converging to $u$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. So, $\left[|\nabla u|^{q+\varepsilon}\right]_{\mathfrak{M}^{\mathcal{G}_{1}(q+\varepsilon)^{\prime}}\left(\Omega_{T}\right)} \leq 2^{q+\varepsilon} c_{2} c_{5}^{q+\varepsilon}[\omega]_{\mathfrak{M}^{\mathcal{G}_{1},(q+\varepsilon)^{\prime}}\left(\Omega_{T}\right)}^{q+\varepsilon}$ and $\left|\nabla u_{n}\right|^{q} \rightarrow|\nabla u|^{q}$ in $L^{1}(\Omega)$ since $\left\{\left|\nabla u_{n}\right|^{q}\right\}$ is equi-integrable. It follows the result by Proposition 3.5 and Theorem 3.6.

Case c. For $n_{0} \in \mathbb{N}$. We prove that the problem (2.53) has a solution with data $\mu=$ $\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$. Now we put

$$
\mathbf{E}_{\Lambda}=\left\{u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right):\left|\|\nabla u \mid\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \leq \Lambda\right\}\right.
$$

where $L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)$ is Lorent space with norm

$$
\|f\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}:=\sup _{0<|D|<\infty}\left(|D|^{-1+\frac{1}{(N+2)(q-1)}} \int_{D \cap \Omega_{T}}|f| d x d t\right) .
$$

By Fatou's lemma, $\mathbf{E}_{\Lambda}$ is closed under the strong topology of $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ and convex. We consider a map $S: \mathbf{E}_{\Lambda} \rightarrow \mathbf{E}_{\Lambda}$ defined for each $v \in \mathbf{E}_{\Lambda}$ by $S(v)=u$, where $u \in$ $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ is the unique solution of problem (9.1). By Theorem 2.20, there exist $\delta=\delta\left(N, \Lambda_{1}, \Lambda_{2}, q\right) \in(0,1)$ and $s_{0}=s_{0}\left(N, \Lambda_{1}, \Lambda_{2}\right)>0$ such that $\Omega$ is $\left(\delta, R_{0}\right)$ - Reifenberg flat domain and $[\mathcal{A}]_{s_{0}}^{R_{0}} \leq \delta$ for some $R_{0}$ we have

$$
\begin{aligned}
\|\nabla u \mid\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} & \leq c_{6}\left\|\mathbb{M}_{1}\left[|\nabla v|^{q}+\omega_{n_{0}}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \\
& \leq c_{6}\left(\left\|\left.\mathbb{M}_{1}\left[|\nabla v|^{q}\right]\right|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}+\right\| \mathbb{M}_{1}\left[\omega_{n_{0}}\right] \|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}\right),
\end{aligned}
$$

where $c_{6}=c_{6}\left(N, \Lambda_{1}, \Lambda_{2}, q, T_{0} / R_{0}\right)$ and $T_{0}=\operatorname{diam}(\Omega)+T^{1 / 2}$.
By Proposition 4.4 we have

$$
\begin{aligned}
\left\|\mathbb{M}_{1}\left[|f|^{q}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{n+1}\right)} & \leq c_{7}\left\|\mathbb{I}_{1}\left[|f|^{q}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{n+1}\right)} \\
& \leq c_{8}\|f\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{n+1}\right)}^{q} \forall f \in L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{n+1}\right),
\end{aligned}
$$

where a constant $c_{8}$ only depends on $N, q$. Thus,

$$
\begin{aligned}
\|\nabla u\| \|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} & \leq c_{6}\left(c_{8}\| \| v\| \|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}^{q}+\left\|\mathbb{M}_{1}\left[\omega_{n_{0}}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}\right) \\
& \leq c_{6}\left(c_{8} \Lambda^{q}+\left\|\mathbb{M}_{1}\left[\omega_{n_{0}}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}\right)
\end{aligned}
$$

which implies that $S$ is well defined with $\left\|\mathbb{M}_{1}\left[\omega_{n_{0}}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \leq c_{9}:=\left(2 c_{6}\right)^{-q^{\prime}} c_{8}^{-\frac{1}{q-1}}$ and $\Lambda:=2 c_{6}| | \mathbb{M}_{1}\left[\omega_{n_{0}}\right] \|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)}$.
As in Case b we can show that $S: \mathbf{E}_{\Lambda} \rightarrow \mathbf{E}_{\Lambda}$ is continuous and $S\left(\mathbf{E}_{\Lambda}\right)$ is pre-compact, thus by Schauder Fixed Point Theorem, $S$ has a fixed point on $\mathbf{E}_{\Lambda}$. Hence the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$.
To continue, we need to show that

$$
\begin{align*}
& \left\|\mathbb{M}_{1}\left[\omega_{n}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)} \\
& \quad \leq c_{10}\left\|\mathbb{I}_{1}[|\mu|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+c_{10}\left\|\mathbf{I}_{(N+2)(q-1)}-1[|\sigma|]\right\|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)}, \tag{9.9}
\end{align*}
$$

for every $n \geq k_{0}$. Where $k_{0}$ is a constant large enough and $c_{10}=c_{10}(N, q)$ Indeed, we have $\mathbb{M}_{1}\left[\omega_{n}\right] \leq c_{11} \mathbb{I}_{1}\left[\varphi_{n} *|\mu|\right]+c_{11} \mathbb{I}_{1}\left[\left(\varphi_{1, n} *|\sigma|\right) \otimes \delta_{\{t=0\}}\right]$. Thus, by Proposition 4.19 we deduce

$$
\begin{aligned}
& \left\|\mathbb{M}_{1}\left[\omega_{n}\right]\right\| \|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)} \\
& \quad \leq c_{11}\left\|\left.\mathbb{I}_{1}\left[\varphi_{n} *|\mu|\right]\right|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+c_{12}\right\| \mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}\left[\varphi_{1, n} *|\sigma|\right] \|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)} \\
& \quad=c_{11}\left\|\varphi_{n} * \mathbb{I}_{1}[|\mu|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+c_{12}\left\|\varphi_{1, n} * \mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]\right\|_{L^{(N+2)(q-1)\left(\mathbb{R}^{N}\right)}} \\
& \quad \rightarrow c_{11}\left\|\mathbb{I}_{1}[|\mu|] \mid\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+c_{12}\left\|\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]\right\|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)} \text { as } n \rightarrow \infty .
\end{aligned}
$$

It implies (9.9).
Now we assume that

$$
\left\|\left.\mathbb{I}_{1}[|\mu|]\right|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)},\right\| \mathbf{I}_{\frac{2}{(N+2)(q-1)}}[|\sigma|] \|_{L^{(N+2)(q-1)}\left(\mathbb{R}^{N}\right)} \leq c_{9}\left(2 c_{10}\right)^{-1}
$$

then $\left\|\mathbb{M}_{1}\left[\omega_{n}\right]\right\|_{L^{(N+2)(q-1), \infty\left(\mathbb{R}^{N+1}\right)}} \leq c_{9}$ for all $n \geq k_{0}$. Consequently, there exists a renormalized solution $u_{n}$ of problem (9.3) satisfied

$$
\begin{aligned}
& \left\|\left\|u_{n}\right\|\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \leq 2 c_{6}\left\|\mathbb{M}_{1}\left[\omega_{n}\right]\right\|_{L^{(N+2)(q-1), \infty}\left(\Omega_{T}\right)} \\
& \quad \leq 2 c_{6} c_{10}\left\|\mathbb{I}_{1}[|\mu|]\right\|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}+2 c_{6} c_{10}\left\|\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]\right\|_{L^{(N+2)(q-1)\left(\mathbb{R}^{N}\right)}}=: C
\end{aligned}
$$

for any $n \geq k_{0}$. Thanks to Proposition 3.5, there exists a subsequence of $\left\{u_{n}\right\}$ converging to $u$ in $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$. So, $\mid\|\nabla u\| \|_{L^{(N+2)(q-1), \infty\left(\Omega_{T}\right)}} \leq C$ and $\left|\nabla u_{n}\right|^{q} \rightarrow|\nabla u|^{q}$ in $L^{1}(\Omega)$ since $\left\{\left|\nabla u_{n}\right|^{q}\right\}$ is equi-integrable.
It follows the result by Proposition 3.5 and Theorem 3.6. This completes the proof.
Proof of Theorem 2.33. Let $\left\{\mu_{n}\right\} \subset C_{c}^{\infty}\left(\Omega_{T}\right), \sigma_{n} \in C_{c}^{\infty}(\Omega)$ be as in the proof of Theorem 2.1. We have $\left|\mu_{n}\right| \leq \varphi_{n} *|\mu|,\left|\sigma_{n}\right| \leq \varphi_{1, n} *|\sigma|$ for any $n \in \mathbb{N},\left\{\varphi_{n}\right\},\left\{\varphi_{1, n}\right\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^{N}$ respectively. We can assume that $\operatorname{supp}\left(\mu_{n}\right) \subset$ $\left(\Omega^{\prime}+B_{d / 4}(0)\right) \times[0, T]$ and $\operatorname{supp}\left(\sigma_{n}\right) \subset \Omega^{\prime}+B_{d / 4}(0)$ for any $n \in \mathbb{N}$. Set $\omega_{n}=\left|\mu_{n}\right|+\left|\sigma_{n}\right| \otimes \delta_{\{t=0\}}$ and $\omega=|\mu|+|\sigma| \otimes \delta_{\{t=0\}}$.
First, we prove that the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$ for $n_{0} \in \mathbb{N}$. By Corollary 4.39 and Remark 4.40, we have

$$
\begin{equation*}
\left[\omega_{n}\right]_{\mathfrak{M}^{\mathcal{G}_{1}, q^{\prime}}} \leq c_{1} \varepsilon_{0} \forall n \in \mathbb{N}, \tag{9.10}
\end{equation*}
$$

where $c_{1}=c_{1}\left(N, q, T_{0}\right)$ and $\varepsilon_{0}=[\omega]_{\mathfrak{M}^{\mathcal{G}_{1}}, q^{\prime}}$. By Proposition 4.36 and Remark 4.37, we have

$$
\begin{gather*}
\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\left(\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n}\right]\right)^{q}\right] \leq c_{2} \varepsilon_{0}^{q-1} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n}\right] \text { a.e in } \mathbb{R}^{N+1} \text { and }  \tag{9.11}\\
\mathbb{I}_{2}\left[\left(\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n}\right]\right)^{q}\right] \leq c_{2} \varepsilon_{0}^{q-1} \mathbb{I}_{2}\left[\omega_{n}\right] \text { a.e in } \mathbb{R}^{N+1}, \tag{9.12}
\end{gather*}
$$

for any $n \in \mathbb{N}$, where $c_{2}=c_{2}\left(N, \delta, q, T_{0}\right)$ and $0<\delta<1$. We set

$$
\mathbf{E}_{\Lambda}=\left\{u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right):|\nabla u| \leq \Lambda \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right]\right\}
$$

Clearly, $\mathbf{E}_{\Lambda}$ is closed under the strong topology of $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ and convex.
We consider a $\operatorname{map} S: \mathbf{E}_{\Lambda} \rightarrow L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ defined for each $v \in \mathbf{E}_{\Lambda}$ by $S(v)=u$, where $u \in L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$ is the unique renormalized solution of problem (9.1). We will show that $S\left(\mathbf{E}_{\Lambda}\right)$ is subset of $\mathbf{E}_{\Lambda}$ for some $\Lambda>0$ and $\varepsilon_{0}$ small enough.
We have

$$
\begin{equation*}
|\nabla v| \leq \Lambda \mathbb{I}_{1}\left[\omega_{n_{0}}\right] . \tag{9.13}
\end{equation*}
$$

In particular, $\left\|\|v v\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)} \leq \Lambda(N+1)^{-1}(d / 2)^{-N-1} \omega_{n_{0}}\left(\overline{\Omega_{T}}\right)\right.$, where $\Omega_{d / 2}=\{x \in$ $\Omega: d(x, \partial \Omega) \leq d / 2\}$.
From (9.11) and (9.12) lead to

$$
\begin{aligned}
& \mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\nabla v|^{q}\right] \leq \Lambda^{q} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\left(\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right]\right)^{q}\right] \leq c_{2} \Lambda^{q} \varepsilon_{0}^{q-1} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right] \text { and } \\
& \mathbb{I}_{2}\left[|\nabla v|^{q}\right] \leq \Lambda^{q} \mathbb{I}_{2}\left[\left(\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right]\right)^{q}\right] \leq c_{2} \Lambda^{q} \varepsilon_{0}^{q-1} \mathbb{I}_{2}\left[\omega_{n_{0}}\right]
\end{aligned}
$$

Clearly, from [27, Theorem 1.2], we have for any $Q_{r}(x, t) \subset \subset \Omega \times(-\infty, T)$ with $r \leq r_{0}$

$$
\begin{align*}
|\nabla u(x, t)| & \leq c_{3} f_{Q_{r}(x, t)}|\nabla u| d y d s+c_{3} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\nabla v|^{q}+\omega_{n_{0}}\right](x, t) \\
& \leq c_{3} f_{Q_{r}(x, t)}|\nabla u| d y d s+c_{3} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\nabla v|^{q}\right](x, t)+c_{3} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t) \\
& \leq c_{3} f_{Q_{r}(x, t)}|\nabla u| d y d s+c_{3}\left(c_{2} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t), \tag{9.14}
\end{align*}
$$

where $c_{3}=c_{3}\left(N, \Lambda_{1}\right)$ and $r_{0}=r_{0}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \beta\right)>0$.
Since $\|\mid \nabla u\|_{L^{1}\left(\Omega_{T}\right)} \leq c_{4} T_{0}\left(\| \| \nabla v \|_{L^{q}\left(\Omega_{T}\right)}^{q}+\omega_{n_{0}}\left(\overline{\Omega_{T}}\right)\right)$, for any $(x, t) \in\left(\Omega \backslash \Omega_{d / 4}\right) \times(-\infty, T)$ where $\Omega_{d / 4}=\{x \in \Omega: d(x, \partial \Omega) \leq d / 4\}$,

$$
\begin{align*}
\frac{1}{\left|Q_{d_{0}}(x, t)\right|} \int_{Q_{d_{0}}(x, t)}|\nabla u| d y d s & \leq c_{5} d_{0}^{-N-2} T_{0}\left(| ||\nabla v| \|_{L^{q}\left(\Omega_{T}\right)}^{q}+\omega_{n_{0}}\left(\overline{\Omega_{T}}\right)\right) \\
& \leq c_{6} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\nabla v|^{q}+\omega_{n_{0}}\right](x, t) \\
& \leq c_{6}\left(c_{2} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \mathbb{I}_{1}^{2 T_{0}, \delta} \mid\left[\omega_{n_{0}}\right](x, t), \tag{9.15}
\end{align*}
$$

where $d_{0}=\min \left\{d / 8, r_{0}\right\}$ and $c_{6}=c_{6}\left(N, p, \Lambda_{1}, \Lambda_{2}, T_{0} / d_{0}\right)$.
By regularity theory, we have

$$
\|\nabla u\|_{L^{\infty}\left(\Omega_{d / 4} \times(0, T)\right)} \leq c_{7}\left(\|u\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)}+\left|\left\|\left.\nabla v\right|^{q}\right\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)}\right)\right.
$$

where $c_{7}=c_{7}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \Omega, T\right)$.
a. Estimate $\left|\left||\nabla v|^{q} \|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)}\right.\right.$. Thanks to (9.13),

$$
\left\||\nabla v|^{q}\right\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)} \leq\left(\Lambda(d / 2)^{-N-1}\left(\omega_{n_{0}}\left(\overline{\Omega_{T}}\right)\right)\right)^{q}
$$

Since $\omega_{n_{0}}\left(\overline{\Omega_{T}}\right) \leq c_{1} \varepsilon_{0} \operatorname{Cap}_{\mathcal{G}_{1}, q^{\prime}}\left(\tilde{Q}_{T_{0}}\left(x_{0}, t_{0}\right)\right)=c_{8}\left(N, q, p, T_{0}\right) \varepsilon_{0}$ with $\left(x_{0}, t_{0}\right) \in \Omega_{T}$, thus

$$
\left\||\nabla v|^{q}\right\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)} \leq c_{9} \Lambda^{q} \varepsilon_{0}^{q-1} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t) \quad \forall(x, t) \in \Omega_{T}
$$

where $c_{9}=c_{9}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, q, d, \Omega, T\right)$.
b. Estimate $\|u\|_{L^{\infty}\left(\Omega_{d / 2}\right)}$. By Theorem 2.1 we have

$$
|u(x, t)| \leq c_{10} \mathbb{I}_{2}\left[|\nabla v|^{q}+\omega_{n_{0}}\right](x, t) \quad \forall(x, t) \in \Omega_{T}
$$

where $c_{10}=c_{10}\left(N, \Lambda_{1}, \Lambda_{2}\right)$. Thus,

$$
\begin{aligned}
|u(x, t)| & \leq c_{10} \mathbb{I}_{2}\left[|\nabla v|^{q}\right](x, t)+c_{10} \mathbb{I}_{2}\left[\omega_{n_{0}}\right](x, t) \\
& \leq c_{10}\left(c_{2} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \mathbb{I}_{2}\left[\omega_{n_{0}}\right](x, t),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|u\|_{L^{\infty}\left(\Omega_{d / 2} \times(0, T)\right)} & \leq c_{11}\left(c_{2} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) d^{-N} \omega_{n_{0}}\left(\overline{\Omega_{T}}\right) \\
& \leq c_{12}\left(c_{2} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t) \quad \forall(x, t) \in \Omega_{T}
\end{aligned}
$$

where $c_{12}=c_{12}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, q, T_{0} / d\right)$. Therefore,

$$
\begin{equation*}
\|\nabla u\|_{L^{\infty}\left(\Omega_{d / 4} \times(0, T)\right)} \leq c_{13}\left(c_{14} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \inf _{(x, t) \in \Omega_{T}} \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t) . \tag{9.16}
\end{equation*}
$$

where $c_{13}=c_{13}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, q, d, \Omega, T\right)$.
Finally from (9.15) (9.16) and (9.14) we get for all $(x, t) \in \Omega_{T}$

$$
|\nabla u(x, t)| \leq c_{14}\left(c_{15} \Lambda^{q} \varepsilon_{0}^{q-1}+1\right) \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t) .
$$

where $c_{14}=c_{14}\left(N, \Lambda_{1}, \Lambda_{2}, \Lambda_{3}, q, d, \Omega, T\right)$ and $c_{15}=c_{15}(N, \delta, q)$.
So, we suppose that $\Lambda=2 c_{14}$ and $\varepsilon_{0} \leq c_{15}^{-\frac{1}{q-1}}\left(2 c_{14}\right)^{-\frac{q}{q-1}}$, it is equivalent to (2.61), (2.62) holding for some $C>0$. Then for any $(x, t) \in \Omega_{T}$

$$
|\nabla u(x, t)| \leq \Lambda \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n_{0}}\right](x, t),
$$

and $S$ is well defined.
On the other hand, we can see that $S: \mathbf{E}_{\Lambda} \rightarrow \mathbf{E}_{\Lambda}$ is continuous and $S(E)$ is pre-compact under the strong topology of $L^{1}\left(0, T, W_{0}^{1,1}(\Omega)\right)$.
Thus, by Schauder Fixed Point Theorem, $S$ has a fixed point on $\mathbf{E}_{\Lambda}$. This means: the problem (2.53) has a solution with data $\mu=\mu_{n_{0}}, \sigma=\sigma_{n_{0}}$.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution $u_{n}$ of problem (9.3) which satisfies

$$
\left|\nabla u_{n}(x, t)\right| \leq \Lambda \mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n}\right](x, t) \forall(x, t) \in \Omega_{T} .
$$

Since $\mathbb{I}_{1}^{2 T_{0}, \delta}\left[\omega_{n}\right](x, t) \leq \varphi_{n} * \mathbb{I}_{1}^{2 T_{0}, \delta}[|\mu|](x, t)+\varphi_{1, n} *\left(\mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\sigma| \otimes \delta_{\{t=0\}}\right](., t)\right)(x)=: A_{n}(x, t)$ and $A_{n}$ converges to $\mathbb{I}_{1}^{2 T_{0}, \delta}[|\mu|]+\mathbb{I}_{1}^{2 T_{0}, \delta}\left[|\sigma| \otimes \delta_{\{t=0\}}\right]$ in $L^{q}\left(\mathbb{R}^{N+1}\right)$, thus $\left|\nabla u_{n}\right|^{q}$ is equi-integrable. As in the proof of Theorem 2.32, we get the result by using Proposition 3.5 and Theorem 3.6. This completes the proof.

### 9.2 Quasilinear Riccati Type Parabolic Equation in $\mathbb{R}^{N} \times(0, \infty)$ and $\mathbb{R}^{N+1}$

In this subsection, we provide the proofs of Theorem 2.37 and 2.38 . In the same way, we can prove Theorem 2.36.
Proof of Theorem 2.37. As in the proof of Theorem 2.25 and Theorem 2.27, we can apply Theorem 2.32 to obtain: there exists a constant $c_{1}=c_{1}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$ that if $[A]_{s_{0}}^{\infty} \leq \delta$ and (2.64) holds with constant $c_{1}$ then we can find a sequence of renormalized solutions $\left\{u_{n_{k}}\right\}$ of

$$
\left\{\begin{array}{l}
\left(u_{n_{k}}\right)_{t}-\operatorname{div}\left(A\left(x, t, \nabla u_{n_{k}}\right)\right)=\left|\nabla u_{n_{k}}\right|^{q}+\chi_{D_{n_{k}-1}} \omega \text { in } D_{n_{k}} \\
u_{n_{k}}=0 \text { on } \partial B_{n_{k}}(0) \times\left(-n_{k}^{2}, n_{k}^{2}\right) \\
u_{n_{k}}\left(-n_{k}^{2}\right)=0 \text { on } B_{n_{k}}(0)
\end{array}\right.
$$

converging to some $u$ in $L_{\text {loc }}^{1}\left(\mathbb{R} ; W_{\text {loc }}^{1,1}\left(\mathbb{R}^{N}\right)\right)$ and satisfying

$$
\left\|\left|\nabla u_{n_{k}}\right|\right\|_{L^{(q-1)(N+2), \infty}\left(D_{n_{k}}\right)} \leq\left. c_{2}| | \mathbb{I}_{1}[|\omega|]\right|_{L^{(N+2)(q-1), \infty}\left(\mathbb{R}^{N+1}\right)}
$$

for some $c_{2}=c_{2}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)$, where $D_{n}=B_{n}(0) \times\left(-n^{2}, n^{2}\right)$. It follows $\left|\nabla u_{n_{k}}\right|^{q} \rightarrow|\nabla u|^{q}$ in $L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{N+1}\right)$. Thus, $u$ is a distribution solution of (2.55) which satisfies (2.63).
Furthermore, if $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then $u_{n_{k}}=0$ in $B_{n_{k}}(0) \times\left(-n_{k}^{2}, 0\right)$. So, $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$. Therefore, clearly $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.54).

Proof of Theorem 2.38. Let $\omega_{n}=\varphi_{n} *\left(\chi_{D_{n-1}} \omega\right)$ for any $n \geq 2$. We have $\mu_{n} \in C_{c}^{\infty}\left(\mathbb{R}^{N+1}\right)$ with $\operatorname{supp}\left(\omega_{n}\right) \subset D_{n}$ and $\omega_{n} \rightarrow \omega$ weakly in $\mathfrak{M}\left(\mathbb{R}^{N+1}\right)$.
According to Corollary 4.39 and Remark 4.40, we have

$$
\left[\omega_{n}\right]_{\mathfrak{M}^{\mathcal{H}_{1}, q^{\prime}}} \leq c_{1} \varepsilon_{0} \quad \forall n \in \mathbb{N}
$$

where $c_{1}=c_{1}(N, q)$ and $[\omega]_{\mathfrak{M}^{\mathcal{H}_{1}, q^{\prime}}} \leq \varepsilon_{0}$. Thus, thanks to Theorem 1.3 we get

$$
\begin{align*}
& \mathbb{I}_{1}\left[\left(\mathbb{I}_{1}\left[\omega_{n}\right]\right)^{q}\right] \leq c_{2} \varepsilon_{0}^{q-1} \mathbb{I}_{1}\left[\omega_{n}\right] \quad \text { and }  \tag{9.17}\\
& \mathbb{I}_{2}\left[\left(\mathbb{I}_{1}\left[\omega_{n}\right]\right)^{q}\right] \leq c_{2} \varepsilon_{0}^{q-1} \mathbb{I}_{2}\left[\omega_{n}\right] \quad \forall n \in \mathbb{N} \tag{9.18}
\end{align*}
$$

where $c_{2}=c_{2}\left(N, q, c_{1}\right)$.
We fix $n_{0} \in \mathbb{N}$, put:

$$
\mathbf{E}_{\Lambda}=\left\{u \in L^{1}\left(-n_{0}^{2}, n_{0}^{2}, W_{0}^{1,1}\left(B_{n_{0}}(0)\right)\right):|\nabla u| \leq \Lambda \mathbb{I}_{1}\left[\omega_{n_{0}}\right] \text { in } B_{n_{0} / 4}(0) \times\left(-n_{0}^{2}, n_{0}^{2}\right)\right\}
$$

By using estimate (5.8) in Remark 5.3, we can apply the argument of the proof of Theorem 2.9 , with problem (6.9) replaced by

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}(A(t, \nabla u))=\chi_{B_{n_{0} / 4}(0) \times\left(-n_{0}^{2}, n_{0}^{2}\right)}|\nabla v|^{q}+\omega_{n_{0}} \text { in } D_{n_{0}} \\
u=0 \quad \text { on } \quad \partial B_{n_{0}}(0) \times\left(-n_{0}^{2}, n_{0}^{2}\right), \\
u\left(-n_{0}^{2}\right)=0 \quad \text { in } \quad B_{n_{0}}(0)
\end{array}\right.
$$

to obtain: the operator $S$ (in the proof of Theorem 2.9) has a fixed point on $\mathbf{E}_{\Lambda}$ for some $\Lambda=\Lambda\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$ and $\varepsilon_{0}=\varepsilon_{0}\left(N, \Lambda_{1}, \Lambda_{2}, q\right)>0$. Therefore, for any $n \in \mathbb{N}$ there exists a solution $u_{n}$ of problem

$$
\left\{\begin{array}{l}
\left(u_{n}\right)_{t}-\operatorname{div}\left(A\left(t, \nabla u_{n}\right)\right)=\chi_{B_{n / 4}(0) \times\left(-n^{2}, n^{2}\right)}\left|\nabla u_{n}\right|^{q}+\omega_{n} \text { in } D_{n}, \\
u_{n}=0 \text { on } \partial B_{n}(0) \times\left(-n^{2}, n^{2}\right), \\
u_{n}\left(-n^{2}\right)=0 \quad \text { in } \quad B_{n}(0),
\end{array}\right.
$$

which satisfies

$$
\left|\nabla u_{n}(x, t)\right| \leq \Lambda \mathbb{I}_{1}\left[\omega_{n}\right](x, t) \quad \forall(x, t) \in B_{n / 4}(0) \times\left(-n^{2}, n^{2}\right)
$$

Moreover, combining this with (9.18) and Theorem 2.1 we also obtain

$$
\begin{aligned}
\left|u_{n}(x, t)\right| & \leq K \mathbb{I}_{2}\left[\chi_{B_{n / 4}(0) \times\left(-n^{2}, n^{2}\right)}\left|\nabla u_{n}\right|^{q}+\left|\omega_{n}\right|\right](x, t) \\
& \leq K \Lambda^{q} \mathbb{I}_{2}\left[\left(\mathbb{I}_{1}\left[\left|\omega_{n}\right|\right]\right)^{q}\right]+K \mathbb{I}_{2}\left[\left|\omega_{n}\right|\right](x, t) \\
& \leq c_{3} \mathbb{I}_{2}\left[\left|\omega_{n}\right|\right](x, t) \\
& \leq c_{3} \varphi_{n} * \mathbb{I}_{2}\left[\left|\chi_{D_{n-1}} \omega\right|\right](x, t),
\end{aligned}
$$

for any $(x, t) \in B_{n}(0) \times\left(-n^{2}, n^{2}\right)$.
Since $\mathbb{I}_{2}[\omega]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$, thus $\sup _{n} \int_{D_{m}} \chi_{D_{n}}\left|u_{n}\right|^{q_{0}} d x d t<\infty$ for all $m \in \mathbb{N}, 1<q_{0}<\frac{N+2}{N}$.
In addition, since $\mathbb{I}_{1}[\omega] \in L_{\mathrm{loc}}^{q}\left(\mathbb{R}^{N+1}\right)$, thus $\varphi_{n} * \mathbb{I}_{1}\left[\left|\chi_{D_{n-1}} \omega\right|\right] \rightarrow \mathbb{I}_{1}[\omega]$ in $L_{\mathrm{loc}}^{q}\left(\mathbb{R}^{N+1}\right)$ and $\left\{\chi_{B_{n / 4}(0) \times\left(-n^{2}, n^{2}\right)}\left|\nabla u_{n}\right|^{q}\right\}$ is equi local integrable in $\mathbb{R}^{N+1}$.
Therefore, we can apply Corollary 3.18 to obtain: $u_{n} \rightarrow u$ in $L_{\mathrm{loc}}^{1}\left(\mathbb{R} ; W_{\mathrm{loc}}^{1,1}\left(\mathbb{R}^{N}\right)\right)$ ( we will
take its subsequence if need) and $u$ satisfies (2.66). Also, $\left|\nabla u_{n}\right|^{q} \rightarrow|\nabla u|^{q}$ in $L_{l o c}^{1}\left(\mathbb{R}^{N+1}\right)$. Finally, we can conclude that $u$ is a distribution solution of problem (2.65). Note that the assumption $[\omega]_{\mathfrak{M}^{\mathcal{H}_{1}, q^{\prime}}} \leq \varepsilon_{0}$ is equivalent to (2.67) holding with $C=\varepsilon_{0}$.
Furthermore, if $\omega=\mu+\sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}\left(\mathbb{R}^{N} \times(0, \infty)\right)$ and $\sigma \in \mathfrak{M}\left(\mathbb{R}^{N}\right)$, then $u_{n}=0$ in $B_{n}(0) \times\left(-n^{2}, a_{n}\right)$ where $\operatorname{supp}\left(\omega_{n}\right) \subset \mathbb{R}^{N} \times\left(a_{n}, \infty\right)$ and $a_{n} \rightarrow 0^{-}$as $n \rightarrow \infty$. So, $u=0$ in $\mathbb{R}^{N} \times(-\infty, 0)$. Therefore, clearly $\left.u\right|_{\mathbb{R}^{N} \times[0, \infty)}$ is a distribution solution to (2.68).
This completes the proof of the Theorem.

## 10 Appendix

Proof of the Remark 2.7. For $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right), 0<\alpha<N+2$ if $\mathbb{I}_{\alpha}[\omega]\left(x_{0}, t_{0}\right)<\infty$ for some $\left(x_{0}, t_{0}\right) \in \mathbb{R}^{N+1}$ then for any $0<\beta \leq \alpha, \mathbb{I}_{\beta}[\omega] \in L_{\text {loc }}^{s}\left(\mathbb{R}^{N+1}\right)$ for any $0<s<\frac{N+2}{N+2-\beta}$. Indeed, by Remark 4.28 we have $\mathbb{I}_{\alpha}[\omega] \in L_{\text {loc }}^{s}\left(\mathbb{R}^{N+1}\right)$ for any $0<s<\frac{N+2}{N+2-\beta}$.
Take $0<\beta \leq \alpha$ and $0<s<\frac{N+2}{N+2-\beta}$. For $R>0$, by Proposition 4.4 we have $\mathbb{I}_{\beta}\left[\chi_{\tilde{Q}_{2 R}(0,0)} \omega\right] \in$ $L_{\text {loc }}^{s}\left(\mathbb{R}^{N+1}\right)$. Thus,

$$
\begin{aligned}
& \int_{\tilde{Q}_{R}(0,0)}\left(\mathbb{I}_{\beta}[\omega](x, t)\right)^{s} d x d t \\
& \quad \leq c_{1} \int_{\tilde{Q}_{R}(0,0)}\left(\mathbb{I}_{\beta}\left[\chi_{\tilde{Q}_{2 R}(0,0)} \omega\right](x, t)\right)^{s} d x d t+c_{1} \int_{\tilde{Q}_{R}(0,0)}\left(\mathbb{I}_{\beta}\left[\chi_{\tilde{Q}_{2 R}(0,0)^{c}} \omega\right](x, t)\right)^{s} d x d t \\
& \quad \leq c_{1} \int_{\tilde{Q}_{R}(0,0)}\left(\mathbb{I}_{\beta}\left[\chi_{\tilde{Q}_{2 R}(0,0)} \omega\right](x, t)\right)^{s} d x d t+c_{1} R^{-s(\alpha-\beta)} \int_{\tilde{Q}_{R}(0,0)}\left(\mathbb{I}_{\alpha}[\omega](x, t)\right)^{s} d x d t \\
& \quad<\infty
\end{aligned}
$$

For $0<\beta<\alpha<N+2$, we consider

$$
\omega(x, t)=\sum_{k=4}^{\infty} \frac{a_{k}}{\left|\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_{k}(0,0)\right|} \chi_{\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_{k}(0,0)}(x, t),
$$

where $a_{k}=2^{n(N+2-\theta)}$ if $k=2^{n}$ and $a_{k}=0$ otherwise with $\theta \in(\beta, \alpha]$.
It is easy to see that $\mathbb{I}_{\alpha}[\omega] \equiv \infty$ and $\mathbb{I}_{\beta}[\omega]<\infty$ in $\mathbb{R}^{N+1}$.
Proof of the Remark 2.26. For $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$, since $\mathbb{I}_{2}[\omega] \leq c_{1} I_{1}\left[I_{1}[\omega]\right]$ thus: If $\mathbb{I}_{1}[\omega] \in L^{s, \infty}\left(\mathbb{R}^{N+1}\right)$ with $1<s<N+2$, then by Proposition 4.4 in next section

$$
\left\|\left.\mathbb{I}_{2}[\omega]\right|_{L^{\frac{s}{N}(N+1)}, \infty\left(\mathbb{R}^{N+1}\right)} \leq c_{1}\right\| \mathbb{I}_{1}[\omega] \|_{L^{s, \infty}\left(\mathbb{R}^{N+1}\right)}<\infty
$$

If $\mathbb{I}_{1}[\omega] \in L^{N+2, \infty}\left(\mathbb{R}^{N+1}\right)$, then by Theorem 4.3,

$$
\mathbb{I}_{2}[\omega] \in L_{\mathrm{loc}}^{s_{0}}\left(\mathbb{R}^{N+1}\right) \quad \forall s_{0}>1
$$

So, $\mathbb{I}_{2}[\omega]<\infty$ a.e in $\mathbb{R}^{N+1}$ if $\mathbb{I}_{1}[\omega] \in L^{s, \infty}\left(\mathbb{R}^{N+1}\right)$ with $1<s \leq N+2$.
For $s>N+2$, there exists $\omega \in \mathfrak{M}^{+}\left(\mathbb{R}^{N+1}\right)$ such that $\mathbb{I}_{2}[\omega] \equiv \infty$ in $\mathbb{R}^{N+1}$ and $\mathbb{I}_{1}[\omega] \in$ $L^{s}\left(\mathbb{R}^{N+1}\right)$. Indeed, consider

$$
\omega(x, t)=\sum_{k=1}^{\infty} \frac{k^{N-1}}{\left|\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_{k}(0,0)\right|} \chi_{\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_{k}(0,0)}(x, t) .
$$

We have for $(x, t) \in \mathbb{R}^{N+1}$ and $n_{0} \in \mathbb{N}$ with $n_{0}>\log _{2}(\max \{|x|, \sqrt{2|t|}\})$

$$
\begin{aligned}
\mathbb{I}_{2}[\omega](x, t) & \geq c_{2} \sum_{n_{0}}^{\infty} \frac{\omega\left(\tilde{Q}_{2^{n}}(x, t)\right)}{2^{n N}} \geq c_{2} \sum_{n_{0}}^{\infty} \frac{\omega\left(\tilde{Q}_{2^{n-1}}(0,0)\right)}{2^{n N}} \\
& \geq c_{2} \sum_{n_{0}}^{\infty} \frac{\sum_{k=1}^{2^{n-1}-1} k^{N-1}}{2^{n N}}=c_{2} \sum_{k=1}^{\infty}\left(\sum_{n_{0}}^{\infty} \chi_{k \leq 2^{n-1}-1} \frac{1}{2^{n N}}\right) k^{N-1} \\
& \geq c_{4} \sum_{k=n_{0}}^{\infty} k^{-1}=\infty
\end{aligned}
$$

On the other hand, for $s_{1}>\frac{N+2}{2}$

$$
\int_{\mathbb{R}^{N+1}} \omega^{s_{1}} d x d t=c_{5} \sum_{k=1}^{\infty} \frac{k^{s(N-1)}}{\left((k+1)^{N+2}-k^{N+2}\right)^{s_{1}-1}} \leq c_{6} \sum_{k=1}^{\infty} \frac{k^{s_{1}(N-1)}}{k^{\left(s_{1}-1\right)(N+1)}}<\infty
$$

since $\left(s_{1}-1\right)(N+1)-s_{1}(N-1)>1$. Thus,

$$
\left\|\left.\mathbb{I}_{1}[\omega]\right|_{L^{s}\left(\mathbb{R}^{N+1}\right)} \leq c_{7}\right\| \omega \|_{L^{\frac{s}{N+N+2)}}\left(\mathbb{R}^{N+1}\right)}<\infty .
$$

Proof of the Proposition 3.16. We will use an idea in $[9,10]$ to prove 3.14. For $S^{\prime} \in W^{1, \infty}(\mathbb{R})$ with $S(0)=0, S^{\prime \prime} \geq 0, S^{\prime}(\tau) \tau \geq 0$ for all $\tau \in \mathbb{R}$ and $\left\|S^{\prime}\right\|_{L^{\infty}(\mathbb{R})} \leq 1$ we have

$$
\begin{aligned}
& -\int_{D} \eta_{t} S(u) d x d t+\int_{D} S^{\prime}(u) A(x, t, \nabla u) \nabla \eta d x d t \\
& \quad+\int_{D} S^{\prime \prime}(u) \eta A(x, t, \nabla u) \nabla u d x d t+\int_{D} S^{\prime}(u) \eta L(u) d x d t=\int_{D} S^{\prime}(u) \eta d \mu
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \Lambda_{2} \int_{D} S^{\prime \prime}(u) \eta|\nabla u|^{2} d x d t \\
& \quad+\int_{D} S^{\prime}(u) \eta L(u) d x d t \leq \Lambda_{1} \int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \eta d|\mu|+\int_{D}\left|\eta_{t}\right||u| d x d t
\end{aligned}
$$

a. We choose $S^{\prime} \equiv \varepsilon^{-1} T_{\varepsilon}$ for $\varepsilon>0$ and let $\varepsilon \rightarrow 0$ we will obtain

$$
\begin{equation*}
\int_{D} \eta|L(u)| d x d t \leq \Lambda_{1} \int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \eta d|\mu|+\int_{D}\left|\eta_{t}\right||u| d x d t \tag{10.1}
\end{equation*}
$$

b. for $S^{\prime}(u)=\left(1-(|u|+1)^{-\alpha}\right) \operatorname{sign}(u)$ for $\alpha>0$ then

$$
\int_{D} \frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \eta d x d t \leq c_{1}\left(\int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \eta d|\mu|+\int_{D}\left|\eta_{t}\right||u| d x d t\right)
$$

Using Holder's inequality, we have

$$
\int_{D}|\nabla u||\nabla \eta| d x d t \leq \frac{1}{2 c_{1}} \int_{D} \frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \eta d x d t+c_{2} \int_{D}(|u|+1)^{q_{0}} \eta d x d t+c_{2} \int_{D}\left|\nabla \eta^{1 / q_{1}}\right|^{q_{1}} d x d t
$$

Hence,

$$
\begin{equation*}
\int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \frac{|\nabla u|^{2}}{(|u|+1)^{\alpha+1}} \eta d x d t \leq c_{3} B \tag{10.2}
\end{equation*}
$$

c. for $S^{\prime}(u)=\frac{-k+\delta+|u|}{2 \delta} \operatorname{sign}(u) \chi_{k-\delta<|u|<k+\delta}+\operatorname{sign}(u) \chi_{|u| \geq k+\delta}, 0<\delta \leq k$ then

$$
\begin{equation*}
\frac{1}{2 \delta} \int_{k-\delta<|u|<k+\delta}|\nabla u|^{2} \eta d x d t \leq c_{4}\left(\int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \eta d|\mu|+\int_{D}\left|\eta_{t}\right||u| d x d t\right) \tag{10.3}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\frac{1}{k} \int_{D}\left|\nabla T_{k}(u)\right|^{2} \eta d x d t \leq c_{5}\left(\int_{D}|\nabla u||\nabla \eta| d x d t+\int_{D} \eta d|\mu|+\int_{D}\left|\eta_{t}\right||u| d x d t\right) \quad \forall k>0 \tag{10.4}
\end{equation*}
$$

Consequently, we deduce (3.14) from (10.1)-(10.4).
Next, take $\varphi \in C_{c}^{\infty}(D)$ and $S^{\prime}(u)=\chi_{|u| \leq k-\delta}+\frac{k+\delta-|u|}{2 \delta} \chi_{k-\delta<|u|<k+\delta}, S(0)=0$ we have

$$
\begin{aligned}
& -\int_{D} \varphi_{t} \eta S(u) d x d t+\int_{D} S^{\prime}(u) \eta A(x, t, \nabla u) \nabla \varphi d x d t+\int_{D} S^{\prime}(u) \varphi A(x, t, \nabla u) \nabla \eta d x d t \\
& \quad-\frac{1}{2 \delta} \int_{k-\delta<|u|<k+\delta} \operatorname{sign}(u) \varphi \eta A(x, t, \nabla u) \nabla u d x d t+\int_{D} S^{\prime}(u) \varphi \eta L(u) d x d t \\
& \quad=\int_{D} S^{\prime}(u) \varphi \eta d \mu+\int_{D} \varphi \eta_{t} S(u) d x d t
\end{aligned}
$$

Combining with (10.1), (10.2) and (10.3), we get

$$
-\int_{D} \varphi_{t} \eta S(u) d x d t+\int_{D} S^{\prime}(u) \eta A(x, t, \nabla u) \nabla \varphi d x d t \leq c_{5}\|\varphi\|_{L^{\infty}(D)} B
$$

Letting $\delta \rightarrow 0$, we get

$$
-\int_{D} \varphi_{t} \eta T_{k}(u) d x d t+\int_{D} \eta A\left(x, t, \nabla T_{k}(u)\right) \nabla \varphi d x d t \leq c_{5}\|\varphi\|_{L^{\infty}(D)} B
$$

By density, we can take $\varphi=T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)$,

$$
\begin{aligned}
& -\int_{D} \frac{\partial}{\partial t}\left(T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)\right) \eta T_{k}(u) d x d t \\
& \quad+\int_{D} \eta A\left(x, t, \nabla T_{k}(u)\right) \nabla T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) d x d t \leq c_{5} \varepsilon B
\end{aligned}
$$

Using integration by part, we have

$$
\begin{aligned}
-\int_{D} \frac{\partial}{\partial t} & \left(T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)\right) \eta T_{k}(u) d x d t \\
& =\frac{1}{2} \int_{D}\left(T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)\right)^{2} \eta_{t} d x d t \\
& +\int_{D} T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)\left\langle T_{k}(w)\right\rangle_{\nu} \eta_{t} d x d t \\
& +\nu \int_{D} \eta\left(T_{k}(w)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) d x d t
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& -\int_{D} \frac{\partial}{\partial t}\left(T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right)\right) \eta T_{k}(u) d x d t \\
& \quad \geq-\varepsilon(1+k)\left\|\eta_{t}\right\|_{L^{1}(D)}+\nu \int_{D} \eta\left(T_{k}(w)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) T_{\varepsilon}\left(T_{k}(u)-\left\langle T_{k}(w)\right\rangle_{\nu}\right) d x d t
\end{aligned}
$$

which follows (3.15).
Proof of the proposition 3.17. Let $S_{k} \in W^{2, \infty}(\mathbb{R})$ such that $S_{k}(z)=z$ if $|z| \leq k$ and $S_{k}(z)=\operatorname{sign}(z) 2 k$ if $|z|>2 k$. For $m \in \mathbb{N}$, let $\eta_{m}$ be the cut off function on $D_{m}$ with respect to $D_{m+1}$. It is easy to see that from the assumption and Remark 3.4, Proposition 3.15 we get $U_{m, n}=\eta_{m} S_{k}\left(v_{n}\right), v_{n}=u_{n}-h_{n}$

$$
\begin{aligned}
& \sup _{n \geq m+1}\left(\left\|\left(U_{m, n}\right)_{t}\right\|_{L^{2}\left(-m^{2}, m^{2}, H^{-1}\left(B_{m}(0)\right)\right)+L^{1}\left(D_{m}\right)}+\left\|U_{m, n}\right\|_{L^{2}\left(-m^{2}, m^{2}, H_{0}^{1}\left(B_{m}(0)\right)\right)}\right. \\
&\left.+\left\|u_{n}\right\|_{L^{1}\left(D_{m}\right)}+\left\|v_{n}\right\|_{L^{1}\left(D_{m}\right)}\right) \leq M_{m}<\infty
\end{aligned}
$$

Thus, $\left\{U_{m, n}\right\}_{n \geq m+1}$ is relatively compact in $L^{1}\left(D_{m}\right)$. On the other hand, for any $n_{1}, n_{2} \geq$ $m+1$

$$
\begin{aligned}
& \left|\left\{\left|v_{n_{1}}-v_{n_{2}}\right|>\lambda\right\} \cap D_{m}\right|=\left|\left\{\left|\eta_{m} v_{n_{1}}-\eta_{m} v_{n_{2}}\right|>\lambda\right\} \cap D_{m}\right| \\
& \quad \leq \frac{1}{k}\left(\left\|v_{n_{1}}\right\|_{L^{1}\left(D_{m}\right)}+\left\|v_{n_{2}}\right\|_{L^{1}\left(D_{m}\right)}\right)+\frac{1}{\lambda}\left\|\eta_{m} S_{k}\left(v_{n_{1}}\right)-\eta_{m} S_{k}\left(v_{n_{2}}\right)\right\|_{L^{1}\left(D_{m}\right)} \\
& \quad \leq \frac{2 M_{m}}{k}+\frac{1}{\lambda}\left\|U_{m, n_{1}}-U_{m, n_{2}}\right\|_{L^{1}\left(D_{m}\right)},
\end{aligned}
$$

and $h_{n}$ is convergent in $L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{N+1}\right)$. So, for any $m \in \mathbb{N}$ there is a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ such that $\left\{u_{n}\right\}$ is a Cauchy sequence (in measure) in $D_{m}$. Therefore, there is a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ such that $\left\{u_{n}\right\}$ converges to $u$ a.e in $\mathbb{R}^{N+1}$ for some $u$. Clearly, $u \in L_{\text {loc }}^{1}\left(\mathbb{R} ; W_{\text {loc }}^{1,1}\left(\mathbb{R}^{N}\right)\right)$. Now, we prove that $\nabla u_{n} \rightarrow \nabla u$ a.e in $\mathbb{R}^{N+1}$. From (3.15) with $D=D_{m+2}, \eta=\eta_{m}$ and $T_{k}(w)=T_{k}\left(\eta_{m+1} u\right)$ we have

$$
\begin{align*}
& \nu \int_{D_{m+2}} \eta_{m}\left(T_{k}\left(\eta_{m+1} u\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right) T_{\varepsilon}\left(T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right) d x d t \\
& \quad+\int_{D_{m+2}} \eta_{m} A\left(x, t, \nabla T_{k}\left(u_{n}\right)\right) \nabla T_{\varepsilon}\left(T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right) d x d t \\
& \quad \leq c_{1} \varepsilon(1+k) B(n, m) \quad \forall n \geq m+2 \tag{10.5}
\end{align*}
$$

where

$$
\begin{aligned}
B(n, m)= & \left\|\left(\eta_{m}\right)_{t}\left(\left|u_{n}\right|+1\right)\right\|_{L^{1}\left(D_{m+2}\right)} \\
& +\int_{D_{m+2}}\left(\left|u_{n}\right|+1\right)^{q_{0}} \eta d x d t+\int_{D_{m+2}}\left|\nabla \eta_{m}^{1 / q_{1}}\right|^{q_{1}} d x d t+\int_{D_{m+2}} \eta_{m} d\left|\mu_{n}\right|
\end{aligned}
$$

with $q_{1}<\frac{q_{0}-1}{2 q_{0}}$. By the assumption, we verify that the right hand side of (10.5) is bounded by $c_{2} \varepsilon$, where $c_{2}$ does not depend on $n$.
Since $\left\{\eta_{m} T_{k}\left(u_{n}\right)\right\}_{n \geq m+2}$ is bounded in $L^{2}\left(-(m+2)^{2},(m+2)^{2} ; H_{0}^{1}\left(B_{m+2}(0)\right)\right)$, thus there is a subsequence of $\left\{u_{n}\right\}$, still denoted by $\left\{u_{n}\right\}$ such that

$$
\lim _{n \rightarrow \infty} \int_{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right| \leq \varepsilon} \eta_{m} A\left(x, t, \nabla T_{k}(u)\right) \nabla\left(T_{k}\left(u_{n}\right)-T_{k}(u)\right) d x d t=0 .
$$

Therefore, thanks to $u_{n} \rightarrow u$ a.e in $D_{m+2}$ and $\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu} \rightarrow T_{k}\left(\eta_{m+1} u\right)$ in $L^{2}(-(m+$ $\left.2)^{2},(m+2)^{2} ; H_{0}^{1}\left(B_{m+2}(0)\right)\right)$, we get

$$
\limsup _{\nu \rightarrow \infty} \limsup _{n \rightarrow \infty} \int_{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right| \leq \varepsilon} \eta_{1, m} \Phi_{n, k} d x d t \leq c_{2} \varepsilon \forall \varepsilon \in(0,1),
$$

where $\Phi_{n, k}=\left(A\left(x, t, T_{k}\left(u_{n}\right)\right)-A\left(x, t, T_{k}(u)\right)\right) \nabla\left(T_{k}\left(u_{n}\right)-T_{k}(u)\right)$. Using Holder inequality,

$$
\begin{aligned}
& \int_{D_{m+2}} \eta_{m} \Phi_{k, n}^{1 / 2} d x d t=\int_{D_{m+2}} \eta_{m} \Phi_{k, n}^{1 / 2} \chi_{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right| \leq \varepsilon} d x d t \\
& \quad+\int_{D_{m+2}} \eta_{m} \Phi_{k, n}^{1 / 2} \chi_{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right|>\varepsilon} d x d t \\
& \quad \leq\left\|\eta_{1, m}\right\|_{L^{1}\left(D_{m+2}\right)}^{1 / 2}\left(\int_{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right| \leq \varepsilon} \eta_{m} \Phi_{n, k} d x d t\right)^{1 / 2} \\
& \quad+\left|\left\{\left|T_{k}\left(u_{n}\right)-\left\langle T_{k}\left(\eta_{m+1} u\right)\right\rangle_{\nu}\right|>\varepsilon\right\} \cap D_{m+1}\right|^{1 / 2}\left(\int_{D_{m+2}} \eta_{m}^{2} \Phi_{k, n} d x d t\right)^{1 / 2} \\
& \quad=A_{n, \nu, \varepsilon}
\end{aligned}
$$

Clearly, $\limsup _{\varepsilon \rightarrow 0} \limsup _{\nu \rightarrow \infty} \limsup _{n \rightarrow \infty} A_{n, \nu, \varepsilon}=0$. It follows

$$
\limsup _{n \rightarrow \infty} \int_{D_{m+2}} \eta_{m} \Phi_{k, n}^{1 / 2} d x d t=0
$$

Since $\Phi_{n, k} \geq \Lambda_{2}\left|\nabla T_{k}\left(u_{n}\right)-\nabla T_{k}(u)\right|^{2}$, thus $\nabla T_{k}\left(u_{n}\right) \rightarrow \nabla T_{k}(u)$ in $L^{1}\left(D_{m}\right)$.
Note that

$$
\begin{aligned}
\left|\left\{\left|\nabla u_{n_{1}}-\nabla u_{n_{2}}\right|>\lambda\right\} \cap D_{m}\right| & \leq \frac{1}{k}\left(\left\|u_{n_{1}}\right\|_{L^{1}\left(D_{m}\right)}+\left\|u_{n_{2}}\right\|_{L^{1}\left(D_{m}\right)}\right) \\
& \left.+\frac{1}{\lambda} \right\rvert\,\left\|\nabla T_{k}\left(u_{n_{1}}\right)-\nabla T_{k}\left(u_{n_{2}}\right)\right\|_{L^{1}\left(D_{m}\right)} \\
& \left.\leq \frac{2 M_{m}}{k}+\frac{1}{\lambda}| | \right\rvert\, \nabla T_{k}\left(u_{n_{1}}\right)-\nabla T_{k}\left(u_{n_{2}}\right) \|_{L^{1}\left(D_{m}\right)}
\end{aligned}
$$

Thus, we can show that there is a subsequence of $\left\{\nabla u_{n}\right\}$ still denoted by $\left\{\nabla u_{n}\right\}$ converging $\nabla u$ a.e in $\mathbb{R}^{N+1}$.

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