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CONCRETE EXAMPLES OF $\mathcal{H}(b)$ SPACES

EMMANUEL FRICAIN, ANDREAS HARTMANN, AND WILLIAM T. ROSS

ABSTRACT. In this paper we give an explicit description of de Branges-Rovnyak spaces $\mathcal{H}(b)$ when b is of the form q^r , where q is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

1. INTRODUCTION

The purpose of this paper is to explicitly describe the elements of the de Branges-Rovnyak space $\mathcal{H}(b)$ for certain $b \in \mathbf{b}(H^\infty)$. Here H^∞ denotes the space of bounded analytic functions on the open unit disk \mathbb{D} normed by $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$, and $\mathbf{b}(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leq 1\}$ is the closed unit ball in H^∞ and, for $b \in \mathbf{b}(H^\infty)$, the *de Branges-Rovnyak space* $\mathcal{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on \mathbb{D} whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

Besides possessing a fascinating internal structure [9], $\mathcal{H}(b)$ spaces play an important role in several aspects of function theory and operator theory, most importantly, in the model theory for many types of contraction operators [3, 4].

Despite the important role $\mathcal{H}(b)$ spaces play in operator theory, the exact contents of $\mathcal{H}(b)$ often remain mysterious. What functions belong to $\mathcal{H}(b)$? Certainly the kernel functions $k_\lambda^b, \lambda \in \mathbb{D}$, do (and have dense linear span). What else?

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In this paper, we give a precise description of the elements of $\mathcal{H}(b)$ for certain relatively simple b , namely positive powers of rational outer functions. Our description needs the following set up. If $b \in \mathbf{b}(H^\infty)$ is a non-extreme point of $\mathbf{b}(H^\infty)$, equivalently, $\log(1 - |b|) \in L^1(\mathbb{T}, m)$ (where $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and m Lebesgue measure on \mathbb{T} normalized so that $m(\mathbb{T}) = 1$), then there exists a unique outer function $a \in \mathbf{b}(H^\infty)$, called the *Pythagorean mate* for b , such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ almost everywhere on \mathbb{T} . The pair (a, b) is said to be a *Pythagorean pair*.

Our first observation says that in certain situations $\mathcal{H}(b^r)$ does not depend on $r > 0$.

Theorem 1.1.

- (1) *Suppose $b \in \mathbf{b}(H^\infty)$ is outer. The following are equivalent:*
- (a) *For any $r > 0$ we have $\mathcal{H}(b^r) = \mathcal{H}(b)$ as sets.*
 - (b) *$\mathcal{H}(b^2) = \mathcal{H}(b)$ as sets.*
 - (c) *$b\mathcal{H}(b) \subset \mathcal{H}(b)$.*
- (2) *If b is non-extreme, i.e., $\log(1 - |b|) \in L^1(\mathbb{T})$, with Pythagorean mate a , then conditions (a), (b), and (c) are equivalent to the condition*

$$(1.2) \quad \inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0.$$

- (3) *If b extreme, i.e., $\log(1 - |b|) \notin L^1(\mathbb{T})$, then conditions (a), (b), and (c) are equivalent to the condition*

$$(1.3) \quad b \text{ is invertible in } H^\infty.$$

Remark 1.4. (1) Since b is outer, it has no zeros on \mathbb{D} and so we can define b^r by taking any logarithm of b . Note that $b^r \in \mathbf{b}(H^\infty)$.

- (2) Statement (a) of Theorem 1.1 says that $\mathcal{H}(b^r) = \mathcal{H}(b)$ as sets. Though the norms on $\mathcal{H}(b^r)$ and $\mathcal{H}(b)$ are different, one sees from the closed graph theorem that they are equivalent.
- (3) Statement (c) of the theorem says that b is a *multiplier* of $\mathcal{H}(b)$. We refer the reader to Sarason's book [9] for further information and references about multipliers of $\mathcal{H}(b)$.
- (4) By Carleson's corona theorem [6], the condition (1.2) is equivalent to existence of $\phi, \psi \in H^\infty$ so that $a\phi + b\psi = 1$ on \mathbb{D} . Such a pair (a, b) satisfying this condition is called a *corona pair*.

When b is a *rational* outer function, or any positive power of a rational function (which is necessarily non-extreme (see Lemma 3.1)), we obtain the following complete description of $\mathcal{H}(b)$ involving the derivatives of the reproducing kernels. Indeed, when $b = q^r$, where q is outer and rational and $r > 0$, we set

$$v_{r,\lambda}^\ell(z) := \frac{d^\ell}{d\bar{\lambda}^\ell} k_\lambda^{q^r}(z) = \frac{d^\ell}{d\bar{\lambda}^\ell} \left(\frac{1 - \overline{q^r(\lambda)} q^r(z)}{1 - \bar{\lambda}z} \right),$$

for any $z \in \mathbb{D}$, $\lambda \in \mathbb{D}^-$, and $\ell \geq 0$. We let H^2 denote the classical Hardy space [6]. By means of the Féjer-Riesz theorem (see Section 6), one can prove that if q is a rational function then so is its Pythagorean mate a . In this case, also notice that for $\zeta \in \mathbb{T}$ we have $|q(\zeta)| = 1$ if and only if $a(\zeta) = 0$.

Theorem 1.5. *Suppose $q \in \mathbf{b}(H^\infty)$ is a rational outer function and r is a positive real number. Then*

- (1) $\mathcal{H}(q^r) = \mathcal{H}(q)$ as sets.
- (2) *If a is the Pythagorean mate for q and a has distinct zeros ζ_1, \dots, ζ_n on \mathbb{T} with corresponding multiplicities m_1, \dots, m_n , then*
 - (a) *the functions $v_{r,j}^\ell := v_{r,\zeta_j}^\ell$ are well-defined and belong to $\mathcal{H}(q^r)$ for $1 \leq j \leq n$ and $0 \leq \ell \leq m_j - 1$. Moreover, they are orthogonal to*

$$aH^2 = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2.$$

- (b) $\mathcal{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \oplus \bigvee \{ v_{r,j}^\ell : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n \},$$

where the orthogonal decomposition is in terms of the inner product in $\mathcal{H}(q^r)$.

Writing $v_j^\ell = v_{1,j}^\ell$, the theorem above implies that $\mathcal{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \dot{+} \bigvee \{ v_j^\ell(z) : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n \},$$

where the sum is no longer necessarily orthogonal.

It was shown in [2], and rediscovered in [1], that

$$(1.6) \quad \mathcal{H}(q) = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \dot{+} \mathcal{P}_{N-1},$$

where $N = \sum_{j=1}^n m_j$, \mathcal{P}_{N-1} is the N -dimensional vector space of polynomials of degree at most $N - 1$, and the sum is an algebraic direct sum (not necessarily an orthogonal one). The novelty of our result is that we can precisely identify the orthogonal complement of $aH^2 = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2$ in $\mathcal{H}(q)$ without using (1.6).

In a recent preprint, Lanucha and Nowak [7] examined when an $\mathcal{H}(b)$ space is isomorphic to a Dirichlet type space. Their discussion naturally leads to the situation when a is a polynomial with simple zeros on \mathbb{T} and a similar description of $\mathcal{H}(b)$ for such a .

A key ingredient used to show statement (1) of Theorem 1.5, and an added bonus to our result, is that if a_r is the Pythagorean mate for q^r then the co-analytic Toeplitz operators $T_{\bar{a}}$ and $T_{\bar{a}_r}$ on H^2 have the same range, namely $\mathcal{H}(q)$.

2. PRELIMINARIES

There are several equivalent definitions of the de Branges-Rovnyak space $\mathcal{H}(b)$. We can, for instance, define it in the standard way [8] as the reproducing kernel Hilbert space associated with the (positive definite) reproducing kernel

$$k_{\lambda}^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

By definition, $f(\lambda) = \langle f, k_{\lambda}^b \rangle_b$ for all $f \in \mathcal{H}(b)$ and $\lambda \in \mathbb{D}$, where $\langle \cdot, \cdot \rangle_b$ represents the scalar product in $\mathcal{H}(b)$.

The space $\mathcal{H}(b)$ can also be defined as the range space $(I - T_b T_{\bar{b}})^{1/2} H^2$ equipped with the norm which makes $(I - T_b T_{\bar{b}})^{1/2}$ a partial isometry. Here T_{φ} is the Toeplitz operator on H^2 with symbol $\varphi \in L^{\infty}(\mathbb{T})$ defined by

$$T_{\varphi} f = P_+(\varphi f), \quad f \in H^2,$$

where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 . The book [9] is the classic reference for $\mathcal{H}(b)$ spaces.

When $\|b\|_\infty < 1$, $\mathcal{H}(b)$ turns out to be a renormed version of H^2 while if b is an inner function, then $\mathcal{H}(b)$ turns out to be one of the classical and well-studied model spaces $H^2 \ominus bH^2$.

When b is non-extreme and a is its Pythagorean mate, two important (not necessarily closed) vector spaces of functions in $\mathcal{H}(b)$ are

$$\mathcal{M}(a) := T_a H^2 \quad \text{and} \quad \mathcal{M}(\bar{a}) := T_{\bar{a}} H^2.$$

It follows from the Douglas factorization theorem and the operator inequalities

$$(2.1) \quad T_a T_{\bar{a}} \leq T_{\bar{a}} T_a \quad \text{and} \quad T_{\bar{a}} T_a = I - T_{\bar{b}} T_b \leq I - T_b T_{\bar{b}}$$

that $\mathcal{M}(a) \subset \mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$ (see [9, p. 24]).

For technical reasons, we will make use of the space $\mathcal{H}(\bar{b})$ which, for any $b \in \mathbf{b}(H^\infty)$, is defined similarly as with $\mathcal{H}(b)$ but as the range space $(I - T_{\bar{b}} T_b)^{1/2} H^2$. The operator inequalities from (2.1) show that $\mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{H}(b)$.

3. CORONA PAIRS

This following lemma is well-known but we record it here along with a proof for the sake of completeness and for the discussion of the examples in Section 6.

Lemma 3.1. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. Then q is non-extreme and, if a is the Pythagorean mate for q , then a is also rational.*

Proof. Since q is rational then $q = p_1/p_2$ where p_1 and p_2 are analytic polynomials and p_2 has no zeros on \mathbb{D}^- . We can, of course, choose p_2 such that $p_2(0) > 0$. Since $q \in \mathbf{b}(H^\infty)$, we see that $1 - |q(e^{i\theta})|^2 \geq 0$ for all θ and so $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is a non-negative trigonometric polynomial. Furthermore, $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is not the zero function since we are assuming that q is not an inner function. By the Féjer-Riesz theorem, $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 = |p(e^{i\theta})|^2$, where p is an analytic polynomial which is zero free in \mathbb{D} and $p(0) > 0$.

Let $a = p/p_2$. Note that a is rational and zero free in \mathbb{D} , hence outer. Moreover, $a(0) > 0$.

Furthermore, on \mathbb{T} we have

$$|a|^2 = \left| \frac{p}{p_2} \right|^2 = \frac{|p_2|^2 - |p_1|^2}{|p_2|^2} = 1 - \left| \frac{p_1}{p_2} \right|^2 = 1 - |q|^2.$$

This means that (a, q) is a Pythagorean pair which, in particular, implies that q is non-extreme. \square

Lemma 3.2. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and r is a positive real number. Then b and b^r are simultaneously non-extreme. Moreover, if a_r is the Pythagorean mate for b^r , the pairs (a, b) and (a_r, b^r) are simultaneously corona.*

Proof. Since

$$(3.3) \quad \frac{1 - x^r}{1 - x} \asymp 1, \quad x \in [0, 1),$$

we see that $1 - |b|^r \asymp 1 - |b|$ when $b \in \mathbf{b}(H^\infty)$, from which we deduce the first part of the Lemma.

Now observe that

$$\frac{|a|^2}{|a_r|^2} = \frac{1 - |b|^2}{1 - |b^r|^2} \asymp 1,$$

and since a and a_r are outer, Smirnov's theorem (which says that if the boundary function for the quotient of two outer functions is bounded on \mathbb{T} , then $f \in H^\infty$), shows that a/a_r is invertible in H^∞ . Thus both expressions

$$\inf_{z \in \mathbb{D}} (|a(z)| + |b(z)|) \quad \text{and} \quad \inf_{z \in \mathbb{D}} (|a_r(z)| + |b^r(z)|)$$

are strictly positive (or not) simultaneously. Indeed, if there is a sequence $\{z_n\}_{n \geq 1}$ in \mathbb{D} such that one expression goes to 0 then, since both $a(z_n)$ and $b(z_n)$ go to zero, the other expression will go to zero as well. \square

A special situation where b forms a corona pair with its Pythagorean mate is when b is rational.

Lemma 3.4. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. If a is the Pythagorean mate for q , then (a, q) is a corona pair.*

Proof. According to the proof of Lemma 3.1, we know that a is rational, $a = p/p_2$, where p and p_2 are polynomials, p_2 has no zeros in \mathbb{D}^- and p is zero free in \mathbb{D} . In particular, a is analytic in an open neighborhood of \mathbb{D}^- and thus has a finite number of zeros on \mathbb{T} , say $\{\zeta_1, \dots, \zeta_n\}$. Note that, due to the identity $|a|^2 + |q|^2 = 1$ on \mathbb{T} , the zeros of a (on \mathbb{T}) must lie where q is unimodular on \mathbb{T} .

Let D_j be disjoint open disks with center at the zeros ζ_j of a and let

$$F = \mathbb{D}^- \setminus \bigcup_{j=1}^n D_j.$$

By making the disks smaller, one can, by using the continuity of $|q|$ on \mathbb{D}^- , arrange things so that $|q| \geq \frac{1}{2}$ on each $D_j \cap \mathbb{D}^-$.

Notice that F is closed and omits all of the zeros of a in \mathbb{D}^- and so

$$\inf_{z \in F} |a(z)| = \delta > 0.$$

Thus

$$\inf_{z \in \mathbb{D}} (|a(z)| + |q(z)|) \geq \min(\frac{1}{2}, \delta) > 0$$

concluding the proof. \square

The first statement of Theorem 1.1 depends on the following two results. The first is from Sarason's book [9, p. 62].

Proposition 3.5. *For $b \in \mathbf{b}(H^\infty)$ and non-extreme, the following are equivalent:*

- (1) (a, b) is a corona pair;
- (2) $\mathcal{H}(b) = \mathcal{M}(\bar{a})$.

The second is the following.

Proposition 3.6. *If $a, a_1 \in H^\infty$ are two outer functions such that a/a_1 and a_1/a belong to L^∞ , then $\mathcal{M}(\bar{a}) = \mathcal{M}(\bar{a}_1)$.*

Proof. Again, by Smirnov's theorem, we know that a/a_1 and a_1/a belong to H^∞ . Thus T_{a/a_1} , and hence $T_{\overline{a/a_1}}$, are invertible operators on H^2 . From here we get

$$\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2 = T_{\bar{a}_1}T_{\overline{a/a_1}}H^2 = T_{\bar{a}_1}H^2 = \mathcal{M}(\bar{a}_1). \quad \square$$

4. $\mathcal{H}(b^r) = \mathcal{H}(b)$ AS SETS

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The implication (a) \implies (b) is trivial.

To show (b) \implies (c) note from [9, I-10] we have

$$\mathcal{H}(b^2) = \mathcal{H}(b) + b\mathcal{H}(b).$$

But since we are assuming that $\mathcal{H}(b^2) = \mathcal{H}(b)$ it follows that $b\mathcal{H}(b) \subset \mathcal{H}(b)$.

For the implication (c) \implies (1.2), we use the fact that b is non-extreme and [9, VIII-1, VIII-7] to see that b being a multiplier of $\mathcal{H}(b)$ is equivalent to (a, b) being a corona pair.

To show that (1.2) \implies (a), we proceed as follows. By Lemma 3.2 we know that since (a, b) is a corona pair, then so is (a_r, b^r) . Thus from Proposition 3.5 we see that $\mathcal{H}(b) = \mathcal{M}(\bar{a})$ and $\mathcal{H}(b^r) = \mathcal{M}(\bar{a}_r)$. As in the proof of Lemma 3.2 a/a_r and a_r/a belong to H^∞ so, by Proposition 3.6, we get $\mathcal{M}(\bar{a}) = \mathcal{M}(\bar{a}_r)$. Putting this all together we get the desired set equality $\mathcal{H}(b^r) = \mathcal{H}(b)$.

For the implication (c) \implies (1.3), we use the fact that b is extreme and [9, VIII-1, VIII-5] to see that b being a multiplier of $\mathcal{H}(b)$ is equivalent to b being an invertible element of H^∞ .

It remains to show (1.3) \implies (a). Assuming that b is invertible in H^∞ , we use, once again, [9, VIII-1] to see that $\mathcal{H}(b) = \mathcal{H}(\bar{b})$. But, since b is invertible in H^∞ , then so is b^r and we thus also have $\mathcal{H}(b^r) = \mathcal{H}(\bar{b}^r)$. Remember that

$$\frac{1 - |b|^2}{1 - |b^r|^2} \asymp 1$$

and thus there are two constants $c_1, c_2 > 0$ such that

$$c_1(I - T_{\bar{b}}T_b) \leq I - T_{\bar{b}^r}T_{b^r} \leq c_2(I - T_{\bar{b}}T_b).$$

The Douglas factorization theorem implies that $\mathcal{H}(\bar{b}) = \mathcal{H}(\bar{b}^r)$ which concludes the proof. \square

Remark 4.1. In Theorem 1.1 we see from the above proofs that one can add the condition $\mathcal{H}(b) = \mathcal{H}(\bar{b})$ to the list of equivalent conditions.

5. THE CONTENTS OF $\mathcal{H}(b)$

We now can give the proof of Theorem 1.5. Indeed, statement (1) of the theorem follows from Lemma 3.4 and Theorem 1.1. Let us consider statement (2).

In [5] it was shown, for an outer function b , that if $\zeta \in \mathbb{T}$ and

$$(5.1) \quad \int_{\mathbb{T}} \frac{|\log |b(w)||}{|w - \zeta|^{2n+2}} dm(w) < \infty,$$

then every function in $\mathcal{H}(b)$, as well as its derivatives up to order n , has a finite non-tangential limit at ζ .

Recalling the notation $v_{r,\lambda}^\ell$ for the ℓ -th derivative in the variable $\bar{\lambda}$ of the reproducing kernel in $\mathcal{H}(q^r)$, the results of [5] also show that $v_{r,\zeta}^\ell \in \mathcal{H}(q^r)$, $0 \leq \ell \leq n$, and

$$f^{(\ell)}(\zeta) = \langle f, v_{r,\zeta}^\ell \rangle_{q^r}, \quad f \in \mathcal{H}(q^r), \quad 0 \leq \ell \leq n.$$

Let us check condition (5.1) for our situation. Since q is rational its Pythagorean mate a is also rational and can be written as

$$(5.2) \quad a(z) = s(z) \prod_{j=1}^n (z - \zeta_j)^{m_j},$$

where s is a rational function whose poles and zeros lie on the complement of \mathbb{D}^- . Pick $w = e^{it}$ near one of the zeros $\zeta_j = e^{i\theta_j}$ of a . Then

$$\begin{aligned} |\log |q^r(e^{it})|| &\asymp |\log |q(e^{it})|^2| = |\log(1 - |a(e^{it})|^2)| \\ &\asymp |a(e^{it})|^2 \asymp |e^{it} - e^{i\theta_j}|^{2m_j} \end{aligned}$$

This means that for t near θ_j we have

$$\frac{|\log |q^r(e^{it})||}{|e^{it} - e^{i\theta_j}|^{2(m_j-1)+2}} \asymp 1$$

and so, by (5.1), every function in $\mathcal{H}(b)$ as well as its derivatives up to the order $m_j - 1$ admits non-tangential limits at ζ_j , and $v_{r,\zeta_j}^\ell \in \mathcal{H}(q^r)$ for all $0 \leq \ell \leq m_j - 1$.

The following interesting observation will be very useful in the proof of our main theorem.

Lemma 5.3. *Suppose $a(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j}$, where $\zeta_j \in \mathbb{T}$ and m_j is the corresponding multiplicity. If the non-tangential limits of $af = T_{\bar{a}}g \in \mathcal{M}(\bar{a})$, along with the non-tangential limits of its derivatives up to order $m_j - 1$, vanish at every point ζ_j , $j = 1, \dots, n$, then*

$$(5.4) \quad \widehat{g}(0) = \widehat{g}(1) = \dots = \widehat{g}(N-1) = 0,$$

where $N = \sum_{j=1}^n m_j$.

Proof of Lemma. We prove (5.4) as follows. Consider the kernels

$$k_{\lambda,\ell}(z) = c_\ell \frac{z^\ell}{(1 - \bar{\lambda}z)^{\ell+1}},$$

where c_ℓ is adjusted so that these are the reproducing kernels for ℓ -th derivatives at point $\lambda \in \mathbb{D}$ in the Hardy space H^2 , that is to say,

$$f^{(\ell)}(\lambda) = \langle f, k_{\lambda, \ell} \rangle_{H^2} = \int_{\mathbb{T}} f(\zeta) \overline{k_{\lambda, \ell}(\zeta)} dm(\zeta), \quad f \in H^2.$$

Observe, for $1 \leq j \leq n$ and $0 \leq \ell \leq m_j - 1$, that

$$\begin{aligned} a(z)k_{t\zeta_j, \ell}(z) &= c_\ell \frac{z^\ell (z - \zeta_j)^{m_j}}{(1 - t\overline{\zeta_j}z)^{\ell+1}} \prod_{k \neq j} (z - \zeta_k)^{m_k} \\ &= c_\ell z^\ell (z - \zeta_j)^{m_j - (\ell+1)} \left(\frac{z - \zeta_j}{1 - t\overline{\zeta_j}z} \right)^{\ell+1} \prod_{k \neq j} (z - \zeta_k)^{m_k}. \end{aligned}$$

Writing

$$\frac{z - \zeta_j}{1 - t\overline{\zeta_j}z} = -\zeta_j \left(1 - \overline{\zeta_j}z \frac{1-t}{1 - t\overline{\zeta_j}z} \right),$$

we see that $a(z)k_{t\zeta_j, \ell}(z)$ is uniformly bounded in $z \in \mathbb{D}$ and $t \in [0, 1)$, and moreover

$$\frac{z - \zeta_j}{1 - t\overline{\zeta_j}z} \rightarrow -\overline{\zeta_j}, \quad t \rightarrow 1,$$

for every z . Thus, by the dominated convergence theorem,

$$ak_{t, \ell} \rightarrow cz^\ell (z - \zeta_j)^{m_j - (\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k}$$

in the norm of H^2 , where c is some non zero constant depending on ℓ and j .

Choose any function $f = T_{\overline{a}}g \in \mathcal{M}(\overline{a})$ with $(T_{\overline{a}}g)^{(\ell)}(\zeta_j) = 0$ for all $1 \leq j \leq n$, $0 \leq \ell \leq m_j - 1$. Recall that $\mathcal{M}(\overline{a}) \subset \mathcal{H}(b)$ and so f , as well as all its derivatives up to order $m_j - 1$, admits non-tangential limits at ζ_j for all $1 \leq j \leq n$. Then

$$\begin{aligned} 0 &= (T_{\overline{a}}g)^{(\ell)}(\zeta_j) = \lim_{t \rightarrow 1^-} (T_{\overline{a}}g)^{(\ell)}(t\zeta_j) = \lim_{t \rightarrow 1^-} \langle T_{\overline{a}}g, k_{t\zeta_j, \ell} \rangle_{H^2} \\ &= \lim_{t \rightarrow 1^-} \langle g, ak_{t\zeta_j, \ell} \rangle_{H^2} \\ &= \overline{c} \langle g, cz^\ell (z - \zeta_j)^{m_j - (\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k} \rangle_{H^2}. \end{aligned}$$

In order to prove the lemma, it suffices to show that the set

$$\left\{ \varphi_{j, \ell}(z) := z^\ell (z - \zeta_j)^{m_j - (\ell+1)} \prod_{k \neq j} (z - \zeta_k)^{m_k} \right\},$$

where $j = 1, \dots, n$ and $\ell = 0, \dots, m_j - 1$, is a basis for the space of polynomials of degree at most $N - 1$. Clearly each $\varphi_{j, \ell}$ is a polynomial

of degree $N - 1$ and there are $N - 1$ functions $\varphi_{j,\ell}$. It remains to show that the elements of this family are linearly independent. Obviously, for fixed $1 \leq r \leq n$ and $0 \leq k \leq m_r - 1$, we have

$$\varphi_{j,\ell}^{(k)}(\zeta_r) = 0, \quad j \neq r, \quad 0 \leq k \leq m_r - 1,$$

and

$$(5.5) \quad \varphi_{r,\ell}^{(k)}(\zeta_r) = 0, \quad 0 \leq k \leq m_r - (\ell + 2).$$

In particular, if $\sum_{j,\ell} \alpha_{j,\ell} \varphi_{j,\ell} = 0$, then, for fixed r and $0 \leq k \leq m_r - 1$, $\sum_{j,\ell} \alpha_{j,\ell} \varphi_{j,\ell}^{(k)}(\zeta_r) = 0$ which reduces to $\sum_{\ell} \alpha_{r,\ell} \varphi_{r,\ell}^{(k)}(\zeta_r) = 0$. Writing $\varphi_{j,\ell}(z) = (z - \zeta_j)^{m_j - (\ell + 1)} p_{j,\ell}(z)$, where $p_{j,\ell}$ does not vanish at ζ_j , Leibniz's formula gives

$$\begin{aligned} \varphi_{j,\ell}(z)^{m_j - (\ell + 1)}(z) &= \sum_{k=0}^{m_j - (\ell + 1)} \binom{m_j - (\ell + 1)}{k} \\ &\quad \times \frac{(m_j - (\ell + 1))!}{(m_j - (\ell + 1) - k)!} (z - \zeta_j)^{m_j - (\ell + 1) - k} p_{j,\ell}^{(m_j - (\ell + 1) - k)}(z). \end{aligned}$$

Evaluating this expression at ζ_j makes all terms of the sum vanish except for $k = m_j - (\ell + 1)$, and thus

$$\varphi_{j,\ell}(z)^{m_j - (\ell + 1)}(\zeta_j) = (m_j - (\ell + 1))! p_{j,\ell}(\zeta_j) \neq 0.$$

This together with (5.5) generates a triangular system of linear equations with non-zero diagonal entries. Thus $\alpha_{r,\ell} = 0$, $0 \leq \ell \leq m_r - 1$. \square

We are now in a position to prove Theorem 1.5.

Our arguments so far yield

$$(5.6) \quad \mathcal{H}(q^r) = \mathcal{M}(\bar{a}_r)$$

and

$$(5.7) \quad \mathcal{M}(\bar{a}_r) = \mathcal{M}(\bar{a}) \supset \mathcal{M}(a) + \bigvee \{v_{r,\zeta_j}^\ell : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1\}.$$

First we show that the sum is orthogonal in the $\mathcal{H}(q^r)$ inner product:

$$v_{r,\zeta_j}^\ell \perp \mathcal{M}(a), \quad 1 \leq j \leq n, \quad 0 \leq \ell \leq m_j - 1.$$

Indeed, for each $f \in \mathcal{H}(q^r)$ the radial limits $f^{(\ell)}(t\zeta_j)$ exist as $t \rightarrow 1^-$. Since

$$f^{(\ell)}(t\zeta_j) = \langle f, v_{r,t\zeta_j}^\ell \rangle_{q^r},$$

we can apply the principle of uniform boundedness to see that $\|v_{r,t\zeta_j}^\ell\|_{q^r}$ is uniformly bounded as $t \rightarrow 1^-$. Since $v_{r,t\zeta_j}^\ell$ converges pointwise to v_{r,ζ_j}^ℓ as $t \rightarrow 1^-$ we see that $v_{r,t\zeta_j}^\ell$ converges weakly to v_{r,ζ_j}^ℓ . Thus, since $v_{r,t\zeta_j}^\ell$

reproduces the ℓ -th derivative of $\mathcal{H}(q^r)$ -functions at point $t\zeta_j$, for any $g \in H^2$, we have

$$\begin{aligned} \langle ag, v_{r,\zeta_j}^\ell \rangle_{q^r} &= \lim_{t \rightarrow 1^-} \langle ag, v_{r,t\zeta_j}^\ell \rangle_{q^r} = \lim_{t \rightarrow 1} (ag)^{(\ell)}(t\zeta_j) \\ &= \lim_{t \rightarrow 1} \sum_{p=0}^{\ell} \binom{\ell}{p} a^{(p)}(t\zeta_j) g^{(\ell-p)}(t\zeta_j). \end{aligned}$$

Using the estimate

$$|a^{(p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p}$$

along with the following standard H^2 estimate on the growth of the derivative of an H^2 function

$$|g^{(\ell-p)}(t\zeta_j)| \lesssim \frac{1}{(1-t)^{(\ell-p)+1/2}},$$

we see that

$$|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p-((\ell-p)+1/2)}.$$

But since $0 \leq \ell \leq m_j - 1$ we see that

$$m_j - p - ((\ell - p) + 1/2) = m_j - \ell - \frac{1}{2} \geq \frac{1}{2}$$

and so

$$\lim_{t \rightarrow 1^-} |a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| = 0.$$

Thus $\langle ag, v_{r,\zeta_j}^\ell \rangle_{q^r} = 0$ and $v_{r,\zeta_j}^\ell \perp \mathcal{M}(a)$ in $\mathcal{H}(q^r)$, for all $0 \leq \ell \leq m_j - 1$.

This upgrades (5.6) and (5.7) to

$$(5.8) \quad \mathcal{H}(q^r) = \mathcal{M}(\bar{a}) \supset \mathcal{M}(a) \oplus \bigvee \{v_{r,\zeta_j}^\ell : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1\},$$

and orthogonality is with respect to the norm in $\mathcal{H}(q^r)$.

To show equality in (5.8), our second step is to show that if $f \in \mathcal{M}(\bar{a})$ and $f \perp v_{r,\zeta_j}^\ell$ for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$, then $f \in \mathcal{M}(a)$. Since $\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2$ this is equivalent to prove that if $g \in H^2$ and

$$0 = (T_{\bar{a}}g)^{(\ell)}(\zeta_j) = \lim_{t \rightarrow 1^-} (T_{\bar{a}}g)^{(\ell)}(t\zeta_j)$$

for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$ then $T_{\bar{a}}g \in \mathcal{M}(a)$. To simplify matters a bit, let us recall the formula for a from (5.2). Since s is a rational function with zeros and poles outside \mathbb{D}^- then certainly the Toeplitz operators $T_{1/s}$ and $T_{\overline{1/s}}$ are invertible, and so $\mathcal{M}(\bar{a}) = \mathcal{M}(\overline{a/s})$. We can therefore make the simplifying assumption that

$$a(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j}.$$

We will show that

$$(5.9) \quad (T_{\bar{a}}g)^{(\ell)}(\zeta_j) = 0, \quad 1 \leq j \leq n, 0 \leq \ell \leq m-1 \implies T_{\bar{a}}g \in aH^2.$$

With $N = \sum_{j=1}^n m_j$, one can verify the identity

$$\overline{a(\zeta)} = \bar{\zeta}^N a(\zeta) \prod_{j=1}^n (-\bar{\zeta}_j)^{m_j}, \quad \zeta \in \mathbb{T}.$$

Thus

$$T_{\bar{a}}g = \prod_{j=1}^n (-\bar{\zeta}_j)^{m_j} P_+(a\bar{\zeta}^N g).$$

By Lemma 5.3, we have $\widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(N-1) = 0$, which shows that $\bar{\zeta}^N g \in H^2$ and so

$$T_{\bar{a}}g = \left(\prod_{j=1}^n (-\bar{\zeta}_j)^{m_j} \right) P_+(a\bar{\zeta}^N g) \in aH^2.$$

This completes the proof. \square

6. EXAMPLES

Example 6.1. Consider the function

$$q(z) = \frac{1}{2}(1+z)$$

and notice that q is outer and $\|q\|_\infty = 1$. One can easily guess the Pythagorean mate for q to be $a(z) = \frac{1}{2}(1-z)$. The function $a(z)$ has one zero of order 1 at $z = 1$ and a computation reveals that

$$v_{1,1}^0(z) = \frac{1 - \overline{q(1)}q(z)}{1-z} = \frac{1}{2}.$$

In this case

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}.$$

Moreover, for any $r > 0$ we get $\mathcal{H}(q^r) = \mathcal{H}(q)$ and

$$\mathcal{H}(q^r) = (z-1)H^2 \dot{+} \mathbb{C} = (z-1)H^2 \oplus \mathbb{C} \frac{1 - \left(\frac{1+z}{2}\right)^r}{1-z}.$$

For more general q we need to review the proof of the Féjer-Riesz theorem which says that if

$$w(e^{i\theta}) = \sum_{j=-n}^n c_j e^{ij\theta}$$

is a non-zero trigonometric polynomial which assumes non-negative values for all θ , then there is an analytic polynomial

$$p(z) = \sum_{j=0}^n a_j z^j$$

so that $w(e^{i\theta}) = |p(e^{i\theta})|^2$. Since the proof gives us the algorithm for computing p , we give a quick sketch. Indeed, as a function of the complex variable z , we see that if

$$w(z) = \sum_{j=-n}^n c_j z^j$$

then $\overline{w(1/\bar{z})} = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$ we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of s occur in the pairs $\alpha, 1/\bar{\alpha}$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^n (z - \alpha_j) \left(\frac{1}{z} - \bar{\alpha}_j \right)$$

for some positive constant c and where $\alpha_1, \dots, \alpha_n$ satisfy $|\alpha_j| \geq 1$ for $1 \leq j \leq n$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^n (z - \alpha_j).$$

Note that p is zero free in \mathbb{D} and we can multiply p by a unimodular constant so that $p(0) > 0$.

Recall from the proof of Lemma (3.1) that if $q = p_1/p_2$ is rational then the Pythagorean mate a for q is given by $a = p/p_2$, where p is the analytic polynomial (guaranteed by the Féjer-Riesz theorem) which satisfies $|p(e^{i\theta})|^2 = w(e^{i\theta}) = |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \geq 0$, and p is chosen to that $a(0) > 0$.

Example 6.2. Consider the function

$$q(z) = \frac{1}{2}(1-z)(1+z)$$

and note that $q \in \mathbf{b}(H^\infty)$ and is outer. A computation shows that

$$1 - |q(e^{it})|^2 = \frac{1}{4}e^{-2it} + \frac{1}{4}e^{2it} + \frac{1}{2}.$$

Define

$$w(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$

and

$$s(z) = z^2 w(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4} = \frac{1}{4}(z-i)^2(z+i)^2.$$

Notice how the zeros occur in pairs $i = 1/\bar{i}$ and $-i = 1/(-\bar{i})$ as guaranteed by the above proof of the F ej er-Riesz theorem. Thus the Pythagorean mate a for q is of the form $a(z) = c(z-i)(z+i)$ for some c adjusted so that $a(0) > 0$ and $1 - |q(e^{i\theta})|^2 = |a(e^{i\theta})|^2$. One can check by direct calculation that $c = 1/2$ works and so $a(z) = \frac{1}{2}(z-i)(z+i)$. Of course the exact value of c is not important for our calculations since we only need to identify the zeros of a along with their multiplicities.

The zeros of a are at $z = i$ and $z = -i$ and each has order one. Thus

$$\mathcal{H}(q) = (z-i)(z+i)H^2 \oplus \bigvee \{v_{1,i}^0, v_{1,-i}^0\},$$

where the kernels can be computed directly as

$$v_{1,i}^0(z) = \frac{1}{2i}(z+i), \quad v_{1,-i}^0(z) = \frac{1}{2i}(z-i).$$

Again, as in the previous example, $\mathcal{H}(q^r) = \mathcal{H}(q)$ and so

$$\mathcal{H}(q^r) = (z-i)(z+i)H^2 \dot{+} \bigvee \{z+i, z-i\}.$$

Example 6.3. Consider the function

$$q(z) = \frac{1}{4}(z+1)^2$$

and note that q is outer and belongs to $\mathbf{b}(H^\infty)$. Following our Fejer-Riesz computations as in the previous example, note that

$$1 - |q(e^{it})|^2 = -\frac{e^{-it}}{4} - \frac{e^{it}}{4} - \frac{1}{16}e^{-2it} - \frac{1}{16}e^{2it} + \frac{5}{8}.$$

Define

$$w(z) = -\frac{z^2}{16} - \frac{1}{16z^2} - \frac{z}{4} - \frac{1}{4z} + \frac{5}{8}$$

and

$$\begin{aligned} s(z) &= z^2 w(z) \\ &= -\frac{z^4}{16} - \frac{z^3}{4} + \frac{5z^2}{8} - \frac{z}{4} - \frac{1}{16} \\ &= -\frac{1}{16}(-1+z)^2(1+6z+z^2). \end{aligned}$$

The zeros of s are at

$$z = -1, z = -1, z = -3-2\sqrt{2} \approx -5.82843, z = -3+2\sqrt{2} \approx -0.171573.$$

Notice how these roots occur in the pairs $\alpha, 1/\bar{\alpha}$. The function a is then $a(z) = c(z-1)(z+3+2\sqrt{2})$ for some appropriate constant c . There is one zero of a at $z=1$ with multiplicity one and so

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}v_{1,1}^0(z).$$

The kernel can be computed to be

$$v_{1,1}^0(z) = \frac{z+3}{4}.$$

As in our previous examples, note that

$$\mathcal{H}(q^r) = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Observe that the q from this example is the square of the q from Example 6.1 and thus the corresponding spaces should be the same. Indeed, a little algebra will show that

$$(z-1)H^2 \dot{+} \mathbb{C} = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Example 6.4. Reversing the roles of a and q in the preceding example:

$$a(z) = \frac{1}{4}(z+1)^2, \quad q(z) = c(z-1)(z+3+2\sqrt{2}),$$

with suitable c so that $\|q\|_\infty = 1$ (the maximum modulus on \mathbb{D}^- being attained at -1 , one has $c = (4(1+\sqrt{2}))^{-1}$, and $q(-1) = -1$ corresponding to the normalization $q(0) > 0$), we obtain a function a with double zero, and so

$$\mathcal{H}(q) = (z+1)^2H^2 \oplus \bigvee \{v_{1,-1}^0, v_{1,-1}^1\}$$

where

$$v_{1,-1}^0(z) = \frac{1 - \overline{q(-1)}q(z)}{1 - \overline{(-1)}z} = \frac{1+q(z)}{1+z}.$$

Using the facts that $q(-1) = -1$, $q'(z) = c(2+2\sqrt{2})$, and $q'(-1) = -1/2$, we obtain

$$v_{1,-1}^1(z) = \frac{\frac{1}{2}q(z)(1+z) + z(1+q(z))}{(1+z)^2} = \frac{1}{2} \frac{q(z)(1+3z) + 2z}{(1+z)^2}.$$

Question 6.5. So far we have computed the exact contents of $\mathcal{H}(b)$ when b outer, rational, and non-extreme. Can one compute the contents of $\mathcal{H}(b)$ when b is outer and extreme. For example if b is the outer function corresponding to the outer function which satisfies $|b(e^{i\theta})| = 1$ for $0 \leq \theta \leq \pi$ and $|b(e^{i\theta})| = \frac{1}{2}$ for $\pi < \theta < 2\pi$, can one describe the functions in $\mathcal{H}(b)$?

REFERENCES

- [1] Alain Blandignères, Emmanuel Fricain, Frédéric Gaunard, Andreas Hartmann, and William Ross. Direct and reverse Carleson measures for $\mathcal{H}(b)$ spaces. Preprint.
- [2] Constantin Costara and Thomas Ransford. Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)? *J. Funct. Anal.*, 265(12):3204–3218, 2013.
- [3] Louis de Branges and James Rovnyak. Canonical models in quantum scattering theory. In *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965)*, pages 295–392. Wiley, New York, 1966.
- [4] Louis de Branges and James Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966.
- [5] Emmanuel Fricain and Javad Mashreghi. Integral representation of the n -th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel. *Ann. Inst. Fourier (Grenoble)*, 58(6):2113–2135, 2008.
- [6] J. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.
- [7] B Lanucha and M. Nowak. De Branges spaces and generalized Dirichlet spaces. Preprint.
- [8] V. Paulsen. *An Introduction to the theory of reproducing kernel Hilbert spaces*. www.math.uh.edu/~vern/rkhs.pdf, 2009.
- [9] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

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