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# Determinantal method for locally modified structures

## Application to the vibration damping of a space launcher

Denis Brizard · Sébastien Besset · Louis Jézéquel · Bernard Troclet

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**Abstract** In order to improve the dynamic behaviour of an existing or already designed structure, local modifications can be performed by taking advantage of the relative displacement between two points of the structure. A stiffener, damper or viscoelastic rod may be added and its effect on the initial structure must be assessed. A new formulation is developed, based on the response of the initial structure at the attachment points of the local modification. A determinantal equation results, whose roots are the eigenvalues of the modified structure. The equation is solved numerically with a dedicated algorithm and it is shown that this is faster than performing an eigenvalue problem reanalysis. The method is able to deal with both undamped and damped systems, and can handle several modifications simultaneously. It is applied on the last stage of a space launcher, along with a double modal synthesis method.

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### Nomenclature

$n$	total number of degrees of freedom
$n_i^{(s)}$	number of internal degrees of freedom of substructure ( $s$ )
$n_j$	number of junction degrees of freedom
$n_I^{(s)}$	number of retained eigenmodes for substructure ( $s$ )
$n_B$	number of retained branch modes
$I$	set of internal degrees of freedom
$J$	set of junction degrees of freedom
$E$	set of excitation degrees of freedom
$\eta^{(s)}$	substructure ( $s$ ) eigenmode generalized coordinate ( $n_I^{(s)} \times 1$ )
$\zeta$	branch mode generalised coordinate ( $n_B \times 1$ )
$\mathbf{F}_j$	junction force ( $n_j \times 1$ )
$\mathbf{q}_j$	junction displacement ( $n_j \times 1$ )
$\Psi_s$	matrix of constraint modes of the global structure ( $n \times n_j$ )
$\mathbf{X}_{Bj}$	matrix of branch modes ( $n_j \times n_B$ )
$\Psi^{(s)}$	substructure ( $s$ ) constraint mode ( $((n_i^{(s)} + n_j) \times n_j)$ )

- $\Psi_i^{(s)}$  restriction of constraint modes  $\Psi^{(s)}$  on  $I$  set  
( $n_i^{(s)} \times n_j$ )
- $\Psi_j$  restriction of constraint modes  $\Psi^{(s)}$  on  $J$  set  
( $n_j \times n_j$ )

## 1 Introduction

In many design processes, the effect of local modifications has to be assessed. This can be at an early stage when the effect of a part on the whole structure is to be determined rapidly. On the contrary, when the design is already set, or when dealing with an existing structure, the addition of a stiffener or a damper may be evaluated for a resonance shifting or a damping purpose.

This is often the case with space launchers: the stages design is already set and may need to be adapted to protect the payload from excessive vibration levels. The influence of local dampers thus needs to be evaluated, so as to optimize their location and properties.

Because industrial structures are generally complex and therefore have an important number of degrees of freedom, a reduction technique is required to describe the dynamic behaviour efficiently with a small model (ie. a small number of generalized degrees of freedom.)

Component Mode Synthesis (CMS) techniques allow to compute structure responses through the use of substructures modes, thus reducing the size of the problem to solve. Basically, each substructure is described with a certain number of eigenmodes and boundary modes are added to compensate the modal truncation. Two main methods may be distinguished. On one hand, the fixed interface based CMS, developed by Craig and Bampton [1], uses fixed interface substructure eigenmodes and constraint modes as boundary modes (these are the substructure response to successive unitary boundary displacement). On the other hand, free interface CMS (see MacNeal [2], Rubin [3] or Craig and Chang [4]) uses free interface substructure eigenmodes and attachment modes (substructure response to successive unitary boundary force).

Because the number of boundary modes equals the number of junction degrees of freedom, the idea of condensing interface coordinates quickly emerged (see Craig and Chang [5]). Sometimes under different names, reduction with “boundary modes” was studied by some authors, among which are Balmès [6] and Tran [7]. Moreover, in addition to boundary reduction, a few authors employed  $\omega^2$  boundary developments to better compensate modal truncation (see Jézéquel and Setio [8] and Besset and Jézéquel [9]) in a method called Double Modal Synthesis; see also Gérardin and Rixen [10] for  $\omega^2$  development aspects.

In a previous paper [11], the addition of a spring between two substructure degrees of freedom – to model an elastic link – was studied with the use of the Double Modal Synthesis. It was shown that only small modifications —ie. small spring stiffnesses— could be well handled. Indeed, the reduction bases were not able to describe big changes in the behaviour of the structure since they do not take into account of the modification; their performances are therefore limited. Having a modal basis capable of describing the behaviour of the structure whatever the modification is quite a challenge and somehow impossible. In addition, since the reduction bases are composed of real modes, they are not able to deal with highly dissipative links containing visous damping or hysteretic damping. That is why a modification technique is employed in combination with the Double Modal Synthesis.

Baldwin and Hutton [12] wrote an interesting review and classified the modification techniques into three categories: the techniques based on small modifications (sensitivity approach, perturbation methods); the techniques based on modal approximation; and the techniques based on localized modifications, which is studied in this paper.

Weissenburger [13] is one of the first to treat the local modification of an undamped system. Pomazal [14] extended Weissenburger’s procedure to deal with the local modification of one of the system elements on linear damped system. At last, Jézéquel [15] proposed a method to diminish the effects of modal truncation. Hallquist [16] discussed the vibration of structures with arbitrary support conditions, using Lagrange multipliers, to get frequencies of the constrained system. In [17], mass and damping modifications in damped systems are evoked, a Newton-Raphson numerical scheme is used to locate the complex roots of the characteristic equation. Jacquot [18] proposed an original approach based on transfer functions to get the response of a modified system, the eigenvalues of the modified system are not computed. Dowel [19] used a Lagrangian formulation to analyse the effect of spring-mass addition to a dynamical system.

All these methods either require the explicit knowledge of all the eigenmodes or the explicit knowledge of the transfer functions. It is therefore difficult to describe the behaviour of the structure with a black-box model – providing the response of the structure for a given frequency – such as the Double Modal Synthesis that is used here.

Even though the aim is different, it is interesting to have a look at the inverse problem approach. Tseui [20] treats the modification of stiffness and mass parameters for desired eigenfrequency of undamped mechan-

ical systems. Yee [21] extended this work to damped systems: a method for modifying a mechanical system and shifting its damped natural frequency to a desired value is developed, the calculation involves iteration and only one modification at a time can be handled. Ram [22] addressed the problem of structural modifications in truncated systems. He also investigated the inverse problem [23], given an incomplete modal model (natural frequencies and modes), to reach a desired dynamic behaviour. Kyprianou [24] explored in detail the assignment of natural frequencies by an added mass and one or more springs.

The aim of this article is to introduce a general formulation, based on transfer functions – which can be black-boxes – and impedances, to determine the eigenvalues of a locally modified structure. The addition of a mechanical impedance models the structural modification (cf. figure 1). It can correspond to the introduction of a viscous damper, a spring, or any combination, in order to modify the characteristics of the system. The method introduced in this article is able to deal with one or more impedance, and the attachment point of the impedance may be modified easily. The advantage of this new formulation is that it can be applied to both undamped and damped structures.

This method is based on the response function of the points where the local modification is performed and on the force-displacement transfers between those points. The impedance equation of the local modification is then used to write a characteristic (determinantal) equation whose roots are the eigenvalues of the modified structure. Since the formulation is exact whatever the value of the added impedance, it is not limited to small perturbations; the only difficulty is to solve this equation. An good reduced model of the structure is of course also required to preserve the accuracy of the method when dealing with a large number of degrees of freedom.

In the following section, the Double Modal Synthesis method is introduced. The determinantal method and the associated solving algorithm are then exposed, along with a fast single mode approach. The application of the method is then illustrated on a few examples and some points of interest are discussed. The single mode approach is developed on the same example with a view to pre-design some damping solutions.

## 2 An improved substructuring method: the Double Modal Synthesis

The structure is supposed to be composed of two substructures, which does not restrict the method. If needed,

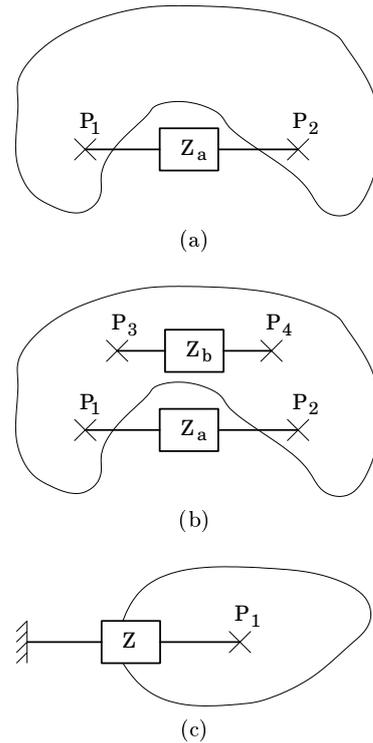


Fig. 1: Local modifications : (a) addition of a mechanical impedance; (b) case of two local impedances; and, (c) addition of a ground link on the structure

a superscript  $(.^{(s)})$  is added to make the distinction between substructures. Subscripts  $i$ ,  $e$  and  $j$  refer to internal  $I$ , excitation  $E$  and junction  $J$  degrees of freedom sets, respectively; they are used in matrix partitions. In case of a barred variable ( $\bar{\cdot}$ ), subscripts  $i$  and  $j$  refer to eigenmodes and branch mode, respectively.

### 2.1 Branch modes

Boundary condensation is made through the use of branch modes : instead of keeping a number of modes equal to the number of boundary degrees of freedom  $n_j$ , only a certain number of boundary modes are kept. These branch modes are representative of the structure behaviour in the frequency range of interest. The branch modes defined in this paper are the modes of the structure condensed on its interfaces thanks to constraint modes.

Constraint modes of the global structure

$$\Psi_s = \begin{bmatrix} \Psi_i^{(1)} \\ \Psi_j \\ \Psi_i^{(2)} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{ii}^{(1)-1} \mathbf{K}_{ij}^{(1)} \\ \mathbf{I}_{jj} \\ -\mathbf{K}_{ii}^{(2)-1} \mathbf{K}_{ij}^{(2)} \end{bmatrix} \quad (1)$$

are computed with the stiffness matrices of each substructure. Branch modes  $x_{Bj}$  are defined on junction  $J$  between the two substructures. They are the solution of the global structure eigenvalue problem projected on constraint modes

$$[{}^t\Psi_s \mathbf{K}_{tot} \Psi_s - \omega^2 {}^t\Psi_s \mathbf{M}_{tot} \Psi_s] \mathbf{x}_{Bj} = 0 \quad (2)$$

The set of all branch modes  $\mathbf{X}_{Bj}$  spans the same subspace as the complete set of constraint modes  $\Psi_s$ . Double Modal Synthesis (DMS) consists in retaining only the first “boundary modes”, called branch modes hereafter. The projection of branch modes on each substructure is achieved through the use of boundary modes  $\omega^2$  developpements.

The internal displacement of a substructure due to imposed boundary displacement  $\Psi_j$  may be written

$$\begin{aligned} \Psi_i^{(s)} &= -\left(\mathbf{D}_{ii}^{(s)}\right)^{-1} \mathbf{D}_{ij}^{(s)} \Psi_j \\ &= \left(-\omega^2 \mathbf{M}_{ii}^{(s)} + \mathbf{K}_{ii}^{(s)}\right)^{-1} \left(-\omega^2 \mathbf{M}_{ij}^{(s)} + \mathbf{K}_{ij}^{(s)}\right) \Psi_j \end{aligned} \quad (3)$$

The following equation

$$\begin{aligned} (\mathbf{K} - \omega^2 \mathbf{M})^{-1} &= \mathbf{K}^{-1} + \omega^2 (\mathbf{K}^{-1} \mathbf{M}) \mathbf{K}^{-1} + \dots \\ &+ (\omega^2 \mathbf{K}^{-1} \mathbf{M})^{n-1} \mathbf{K}^{-1} \\ &+ (\omega^2 \mathbf{K}^{-1} \mathbf{M})^n (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \end{aligned} \quad (4)$$

is valid for all integer  $n \geq 0$ . It is used to calculate the inverse required for boundary modes in Eq. (3).

An imposed boundary displacement  $\Psi_j = \mathbf{I}_{jj}$  (identity matrix of size  $n_j$ ) is considered, which corresponds to successive unit displacement of boundary degrees of freedom, while the others remain clamped. Going up to the second order, on gets from Eqs. (3) and (4)

$$\begin{aligned} \Psi_i^{(s)}(\omega) &\simeq -\mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ij}^{(s)} \\ &+ \omega^2 \left[ -(\mathbf{K}_{ii}^{(s)-1} \mathbf{M}_{ii}^{(s)}) \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ij}^{(s)} + \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ij}^{(s)} \right] \\ &+ \omega^4 \left[ -(\mathbf{K}_{ii}^{(s)-1} \mathbf{M}_{ii}^{(s)})^2 \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ij}^{(s)} \right. \\ &\quad \left. + (\mathbf{K}_{ii}^{(s)-1} \mathbf{M}_{ii}^{(s)}) \mathbf{K}_{ii}^{(s)-1} \mathbf{M}_{ij}^{(s)} \right] \end{aligned} \quad (5)$$

Order 2 boundary modes may be written in a more condensed way

$$\Psi^{(s)} = \begin{bmatrix} \Psi_i^{(s)}(\omega) \\ \Psi_j \end{bmatrix} = \begin{bmatrix} \Psi_{i0}^{(s)} + \omega^2 \Psi_{i1}^{(s)} + \omega^4 \Psi_{i2}^{(s)} \\ \mathbf{I}_{jj} \end{bmatrix} \quad (6)$$

It should be noticed that only the internal mode displacement is affected by the  $\omega^2$  developpement.

Branch modes, once projected on substructure  $(s)$ , may thus be written

$$\begin{aligned} \mathbf{X}_{Bi}^{(s)}(\omega) &= \Psi_i^{(s)}(\omega) \mathbf{X}_{Bj} = \mathbf{X}_{B0}^{(s)} + \omega^2 \mathbf{X}_{B1}^{(s)} + \omega^4 \mathbf{X}_{B2}^{(s)} \\ \mathbf{X}_{Bj}^{(s)} &= \mathbf{I}_{jj} \mathbf{X}_{Bj} = \mathbf{X}_{Bj} \end{aligned} \quad (7)$$

## 2.2 Substructure reduced basis

In this subsection, all the variables are related to one of the substructures: the superscript  $(\cdot)^{(s)}$  is therefore omitted.

Substructure displacements are described by a set of fixed interface eigenmodes  $\Phi$  and a set of branch modes  $X_B$  with  $\omega^2$  developpements, to compensate modal truncation. To take the local flexibility at excitation point into account, excitation set  $E$  is integrated in the boundary, which already contains  $J$ , the junction set between substructures. Branch modes are thus computed on the frontier  $F = J \cup E$  and fixed interface substructure eigenmodes correspond to the modes of the substructure being clamped at the frontier  $F$ . The reduced basis is therefore

$$\mathbf{T} = [\Phi \mathbf{X}_B] = \begin{bmatrix} \Phi_i & \mathbf{X}_{Bi}(\omega) \\ 0 & \mathbf{X}_{Be} \\ 0 & \mathbf{X}_{Bj} \end{bmatrix} \quad (8)$$

Let  $\eta$  and  $\zeta$  be the generalized coordinates vectors associated with substructure eigenmodes and branch modes, respectively. Let  $\bar{\mathbf{M}} = {}^t \mathbf{T} \mathbf{M} \mathbf{T}$  and  $\bar{\mathbf{K}} = {}^t \mathbf{T} \mathbf{K} \mathbf{T}$  be the reduced matrices of a given substructure. The reduced matrix  $\bar{\mathbf{M}}$  is given by

$$\bar{\mathbf{M}} = \begin{bmatrix} {}^t \Phi \mathbf{M} \Phi & {}^t \Phi \mathbf{M} \mathbf{X}_B \\ {}^t \mathbf{X}_B \mathbf{M} \Phi & {}^t \mathbf{X}_B \mathbf{M} \mathbf{X}_B \end{bmatrix} \quad (9)$$

In order to show boundary modes  $\omega^2$  developpements, reduction matrix  $\mathbf{T}$  is written as

$$\mathbf{T} = \mathbf{T}_0 + \omega^2 \mathbf{T}_1 + \omega^4 \mathbf{T}_2 \quad (10)$$

Only  $\mathbf{T}_0$  is a full matrix, the next matrices have non-zero terms on lines and columns corresponding to internal DOFs and branch modes, respectively

$$\mathbf{T}_k = \begin{bmatrix} 0 & \mathbf{X}_{Bk} \\ 0 & 0 \end{bmatrix} \quad \forall k > 0 \quad (11)$$

The reduced mass matrix is expanded, taking the second order of  $\omega^2$  developpement of reduction matrix  $\mathbf{T}$ ,

$$\begin{aligned} {}^t \mathbf{T} \mathbf{M} \mathbf{T} &= {}^t \mathbf{T}_0 \mathbf{M} \mathbf{T}_0 + \omega^2 [{}^t \mathbf{T}_1 \mathbf{M} \mathbf{T}_0 + {}^t \mathbf{T}_0 \mathbf{M} \mathbf{T}_1] \\ &+ \omega^4 [{}^t \mathbf{T}_1 \mathbf{M} \mathbf{T}_1 + {}^t \mathbf{T}_2 \mathbf{M} \mathbf{T}_0 + {}^t \mathbf{T}_0 \mathbf{M} \mathbf{T}_2] \\ &+ \omega^6 [{}^t \mathbf{T}_1 \mathbf{M} \mathbf{T}_2 + {}^t \mathbf{T}_1 \mathbf{M} \mathbf{T}_2] + \omega^8 [{}^t \mathbf{T}_2 \mathbf{M} \mathbf{T}_2] \end{aligned} \quad (12)$$

By definition of constraint modes, it can be shown that any mode having null boundary displacement is orthogonal, with respect to the stiffness matrix, to the set of constraint modes  $\Psi_0$ . Substructure eigenmodes are fixed interface modes and branch modes are expanded with constraint modes on substructures. This involves

$${}^t\Phi \mathbf{K} \mathbf{X}_{B0} = 0 \quad (13)$$

Substructure's mass and stiffness matrices are reduced using  $\mathbf{T}$  and the previous orthogonality

$$\bar{\mathbf{M}} = \begin{bmatrix} \bar{\mathbf{M}}_{ii} & \bar{\mathbf{M}}_{ij} \\ \bar{\mathbf{M}}_{ji} & \bar{\mathbf{M}}_{jj} \end{bmatrix} + \begin{bmatrix} 0 & \bar{\mathbf{M}}_{ij}(\omega) \\ \bar{\mathbf{M}}_{ji}(\omega) & \bar{\mathbf{M}}_{jj}(\omega) \end{bmatrix} \quad (14)$$

$$\bar{\mathbf{K}} = \begin{bmatrix} \bar{\mathbf{K}}_{ii} & 0 \\ 0 & \bar{\mathbf{K}}_{jj} \end{bmatrix} + \begin{bmatrix} 0 & \bar{\mathbf{K}}_{ij}(\omega) \\ \bar{\mathbf{K}}_{ji}(\omega) & \bar{\mathbf{K}}_{jj}(\omega) \end{bmatrix} \quad (15)$$

In these reduced matrices, block matrices  $\bar{\mathbf{M}}_{ii}$  and  $\bar{\mathbf{K}}_{ii}$  are diagonal and equal to the modal mass and stiffness of the retained fixed interface eigenmodes. Keeping only order 0 amounts to use Craig and Bampton's component mode synthesis.

External forces on substructure only apply on excitation set  $E$  and on junction set  $J$ . Hence the force vector in the reduced basis is

$$\bar{\mathbf{F}} = {}^t\mathbf{T}\mathbf{F} = {}^t\mathbf{T} \begin{Bmatrix} 0 \\ \mathbf{F}_e \\ \mathbf{F}_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ {}^t\mathbf{X}_{Be}\mathbf{F}_e + {}^t\mathbf{X}_{Bj}\mathbf{F}_j \end{Bmatrix} \quad (16)$$

### 2.3 Substructure equilibrium equations

The substructure equilibrium equation  $(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{q} = \mathbf{F}$  is reduced with the reduction basis  $\mathbf{T}$  and the associated generalized displacement vector  ${}^t\mathbf{q}_R = \{{}^t\boldsymbol{\eta} \ {}^t\boldsymbol{\zeta}\}$

$${}^t\mathbf{T}(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{T}\mathbf{q}_R = {}^t\mathbf{T}\mathbf{F} \quad (17)$$

The matricial equilibrium is divided according to generalized displacements. The first equation of size  $n_i$ , the number of retained fixed interface eigenmodes, is

$$\begin{aligned} & \left[ -\omega^2\bar{\mathbf{M}}_{ii}^{(s)} + \bar{\mathbf{K}}_{ii}^{(s)} \right] \boldsymbol{\eta}^{(s)} \\ & + \left[ -\omega^2 \left( \bar{\mathbf{M}}_{ij}^{(s)} + \bar{\mathbf{M}}_{ij}^{(s)}(\omega) \right) + \bar{\mathbf{K}}_{ij}^{(s)}(\omega) \right] \boldsymbol{\zeta}^{(s)} = 0 \end{aligned} \quad (18)$$

The second one, of size  $n_B$  the number of retained branch modes, is

$$\begin{aligned} & \left[ -\omega^2 \left( \bar{\mathbf{M}}_{ji}^{(s)} + \bar{\mathbf{M}}_{ji}^{(s)}(\omega) \right) + \bar{\mathbf{K}}_{ji}^{(s)}(\omega) \right] \boldsymbol{\eta}^{(s)} \\ & + \left[ -\omega^2 \left( \bar{\mathbf{M}}_{jj}^{(s)} + \bar{\mathbf{M}}_{jj}^{(s)}(\omega) \right) + \bar{\mathbf{K}}_{jj}^{(s)} + \bar{\mathbf{K}}_{jj}^{(s)}(\omega) \right] \boldsymbol{\zeta}^{(s)} \\ & = {}^t\mathbf{X}_{Be}\mathbf{F}_e + {}^t\mathbf{X}_{Bj}\mathbf{F}_j \end{aligned} \quad (19)$$

### 2.4 Substructure coupling

Let the two substructures (1) and (2) be coupled through a common boundary  $J$ . Branch modes definition Eq. (7) implies

$$\mathbf{X}_{Bj}^{(1)} = \mathbf{X}_{Bj}^{(2)} = \mathbf{X}_{Bj} \quad (20)$$

Therefore, boundary nodal equilibrium  $\mathbf{F}_j^{(1)} + \mathbf{F}_j^{(2)} = 0$  becomes

$${}^t\mathbf{X}_{Bj}\mathbf{F}_j^{(1)} + {}^t\mathbf{X}_{Bj}\mathbf{F}_j^{(2)} = 0 \quad (21)$$

Because internal modes are fixed interface modes (c.f. Eq. (8)), substructure ( $s$ ) junction displacements are given by

$$\mathbf{q}_j^{(s)} = \mathbf{X}_{Bj}^{(s)}\boldsymbol{\zeta}^{(s)} \quad (22)$$

Compatibility of interface displacements  $\mathbf{q}_j^{(1)} = \mathbf{q}_j^{(2)}$  and Eqs. (22) and (20) thus give

$$\boldsymbol{\zeta}^{(1)} = \boldsymbol{\zeta}^{(2)} = \boldsymbol{\zeta} \quad (23)$$

Matrices  $\bar{\mathbf{M}}_{ii}^{(s)}$  and  $\bar{\mathbf{K}}_{ii}^{(s)}$  are diagonal because of substructures eigenmodes orthogonality, Eq. (18) is thus easily converted into

$$\boldsymbol{\eta}^{(s)} = - \left( \bar{\mathbf{D}}_{ii}^{(s)} \right)^{-1} \bar{\mathbf{D}}_{ij}^{(s)} \boldsymbol{\zeta} \quad (24)$$

The equilibrium Eq. (21) couples Eq. (19) written for each substructure – (1) and (2) – and  $\boldsymbol{\eta}^{(s)}$  is replaced by the previous expression. A matricial equation of size  $n_B$ , governing branch modes contributions  $\boldsymbol{\zeta}$ , is obtained

$$\begin{aligned} & \left[ \bar{\mathbf{D}}_{jj}^{(1)} - \bar{\mathbf{D}}_{ji}^{(1)} \left( \bar{\mathbf{D}}_{ii}^{(1)} \right)^{-1} \bar{\mathbf{D}}_{ij}^{(1)} \right. \\ & \left. + \bar{\mathbf{D}}_{jj}^{(2)} - \bar{\mathbf{D}}_{ji}^{(2)} \left( \bar{\mathbf{D}}_{jj}^{(2)} \right)^{-1} \bar{\mathbf{D}}_{ij}^{(2)} \right] \boldsymbol{\zeta} = {}^t\mathbf{X}_{Be}\mathbf{F}_e \end{aligned} \quad (25)$$

where a dynamic stiffness matricial notation is adopted

$$\begin{aligned} \bar{\mathbf{D}}_{ii}^{(s)} &= -\omega^2\bar{\mathbf{M}}_{ii}^{(s)} + \bar{\mathbf{K}}_{ii}^{(s)} \\ \bar{\mathbf{D}}_{jj}^{(s)} &= -\omega^2 \left( \bar{\mathbf{M}}_{jj}^{(s)} + \bar{\mathbf{M}}_{jj}^{(s)}(\omega) \right) + \bar{\mathbf{K}}_{jj}^{(s)} + \bar{\mathbf{K}}_{jj}^{(s)}(\omega) \\ \bar{\mathbf{D}}_{kl}^{(s)} &= -\omega^2 \left( \bar{\mathbf{M}}_{kl}^{(s)} + \bar{\mathbf{M}}_{kl}^{(s)}(\omega) \right) + \bar{\mathbf{K}}_{kl}^{(s)}(\omega) \text{ for } kl = ij, ji \end{aligned} \quad (26)$$

Solving Eq. (22) and then replacing  $\boldsymbol{\zeta}$  in Eq. (24) allows to come back to substructures physical displacements at any point. If the observation point is integrated in the boundary (in the  $E$  set), only Eq. (25) need to be solved and the displacement is known through Eq. (22). This will be used in the following determinantal method since only the displacements of the link attachment points are required; these points will therefore be integrated in the boundary.

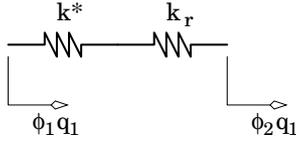


Fig. 2: Rheological model ( $q_1$  is the generalised coordinate associated with mode  $\phi$ )

### 3 Analysis of locally modified structures

In this section, bold type will be used to distinguish matrices and vectors from scalar quantities.

#### 3.1 A single mode approach

In the context of pre-design of structures, it is interesting to develop a specific method to get fastly the approximate characteristics of a locally modified structure (addition of damping elastomer). It is based on a single mode vision, in the case of hysteretic damping. The mass of the modification is supposed to be negligible and the modal displacement of the attachment points  $\phi_1$  and  $\phi_2$  remain unchanged by the modification of complex stiffness  $k^* = e(1+i\beta)$  (cf. figure 2). According to the method proposed by Jézéquel [15], the residual stiffness  $k_r$  is introduced in series with the modification, it corresponds to the static flexibility of the initial structure, for the considered mode, at the location of the attachment points. The equivalent impedance of the model given in figure 2 may be written  $Z_m = a + ib$ .

The initial eigenvalue of the unmodified structure is written as

$$\tilde{\lambda} = -\tilde{\omega}^2 (1 + i\eta) \quad (27)$$

According to the first order sensitivity analysis [15], the modified eigenvalue is

$$\tilde{\lambda}_m = \tilde{\lambda} - Z_m (\phi_2 - \phi_1)^2 \quad (28)$$

$$\tilde{\lambda}_m = -\tilde{\omega}^2 (1 + i\eta) - (a + ib) (\phi_2 - \phi_1)^2 \quad (29)$$

which gives the pulsation and the damping factor of the modified eigenvalue

$$\tilde{\omega}_m^2 = -\Re(\tilde{\lambda}_m) = \tilde{\omega}^2 + a (\phi_2 - \phi_1)^2 \quad (30)$$

$$\eta_m = \frac{\Im(\tilde{\lambda}_m)}{\Re(\tilde{\lambda}_m)} = \frac{\tilde{\omega}^2 \eta + b (\phi_2 - \phi_1)^2}{\tilde{\omega}^2 + a (\phi_2 - \phi_1)^2} \quad (31)$$

By blocking the displacement between the attachment points of  $k^*$ , one can have the pulsation  $\tilde{\omega}_\infty$  of

the blocked system. It is therefore possible to get the residual stiffness by using Eq. (30)

$$k_r \simeq \frac{\tilde{\omega}_\infty^2 - \tilde{\omega}^2}{(\phi_2 - \phi_1)^2} \quad (32)$$

As it is a simple rule with restrictive hypothesis, its validity may be limited. A more precise method needs to be developed, which can also handle greater local changes. This is detailed in the following section.

#### 3.2 Determinantal method with one added impedance

Assuming a general linear dynamic system, the equation of motion in the frequency domain may be written

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) \mathbf{U} = \mathbf{F} \quad (33)$$

where  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are the mass, damping and stiffness matrices, respectively;  $\mathbf{U}$  is the displacement vector and  $\mathbf{F}$  is the vector of external forces. Eq. (33) can be rewritten as

$$\mathbf{U} = \mathbf{H}(\lambda) \mathbf{F}, \text{ with } \lambda = \alpha + i\omega \quad (34)$$

where  $\mathbf{H}(\lambda)$  is the dynamic flexibility or frequency response function; the term transfer function will be used thereafter. In order to simplify the notations, the dependency in  $\lambda$  will be omitted in the rest of the article.

Let the flexibility, or transfer function, between points  $P_k$  and  $P_l$  be defined by

$$H_{kl} = \frac{u_k}{F_l} \quad (35)$$

where  $u_i$  is the scalar displacement along the direction defined by  $\overline{P_k P_l}$  and  $F_l$  is the external force along the same direction. It is therefore possible to write the transfers between  $P_1$  and  $P_2$  (cf. figure 1a) in a matrixial way

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (36)$$

The initial structure is modified by the addition of a scalar mechanical impedance  $Z$  between two points  $P_1$  and  $P_2$  (cf. figure 1a), which gives the following force-displacement relationship

$$F_1 = Z(u_1 - u_2) \quad (37)$$

Using Eq. (36),  $u_1$  and  $u_2$  are eliminated from the previous equation

$$Z^{-1} F_1 = (H_{11} - H_{21}) F_1 - (H_{22} - H_{12}) F_2 \quad (38)$$

The force equilibrium in the added impedance may be written

$$F_1 + F_2 = 0 \quad (39)$$

and therefore Eq. (38) becomes

$$((H_{11} + H_{22} - H_{21} - H_{12}) - Z^{-1}) F_1 = 0 \quad (40)$$

Since we are not concerned with the case where the force in the impedance is zero, non-trivial solutions are sought. The eigenvalues of the modified structure may thus be found by solving the one line determinantal equation

$$\Delta = |H_{11} + H_{22} - H_{21} - H_{12} - Z^{-1}| = 0 \quad (41)$$

So instead of solving an eigenvalue problem of size  $n$ , the roots of a determinant give the eigenvalues of the modified structure. This is done through the use of some particular transfer functions of the initial system: the transfer functions between attachment points  $P_1$  and  $P_2$ . It is therefore possible to follow the frequencies of the modes while the impedance varies; this is detailed in section 3.6.

### 3.3 Adding more than one impedance

This method is not limited to only one structural modification. It is developed here for two local modifications – the addition of two distinct dampers on the structure for instance – but it can easily be generalised to a number  $N$  of local modifications.

In the case of two local impedances between points  $P_1$  and  $P_2$  and between points  $P_3$  and  $P_4$  (cf. figure 1b), the transfers between all these attachment points may be written

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \quad (42)$$

The impedance  $Z_a$  between  $P_1$  and  $P_2$  gives

$$F_1 = Z_a (u_1 - u_2) \quad (43)$$

$$F_1 + F_2 = 0 \quad (44)$$

and the impedance  $Z_b$  between  $P_3$  and  $P_4$  leads to

$$F_3 = Z_b (u_3 - u_4) \quad (45)$$

$$F_3 + F_4 = 0 \quad (46)$$

By using Eq. (42), Eq. (43) can be re-written as

$$Z_a^{-1} F_1 = (H_{11} - H_{21}) F_1 + (H_{12} - H_{22}) F_2 + (H_{13} - H_{23}) F_3 + (H_{14} - H_{24}) F_4 \quad (47)$$

and then simplified with Eq. (44)

$$Z_a^{-1} F_1 = (H_{11} + H_{22} - H_{21} - H_{12}) F_1 + (H_{13} + H_{24} - H_{23} - H_{14}) F_3 \quad (48)$$

In a similar way, we get for impedance  $Z_b$

$$Z_b^{-1} F_3 = (H_{31} - H_{41}) F_1 + (H_{32} - H_{42}) F_2 + (H_{33} - H_{43}) F_3 + (H_{34} - H_{44}) F_4 \quad (49)$$

$$Z_b^{-1} F_1 = (H_{31} + H_{42} - H_{41} - H_{32}) F_1 + (H_{33} + H_{44} - H_{43} - H_{34}) F_3 \quad (50)$$

Let the following mechanical transfers

$$T_{aa} = H_{11} + H_{22} - H_{21} - H_{12} \quad (51)$$

$$T_{ab} = H_{13} + H_{24} - H_{23} - H_{14} \quad (52)$$

$$T_{bb} = H_{33} + H_{44} - H_{43} - H_{34} \quad (53)$$

$$T_{ba} = H_{31} + H_{42} - H_{41} - H_{32} \quad (54)$$

Gathering Eqs. (48) and (50) gives

$$\begin{bmatrix} T_{aa} - Z_a^{-1} & T_{ab} \\ T_{ba} & T_{bb} - Z_b^{-1} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (55)$$

The non-trivial solutions of the previous system of two equations are found by cancelling the two by two determinant

$$\Delta = \begin{vmatrix} T_{aa} - Z_a^{-1} & T_{ab} \\ T_{ba} & T_{bb} - Z_b^{-1} \end{vmatrix} = 0 \quad (56)$$

The eigenvalues of the modified structure are therefore the solutions of the equation

$$(T_{aa} - Z_a^{-1})(T_{bb} - Z_b^{-1}) - T_{ab}T_{ba} = 0 \quad (57)$$

Eq. (56) can be easily generalised for the case of  $N$  local impedances.

### 3.4 Case of a ground link

Treating a ground link, represented by the figure 1c, just simplifies the formulation. The response of the structure at point  $P_1$  in the direction of the link is

$$u_1 = H_{11} F_1 \quad (58)$$

The impedance equation of the ground link gives

$$u_1 = Z^{-1} F_1 \quad (59)$$

which leads to

$$(H_{11} - Z^{-1}) F_1 = 0 \quad (60)$$

The eigenvalues of the modified structure are thus given by the non-trivial solutions

$$\Delta = |H_{11} - Z^{-1}| = 0 \quad (61)$$

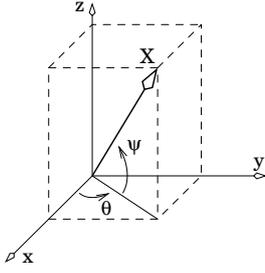


Fig. 3: Coordinate system change:  $\theta$  and  $\psi$  angles to go from the global frame  $(x, y, z)$  to the link frame  $(X, Y, Z)$

### 3.5 From global frame to link frame

In the previous sections, the equations were written with  $F$  being the force along the link. However, the direction of the link may not correspond to the global frame and thus may not be aligned with the degrees of freedom. The force therefore needs to be written in the link coordinate system.

In the present section, the upper case letters refer to the variables in the link coordinate system and the lower case letters refer to these variables in the global coordinate system.

Let  $u$ ,  $v$  and  $w$  be the displacements respectively along the  $\vec{x}$ ,  $\vec{y}$  and  $\vec{z}$  axes (see figure 3). The link frame is such than the displacement  $U$  is along the  $\vec{X}$  axis of this new frame. The coordinate system change is written

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (62)$$

where  $\mathbf{R} = R(\psi)R(\theta)$  and

$$R(\psi) = \begin{bmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{bmatrix} \quad (63)$$

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (64)$$

$$\mathbf{R} = \begin{bmatrix} \cos \psi \cos \theta & \cos \psi \sin \theta & \sin \psi \\ -\sin \theta & \cos \theta & 0 \\ -\sin \psi \cos \theta & -\sin \psi \sin \theta & \cos \psi \end{bmatrix} \quad (65)$$

The transfers between points  $P_1$  and  $P_2$  in the global coordinate system are

$$\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \mathbf{h}_{11} \begin{Bmatrix} f_1 \\ g_1 \\ e_1 \end{Bmatrix} + \mathbf{h}_{12} \begin{Bmatrix} f_2 \\ g_2 \\ e_2 \end{Bmatrix} \quad (66)$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ w_2 \end{Bmatrix} = \mathbf{h}_{21} \begin{Bmatrix} f_1 \\ g_1 \\ e_1 \end{Bmatrix} + \mathbf{h}_{22} \begin{Bmatrix} f_2 \\ g_2 \\ e_2 \end{Bmatrix} \quad (67)$$

where  $\mathbf{h}_{kl}$  is a 3-by-3 transfer matrix between forces at point  $P_l$  ( $f_l$ ,  $g_l$  and  $e_l$  along respectively the  $x$ ,  $y$  and  $z$  axes of the global frame) and displacements at point  $P_k$  ( $u_k$ ,  $v_k$  and  $w_k$  along respectively the  $x$ ,  $y$  and  $z$  axes)

$$\mathbf{h}_{kl} = \begin{bmatrix} r_{kl} & o_{kl} & p_{kl} \\ o'_{kl} & s_{kl} & q_{kl} \\ p'_{kl} & q'_{kl} & t_{kl} \end{bmatrix} \quad (68)$$

In the coordinate system of the link we get

$$\begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix} = \mathbf{H}_{11} \begin{Bmatrix} F_1 \\ G_1 \\ E_1 \end{Bmatrix} + \mathbf{H}_{12} \begin{Bmatrix} F_2 \\ G_2 \\ E_2 \end{Bmatrix} \quad (69)$$

$$\begin{Bmatrix} U_2 \\ V_2 \\ W_2 \end{Bmatrix} = \mathbf{H}_{21} \begin{Bmatrix} F_1 \\ G_1 \\ E_1 \end{Bmatrix} + \mathbf{H}_{22} \begin{Bmatrix} F_2 \\ G_2 \\ E_2 \end{Bmatrix} \quad (70)$$

where  $\mathbf{H}_{kl} = \mathbf{R} \mathbf{h}_{kl} {}^t \mathbf{R}$ ;  $U_k$ ,  $V_k$  and  $W_k$  are the displacements at point  $P_k$  along the  $X$ ,  $Y$  and  $Z$  axes of the link frame; and  $F_k$ ,  $G_k$  and  $E_k$  are the forces at point  $P_k$  along the  $X$ ,  $Y$  and  $Z$  axes. Indeed, since  $\mathbf{R}$  is a rotation matrix,  $\mathbf{R}^{-1} = {}^t \mathbf{R}$ .

We now have the relationship between forces and displacements along the link direction defined by  $\overline{P_1 P_2}$ , whatever the orientation of this link

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{bmatrix} (\mathbf{H}_{11})_{11} & (\mathbf{H}_{12})_{11} \\ (\mathbf{H}_{21})_{11} & (\mathbf{H}_{22})_{11} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (71)$$

where only the upper left term of matrix  $\mathbf{H}_{kl}$  needs to be computed

$$\begin{aligned} (\mathbf{H}_{kl})_{11} &= (\mathbf{R} \mathbf{h}_{kl} {}^t \mathbf{R})_{11} = r_{kl} (\cos \psi \cos \theta)^2 + \\ &o_{kl} \sin \theta \cos \theta (\cos \psi)^2 + p_{kl} \cos \theta \sin \psi \cos \psi \\ &+ o'_{kl} \sin \theta \cos \theta (\cos \psi)^2 + s_{kl} (\sin \theta \cos \theta)^2 \\ &+ q_{kl} \sin \theta \sin \psi \cos \psi + p'_{kl} \cos \theta \sin \psi \cos \psi \\ &+ q'_{kl} \sin \theta \sin \psi \cos \psi + t_{kl} (\sin \psi)^2 \end{aligned} \quad (72)$$

Now consider the case where we have two links on the structure, the first between points  $P_1$  and  $P_2$  and a second one between points  $P_3$  and  $P_4$ . The following transfer matrix is required

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \quad (73)$$

with  $U_1, U_2, F_1$  and  $F_2$  the displacements and forces along the  $\overline{P_1P_2}$  direction and  $U_3, U_4, F_3$  and  $F_4$  the displacements and forces along the  $\overline{P_3P_4}$  direction. A second coordinate system change, from the global frame to the frame aligned on  $\overline{P_3P_4}$ , is defined with the corresponding  $\theta'$  and  $\psi'$  angles

$$\mathbf{R}' = R(\psi') R(\theta') \quad (74)$$

Two different coordinate system changes are now involved; the first line of matricial Eq. (73) comes from the first line of

$$\begin{aligned} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix} &= \mathbf{R} \mathbf{h}_{11} {}^t\mathbf{R} \begin{Bmatrix} F_1 \\ G_1 \\ E_1 \end{Bmatrix} + \mathbf{R} \mathbf{h}_{12} {}^t\mathbf{R} \begin{Bmatrix} F_2 \\ G_2 \\ E_2 \end{Bmatrix} \\ &+ \mathbf{R} \mathbf{h}_{13} {}^t\mathbf{R}' \begin{Bmatrix} F_3 \\ G_3 \\ E_3 \end{Bmatrix} + \mathbf{R} \mathbf{h}_{14} {}^t\mathbf{R}' \begin{Bmatrix} F_4 \\ G_4 \\ E_4 \end{Bmatrix} \end{aligned} \quad (75)$$

which is the extension of Eq. (69) with the added excitation points  $P_3$  and  $P_4$ .

Again, only the first line of this equation needs to be computed

$$\begin{aligned} S_{kl} &= (\mathbf{R} \mathbf{h}_{kl} {}^t\mathbf{R}')_{11} = r_{kl} \cos \psi' \cos \theta' \cos \psi \cos \theta \\ &+ o_{kl} \cos \psi' \sin \theta' \cos \psi \cos \theta + p_{kl} \sin \psi' \cos \psi \cos \theta \\ &+ o'_{kl} \cos \psi' \cos \theta' \cos \psi \sin \theta + s_{kl} \cos \psi' \sin \theta' \cos \psi \sin \theta \\ &+ q_{kl} \sin \psi' \cos \psi' \cos \psi \sin \theta + p'_{kl} \cos \psi' \cos \theta' \sin \psi \\ &+ q'_{kl} \cos \psi' \sin \theta' \sin \psi + t_{kl} \sin \psi' \sin \psi \end{aligned} \quad (76)$$

for  $kl = 13, 14, 23, 24$ . For  $kl = 11, 12, 21, 22$ , only one coordinate change is involved: Eq. (72) is still relevant. For  $kl = 33, 34, 43, 44$  again there is only one coordinate change involved, but this time with angles  $\theta'$  and  $\psi'$ .

Since Eqs. (72) and (76) involve many different transfers, it may lengthen computation times needlessly; indeed, for some particular angles  $\theta$  and  $\psi$  these equations can be simplified. So only the transfers that are not cancelled by a zero sine or cosine are actually computed.

### 3.6 Numerical considerations

The distinction between two kinds of problem can be made, which influences the way Eq. (41)  $\Delta = 0$  is solved. On one side, there is the purely real value problem: a spring is added to a linear undamped structure. The modified eigenvalues  $\lambda$  remain purely imaginary and there is only one unknown which is the pulsation  $\omega$ , or circular frequency

$$\omega \in \mathbf{R} \mid \Delta(\omega) = 0 \quad (77)$$

On the other side, the problem can be complex: either the structure is already damped, or a damper may be added to a linear undamped structure. In both cases, the new eigenvalues  $\lambda$  must be sought in the complex plane, which means there are two unknown: the damping coefficient  $\alpha$  (real part of  $\lambda$ ) and the pulsation  $\omega$  (imaginary part of  $\lambda$ )

$$\lambda \in \mathbf{C}, \lambda = \alpha + i\omega \mid \Delta(\lambda) = 0 \quad (78)$$

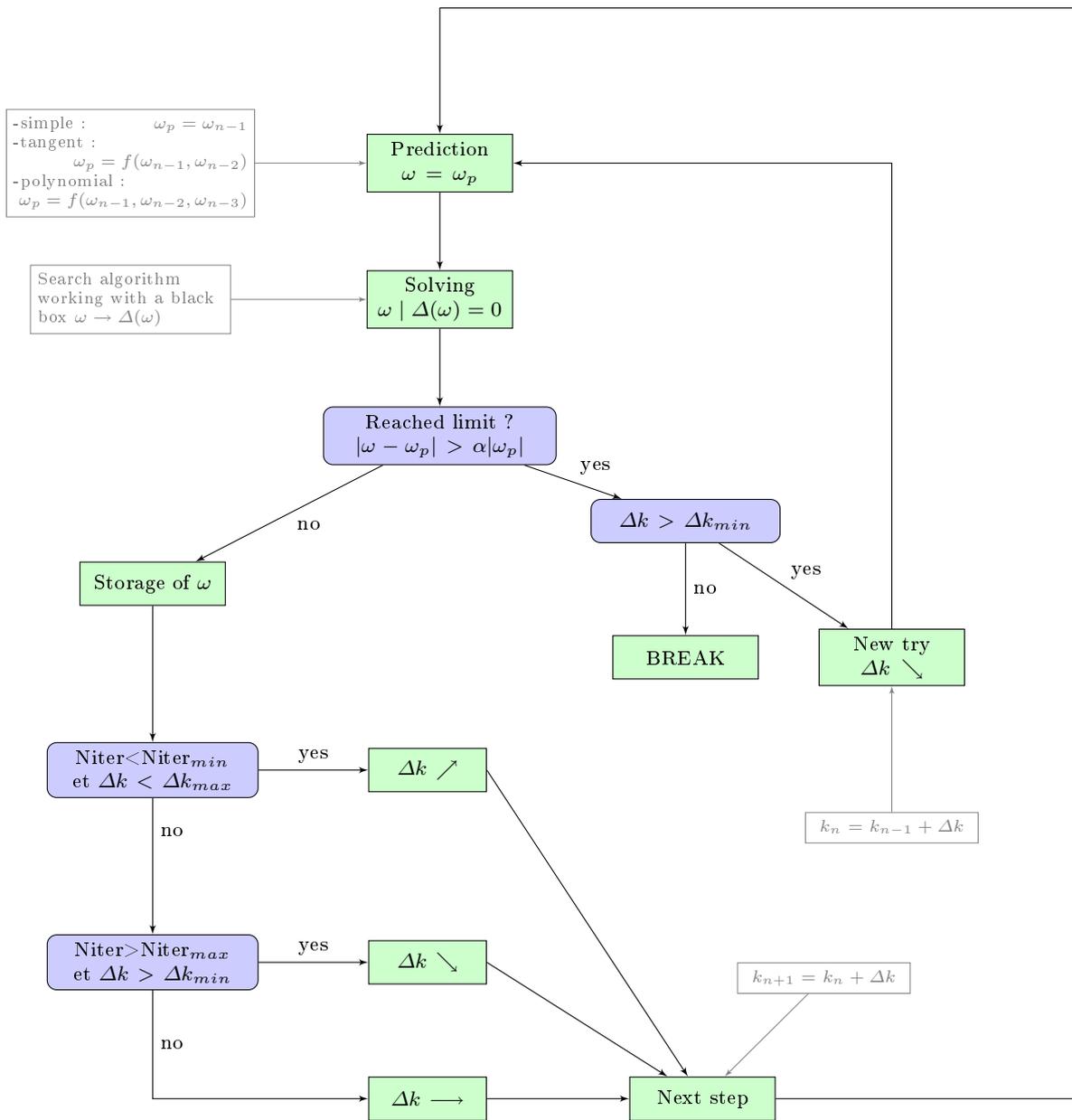


Fig. 4: Variable step algorithm for following the evolution of an eigenvalue

A particular continuation algorithm is developed to follow the solutions of the determinant while the impedance of the link(s) varies. The diagram of figure 4 details its steps. A first prediction of the solution is made; depending on how many points were previously computed, the prediction can be simple (same solution as previous point), tangent (with the use of the last two points) or polynomial of degree  $n$  (with the use of the  $n + 1$  previous points.) Note that a tangent prediction is the same as a polynomial prediction of degree 1. It is not useful to increase the order of the prediction beyond 2 since this can lead to oscillations and thus may not improve the prediction; moreover, the trajectories of the solutions remain most of the time relatively simple (see the following examples of section 4).

Once the solution is predicted, the exact solution is sought around this point with a search algorithm. The search algorithm is `fminbnd`<sup>1</sup> for the real value problem and `fsolve`<sup>2</sup> for the complex problem; both functions belong to the Matlab Optimization Toolbox. A research bound  $\alpha$ -parameter is defined to make sure that the found solution is not too far from the prediction.

Before going to the next step, the impedance increase is adjusted: if the solution was easily found, then the impedance step  $\Delta k$  is augmented; if the algorithm had difficulties in finding the solution, the impedance step  $\Delta k$  is decreased. Between these two cases, the impedance step remains constant. On the other side, when the solution is not found, the impedance step is decreased before making a new attempt. Indeed, it is expected that the closer from the previous point, the easier to find the next point.

A maximum and a minimum impedance step are defined. The minimum impedance step mainly serves as a stopping criterion if the algorithm ever fails to find a solution. This may happen if we are trying to observe a non-controllable mode, i.e. a mode which is not at all influenced by the structural modification. In fact, for the non-controllable mode, the force in the link is zero: this is the trivial solution of Eq. (40) and the determinant of Eq. (41) does not have a zero for the corresponding mode. The maximum impedance step may be used to restrict the distance between the points.

The advantage of this variable step algorithm is that it is able to deal with large impedance changes without having to search too many solutions. It is therefore possible to handle variations of impedance of several orders of magnitude: when the solution greatly varies while the impedance increases, the impedance step automatically decreases; on the contrary, when the solu-

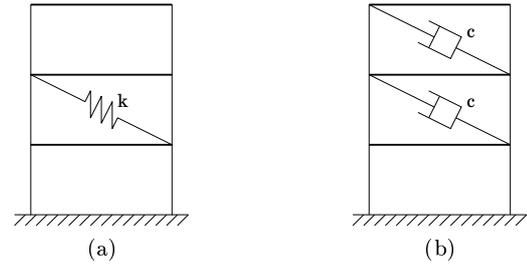


Fig. 5: Local modification of the frame structure: (a) stiffness modification; and, (b) damping modification

Characteristic	Value
storey height	2.236 m
storey width	4.472 m
vertical beam section	0.04 m $\times$ 0.04 m
horizontal beam section	0.08 m $\times$ 0.08 m
Young modulus E	210 GPa
Poisson modulus $\nu$	0.285
density $\rho$	7800 kg m <sup>-3</sup>

Table 1: Characteristics of the three-storey building

tion evolves slowly, then the impedance step automatically increases since there is no difficulty in finding the solution, thanks to the prediction step which becomes very efficient. This will be illustrated by the following examples.

## 4 Numerical examples

### 4.1 A first simple frame example

A beam structure modeling the frame of a building is studied. The geometry of the frame is given in figure 5 and its characteristics are given in table 1. Each floor is composed of 8 elements, the vertical beams between two floors are composed of 4 elements, leading to a 135 degrees of freedom structure. The first three modes are global flexion modes of the building; the respective pulsations are 7.88 rad s<sup>-1</sup>, 22.9 rad s<sup>-1</sup> and 34.6 rad s<sup>-1</sup>.

The determinantal method developed in the previous section is first applied on the stiffness modified frame of figure 5a. Since the structure is very simple, the Double Modal Synthesis was not employed in this first example; the response of the structure is calculated with the first 20 modes of the initial frame

$$H_{ij}(\omega) = \sum_{k=1}^{n_m=20} \frac{X_{ik}^T X_{kj}}{-\omega^2 + \omega_k^2} \quad (79)$$

<sup>1</sup> Find minimum of a single-variable function on fixed interval

<sup>2</sup> Solve system of nonlinear equations

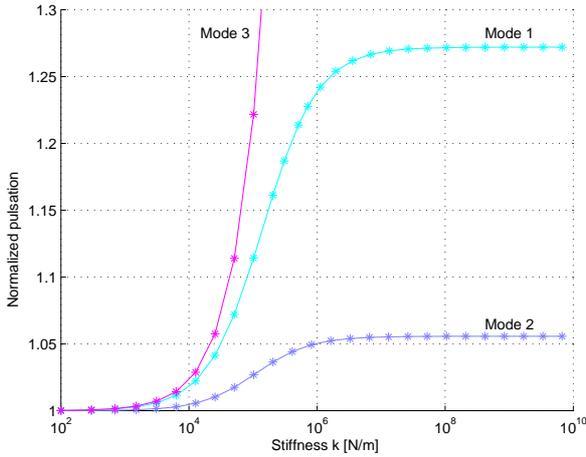


Fig. 6: Evolution, with respect to  $k$ , of the first three eigenfrequencies of the stiffness modified frame

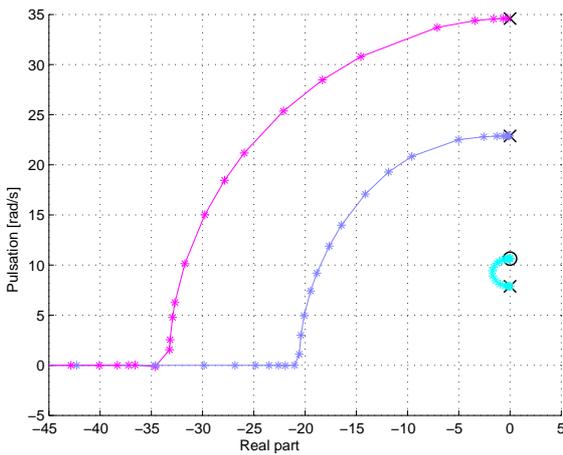


Fig. 7: Evolution in the complex plane of the first three eigenvalues of the frame with varying damping  $c$  ( $\times$  poles,  $\circ$  zeros)

It is important to notice that the same basis is kept whatever the importance of the modification because the determinantal method is based on the response of the unmodified structure.

The evolution of the first three eigenfrequencies with respect to  $k$  is illustrated in figure 6; the eigenfrequencies are normalised by their initial – unmodified – values for ease of comparison. The first flexion mode is more influenced by the modification than the second mode; the third mode frequency suddenly increases when the stiffness of the modification becomes comparable to the stiffness of the frame. Indeed the modification tends to

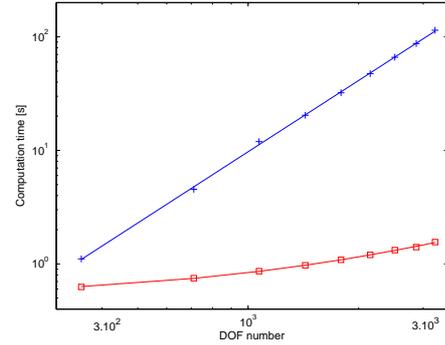


Fig. 8: Computation times for the frequency evolution, on 20 steps, of ten modes ( $\square$  determinantal method,  $+$  eigenvalue analysis,  $-$  regression)

block the relative horizontal displacement between the first and the second floor; yet the third mode cannot exist without this relative displacement, consecutively, the frequency of the third mode tends toward infinity.

The efficiency of the proposed method is also tested on a complex example having two viscous damping links (see figure 5b). This is illustrated in the root locus plot figure 7. The poles  $\times$  correspond to the unmodified eigenvalues and the zeros  $\circ$  are the eigenvalues of the blocked system<sup>3</sup>.

#### 4.2 Computation times

In order to see the efficiency of the proposed method, we compare the computation times of the determinantal method and the classical eigenvalue analysis for a varying number of degrees of freedom, in the case of one added link. The frame structure is kept since its degree of freedom number can be easily parameterized. The Double Modal Synthesis is not employed here because the aim is to compare only the local modification method.

The eigenvalue analysis consists in computing the first ten modes of the frame using its sparse mass and stiffness matrices thanks to the Matlab function `eigs`<sup>4</sup>. On the other side, the time taken by the determinantal method to follow the evolution of a mode is noticed to be independent of the mode number. Thus, to be compared with the eigenvalue analysis computation time, the determinantal method computation time for one

<sup>3</sup> The reader is invited to refer to root locus plot methods if more details about the construction and interpretation of this kind of graph are needed.

<sup>4</sup> Function to find a few eigenvalues and eigenvectors of a matrix using ARPACK library.

mode is multiplied by ten. As the number of degrees of freedom of the structure increases, the advantage of the proposed method for determining the modified eigenvalues becomes clear (see figure 8).

The computation time for the determinantal method mainly depends on:

- the time needed to evaluate the determinant  $\Delta(\lambda)$  for a certain value of  $\lambda$ ; when computing the response on the first  $n_m$  modes, this computation cost is linked to the degree of freedom number  $n$  and the number of retained modes  $n_m$ ;
- the number of time the determinant is computed to find its root for a particular impedance value, which depends on the algorithm and its solving parameters.

A power regression is performed on the eigenvalue computation time and it is found to be proportional to  $n^{2.09}$ . On the other side, a linear regression on the determinantal method computation time shows that this time is proportional to  $n$ . In both cases, the regression fits very well to the data (cf. figure 8).

The determinantal method is an exact formulation: if we are able to solve the determinantal equation, the exact eigenvalues of the modified structure are obtained. However, the precision of the method depends on two things: the precision of model of the structure, i.e. the expression of the transfer functions, which can be truncated or approximated in some way; the way the determinant is solved numerically, which can also lead to approximations. As a consequence it is possible to have as precise eigenvalues as possible provided that the model is exact and that the solving method is sufficiently accurate. The point in this article is not to give a fully optimized algorithm to solve Eq. (57), it is to introduce a new formalism, both simple and general, which can be applied on concrete cases.

### 4.3 Single mode approach application

The single mode method developed in section 3.1 is applied on the frame structure. The aim is to find the optimal stiffness  $e$  of the hysteretic link  $k^* = e(1 + i\beta)$  between points 2 and 3, for a given hysteretic damping in the link  $\beta = 0.3$ , which gives the maximum damping on the first mode of the frame. The whole structure has a hysteretic damping of 0.02. This analysis is analogous to the classic optimization rule for a tuned mass damper.

Equations 30 and 31 are rewritten, using Eq. (32) and the impedance  $Z_m$  ( $k^*$  and  $k_r$  in series), it gives

$$\tilde{\omega}_m^2 = \frac{(\chi + 1 + \beta^2) \tilde{\omega}_\infty^2 + (\chi + 1) \chi \tilde{\omega}^2}{(\chi + 1)^2 + \beta^2} \quad (80)$$

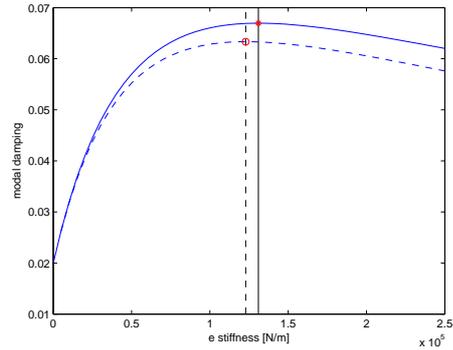


Fig. 9: First mode damping  $\eta$  versus modification stiffness  $e$ : — reference solution, - - single mode approach (the vertical lines indicate the optimal modification stiffness)

$$\eta_m = \frac{\beta\chi(\tilde{\omega}_\infty^2 - \tilde{\omega}^2) + ((\chi + 1)^2 + \beta^2)\tilde{\omega}^2\eta}{(\chi + 1 + \beta^2)\tilde{\omega}_\infty^2 + (\chi + 1)\chi\tilde{\omega}^2} \quad (81)$$

where  $\chi = \frac{k_r}{e}$ . The study of the function defined by Eq. (81) shows that the maximum damping is reached for

$$\chi_{\text{opt}} = \frac{1 + \beta^2 + \sqrt{\mu(1 + \beta^2)(\beta\eta + \mu\Omega - \Omega + 1)}}{\mu - 1} \quad (82)$$

where  $\mu = \frac{\beta}{\eta}$  and  $\Omega = \frac{\tilde{\omega}_\infty^2}{\tilde{\omega}^2}$ .

The pulsation of the first mode of the frame is  $\tilde{\omega} = 7.88 \text{ rad s}^{-1}$ ; the pulsation of the blocked system is  $\tilde{\omega}_\infty = 11.03 \text{ rad s}^{-1}$ , leading to a residual stiffness of  $k_r = 1.988 \times 10^5 \text{ N m}^{-1}$  with Eq. (32).

Finally, the optimal stiffness, with the single mode approach, is  $e_{\text{opt}} = 1.22 \times 10^5 \text{ N m}^{-1}$ . The comparison with the reference optimal stiffness  $e_{\text{opt}}^{\text{ref}} = 1.31 \times 10^5 \text{ N m}^{-1}$  gives a 6.20% relative error. Figure 9 illustrates the evolution of the first mode damping with varying modification stiffness. The curve for the single mode approach tends to move away from the reference solution as the stiffness of the local modification increases; however, the single mode approach gives a relatively good approximation of the optimal stiffness with very low computational effort. Indeed the reference solution requires the computation of the first eigenvalue on a wide range of stiffnesses, whereas Eq. (82) gives immediately a good approximation for the optimal stiffness.

This single mode approach is however limited to lightly damped structures on which a hysteretic damper is added. Nevertheless it may be considered as a complementary method to get a preliminary design before exploring more deeply some modification.

Characteristic	Value
stage height	1.2 m
total height	2.1 m
stage diameter	1.220 m
inner tube diameter	0.600 m
total mass	1500 kg
engine mass	880 kg
Young modulus $E$	210 GPa
Poisson modulus $\nu$	0.285
density $\rho$	$7800 \text{ kg m}^{-3}$

Table 2: Main characteristics of the launcher stage

#### 4.4 Local modification of a launcher's last stage

A structure representative of a launcher's last stage with its payload is considered. Figure 10 illustrates the geometry of the structure and the corresponding meshing; its main characteristics are given in table 2. The stage is composed of an outer structural cylinder; the inner cylinder and the attached mass represents the engine. Above this stage, a beam models the payload; it is connected to the stage through three columns (the attachment and spacing system of the payload).

This structure is well adapted to modal synthesis since it is clearly composed of two substructures: the stage; and the payload and its support. The stage substructure has 11 100 degrees of freedom and the payload substructure has 1550 degrees of freedom. Again, the Double Modal Synthesis only needs to be computed once since the determinantal method requires the initial behaviour. The base of the stage is considered to be clamped, since the connection interface of the launcher is very stiff.

The idea is to use the relative displacement between the engine and the outer shell to damp vibrations. Three dissipative links, with viscous damping  $c$ , are therefore added between the engine mass and the locally reinforced outer shell (●—● on figure 10b). Figure 11a shows the root locus plot of the system for the first four modes. The first mode is a flexion mode of the payload on its support; the second mode is a flexion mode of the inner tube; the third mode is a longitudinal mode, the payload and the engine mass vibrate vertically in phase; the fourth mode is again a longitudinal mode, but it is antisymmetrical. The mainly affected mode is mode two, however the other modes are also lightly damped (see figure 11b for a more detailed graph) because the added links are not exactly horizontal – which influences longitudinal modes – and the displacement of the engine mass is not zero for the payload modes.

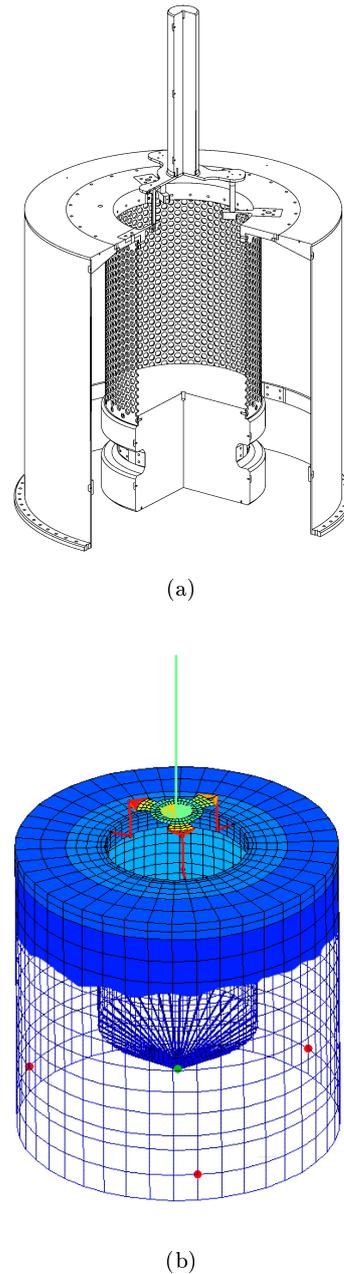
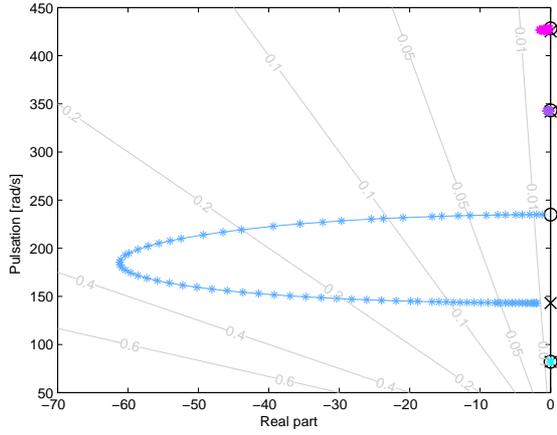


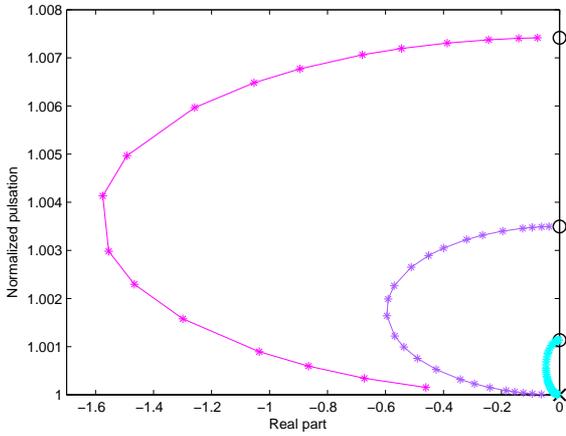
Fig. 10: Model of the launcher stage: (a) cross-section view; and, (b) meshing of the stage

Figure 12 gives the evolution of the real part of these three lightly damped modes with respect to link damping  $c$ . The increase of the impedance step is noticeable (the points are nearly linearly spaced on the vertical log axis). This figure also shows that the optimal damping for each mode may not be for the same value of  $c$ .

At last, the proposed method is applied on the launcher equipped with three equally spaced horizontal links between the engine (● on figure 10b) and the outer shell.



(a)



(b)

Fig. 11: Evolution in the complex plane of the first eigenvalues of the damped stage: (a) the four first modes, — curves of equal modal damping,  $\times$  poles and  $\circ$  zeros ; and, (b) detail of the three least damped modes

These links are composed of a spring in series with a viscous damper. The influence of both the stiffness  $k$  and the damping  $c$  of the links is illustrated by figure 13; the **blue curves** correspond to the evolution, in the complex plane, of the second pole with a damping parameter  $c$  varying from 0 (initial pole  $\times$ ) to  $\infty$  (zero  $\circ$ , corresponding to the blocked pulsation) and a fixed stiffness parameter; the **red curves** correspond to the evolution of the pole with a fixed damping parameter and a stiffness parameter decreasing from  $\infty$  to 0 (from left to right on figure 13). When the stiffness of the link tends toward zero, it amounts to remove the link: that is why the pole tends to return to its initial – unmodified – value. This kind of graphic allows to evaluate the

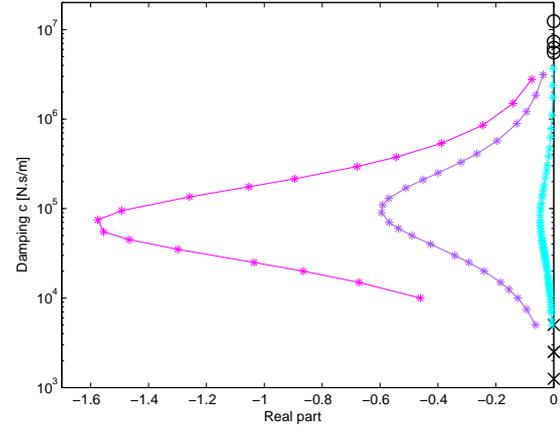


Fig. 12: Evolution of the real part of the eigenvalues with respect to viscous damping  $c$  in the three links

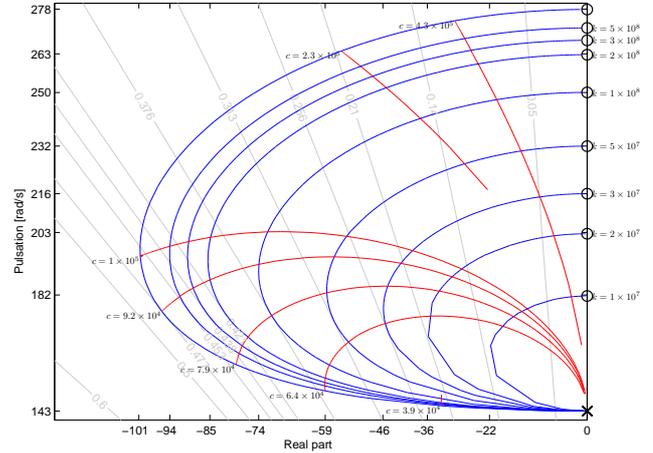


Fig. 13: Evolution in the complex plane of the damped stage second eigenvalue for varying  $k$  and  $c$  parameters

sensitivity of the pole to the parameters of the links, which may be crucial at a pre-design stage.

## 5 Conclusion

The proposed determinantal method appears to be promising to follow the evolution of the eigenvalues of a locally modified structure. The formalism and the underlying equations are relatively simple and it gives very accurate results. In addition to this, a single mode approach may be used to have a first approximation when dealing with the hysteretic damping optimization of a particular mode.

With the determinantal method, the same initial modal basis is kept through the exploration of the de-

sign parameters whatever the local modification is. This can be useful to determine the best place, according to a certain criterion, for a given impedance. In addition, this method also works with a black-box model of the initial structure.

A few numerical examples were developed to prove the efficiency of the proposed method. To deal efficiently with structures having a large number of degrees of freedom, a Double Modal Synthesis method is introduced. The new eigenvalues are then sought thanks to a continuation algorithm. This algorithm for solving the determinantal equation already shows good performances, it may however be further improved to reduce the computational cost.

Eventually, it would be interesting to apply the determinantal method to real structures through the use of experimental measurements of transfer functions at the location of the attachment points of the modification. However, this is far beyond the scope of this paper; measurement uncertainties and noise problems could arise and more work has to be done to be able to extract, if possible,  $H(\lambda)$  from experimental tests. Indeed, the knowledge of  $H(\omega)$  for a lightly damped structure only allows to calculate the effect of stiffness modifications.

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