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CONCRETE EXAMPLES OF $\mathcal{H}(b)$ SPACES

EMMANUEL FRICAIN, ANDREAS HARTMANN, AND WILLIAM T. ROSS

ABSTRACT. In this paper we give an explicit description of de Branges-Rovnyak spaces $\mathcal{H}(b)$ when b is of the form q^r , where q is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

1. INTRODUCTION

The purpose of this paper is to explicitly describe the elements of the de Branges-Rovnyak space $\mathcal{H}(b)$ for certain $b \in \mathbf{b}(H^\infty)$. Here H^∞ is the algebra of bounded analytic functions on the open unit disk \mathbb{D} normed by

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|,$$

and

$$\mathbf{b}(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leq 1\}$$

is the closed unit ball in H^∞ and, for $b \in \mathbf{b}(H^\infty)$, the *de Branges-Rovnyak space* $\mathcal{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on \mathbb{D} whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

The spaces $\mathcal{H}(b)$, and their vector-valued analogues, are the setting for the operator model theory of de Branges and Rovnyak [3, 4]. Besides possessing a fascinating internal structure, it also became clear that $\mathcal{H}(b)$ spaces have a key role to play in several questions in function theory and operator theory, most importantly, in the model theory for many types of contraction operators.

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Nevertheless, the elements of $\mathcal{H}(b)$ are rather opaque and besides the kernel functions k_λ^b , $\lambda \in \mathbb{D}$ (whose linear span is indeed dense in $\mathcal{H}(b)$), the exact contents of $\mathcal{H}(b)$ is mysterious.

In this paper, we give a precise description of the elements of $\mathcal{H}(b)$ for certain relatively simple b , namely roots of rational outer functions. Our description needs the following set up. If $b \in \mathbf{b}(H^\infty)$ satisfies

$$\int_{\mathbb{T}} \log(1 - |b|) dm > -\infty,$$

(such functions are called *non-extreme elements* of the unit ball in H^∞) where $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and m Lebesgue measure on \mathbb{T} normalized so that $m(\mathbb{T}) = 1$, then there exists a unique outer function $a \in \mathbf{b}(H^\infty)$, called the *Pythagorean mate* for b , such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ almost everywhere on \mathbb{T} . The pair (a, b) is said to be a *Pythagorean pair*. By means of the Fejer-Riesz theorem (we give a proof of this in Section 5), one can prove that if b is a rational (or polynomial) function then so is a . In this case, also notice that for $\zeta \in \mathbb{T}$, we have $|b(\zeta)| = 1$ if and only if $a(\zeta) = 0$.

Our first observation is that in certain situations $\mathcal{H}(b^r)$ does not depend on $r > 0$.

Theorem 1.1. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and non extreme with Pythagorean mate a . If (a, b) is a corona pair, i.e.,*

$$\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0,$$

then for any $r > 0$ we have $\mathcal{H}(b^r) = \mathcal{H}(b)$.

Note that since b is outer it has no zeros on \mathbb{D} and we can define b^r by taking any logarithm of b .

In the special situation when b is a *rational* outer function, or any positive power of a rational function, we get the following complete description of $\mathcal{H}(b)$. In order to state the result, we also need derivatives of kernels. When $b = q^r$, where q is outer and rational and $r > 0$, we set

$$v_{r,\lambda}^\ell(z) := \frac{d^\ell}{d\bar{\lambda}^\ell} k_\lambda^{q^r}(z) = \frac{d^\ell}{d\bar{\lambda}^\ell} \left(\frac{1 - \overline{q^r(\lambda)} q^r(z)}{1 - \bar{\lambda}z} \right),$$

for any $z \in \mathbb{D}$, $\lambda \in \mathbb{D}^-$, and $\ell = 0, 1, 2, \dots$

We let H^2 denote the classical Hardy space in \mathbb{D} [6].

Theorem 1.2. *Suppose $q \in \mathbf{b}(H^\infty)$ is a rational outer function and r is a positive real number. Then*

- (1) $\mathcal{H}(q^r) = \mathcal{H}(q)$.
- (2) If a is the Pythagorean mate for q and a has distinct zeros ζ_1, \dots, ζ_n on \mathbb{T} with corresponding multiplicities m_1, \dots, m_n , then
- (a) the functions $v_{r,j}^\ell := v_{r,\zeta_j}^\ell$ are well-defined and belong to $\mathcal{H}(q^r)$ for $1 \leq j \leq n$ and $0 \leq \ell \leq m_j - 1$. Moreover, they are orthogonal to

$$aH^2 = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2.$$

- (b) $\mathcal{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \dot{+} \bigvee \{v_{r,j}^\ell : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n\},$$

where the orthogonal decomposition is in terms of the inner product in $\mathcal{H}(q^r)$.

Writing $v_j^\ell = v_{1,j}^\ell$, the theorem above implies that $\mathcal{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \dot{+} \bigvee \{v_j^\ell(z) : 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n\},$$

where the sum is no longer necessarily orthogonal.

It was shown in [2], and rediscovered in [1], that

$$(1.3) \quad \mathcal{H}(q) = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2 \dot{+} \mathcal{P}_{n-1},$$

where \mathcal{P}_{n-1} is the n -dimensional space of polynomials of degree at most $n - 1$ and the sum is an algebraic direct sum (not necessarily an orthogonal one). The novelty here is that we can precisely identify the orthogonal complement of $aH^2 = \left(\prod_{j=1}^n (z - \zeta_j)^{m_j} \right) H^2$ in $\mathcal{H}(q)$ without using (1.3).

A key ingredient in showing statement (1) of Theorem 1.2, and an added bonus to our result, is that if a_r is the Pythagorean mate for q^r then the co-analytic Toeplitz operators $T_{\bar{a}}$ and $T_{\bar{a}_r}$ on the Hardy space H^2 have the same range and this range can be precisely identified as $\mathcal{H}(q)$.

2. PRELIMINARIES

There are several equivalent definitions of the de Branges-Rovnyak space $\mathcal{H}(b)$ for $b \in \mathbf{b}(H^\infty)$. We can for instance define it in the standard way (see [7]) as the reproducing kernel Hilbert space associated with the (positive definite) reproducing kernel

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

By definition,

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b, \quad f \in \mathcal{H}(b), \lambda \in \mathbb{D},$$

where $\langle \cdot, \cdot \rangle_b$ stands for the scalar product in $\mathcal{H}(b)$.

The space $\mathcal{H}(b)$ can also be defined as the range space $(I - T_b T_{\overline{b}})^{1/2} H^2$ equipped with the norm making $(I - T_b T_{\overline{b}})^{1/2}$ a partial isometry. Here T_φ is the Toeplitz operator on H^2 with symbol $\varphi \in L^\infty(\mathbb{T})$ defined by

$$T_\varphi f = P_+(\varphi f), \quad f \in H^2,$$

where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 . The book [8] is the classic reference for $\mathcal{H}(b)$ spaces.

When $\|b\|_\infty < 1$, $\mathcal{H}(b)$ turns out to be a reformed version of H^2 while, on the other extreme, if b is an inner function, then $\mathcal{H}(b)$ turns out to be one of the classical and well-studied model spaces $H^2 \ominus bH^2$.

When b is non-extreme, two important (not necessarily closed) vector spaces of functions in $\mathcal{H}(b)$ are

$$\mathcal{M}(a) := T_a H^2 \quad \text{and} \quad \mathcal{M}(\overline{a}) := T_{\overline{a}} H^2.$$

It is known (see [8, page 24]) that

$$\mathcal{M}(a) \subset \mathcal{M}(\overline{a}) \subset \mathcal{H}(b).$$

3. CORONA PAIRS

This following lemma is well-known but we record it here along with a proof for the sake of completeness and for the discussion of the examples in Section 5.

Lemma 3.1. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. Then q is non-extreme and if a is the Pythagorean mate for q then a is also rational.*

Proof. Since q is rational then $q = p_1/p_2$ where p_1 and p_2 are analytic polynomials and p_2 has no zeros on \mathbb{D}^- . We can of course choose p_2 such that $p_2(0) > 0$. Since $q \in \mathbf{b}(H^\infty)$ we see that $1 - |q(e^{i\theta})|^2 \geq 0$ for all θ and so $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is a non-negative trigonometric polynomial, which, by the Fejer-Riesz theorem, must be equal to $|p(e^{i\theta})|^2$, where p is an analytic polynomial which is zero free in \mathbb{D} and $p(0) > 0$. Note that $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is non zero because q is assumed to be not inner.

Let

$$a = \frac{p}{p_2}.$$

Note that a is rational and zero free in \mathbb{D} , hence outer. Moreover $a(0) > 0$.

Furthermore, on \mathbb{T} we have

$$|a|^2 = \left| \frac{p}{p_2} \right|^2 = \frac{|p_2|^2 - |p_1|^2}{|p_2|^2} = 1 - \left| \frac{p_1}{p_2} \right|^2 = 1 - |q|^2.$$

This means that (a, q) is a Pythagorean pair which, in particular, implies that q is non-extreme. \square

Two functions $u, v \in H^\infty$ which satisfy

$$\inf_{z \in \mathbb{D}} (|u(z)| + |v(z)|) > 0$$

are said to be a corona pair. Carleson's corona theorem [6] says that if u, v form a corona pair then there are function $\phi, \psi \in H^\infty$ such that

$$\phi u + \psi v = 1.$$

Lemma 3.2. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and r is a positive real number. Then b and b^r are simultaneously non-extreme. Moreover, if a_r is the Pythagorean mate for b^r , the pairs (a, b) and (a_r, b^r) are simultaneously corona.*

Proof. Since

$$(3.3) \quad \frac{1 - x^r}{1 - x} \asymp 1, \quad x \in [0, 1],$$

we see that $1 - |b|^r \asymp 1 - |b|$ when $b \in \mathbf{b}(H^\infty)$. This proves the first part of the Lemma.

Now observe that

$$\frac{|a|^2}{|a_r|^2} = \frac{1 - |b|^2}{1 - |b^2|^r} \asymp 1,$$

and since a and a_r are outer, by Smirnov's theorem (which says that if the boundary function of $f \in H^2$ is bounded, then $f \in H^\infty$), a/a_r is invertible in H^∞ . Thus both expressions

$$\inf_{z \in \mathbb{D}} (|a(z)| + |b(z)|) \quad \text{and} \quad \inf_{z \in \mathbb{D}} (|a_r(z)| + |b^r(z)|)$$

are strictly positive simultaneously. Indeed, if there is a sequence $\{z_n\}_{n \geq 1}$ in \mathbb{D} such that one expression goes to 0 then, since both $a(z_n)$ and $b(z_n)$ go to zero, the other expression will go to zero as well. \square

A special situation where b forms a corona pair with its Pythagorean mate is when b is rational, as explained in the following lemma.

Lemma 3.4. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. If a is the Pythagorean mate for q , then (a, q) is a corona pair.*

Proof. According to the proof of Lemma 3.1, we know that a is rational, $a = p/p_2$, where p and p_2 are polynomials, p_2 has no zeros in \mathbb{D}^- and p is zero free in \mathbb{D} . In particular, a is continuous on \mathbb{D}^- and it has a finite number of zeros on \mathbb{T} , say $\{\zeta_1, \dots, \zeta_M\}$. Note that, due to the identity $|a|^2 + |q|^2 = 1$ on \mathbb{T} , the zeros of a (on \mathbb{T}) must lie where q is unimodular on \mathbb{T} .

Let D_j be open disjoint disks with center at the zeros ζ_j of a and let

$$F = \mathbb{D}^- \setminus \bigcup_{j=1}^M D_j.$$

By making the disks smaller, one can, by using the continuity of $|q|$ on \mathbb{D}^- , arrange things so that $|q| \geq \frac{1}{2}$ on each $D_j \cap \mathbb{D}^-$.

Notice that F is closed and omits all of the zeros of a in \mathbb{D}^- and so

$$\inf_{z \in F} |a(z)| = \delta > 0.$$

Thus

$$\inf_{z \in \mathbb{D}} (|a(z)| + |q(z)|) \geq \min(\frac{1}{2}, \delta) > 0$$

concluding the proof. \square

The first statement of our main theorem (Theorem 1.2) $\mathcal{H}(q) = \mathcal{H}(q^r)$ depends of the following two results. The first is from Sarason's book [8, p. 62].

Proposition 3.5. *For $b \in \mathbf{b}(H^\infty)$ and non-extreme, the following are equivalent:*

- (1) (a, b) is a corona pair;
- (2) $\mathcal{H}(b) = \mathcal{M}(\bar{a})$.

The second is the following.

Proposition 3.6. *If $a, a_1 \in H^\infty$ are two outer functions such that a/a_1 and a_1/a belong to L^∞ , then $\mathcal{M}(\bar{a}) = \mathcal{M}(\bar{a}_1)$.*

Proof. Again, by Smirnov's theorem, we know that a/a_1 and a_1/a belong to H^∞ , so that T_{a/a_1} and hence $T_{\frac{1}{a/a_1}}$ are invertible. Then we get

$$\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2 = T_{\bar{a}_1}T_{\frac{1}{a/a_1}}H^2 = T_{\bar{a}_1}H^2 = \mathcal{M}(\bar{a}_1). \quad \square$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2 we know that since (a, b) is a corona pair, then so is (a_r, b^r) . Thus from Proposition 3.5 we see that $\mathcal{H}(b) = \mathcal{M}(\bar{a})$ and $\mathcal{H}(b^r) = \mathcal{M}(\bar{a}_r)$. As in the proof of Lemma 3.2 a/a_r and a_r/a are in H^∞ so that by Proposition 3.6, we get $\mathcal{M}(\bar{a}) = \mathcal{M}(\bar{a}_r)$ which concludes the proof. \square

4. PROOF OF THE MAIN THEOREM 1.2

Statement (1) of the theorem follows from Lemma 3.4 and Theorem 1.1.

Let us consider statement (2).

In [5] it was shown that if $\zeta \in \mathbb{T}$ and

$$(4.1) \quad \int_{\mathbb{T}} \frac{|\log |b(w)||}{|w - \zeta|^{2n+2}} dm(w) < \infty,$$

then every function in $\mathcal{H}(b)$, as well as their derivatives up to order n , has a finite non-tangential limit at ζ .

Recalling the notation $v_{r,\lambda}^\ell$ for the ℓ -th derivative in the variable $\bar{\lambda}$ of the reproducing kernel in $\mathcal{H}(q^r)$, under (4.1), we have $v_{r,\zeta}^\ell \in \mathcal{H}(q^r)$, $0 \leq \ell \leq n$, and

$$f^{(\ell)}(\zeta) = \langle f, v_{r,\zeta}^\ell \rangle_{q^r}, \quad f \in \mathcal{H}(q^r), \quad 0 \leq \ell \leq n.$$

Remark 4.2. In the above, we are using $\langle \cdot, \cdot \rangle_b$ to denote the inner product in the space $\mathcal{H}(b)$. Note that $\mathcal{H}(0) = H^2$ with the integral inner product

$$\langle f, g \rangle_0 = \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} dm(\zeta).$$

When the contact is clear, we will drop the subscript b from the inner product.

Let us check condition (4.1) for our situation. Since q is rational then its Pythagorean mate a is also rational and can be written as

$$(4.3) \quad a(z) = s(z) \prod_{j=1}^n (z - \zeta_j)^{m_j},$$

where s is a rational function whose poles and zeros lie on the complement of \mathbb{D}^- . Pick $w = e^{it}$ near one of the zeros $\zeta_j = e^{i\theta_j}$ of a , then

$$\begin{aligned} |\log |q^r(e^{it})|| &\asymp |\log |q(e^{it})|^2| = |\log(1 - |a(e^{it})|^2)| \\ &\asymp |a(e^{it})|^2 \asymp |e^{it} - e^{i\theta_j}|^{2m_j} \end{aligned}$$

This means that for t near θ_j we have

$$\frac{|\log |q^r(e^{it})||}{|e^{it} - e^{i\theta_j}|^{2(m_j-1)+2}} \asymp 1$$

and so, by (4.1),

$$v_{r, \zeta_j}^\ell \in \mathcal{H}(q^r), \quad 0 \leq \ell \leq m_j - 1.$$

So far we know that

$$(4.4) \quad \mathcal{H}(q^r) = \mathcal{M}(\overline{a_r}) = \mathcal{M}(\overline{a}) \supset \mathcal{M}(a) + \bigvee \{v_{r, \zeta_j}^\ell : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1\}.$$

First we show that

$$v_{r, \zeta_j}^\ell \perp \mathcal{M}(a), \quad 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1.$$

Indeed, for each $f \in \mathcal{H}(q^r)$ the radial limits $f^{(\ell)}(t\zeta_j)$ exist as $t \rightarrow 1^-$. Since

$$f^{(\ell)}(t\zeta_j) = \langle f, v_{r, t\zeta_j}^\ell \rangle_{q^r},$$

we can apply the principle of uniform boundedness to see that $\|v_{r, t\zeta_j}^\ell\|_{q^r}$ is uniformly bounded as $t \rightarrow 1^-$. Since $v_{r, t\zeta_j}^\ell$ converges pointwise to v_{r, ζ_j}^ℓ as $t \rightarrow 1^-$ we see that $v_{r, t\zeta_j}^\ell$ converges weakly to v_{r, ζ_j}^ℓ . Thus, since $v_{r, t\zeta_j}^\ell$

reproduces the ℓ -th derivative of $\mathcal{H}(q^r)$ -functions at point $t\zeta_j$, for any $g \in H^2$, we have

$$\begin{aligned} \langle ag, v_{r,\zeta_j}^\ell \rangle_{q^r} &= \lim_{t \rightarrow 1^-} \langle ag, v_{r,t\zeta_j}^\ell \rangle_{q^r} = \lim_{t \rightarrow 1} (ag)^{(\ell)}(t\zeta_j) \\ &= \lim_{t \rightarrow 1} \sum_{p=0}^{\ell} \binom{\ell}{p} a^{(p)}(t\zeta_j) g^{(\ell-p)}(t\zeta_j). \end{aligned}$$

Using the estimate

$$|a^{(p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p}$$

along with the following standard H^2 estimate on the growth of the derivative of an H^2 function

$$|g^{(\ell-p)}(t\zeta_j)| \lesssim \frac{1}{(1-t)^{(\ell-p)+1/2}},$$

we see that

$$|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p-((\ell-p)+1/2)}.$$

But since $0 \leq \ell \leq m_j - 1$ we see that

$$m_j - p - ((\ell - p) + 1/2) = m_j - \ell - \frac{1}{2} \geq \frac{1}{2}$$

and so

$$\lim_{t \rightarrow 1^-} |a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| = 0.$$

Thus $\langle ag, v_{r,\zeta_j}^\ell \rangle_{q^r} = 0$ and $v_{r,\zeta_j}^\ell \perp \mathcal{M}(a)$ in $\mathcal{H}(q^r)$, for all $0 \leq \ell \leq m_j - 1$.

This upgrades (4.4) to

$$(4.5) \quad \mathcal{H}(q^r) = \mathcal{M}(\bar{a}) \supset \mathcal{M}(a) \oplus \bigvee \{v_{r,\zeta_j}^\ell : 1 \leq j \leq n, 0 \leq \ell \leq m_j - 1\},$$

and orthogonality is with respect to $\mathcal{H}(q^r)$.

To show equality in (4.5), our second step is to show that if $f \in \mathcal{M}(\bar{a})$ and $f \perp v_{r,\zeta_j}^\ell$ for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$ then $f \in \mathcal{M}(a)$. Since $\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2$ this is equivalent to proving that if $g \in H^2$ and

$$0 = (T_{\bar{a}}g)^{(\ell)}(\zeta_j) = \lim_{t \rightarrow 1^-} (T_{\bar{a}}g)^{(\ell)}(t\zeta_j)$$

for all $1 \leq j \leq n, 0 \leq \ell \leq m_j - 1$ then $T_{\bar{a}}g \in \mathcal{M}(a)$. To simplify matters a bit, let us recall the formula for a from (4.3). Since s is a rational function with zeros and poles outside \mathbb{D}^- then certainly the Toeplitz operators $T_{1/s}$ and $T_{\overline{1/s}}$ are invertible, so that $\mathcal{M}(\bar{a}) = \mathcal{M}(\overline{a/s})$. We can thus assume that

$$a(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j}.$$

A preliminary step towards proving what we want is to show it for a special case. Indeed we will show that if $a(z) = (z - 1)^m$ and $g \in H^2$ then

$$(4.6) \quad (T_{\bar{a}}g)^{(\ell)}(1) = 0, \quad 0 \leq \ell \leq m - 1 \implies T_{\bar{a}}g \in aH^2.$$

Notice that

$$\overline{a(\zeta)} = (-1)^m \bar{\zeta}^m a(\zeta), \quad \zeta \in \mathbb{T},$$

and thus

$$T_{\bar{a}}g = (-1)^m P_+(a\bar{\zeta}^m g).$$

We claim that

$$(4.7) \quad \widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(m - 1) = 0.$$

This will show that $\bar{\zeta}^m g \in H^2$ and so

$$T_{\bar{a}}g = (-1)^m P_+(a\bar{\zeta}^m g) \in aH^2.$$

We prove the claim (4.7) as follows. Consider the kernels

$$k_{\lambda, \ell}(z) = c_\ell \frac{z^\ell}{(1 - \bar{\lambda}z)^{\ell+1}},$$

where c_ℓ is adapted so that these are the reproducing kernels for ℓ -th derivatives at point $\lambda \in \mathbb{D}$ in the Hardy space H^2 , that is to say,

$$f^{(\ell)}(\lambda) = \langle f, k_{\lambda, \ell} \rangle_2, \quad f \in H^2.$$

Observe that for $0 \leq \ell \leq m - 1$

$$\begin{aligned} a(z)k_{t, \ell}(z) &= c_\ell \frac{z^\ell (1 - z)^m}{(1 - tz)^{\ell+1}} \\ &= c_\ell z^\ell (1 - z)^{m - (\ell+1)} \left(\frac{1 - z}{1 - tz} \right)^{\ell+1} \\ &= c_\ell z^\ell (1 - z)^{m - (\ell+1)} \left(1 + z \frac{t - 1}{1 - tz} \right)^{\ell+1}. \end{aligned}$$

One can see that $ak_{t, \ell}$ converges pointwise to $c_\ell z^\ell (1 - z)^{m - \ell - 1}$ as $t \rightarrow 1^-$ and is uniformly bounded on \mathbb{T} . Thus by the dominated convergence theorem, $ak_{t, \ell} \rightarrow c_\ell z^\ell (1 - z)^{m - \ell - 1}$ in the norm of H^2 . Now pick a function $f = T_{\bar{a}}g \in \mathcal{M}(\bar{a}) = \mathcal{H}(q^r)$ with $T_{\bar{a}}g^{(\ell)}(1) = 0$ for all $0 \leq \ell \leq$

$m - 1$, then

$$\begin{aligned}
 0 &= (T_{\bar{a}}g)^{(\ell)}(1) = \lim_{t \rightarrow 1^-} (T_{\bar{a}}g)^{(\ell)}(t) = \lim_{t \rightarrow 1^-} \langle T_{\bar{a}}g, k_{t,\ell} \rangle_0 = \lim_{t \rightarrow 1^-} \langle g, ak_{t,\ell} \rangle_0 \\
 &= \bar{c}_\ell \langle g, z^\ell (1 - z)^{m-\ell-1} \rangle_0 \\
 &= \bar{c}_\ell \left\langle g, \sum_{j=0}^{m-\ell-1} \binom{m-\ell-1}{j} (-1)^j z^{j+\ell} \right\rangle_0 \\
 &= \bar{c}_\ell \sum_{j=0}^{m-\ell-1} \binom{m-\ell-1}{j} (-1)^j \widehat{g}(j+k) \\
 &= \sum_{p=\ell}^{m-1} \binom{m-\ell-1}{p-\ell} (-1)^{p-\ell} \widehat{g}(p).
 \end{aligned}$$

If

$$A_{i,j} = \binom{m-i-1}{j-i} (-1)^{j-i},$$

then $A = (A_{i,j})_{0 \leq i,j \leq m-1}$ is an upper triangular invertible matrix and hence the above system of equations can be written as

$$A(\widehat{g}(0), \widehat{g}(1), \dots, \widehat{g}(m-1))^T = (0, \dots, 0),$$

which proves (4.7) and thus (4.6).

We finish the proof by establishing the following claim. If

$$a(z) = \prod_{j=1}^n (z - \zeta_j)^{m_j},$$

and $g \in H^2$ is such that, for any $1 \leq j \leq n$, and any $0 \leq \ell \leq m_j - 1$, we have

$$(4.8) \quad (T_{\bar{a}}g)^\ell(\zeta_j) = 0,$$

then $T_{\bar{a}}g \in aH^2$.

We argue by induction on n . If $n = 1$, then we can of course assume without loss of generality that $\zeta_1 = 1$ and then, the proof of the claim is given by (4.6).

Assume the claim is true for $n - 1$ points. Suppose $g \in H^2$ satisfies (4.8). Decompose a as $a = a_1 a_2$ where $a_1(z) = (z - \zeta_1)^{m_1}$ and

$$a_2(z) = \prod_{j=2}^n (z - \zeta_j)^{m_j}.$$

Since

$$T_{\bar{a}} = T_{\bar{a}_1}T_{\bar{a}_2} = T_{\bar{a}_2}T_{\bar{a}_1},$$

we easily see that the induction hypothesis implies that the function $h := T_{\bar{a}}g$ belongs to $a_1H^2 \cap a_2H^2$. In other words, there exists two functions $\varphi, \psi \in H^2$ such that

$$h = a_1\varphi = a_2\psi.$$

Now note that (a_1, a_2) is a corona pair. Hence, there exists $u, v \in H^\infty$ such that

$$a_1u + a_2v = 1.$$

Multiply this equation by ψ gives

$$\psi = a_1\psi u + a_2\psi v = a_1\psi u + a_1\varphi v = a_1(\psi u + \varphi v).$$

Therefore, we obtain

$$h = a_2\psi = a_2a_1(\psi u + \varphi v),$$

which shows that $h \in a_1a_2H^2 = aH^2$. The claim is established and the proof of Theorem 1.2 is complete.

5. EXAMPLES

Example 5.1. Consider the function

$$q(z) = \frac{1}{2}(1+z)$$

and notice that q is outer and $\|q\|_\infty = 1$. One can easily guess the Pythagorean mate for q to be

$$a(z) = \frac{1}{2}(1-z).$$

The function $a(z)$ has one zero of order 1 at $z = 1$ and a computation reveals that

$$v_{1,1}^0(z) = \frac{1 - \overline{q(1)}q(z)}{1-z} = \frac{1}{2}.$$

In this case

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}$$

and so for any positive real number we get $\mathcal{H}(q^r) = \mathcal{H}(q)$ and

$$\mathcal{H}(q^r) = (z-1)H^2 \dot{+} \mathbb{C} = (z-1)H^2 \oplus \mathbb{C} \frac{1 - \left(\frac{1+z}{2}\right)^r}{1-z}.$$

For more general q we need to review the proof of the Fejer-Riesz theorem which says that if

$$w(e^{i\theta}) = \sum_{j=-n}^n c_j e^{ij\theta}$$

is a non-zero trigonometric polynomial which assumes non-negative values for all θ , then there is an analytic polynomial

$$p(z) = \sum_{j=0}^n a_j z^j$$

so that

$$w(e^{i\theta}) = |p(e^{i\theta})|^2.$$

Since the proof gives us the algorithm for computing p , we give a quick sketch. Indeed, as a function of the complex variable z , we see that if

$$w(z) = \sum_{j=-n}^n c_j z^j$$

then $\overline{w(1/\bar{z})} = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$ we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of s occur in pairs $\alpha, 1/\bar{\alpha}$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^n (z - \alpha_j) \left(\frac{1}{z} - \bar{\alpha}_j \right)$$

for some positive constant c and where $\alpha_1, \dots, \alpha_n$ satisfy $|\alpha_j| \geq 1$ for $1 \leq j \leq n$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^n (z - \alpha_j).$$

Note that p is zero free in \mathbb{D} and we can multiply p by a unimodular constant so that $p(0) > 0$.

Recall from the proof of Lemma (3.1) that if $q = p_1/p_2$ is rational then the Pythagorean mate a for q is given by

$$a = \frac{p}{p_2},$$

where p is the analytic polynomial (guaranteed by the Fejer-Riesz theorem) which satisfies $|p(e^{i\theta})|^2 = w(e^{i\theta}) = |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \geq 0$, and p is chosen so that $a(0) > 0$.

Example 5.2. Consider the function

$$q(z) = \frac{1}{2}(1-z)(1+z)$$

and note that $q \in \mathbf{b}(H^\infty)$ and is outer. A computation shows that

$$1 - |q(e^{it})|^2 = \frac{1}{4}e^{-2it} + \frac{1}{4}e^{2it} + \frac{1}{2}.$$

Define

$$w(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$

and

$$s(z) = z^2 w(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4}.$$

The polynomial s factors as

$$s(z) = \frac{1}{4}(z-i)^2(z+i)^2.$$

Notice how the zeros occur in pairs $i = 1/\bar{i}$ and $-i = 1/\overline{-i}$ as guaranteed by the above proof of the Fejer-Riesz theorem. Thus the Pythagorean mate a for q is of the form

$$a(z) = c(z-i)(z+i)$$

for some c adjusted so that $a(0) > 0$ and $1 - |q(e^{i\theta})|^2 = |a(e^{i\theta})|^2$. One can check by direct calculation that $c = 1/2$ works and so

$$a(z) = \frac{1}{2}(z-i)(z+i).$$

Of course the exact value of c is not important for our calculations since we only need to identify the zeros of a .

The zeros of a are at $z = i$ and $z = -i$ and have order one and so

$$\mathcal{H}(q) = (z-i)(z+i)H^2 \oplus \bigvee \{v_{1,i}^0, v_{1,-i}^0\},$$

where the kernels can be computed directly as

$$v_{1,i}^0(z) = \frac{1}{2i}(z+i), \quad v_{1,-i}^0(z) = \frac{1}{2i}(z-i).$$

Again, as in the previous example, $\mathcal{H}(q^r) = \mathcal{H}(q)$ and so

$$\mathcal{H}(q^r) = (z-i)(z+i)H^2 \dot{+} \bigvee \{z+i, z-i\}.$$

Example 5.3. Consider the function

$$q(z) = \frac{1}{4}(z+1)^2$$

and note that q is outer and belongs to $\mathbf{b}(H^\infty)$. Following our Fejer-Riesz computations as in the previous example, note that

$$1 - |q(e^{it})|^2 = -\frac{e^{-it}}{4} - \frac{e^{it}}{4} - \frac{1}{16}e^{-2it} - \frac{1}{16}e^{2it} + \frac{5}{8}.$$

Define

$$w(z) = -\frac{z^2}{16} - \frac{1}{16z^2} - \frac{z}{4} - \frac{1}{4z} + \frac{5}{8}$$

and

$$\begin{aligned} s(z) &= z^2 w(z) \\ &= -\frac{z^4}{16} - \frac{z^3}{4} + \frac{5z^2}{8} - \frac{z}{4} - \frac{1}{16} \\ &= -\frac{1}{16}(-1+z)^2(1+6z+z^2). \end{aligned}$$

The zeros of s are at $z = -1, z = -1, z = -3 - 2\sqrt{2} \approx -5.82843, z = -3 + 2\sqrt{2} \approx -0.171573$. Notice how these roots occur in the pairs $\alpha, 1/\bar{\alpha}$. The function a is then

$$a(z) = c(z-1)(z+3+2\sqrt{2})$$

for some appropriate constant c . There is one zero of a at $z = 1$ of multiplicity one and so

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}v_{1,1}^0(z),$$

where the kernel can be computed to be

$$v_{1,1}^0(z) = \frac{z+3}{4}.$$

As in our previous examples, note that

$$\mathcal{H}(q^r) = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Notice that the q from this example is the square of the q from our first example and thus the corresponding spaces should be the same. Indeed a little algebra will show that

$$(z-1)H^2 \dot{+} \mathbb{C} = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Example 5.4. Reversing the roles of a and q in the preceding example:

$$a(z) = \frac{1}{4}(z+1)^2, \quad q(z) = c(z-1)(z+3+2\sqrt{2}),$$

with suitable c so that $\|q\|_\infty = 1$ (the maximum modulus on \mathbb{D}^- being attained in -1 , one has $c = (4(1+\sqrt{2}))^{-1}$, and $q(-1) = -1$ corresponding to the normalization $q(0) > 0$), we obtain a function a with double zero, and so

$$\mathcal{H}(q) = (z+1)^2 H^2 \oplus \bigvee \{v_{1,-1}^0, v_{1,-1}^1\}$$

where

$$v_{1,-1}^0(z) = \frac{1 - \overline{q(-1)}q(z)}{1 - \overline{(-1)}z} = \frac{1+q(z)}{1+z},$$

and using $q(-1) = -1$ as well as $q'(z) = c(2+2\sqrt{2})$ so that $q'(-1) = -1/2$, we have

$$v_{1,-1}^1(z) = \frac{\frac{1}{2}q(z)(1+z) + z(1+q(z))}{(1+z)^2} = \frac{1}{2} \frac{q(z)(1+3z) + 2z}{(1+z)^2}.$$

Example 5.5. Here is an example of a rational function:

$$q(z) = \frac{3(z+1)}{2(z+2)}.$$

Then, as it turns out,

$$a(z) = c \frac{z-1}{z+2}$$

which has a zero of order one at $z = 1$. From here we get

$$\mathcal{H}(q) = (z-1)H^2 \oplus \mathbb{C}v_{1,1}^0(z) = (z-1)H^2 \oplus \mathbb{C} \frac{1}{4+2z}.$$

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