Approval Voting for Committee Elections : a General Family of Rules

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Résumé

Le vote par approbation est une procédure de vote utilisée, entre autres, pour élire des comités et qui permet aux votants de voter pour ("d'approuver"), le nombre de candidats qu'ils souhaitent. Deux règles de vote ont été particulièrement utilisées pour élire des comités à l'aide du vote par approbation. La règle usuelle, appelée aussi minisum, choisit l'ensemble des candidats (éventuellement soumis à une contrainte de cardinalité) ayant été le plus approuvés par les votants. La règle minimax élit un ensemble de candidats qui minimise le maximum, sur l'ensemble des votants, de la distance de Hamming à chaque vote.
*Comme ces deux règles semblent trop extrêmes, nous les généralisons en un ensemble continu de règles de vote, par l'utilisation de l'opérateur de moyenne pondérée ordonnée (*ordered weighted averaging *OWA). Cette règle est paramétrée par un vecteur de poids, noté W, qui nous permet de modéliser des procédures de votes entre minisum et minimax. Nous nous intéressons aux vecteurs de poids non-décroissants, et en particulier aux vecteurs de la forme* $W(i) = (0, .., 0, 1, .., 1)$*, où* i *représente le nombre de 0. Nous étudions la complexité de la détermination d'un comité gagnant, et de l'ensemble des comités gagnants pour des règles associées aux vecteurs* $W(i)$*. Nous montrons qu'il est difficile de trouver l'ensemble des comités gagnants pour ces règles, sauf pour minisum avec un nombre impair de votants pour laquelle cela est facile. Enfin, nous prouvons la manipulabilité de ces règles quand elles sont paramétrées par des vecteurs non-décroissants, et strictement croissants.*

Mots Clef

Choix social computationnel, Vote d'approbation, Moyenne pondérée ordonnée, Complexité, Manipulation.

Abstract

Approval voting is a well-known voting procedure used, among others, for electing committees, where each voter casts a ballot consisting of a set of approved candidates (without any cardinality constraint). Two prominent rules for electing committees using approval voting are the standard rule (also called minisum), which selects the set of candidates (possibly subject to some cardinality constraint) with the highest number of approvals, and the minimax rule, where the set of elected candidates minimizes the maximum, over all voters, of the Hamming distance to the voter's ballot.
As these two rules are in some way too extreme, we generalize them into a continuum of rules, by using ordered weighted averaging operators (OWA). The rule is parameterized by a weight vector W, which allows us to model voting procedures between minisum and minimax. We focus on non-decreasing weight vectors, and in particular, vectors of the form W(i)=(0,..,0,1,..,1), where i is the number of 0's. We address the computational aspects of finding a winning committee and all the winning committees for rules associated with the W(i) vectors. We show that finding a winning committee for these rules is NP-hard whereas it is computationally easy for minisum. Finally, we address the issue of manipulating the rules when parameterized by non-decreasing and strictly increasing weight vectors.

Keywords

Computational social choice, Approval voting, Ordered weighted averaging, Complexity, Manipulation.

1 Introduction

Approval voting is a well-known voting procedure used for electing a single winner or a committee [2]. Voters cast approval ballots, which consist each of a set of approved candidates (without any cardinality constraint). An approval ballot can be seen as a binary vector indicating the candidates approved by a voter. Single-winner approval voting elects the candidate with the highest number of approvals (using some tie-breaking if necessary). As for multi-winner approval voting, there are several different ways of finding the winning committee, i.e., the winning set of candidates.

The most standard way consists in choosing the candidates according to their approval scores : in the case where the committee must be of size K, the candidates with the K highest number of approvals are elected ; or, if the size of the committee is not subject to any constraint, the candidates approved by a majority of voters are elected.

As argued in [3], this rule can be unfair to some voters, who may completely disagree with the elected committee. To remedy this, they define *minimax* approval voting, that selects a committee that minimizes the maximum, over all voters, of the Hamming distance to the voter's ballot, seen as a binary vector. The minimax outcome tends to be more widely acceptable than the minisum outcome, as it minimizes the disagreement of the least satisfied voter. Replacing max by sum in the definition of minimax approval voting (thus electing a committee that minimizes the sum of the Hamming distances to the ballots) leads back to the standard rule, that the authors rename *minisum*.

As these two rules (minimax and minisum) are in some way too extreme, we generalize them into a continuum of rules by using *ordered weighted averaging* (OWA). The OWA operator is an aggregation operator parameterized by a weight vector W, which enables us to model various operators like maximum, minimum, or arithmetic average [15, 16]. In OWA aggregation, the weights are associated with ordered values of the input, and the aggregated score corresponds to the ordered weighted sum.

We introduce multi-winner OWA approval rules, denoted by AV_W. A winning committee for AV_W is a committee minimizing the AV_W score, which corresponds to the aggregated score of OWA-W. We study a subset of AV_W parameterized by non-decreasing weight vectors which can model voting rules between minisum and minimax. First, we focus on a simple family of non-decreasing weight vectors $W(i) = (0, \ldots, 0, 1, \ldots 1)$, where i represents the number of 0's, and present some properties verified by winning committees. We address the computational aspects of finding a winning committee and finding the co-winner set for $AV_{W(i)}$. Furthermore, we give manipulation results on AV_W parameterized by non-decreasing and strictly increasing weight vectors.

This work is related to at least three research streams. The first of these is a series of works in social choice theory that study multi-winner approval voting through the two solution concepts minisum and minimax, [4, 3, 14]. One can find a review of the procedures that can be used for electing committees using approval voting in [7]. More generally, our paper relates to computational aspects of multi-winner elections [13].

The second related stream is the study of ordered weighted averaging, introduced in [15]. OWA has been studied in several domains, especially in multi-criteria decision making and decision under uncertainty [17].

The third related research stream is a series of works that study manipulation issues and strategic behaviour in multi-winner elections [12]. The computational aspects of strategic behavior in standard multiwinner approval voting have been studied in [1]. The works [5, 10] both address the computational aspects of minimax approval voting (see Section 2 for more details) and study the conditions under which an approximation of minimax is (or not) sensitive to manipulation. It relates mainly to the section 5, where we explore manipulation of AV_W.

Minimax approval voting also relates to belief merging, which aims at combining several pieces of information coming from different sources [8]. More details on this connection as well as on the computational aspects of multi-winner elections can be found in [9].

The paper is organized as follows. In Section 2 we introduce the necessary background on multi-winner approval voting and ordered weighted averaging. In Section 3 we give some properties on winning committees for $AV_{W(i)}$. In Section 4 we present some computational results on determining a winning committee for $AV_{W(i)}$. Section 5 addresses manipulation issues for AV_W parameterized by non-decreasing vectors. In Section 6 we present further research directions.

2 Preliminaries

We are given an election E with m candidates $X = \{x_1, \ldots, x_m\}$ and n voters $N = \{1, \ldots, n\}$. The *approval ballot* of a voter i is a subset of X, represented by a binary vector $v \in \{0,1\}^m$, where the j^{th} bit of v is 1 if the voter i approves the candidate j, 0 otherwise. An *approval profile* $P = (P_i)_{i \in N}$ is a collection of approval ballots. For a m-vector v, we denote by v_i, the i^{th} coordinate of the vector v, $i = 1, \ldots, m$. For two binary vectors $v, v' \in \{0,1\}^m$, we denote by $\mathrm{H}(v, v')$ their *Hamming distance*, i.e. the numbers of bits on which they differ. Given a binary vector $c \in \{0,1\}^m$, called a *committee*, and an approval profile P, the vector $\mathrm{H}(c, P) = (\mathrm{H}(c, P_i))_{i \in N}$ denotes the vector of Hamming distances between c and each ballot of P. The *approval score* of a candidate x, denoted by $app(x)$, is the number of voters who approve x.

In multi-winner approval voting, the goal is to elect a comittee, that is, a subset of X. Sometimes the size of the committee is fixed to some integer value K ; sometimes it is not fixed and any committee is feasible (see [3]).

The *minisum* rule consists in electing a committee that minimizes the sum of the Hamming distances to the ballots. Formally, $c^* \in \{0,1\}^m$ is a winning committee for minisum if and only if $\sum_{i \in N} \mathrm{H}(c^*, P_i) = \min_{c \in \{0,1\}^m} \sum_{i \in N} \mathrm{H}(c, P_i)$. A *majority voting committee* is a committee containing all the candidates approved by strictly more than $n/2$ voters, and no candidates approved by strictly less than $n/2$ voters. It is known that a winning committee for minisum is equivalent to a majority voting committee [3]. Thus, given an approval profile P, a winning committee for minisum can be easily computed.

The *minimax* rule elects a committee that minimizes the maximum of its Hamming distances to the ballots, i.e. c^* is a winning committee for minimax if and only if $\max_{i \in N} \mathrm{H}(c^*, P_i) = \min_{c \in \{0,1\}^m} \{\max_{i \in N} \mathrm{H}(c, P_i)\}$.

Given an approval profile P, finding a winning committee for minimax is NP-hard, since it is equivalent to the closest string problem in coding theory [6, 11].

The following example illustrates these two rules :

Example Consider an election E, with 6 voters $\{1, 2, 3, 4, 5, 6\}$, and 4 candidates $\{x_1, x_2, x_3, x_4\}$, and the following profile P :

$$\begin{array}{ll} P_1 & (0110) \\ P_2 & (0100) \\ P_3 & (0101) \\ P_4 & (0011) \\ P_5 & (1001) \\ P_6 & (0001) \end{array}$$

Figure 1 shows the Hamming distance, the minisum score and the minimax score for some committees. One can verify that the committees that are not mentioned are not winning committees for neither minisum nor minimax. The winning committees for minisum are c_2 and c_3, whereas they are c_1 and c_3 for minimax.

$H(c, i)$	1	2	3	4	5	6	sum	max
$c_1 = 0000$	2	1	2	2	2	1	10	2
$c_2 = 0001$	3	2	1	1	1	0	8	3
$c_3 = 0101$	2	1	0	2	2	1	8	2

FIGURE 1 – Minisum and minimax scores

We generalize minisum and minimax into a continuum of rules by using *ordered weighted averaging* (OWA) for multi-winner approval voting. Given a vector H, we can order its coordinates in a non-decreasing way and the ordered vector will be denoted by $H^{\uparrow}$. OWA is a family of functions, $O_W : \mathbb{R}_n \mapsto \mathbb{R}$, parameterized by a vector W of size n. It maps a n-vector of scores H to an agregated score, called OWA score : $H \mapsto O_W(H) = W \times H^{\uparrow}$. Note that the definition of OWA differs from the original introduced in [15] in two respects : we use an opposite ordering of vectors H, and we do not require any normalization of weights.

We now introduce multi-winner OWA approval rules, denoted by AV_W. A committee c is a *winning comittee* for AV_W if and only if $O_W(\mathrm{H}(c, P)) = \min_{c \in \{0,1\}^m}\{O_W(\mathrm{H}(c, P))\}$. We call *co-winner set* the set of all winning committees for an approval profile P. The following example illustrates OWA approval rules :

Example Consider the election of the previous example. Figure 2 shows the AV_W score for some committees.
Thus, if we take $W = (1, 2, 3, 4, 5, 6)$, the scores of c_1, c_2 and c_3 are respectively 39, 37 and 35 ; we can check that there is no better committee and that the winning committee is c_3. If we take $W = (0, 0, 0, 1, 1, 1)$, then the scores of c_1, c_2 and c_3 are respectively 6,6,6 and c_1, c_2 and c_3 are all winning committees. If we take $W = (0, 0, 1, 1, 1, 1)$, then the scores of c_1, c_2 and c_3 are respectively 8,7,7 ; c_2 and c_3 are all winning committees.

c	$W \times H^{\uparrow}(c, P)$
c_1	$1 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 2 \cdot W_4 + 2 \cdot W_5 + 2 \cdot W_6$
c_2	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 1 \cdot W_4 + 2 \cdot W_5 + 3 \cdot W_6$
c_3	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4 + 2 \cdot W_5 + 2 \cdot W_6$

FIGURE 2 – Minisum and minimax scores

For fairness issues, we will focus on *non-decreasing weight vectors* W, which give more weight to more unsatisfied voters than to more satisfied voters. A vector W is non-decreasing (respectively, strictly increasing) if $W_i \leq W_{i+1}$ (resp. $W_i < W_{i+1}$), for all $i = 1, \ldots, n-1$. AV_W parameterized by non-decreasing vectors allows us to model a range of voting rules between minisum and minimax, which can be represented respectively by the vectors $W = (1, \ldots, 1)$ and $W = (0, \ldots, 0, 1)$. An interesting family of vectors is the family $W(i)$ defined by $W(i) = (0, \ldots, 0, 1, \ldots, 1)$, where i is the number of 0's, for $i = 0, \ldots, n-1$; this family ranges from minisum approval voting (corresponding to $W(0)$) to minimax approval voting (corresponding to $W(n-1)$).

3 Winning committees for $AV_{W(i)}$

In this section, we focus on non-decreasing weight vectors $W(i) = (0, \ldots, 0, 1, \ldots, 1)$, where i is the number of 0's, in order to explore properties verified by winning committees.
First, we give simple sufficient conditions for a specific candidate to be a member of all winning committees, or not to be a member of any winning committee.

Proposition 3.1 *For $AV_{W(i)}$, $i \in N$, every candidate approved by at least $(n + i + 1)/2$ voters is included in all winning committees.*

Proof Consider an election E and an $AV_{W(i)}$ rule. Suppose there exists a candidate x approved by at least $(n + i + 1)/2$ voters, such that x is not included in a winning committee c. Then, the approval score of x verifies the following inequations :

$$\begin{array}{rcl} app(x) & \geq & (n+i+1)/2 \\ 2 \times app(x) - n & \geq & i+1 \\ app(x) - (n - app(x)) & \geq & i+1 \end{array} \qquad (1)$$

Now we study the $AV_{W(i)}$ score of the committee $c' = c \cup \{x\}$. Compared to c, there are $app(x)$ voters who have their Hamming distance reduced by 1, and $(n - app(x))$ voters who have their Hamming distance increased by 1. Since we consider the weight vector $W(i)$, it follows from inequality 1 that the $AV_{W(i)}$ score of c' is strictly less than the $AV_{W(i)}$ score of c, a contradiction. ∎

With a similar argument to the proof of Proposition 3.1 we can prove the following proposition.

Proposition 3.2 *For $AV_{W(i)}$, $i \in N$, every candidate approved by less than $(n-(i+1))/2$ voters is not included in any winning committee.*

Notice that Propositions 3.1 and 3.2 generalize the result stating that each majority voting committee is a winning committee for minisum [3]. These two propositions enable us to establish the following links between minisum and $AV_{W(1)}$ and between $AV_{W(n-2)}$ and minimax.

Proposition 3.3 *There always exists a committee c such that c is a winning committee for both minisum and $AV_{W(1)}$.*

Proof We distinguish two cases.
First consider an election E with an odd number of voters. Then we build the following winning committee c for the $AV_{W(1)}$ rule :

- Candidates that are approved by more than $(n+3)/2$ voters are in c. (prop. 3.1)
- Candidates that are approved by less than $(n-3))/2$ voters are in not c. (prop. 3.2)
- Candidates that are approved by exactly $(n+1)/2$ voters are in c. Clearly, since we consider $AV_{W(1)}$, removing such candidates from the committee will not decrease its score.
- Candidates that are approved by exactly $(n-1)/2$ voters are not in c. Clearly, since we consider $AV_{W(1)}$, adding such candidates to the committee will not decrease its score.

Furthermore, c is clearly a majority voting committee, hence c is a winning committee for minisum.
Now consider an election E with an even number of voters. Let c be a winning committee for $AV_{W(1)}$. Then it follows from Proposition 3.1 and 3.2 that :

- Candidates that are approved by more than $(n+2)/2$ voters are in c.
- Candidates that are approved by less than $(n-2)/2$ voters are in not c

Thus, by definition, the committee c is a majority voting committee, so c is also a winning committee for minisum. ∎

However, the result of Proposition 3.3 cannot be generalized to $AV_{W(1)}$ and $AV_{W(2)}$, as the following example (found by a computer program) shows.

Example Consider an election E with 7 voters $\{1,\ldots,7\}$, 5 candidates $\{x_1,\ldots,x_5\}$ and an approval profile defined by :

P_1	(01111)
P_2	(01111)
P_3	(01110)
P_4	(11111)
P_5	10000)
P_6	(10000)
P_7	(01011)

There is unique winning committee for $AV_{W(1)}$ which is (01111), and a unique winning committee for $AV_{W(2)}$ which is (11010).

Notice that we were not able to find an example for $AV_{W(1)}$ and $AV_{W(2)}$ when there is an even number of voters. Thus it remains open if the property holds for $AV_{W(1)}$ and $AV_{W(2)}$ and an even number of voters. Also, we conjecture that the counterexample generalizes to any pair $(i, i+1)$ for $2 \leq i \leq n-3$, that is, that for all i such that $2 \leq i \leq n-3$, there exists a profile P such that $AV_{W(i)}(P) \cap AV_{W(i+1)}(P) = \emptyset$. On the other hand, for $i = n-2$ we get a positive result again :

Proposition 3.4 *There always exists a committee c such that c is a winning committee for both $AV_{W(n-1)}$, that is to say minimax, and $AV_{W(n-2)}$.*

Proof Let E be an election with n voters and m candidates. Let c be a winning committee for minimax. Also, let c' be a winning committee for $AV_{W(n-2)}$ which maximizes the $(n-1)^{th}$ coordinate of the ordered vector of the Hamming distances.
We claim that c' is a winning committee for minimax or c is a winning committee for $AV_{W(n-2)}$. Indeed, suppose that c' is not a winning committee for minimax and c is not a winning committee for $AV_{W(n-2)}$. The fact that c' is not a winning committee for minimax implies that the $AV_{W(n-1)}$ score of c' is strictly larger than the $AV_{W(n-1)}$ score of c :

$$\begin{aligned} O_{W(n-1)}(\mathrm{H}(c', P)) &> O_{W(n-1)}(\mathrm{H}(c, P)) \\ H^{\uparrow}(c', P)_n &\geq H^{\uparrow}(c, P)_n + 1 \end{aligned} \tag{2}$$

Also, the fact that c' is not a winning committee for $AV_{W(n-2)}$ implies that the $AV_{W(n-2)}$ score of c is strictly larger than the $AV_{W(n-2)}$ score of c' :

$$\begin{aligned} O_{W(n-2)}(\mathrm{H}(c, P)) &> O_{W(n-2)}(\mathrm{H}(c', P)) \\ \sum_{j=n-1}^{n} H^{\uparrow}(c, P)_j &\geq \sum_{j=n-1}^{n} H^{\uparrow}(c', P)_j + 1 \end{aligned} \tag{3}$$

From (2) and (3) we obtain :

$$H^{\uparrow}(c', P)_{n-1} + 2 \leq H^{\uparrow}(c, P)_{n-1},$$

Therefore, we have :

$$H^{\uparrow}(c', P)_{n-1} + 2 \leq H^{\uparrow}(c, P)_{n},$$

This, together with (2), implies that :

$$\begin{aligned} H^{\uparrow}(c', P)_{n-1} + 2 &\leq H^{\uparrow}(c', P)_n - 1, \\ H^{\uparrow}(c', P)_{n-1} + 3 &\leq H^{\uparrow}(c', P)_n \end{aligned} \tag{4}$$

Inequality 4 implies that there exists at least one candidate x such that the voter corresponding to the $(n-1)^{th}$ largest

Hamming distance agrees with c' and the voter corresponding to the n^{th} largest Hamming distance disagrees with c' on x.
First assume that x is a member of c'. Now if we consider the committee $d = c' \backslash \{x\}$, then $H^{\uparrow}(d,P)_{n-1}$ increases by one, and $H^{\uparrow}(d,P)_n$ decreases by one compared to c'. So, d is also a winning committee for $AV_{W(n-2)}$, but $H^{\uparrow}(d,P)_{n-1} > H^{\uparrow}(c',P)_{n-1}$ which contradicts the definition of c'.
With the same argument, assuming that x is a not member of c' contradicts the definition of c'. ∎

4 Computational aspects

What we know about the complexity of computing winning committees for AV_W is that (a) finding a winning committee for minisum approval voting is polynomial, and it is also easy to give a simple polynomial characterization of all winning committees ; (b) finding a winning committee for minimax approval voting is NP-hard. Our conjecture is that finding a winning committee for AV_W is hard for "most" vectors W. We start by showing that, at least under the restriction to an odd number of voters, minisum is not the only AV_W committee election rule for which winner determination is polynomial : indeed, when the number of voters is odd, finding a winning committee is easy for $AV_{W(1)}$.

Proposition 4.1 *When n is odd, finding a winning committee for $AV_{W(1)}$ is polynomial time solvable.*

Proof Consider an election E, with an odd number of voters n, and the following committee $c = \{x \in X$, such that $app(x) \geq (n+1)/2$ }. As we have already shown in the proof of Proposition 3.3, c is a winning committee for $AV_{W(1)}$. ∎

The positive result of Proposition 4.1 could let us think that finding a winning committee is easy for $AV_{W(1)}$ in general. But the proof of Proposition 4.1 is based on a case study that cannot be used when there is an even number of voters. Indeed in that case, there are candidates with exactly $n/2$ votes, who could have either a positive or a negative effect on the score of a committee. Thus the complexity of finding a winning committee for $AV_{W(1)}$ remains an open question (and we do not have any conjecture about it). The complexity of finding a winning committee for $AV_{W(k)}$ for any $k \geq 2$ is also open ; we conjecture it is NP-hard. But so far, the only NP-hardness we have is for values of k that are very close to n :

Proposition 4.2 *For any fixed i, finding a winning committee for $AV_{W(n-i)}$ is NP-hard.*

Proof Our proof is based on a reduction from winner determination in minimax approval voting. Consider an election E' for minimax approval voting, $E' = (X', N', P')$ with $|X'| = m'$, $|N'| = n'$. We construct an election E for $AV_{W(n-i)}$ with $X = X'$, $N = i \times N'$ and $P = i \times P'$ meaning that we create i copies of each voter of N'. We claim that a winning committee for E is winning committee for E' and vice-versa. To prove this claim, it is sufficient to show that the $AV_{W(n-i)}$ score of a committee c in E is i times its minimax score in E'. Clearly for each voter in N' with a Hamming distance h, there exist i voters in N having the same Hamming distance h. ∎

We may wonder about the complexity of outputting *all* winning committees for $AV_{W(i)}$. We start by remarking that clearly, for minisum with an even number of voters, the size of the co-winner set can be exponential in the number of candidates (consider an election where all the candidates are approved by half of the voters ; then all the committees are winning committees). On the other hand, with an odd number of voters, there are no candidates approved by exactly half of voters, hence there is exactly one winning committee. We may wonder whether this last result extends to $AV_{W(i)}$ for $i \geq 1$; actually, it does not :

Proposition 4.3 *There exists a collection of elections such that the size of the co-winner set for any $i \geq 1$, $AV_{W(i)}$ is exponential in the number of candidates, even when n is odd.*

Proof We study an infinite family of elections $(E_{n\times m})_{n,m\in N}$, with an even number of voters, such that for each of its elements, the size of the co-winner set for any $AV_{W(i)}$ is in $\Theta(2^m)$. Consider the infinite family of elections $(E_{n\times m})_{n,m\in N} = (X, N, P)$, where P is the approval profile defined as follows : $n/2$ voters approve all the candidates, and the other $n/2$ voters approve no candidates.
Remark that, by construction of such an election, the ordered vector of Hamming distances, $\mathrm{H}^{\uparrow}(c,P)$, of any committee c to the profile P verifies : $\mathrm{H}^{\uparrow}(c,P) = (\alpha_c, \ldots, \alpha_c, \beta_c, \ldots, \beta_c)$, with $\alpha_c, \beta_c \in \mathbb{N}$, $n/2$ occurrences of each α_c and β_c, $\alpha_c \leq \beta_c$ and $\alpha_c + \beta_c = m$.
Now, consider an election $e_{n\times m}$ of this family, a vector $W(i)$, $i = 1 \ldots n-1$, and a committee c. We distinguish two cases :

– if $i \geq n/2$, then

$$O_{W(i)}(\mathrm{H}(c,P)) = (n-i) \times \beta_c$$

– if $i < n/2$, then

$$\begin{aligned} O_{W(i)}(\mathrm{H}(c,P)) &= n/2 \times \beta_c + (n/2 - i) \times \alpha_c \\ &= n/2 \times (\beta_c + \alpha_c) - i \times (m - \beta_c) \\ &= n/2 \times m - i \times m + i \times \beta_c \\ &= (n/2 - i)m + i \times \beta_c \end{aligned}$$

With these two results we see that finding the minimum of $O_{W(i)}(\mathrm{H}(c,P))$ is equivalent to minimizing the the value of β_c. The committees that have a minimal value of β_c

are the committees with $m/2$ candidates if m is even, and $\lfloor m/2 \rfloor$ or $\lfloor m/2 \rfloor + 1$ if m is odd. Thus, there are $2^{m/2}$ of such committees.

With the same argument, we can construct a similar family of elections $(E_{n \times m})_{n,m \in N} = (X, V, P)$, with an odd number of voters, such that for each of its element, the size of the co-winner set for an $AV_{W(i)}$ is in $\Theta(2^m)$, for $i \geq 1$. ∎

What this result does not say is whether we can find a simple, succinct characterization of all winning committees, as it is the case for minisum (recall that for minisum, the winning committees are all subsets of candidates that contain all candidates approved by a strict majority of voters and do not contain any candidate disapproved by a strict majority of voters). This question is left for further research.

5 Manipulability

In this section, we study the manipulability of the AV_W rules. Before starting, note that so far we consider the AV_W rules as *nondeterministic rules*, that is, mapping profiles into *sets of winning committees*. In this section, we consider *deterministic* rules (because studying the manipulability of nondeterministic rules requires much more background and definitions, that we do not have the space to expose here). A deterministic multi-winner OWA-approval rule is obtained as the composition of a nondeterministic AV_W rule and a tie-breaking mechanism ; the obtained deterministic rule is denoted by AV_W^R. As we do not want to sacrifice anonymity, a tie-breaking mechanism is a priority relation R over all possible committees [1] ; thus, in case there is more than one committee with a minimal score, the output of a deterministic rule AV_W^R will be the most prioritary committee (according to R) among the committees with minimal score.

We now discuss the manipulability of deterministic AV_W rules. A (deterministic) voting rule is said to be *manipulable* if a voter (or a group of voters) can change the result of the election in his/her favor by not expressing his/her true preferences. So far, we know that minimax approval voting is manipulable and that minisum approval voting is not [3]. How does this extend to AV_W rules in general ? We focus here in non-decreasing vectors (for reasons explained above) and we wonder whether minisum is in fact the only nonmanipulable rule of the family of AV_W rules with nondecreasing W. The following result gives an answer.

Proposition 5.1 *The AV_W rule is manipulable for any non-decreasing vector W such that $W_1 < W_n$.*

1. For instance, a tie-breaking rule could consist in choosing among the co-winning committees, the committee that contains the first candidate in a lexicographic order. If there are several such co-winning committees, then we choose among them the committee that contains the second candidate in a lexicographic order, and so on, until, there is a unique winning committee.

Proof Consider an election E, with 4 voters $\{1, 2, 3, 4\}$, and 4 candidates $\{x_1, x_2, x_3, x_4\}$, and the following profile P :

$$P_1 : (0100); P_2 : (0101); P_3 : (0110); P_4 : (0111)$$

Figure 3 shows the Hamming distance vectors for some committees. One can easily verify that committees that are not mentioned have AV_W scores larger than the AV_W scores of co-winning committees. The AV_W scores of

c	$W \times H^{\uparrow}(c, P)$
0100	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4$
0101	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4$
0110	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4$
0111	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 2 \cdot W_4$

FIGURE 3 – AV_W scores with true preferences

these committees are equal and minimal, so there are 4 co-winning committees. Then, according to the tie-breaking rule, the committee (0111) is elected. The manipulation comes from voter 1 by voting (1000) instead of his true preferences. With this new vote, we have new AV_W scores summarized in Figure 4. Then, with a non-decreasing W

c	$W \times H^{\uparrow}(c, P)$
0100	$1 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 2 \cdot W_4$
0101	$0 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 3 \cdot W_4$
0110	$0 \cdot W_1 + 1 \cdot W_2 + 2 \cdot W_3 + 3 \cdot W_4$
0111	$0 \cdot W_1 + 1 \cdot W_2 + 1 \cdot W_3 + 4 \cdot W_4$

FIGURE 4 – AV_W scores with manipulation of voter 1

such that $W_1 < W_n$, we obtain an unique winning committee (0100). ∎

This proof can be extended to any numbers of candidates by adding dummy candidates approved by all the voters, and to even numbers of voters by adding copies of voters 1 and 4.

6 Conclusion

We have generalized minisum and minimax approval voting by defining a continuous family of committee voting rules that contain minisum and minimax as special cases. We have established a few preliminary results about computational complexity of determining a winning committee. We have also shown that among our rules, only minisum is nonmanipulable. Further research obviously starts by establishing further complexity results, by trying to prove that "almost" all rules of our families are hard to compute.

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