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CONCRETE EXAMPLES OF $\mathscr{H}(b)$ SPACES

EMMANUEL FRICAIN, ANDREAS HARTMANN, AND WILLIAM T. ROSS

Abstract. In this paper we give an explicit description of de Branges-Rovnyak spaces $\mathscr{H}(b)$ when b is of the form q^r, where q is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

1. Introduction

The purpose of this paper is to explicitly describe the elements of the de Branges-Rovnyak space $\mathscr{H}(b)$ for certain $b \in \mathbf{b}(H^\infty)$. Here H^∞ is the algebra of bounded analytic functions on the open unit disk $\mathbb{D}$ normed by

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|,$$

and

$$\mathbf{b}(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leqslant 1\}$$

is the closed unit ball in H^∞ and, for $b \in \mathbf{b}(H^\infty)$, the *de Branges-Rovnyak space* $\mathscr{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

The spaces $\mathscr{H}(b)$, and their vector–valued analogues, are the setting for the operator model theory of de Branges and Rovnyak [3, 4]. Besides possessing a fascinating internal structure, it also became clear that $\mathscr{H}(b)$ spaces have a key role to play in several questions in function theory and operator theory, most importantly, in the model theory for many types of contraction operators.

2010 *Mathematics Subject Classification.* 30J05, 30H10, 46E22.

Key words and phrases. de Branges-Rovnyak spaces, non-extreme points, kernel functions, corona pairs.

This work was initiated while the first two authors were staying at the University of Richmond. These authors would like to thank that institution for the great hospitality. Work supported by Labex CEMPI (ANR-11-LABX-0007-01).

Nevertheless, the elements of $\mathscr{H}(b)$ are rather opaque and besides the kernel functions $k_\lambda^b, \lambda \in \mathbb{D}$ (whose linear span is indeed dense in $\mathscr{H}(b)$), the exact contents of $\mathscr{H}(b)$ is mysterious.

In this paper, we give a precise description of the elements of $\mathscr{H}(b)$ for certain relatively simple b, namely roots of rational outer functions. Our description needs the following set up. If $b \in \mathbf{b}(H^\infty)$ satisfies

$$\int_{\mathbb{T}} \log(1-|b|)dm > -\infty,$$

(such functions are called *non-extreme elements* of the unit ball in H^∞) where $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and m Lebesgue measure on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$, then there exists a unique outer function $a \in \mathbf{b}(H^\infty)$, called the *Pythagorean mate* for b, such that $a(0) > 0$ and $|a|^2+|b|^2 = 1$ almost everywhere on $\mathbb{T}$. The pair (a, b) is said to be a *Pythagorean pair*. By means of the Fejer-Riesz theorem (we give a proof of this in Section 5), one can prove that if b is a rational (or polynomial) function then so is a. In this case, also notice that for $\zeta \in \mathbb{T}$, we have $|b(\zeta)| = 1$ if and only if $a(\zeta) = 0$.

Our first observation is that in certain situations $\mathscr{H}(b^r)$ does not depend on $r > 0$.

Theorem 1.1. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and non extreme with Pythagorean mate a. If (a, b) is a corona pair, i.e.,*

$$\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0,$$

then for any $r > 0$ we have $\mathscr{H}(b^r) = \mathscr{H}(b)$.

Note that since b is outer it has no zeros on $\mathbb{D}$ and we can define b^r by taking any logarithm of b.

In the special situation when b is a *rational* outer function, or any positive power of a rational function, we get the following complete description of $\mathscr{H}(b)$. In order to state the result, we also need derivatives of kernels. When $b = q^r$, where q is outer and rational and $r > 0$, we set

$$v_{r,\lambda}^\ell(z) := \frac{d^\ell}{d\overline{\lambda}^\ell} k_\lambda^{q^r}(z) = \frac{d^\ell}{d\overline{\lambda}^\ell}\left(\frac{1-\overline{q^r(\lambda)}q^r(z)}{1-\overline{\lambda}z}\right),$$

for any $z \in \mathbb{D}$, $\lambda \in \mathbb{D}^-$, and $\ell = 0, 1, 2, \ldots$.

We let H^2 denote the classical Hardy space in $\mathbb{D}$ [6].

Theorem 1.2. *Suppose $q \in \mathbf{b}(H^\infty)$ is a rational outer function and r is a positive real number. Then*

(1) $\mathscr{H}(q^r) = \mathscr{H}(q)$.

(2) *If a is the Pythagorean mate for q and a has distinct zeros $\zeta_1, \ldots, \zeta_n$ on $\mathbb{T}$ with corresponding multiplicities $m_1, \ldots, m_n$, then*

   (a) *the functions $v_{r,j}^{\ell} := v_{r,\zeta_j}^{\ell}$ are well-defined and belong to $\mathscr{H}(q^r)$ for $1 \leqslant j \leqslant n$ and $0 \leqslant \ell \leqslant m_j - 1$. Moreover, they are orthogonal to*

$$aH^2 = \left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2.$$

   (b) *$\mathscr{H}(q^r)$ is equal to*

$$\left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2 \oplus \bigvee \left\{v_{r,j}^{\ell} : 0 \leqslant \ell \leqslant m_j - 1, 1 \leqslant j \leqslant n\right\},$$

   where the orthogonal decomposition is in terms of the inner product in $\mathscr{H}(q^r)$.

Writing $v_j^{\ell} = v_{1,j}^{\ell}$, the theorem above implies that $\mathscr{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2 \dotplus \bigvee \left\{v_j^{\ell}(z) : 0 \leqslant \ell \leqslant m_j - 1, 1 \leqslant j \leqslant n\right\},$$

where the sum is no longer necessarily orthogonal.

It was shown in [2], and rediscovered in [1], that

$$\mathscr{H}(q) = \left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2 \dotplus \mathscr{P}_{n-1}, \tag{1.3}$$

where $\mathscr{P}_{n-1}$ is the n-dimensional space of polynomials of degree at most $n-1$ and the sum is an algebraic direct sum (not necessarily an orthogonal one). The novelty here is that we can precisely identify the orthogonal complement of $aH^2 = \left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2$ in $\mathscr{H}(q)$ without using (1.3).

A key ingredient in showing statement (1) of Theorem 1.2, and an added bonus to our result, is that if a_r is the Pythagorean mate for q^r then the co-analytic Toeplitz operators $T_{\overline{a}}$ and $T_{\overline{a_r}}$ on the Hardy space H^2 have the same range and this range can be precisely identified as $\mathscr{H}(q)$.

2. Preliminaries

There are several equivalent definitions of the de Branges-Rovnyak space $\mathscr{H}(b)$ for $b \in \mathbf{b}(H^\infty)$. We can for instance define it in the standard way (see [7]) as the reproducing kernel Hilbert space associated with the (positive definite) reproducing kernel

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

By definition,

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b, \qquad f \in \mathscr{H}(b), \lambda \in \mathbb{D},$$

where $\langle \cdot, \cdot \rangle_b$ stands for the scalar product in $\mathscr{H}(b)$.

The space $\mathscr{H}(b)$ can also be defined as the range space $(I - T_b T_{\overline{b}})^{1/2} H^2$ equipped with the norm making $(I - T_b T_{\overline{b}})^{1/2}$ a partial isometry. Here T_φ is the Toeplitz operator on H^2 with symbol $\varphi \in L^\infty(\mathbb{T})$ defined by

$$T_\varphi f = P_+(\varphi f), \qquad f \in H^2,$$

where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. The book [8] is the classic reference for $\mathscr{H}(b)$ spaces.

When $\|b\|_\infty < 1$, $\mathscr{H}(b)$ turns out to be a reformed version of H^2 while, on the other extreme, if b is an inner function, then $\mathscr{H}(b)$ turns out to be one of the classical and well-studied model spaces $H^2 \ominus bH^2$.

When b is non-extreme, two important (not necessarily closed) vector spaces of functions in $\mathscr{H}(b)$ are

$$\mathscr{M}(a) := T_a H^2 \quad \text{and} \quad \mathscr{M}(\overline{a}) := T_{\overline{a}} H^2.$$

It is known (see [8, page 24]) that

$$\mathscr{M}(a) \subset \mathscr{M}(\overline{a}) \subset \mathscr{H}(b).$$

3. Corona pairs

This following lemma is well-known but we record it here along with a proof for the sake of completeness and for the discussion of the examples in Section 5.

Lemma 3.1. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. Then q is non-extreme and if a is the Pythagorean mate for q then a is also rational.*

Proof. Since q is rational then $q = p_1/p_2$ where p_1 and p_2 are analytic polynomials and p_2 has no zeros on $\mathbb{D}^-$. We can of course choose p_2 such that $p_2(0) > 0$. Since $q \in \mathbf{b}(H^\infty)$ we see that $1 - |q(e^{i\theta})|^2 \geqslant 0$ for all θ and so $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is a non-negative trigonometric polynomial, which, by the Fejer-Riesz theorem, must be equal to $|p(e^{i\theta})|^2$, where p is an analytic polynomial which is zero free in $\mathbb{D}$ and $p(0) > 0$. Note that $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is non zero because q is assumed to be not inner.

Let

$$a = \frac{p}{p_2}.$$

Note that a is rational and zero free in $\mathbb{D}$, hence outer. Moreover $a(0) > 0$.

Furthermore, on $\mathbb{T}$ we have

$$|a|^2 = \left|\frac{p}{p_2}\right|^2 = \frac{|p_2|^2 - |p_1|^2}{|p_2|^2} = 1 - \left|\frac{p_1}{p_2}\right|^2 = 1 - |q|^2.$$

This means that (a, q) is a Pythagorean pair which, in particular, implies that q is non-extreme. □

Two functions $u, v \in H^\infty$ which satisfy

$$\inf_{z\in\mathbb{D}}(|u(z)| + |v(z)|) > 0$$

are said to be a corona pair. Carleson's corona theorem [6] says that if u, v form a corona pair then there are function $\phi, \psi \in H^\infty$ such that

$$\phi u + \psi v = 1.$$

Lemma 3.2. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and r is a positive real number. Then b and b^r are simultaneously non-extreme. Moreover, if a_r is the Pythagorean mate for b^r, the pairs (a, b) and (a_r, b^r) are simultaneously corona.*

Proof. Since

$$\frac{1 - x^r}{1 - x} \asymp 1, \quad x \in [0, 1], \tag{3.3}$$

we see that $1 - |b|^r \asymp 1 - |b|$ when $b \in \mathbf{b}(H^\infty)$. This proves the first part of the Lemma.

Now observe that

$$\frac{|a|^2}{|a_r|^2} = \frac{1 - |b|^2}{1 - |b^2|^r} \asymp 1,$$

and since a and a_r are outer, by Smirnov's theorem (which says that if the boundary function of $f \in H^2$ is bounded, then $f \in H^\infty$), a/a_r is invertible in H^∞. Thus both expressions

$$\inf_{z\in\mathbb{D}}(|a(z)| + |b(z)|) \quad \text{and} \quad \inf_{z\in\mathbb{D}}(|a_r(z)| + |b^r(z)|)$$

are strictly positive simultaneously. Indeed, if there is a sequence $\{z_n\}_{n\geqslant 1}$ in $\mathbb{D}$ such that one expression goes to 0 then, since both $a(z_n)$ and $b(z_n)$ go to zero, the other expression will go to zero as well. □

A special situation where b forms a corona pair with its Pythagorean mate is when b is rational, as explained in the following lemma.

Lemma 3.4. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. If a is the Pythagorean mate for q, then (a, q) is a corona pair.*

Proof. According to the proof of Lemma 3.1, we know that a is rational, $a = p/p_2$, where p and p_2 are polynomials, p_2 has no zeros in $\mathbb{D}^-$ and p is zero free in $\mathbb{D}$. In particular, a is continuous on $\mathbb{D}^-$ and it has a finite number of zeros on $\mathbb{T}$, say $\{\zeta_1, \ldots, \zeta_M\}$. Note that, due to the identity $|a|^2 + |q|^2 = 1$ on $\mathbb{T}$, the zeros of a (on $\mathbb{T}$) must lie where q is unimodular on $\mathbb{T}$.

Let D_j be open disjoint disks with center at the zeros ζ_j of a and let

$$F = \mathbb{D}^- \setminus \bigcup_{j=1}^{M} D_j.$$

By making the disks smaller, one can, by using the continuity of $|q|$ on $\mathbb{D}^-$, arrange things so that $|q| \geqslant \frac{1}{2}$ on each $D_j \cap \mathbb{D}^-$.

Notice that F is closed and omits all of the zeros of a in $\mathbb{D}^-$ and so

$$\inf_{z\in F} |a(z)| = \delta > 0.$$

Thus

$$\inf_{z\in\mathbb{D}}(|a(z)| + |q(z)|) \geqslant \min(\tfrac{1}{2}, \delta) > 0$$

concluding the proof. □

The first statement of our main theorem (Theorem 1.2) $\mathscr{H}(q) = \mathscr{H}(q^r)$ depends of the following two results. The first is from Sarason's book [8, p. 62].

Proposition 3.5. *For $b \in \mathbf{b}(H^\infty)$ and non-extreme, the following are equivalent:*

(1) (a, b) *is a corona pair;*

(2) $\mathscr{H}(b) = \mathscr{M}(\overline{a})$.

The second is the following.

Proposition 3.6. *If $a, a_1 \in H^\infty$ are two outer functions such that a/a_1 and a_1/a belong to L^∞, then $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_1})$.*

Proof. Again, by Smirnov's theorem, we know that a/a_1 and a_1/a belong to H^∞, so that T_{a/a_1} and hence $T_{\overline{a/a_1}}$ are invertible. Then we get

$$\mathscr{M}(\overline{a}) = T_{\overline{a}} H^2 = T_{\overline{a_1}} T_{\overline{a/a_1}} H^2 = T_{\overline{a_1}} H^2 = \mathscr{M}(\overline{a_1}). \quad \square$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2 we know that since (a, b) is a corona pair, then so is (a_r, b^r). Thus from Proposition 3.5 we see that $\mathscr{H}(b) = \mathscr{M}(\overline{a})$ and $\mathscr{H}(b^r) = \mathscr{M}(\overline{a_r})$. As in the proof of Lemma 3.2 a/a_r and a_r/a are in H^∞ so that by Proposition 3.6, we get $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_r})$ which concludes the proof. $\square$

4. Proof of the main theorem 1.2

Statement (1) of the theorem follows from Lemma 3.4 and Theorem 1.1.

Let us consider statement (2).

In [5] it was shown that if $\zeta \in \mathbb{T}$ and

$$\int_{\mathbb{T}} \frac{|\log|b(w)||}{|w-\zeta|^{2n+2}} dm(w) < \infty, \tag{4.1}$$

then every function in $\mathscr{H}(b)$, as well as their derivatives up to order n, has a finite non-tangential limit at ζ.

Recalling the notation $v^\ell_{r,\lambda}$ for the ℓ-th derivative in the variable $\overline{\lambda}$ of the reproducing kernel in $\mathscr{H}(q^r)$, under (4.1), we have $v^\ell_{r,\zeta} \in \mathscr{H}(q^r)$, $0 \leqslant \ell \leqslant n$, and

$$f^{(\ell)}(\zeta) = \langle f, v^\ell_{r,\zeta} \rangle_{q^r}, \quad f \in \mathscr{H}(q^r),\ 0 \leqslant \ell \leqslant n.$$

Remark 4.2. In the above, we are using $\langle \cdot, \cdot \rangle_b$ to denote the inner product in the space $\mathscr{H}(b)$. Note that $\mathscr{H}(0) = H^2$ with the integral inner product

$$\langle f, g \rangle_0 = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} dm(\zeta).$$

When the contact is clear, we will drop the subscript b from the inner product.

Let us check condition (4.1) for our situation. Since q is rational then its Pythagorean mate a is also rational and can be written as

$$a(z) = s(z) \prod_{j=1}^{n} (z - \zeta_j)^{m_j}, \tag{4.3}$$

where s is a rational function whose poles and zeros lie on the complement of $\mathbb{D}^-$. Pick $w = e^{it}$ near one of the zeros $\zeta_j = e^{i\theta_j}$ of a, then

$$\begin{aligned} |\log|q^r(e^{it})|| &\asymp |\log|q(e^{it})|^2| = |\log(1 - |a(e^{it})|^2)| \\ &\asymp |a(e^{it})|^2 \asymp |e^{it} - e^{i\theta_j}|^{2m_j} \end{aligned}$$

This means that for t near θ_j we have

$$\frac{|\log|q^r(e^{it})||}{|e^{it} - e^{i\theta_j}|^{2(m_j-1)+2}} \asymp 1$$

and so, by (4.1),

$$v^{\ell}_{r,\zeta_j} \in \mathscr{H}(q^r), \qquad 0 \leqslant \ell \leqslant m_j - 1.$$

So far we know that

$$\mathscr{H}(q^r) = \mathscr{M}(\overline{a_r}) = \mathscr{M}(\overline{a}) \supset \mathscr{M}(a) + \bigvee \{v^{\ell}_{r,\zeta_j} : 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1\}. \tag{4.4}$$

First we show that

$$v^{\ell}_{r,\zeta_j} \perp \mathscr{M}(a), \qquad 1 \leqslant j \leqslant n,\ 0 \leqslant \ell \leqslant m_j - 1.$$

Indeed, for each $f \in \mathscr{H}(q^r)$ the radial limits $f^{(\ell)}(t\zeta_j)$ exist as $t \to 1^-$. Since

$$f^{(\ell)}(t\zeta_j) = \langle f, v^{\ell}_{r,t\zeta_j} \rangle_{q^r},$$

we can apply the principle of uniform boundedness to see that $\|v^{\ell}_{r,t\zeta_j}\|_{q^r}$ is uniformly bounded as $t \to 1^-$. Since $v^{\ell}_{r,t\zeta_j}$ converges pointwise to v^{ℓ}_{r,ζ_j} as $t \to 1^-$ we see that $v^{\ell}_{r,t\zeta_j}$ converges weakly to v^{ℓ}_{r,ζ_j}. Thus, since $v^{\ell}_{r,t\zeta_j}$

reproduces the ℓ-th derivative of $\mathscr{H}(q^r)$-functions at point $t\zeta_j$, for any $g \in H^2$, we have

$$\begin{aligned}\langle ag, v^\ell_{r,\zeta_j}\rangle_{q^r} &= \lim_{t\to 1^-}\langle ag, v^\ell_{r,t\zeta_j}\rangle_{q^r} = \lim_{t\to 1}(ag)^{(\ell)}(t\zeta_j)\\ &= \lim_{t\to 1}\sum_{p=0}^{\ell}\binom{\ell}{p}a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j).\end{aligned}$$

Using the estimate

$$|a^{(p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p}$$

along with the following standard H^2 estimate on the growth of the derivative of an H^2 function

$$|g^{(\ell-p)}(t\zeta_j)| \lesssim \frac{1}{(1-t)^{(\ell-p)+1/2}},$$

we see that

$$|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p-((\ell-p)+1/2)}.$$

But since $0 \leqslant \ell \leqslant m_j - 1$ we see that

$$m_j - p - ((\ell-p)+1/2) = m_j - \ell - \tfrac{1}{2} \geqslant \tfrac{1}{2}$$

and so

$$\lim_{t\to 1^-}|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| = 0.$$

Thus $\langle ag, v^\ell_{r,\zeta_j}\rangle_{q^r} = 0$ and $v^\ell_{r,\zeta_j} \perp \mathscr{M}(a)$ in $\mathscr{H}(q^r)$, for all $0 \leqslant \ell \leqslant m_j - 1$.

This upgrades (4.4) to

$$\mathscr{H}(q^r) = \mathscr{M}(\overline{a}) \supset \mathscr{M}(a) \oplus \bigvee\{v^\ell_{r,\zeta_j} : 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1\}, \tag{4.5}$$

and orthogonality is with respect to $\mathscr{H}(q^r)$.

To show equality in (4.5), our second step is to show that if $f \in \mathscr{M}(\overline{a})$ and $f \perp v^\ell_{r,\zeta_j}$ for all $1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1$ then $f \in \mathscr{M}(a)$. Since $\mathscr{M}(\overline{a}) = T_{\overline{a}}H^2$ this is equivalent to proving that if $g \in H^2$ and

$$0 = (T_{\overline{a}}g)^{(\ell)}(\zeta_j) = \lim_{t\to 1^-}(T_{\overline{a}}g)^{(\ell)}(t\zeta_j)$$

for all $1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1$ then $T_{\overline{a}}g \in \mathscr{M}(a)$. To simplify matters a bit, let us recall the formula for a from (4.3). Since s is a rational function with zeros and poles outside $\mathbb{D}^-$ then certainly the Toeplitz operators $T_{1/s}$ and $T_{\overline{1/s}}$ are invertible, so that $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a/s})$. We can thus assume that

$$a(z) = \prod_{j=1}^{n}(z-\zeta_j)^{m_j}.$$

A preliminary step towards proving what we want is to show it for a special case. Indeed we will show that if $a(z) = (z-1)^m$ and $g \in H^2$ then

$$(T_{\overline{a}}g)^{(\ell)}(1) = 0,\ 0 \leqslant \ell \leqslant m-1 \implies T_{\overline{a}}g \in aH^2. \tag{4.6}$$

Notice that

$$\overline{a(\zeta)} = (-1)^m \overline{\zeta}^m a(\zeta), \qquad \zeta \in \mathbb{T},$$

and thus

$$T_{\overline{a}}g = (-1)^m P_+(a\overline{\zeta}^m g).$$

We claim that

$$\widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(m-1) = 0. \tag{4.7}$$

This will show that $\overline{\zeta}^m g \in H^2$ and so

$$T_{\overline{a}}g = (-1)^m P_+(a\overline{\zeta}^m g) \in aH^2.$$

We prove the claim (4.7) as follows. Consider the kernels

$$k_{\lambda,\ell}(z) = c_\ell \frac{z^\ell}{(1-\overline{\lambda}z)^{\ell+1}},$$

where c_ℓ is adapted so that these are the reproducing kernels for ℓ-th derivatives at point $\lambda \in \mathbb{D}$ in the Hardy space H^2, that is to say,

$$f^{(\ell)}(\lambda) = \langle f, k_{\lambda,\ell}\rangle_2, \qquad f \in H^2.$$

Observe that for $0 \leqslant \ell \leqslant m-1$

$$\begin{aligned} a(z)k_{t,\ell}(z) &= c_\ell \frac{z^\ell (1-z)^m}{(1-tz)^{\ell+1}} \\ &= c_\ell z^\ell (1-z)^{m-(\ell+1)} \left(\frac{1-z}{1-tz}\right)^{\ell+1} \\ &= c_\ell z^\ell (1-z)^{m-(\ell+1)} \left(1 + z\frac{t-1}{1-tz}\right)^{\ell+1}. \end{aligned}$$

One can see that $ak_{t,\ell}$ converges pointwise to $c_\ell z^\ell (1-z)^{m-\ell-1}$ as $t \to 1^-$ and is uniformly bounded on $\mathbb{T}$. Thus by the dominated convergence theorem, $ak_{t,\ell} \to c_\ell z^\ell (1-z)^{m-\ell-1}$ in the norm of H^2. Now pick a function $f = T_{\overline{a}}g \in \mathscr{M}(\overline{a}) = \mathscr{H}(q^r)$ with $T_{\overline{a}}g^{(\ell)}(1) = 0$ for all $0 \leqslant \ell \leqslant$

$m-1$, then

$$\begin{aligned}
0 &= (T_{\overline{a}}g)^{(\ell)}(1) = \lim_{t\to 1^-}(T_{\overline{a}}g)^{(\ell)}(t) = \lim_{t\to 1^-}\langle T_{\overline{a}}g, k_{t,\ell}\rangle_0 = \lim_{t\to 1^-}\langle g, ak_{t,\ell}\rangle_0 \\
&= \overline{c_\ell}\langle g, z^\ell(1-z)^{m-\ell-1}\rangle_0 \\
&= \overline{c_\ell}\left\langle g, \sum_{j=0}^{m-\ell-1}\binom{m-\ell-1}{j}(-1)^j z^{j+\ell}\right\rangle_0 \\
&= \overline{c_\ell}\sum_{j=0}^{m-\ell-1}\binom{m-\ell-1}{j}(-1)^j\widehat{g}(j+k) \\
&= \sum_{p=\ell}^{m-1}\binom{m-\ell-1}{p-\ell}(-1)^{p-\ell}\widehat{g}(p).
\end{aligned}$$

If

$$A_{i,j} = \binom{m-i-1}{j-i}(-1)^{j-i},$$

then $A = (A_{i,j})_{0\leqslant i,j\leqslant m-1}$ is an upper triangular invertible matrix and hence the above system of equations can be written as

$$A(\widehat{g}(0), \widehat{g}(1), \ldots, \widehat{g}(m-1))^T = (0, \ldots, 0),$$

which proves (4.7) and thus (4.6).

We finish the proof by establishing the following claim. If

$$a(z) = \prod_{j=1}^{n}(z-\zeta_j)^{m_j},$$

and $g \in H^2$ is such that, for any $1 \leqslant j \leqslant n$, and any $0 \leqslant \ell \leqslant m_j - 1$, we have

$$(T_{\overline{a}}g)^\ell(\zeta_j) = 0, \tag{4.8}$$

then $T_{\overline{a}}g \in aH^2$.

We argue by induction on n. If $n = 1$, then we can of course assume without loss of generality that $\zeta_1 = 1$ and then, the proof of the claim is given by (4.6).

Assume the claim is true for $n-1$ points. Suppose $g \in H^2$ satisfies (4.8). Decompose a as $a = a_1a_2$ where $a_1(z) = (z-\zeta_1)^{m_1}$ and

$$a_2(z) = \prod_{j=2}^{n}(z-\zeta_j)^{m_j}.$$

Since

$$T_{\overline{a}} = T_{\overline{a}_1} T_{\overline{a}_2} = T_{\overline{a}_2} T_{\overline{a}_1},$$

we easily see that the induction hypothesis implies that the function $h := T_{\bar{a}} g$ belongs to $a_1 H^2 \cap a_2 H^2$. In other words, there exists two functions $\varphi, \psi \in H^2$ such that

$$h = a_1 \varphi = a_2 \psi.$$

Now note that (a_1, a_2) is a corona pair. Hence, there exists $u, v \in H^\infty$ such that

$$a_1 u + a_2 v = 1.$$

Multiply this equation by ψ gives

$$\psi = a_1 \psi u + a_2 \psi v = a_1 \psi u + a_1 \varphi v = a_1 (\psi u + \varphi v).$$

Therefore, we obtain

$$h = a_2 \psi = a_2 a_1 (\psi u + \varphi v),$$

which shows that $h \in a_1 a_2 H^2 = a H^2$. The claim is established and the proof of Theorem 1.2 is complete.

5. Examples

Example 5.1. Consider the function

$$q(z) = \frac{1}{2}(1 + z)$$

and notice that q is outer and $\|q\|_\infty = 1$. One can easily guess the Pythagorean mate for q to be

$$a(z) = \frac{1}{2}(1 - z).$$

The function $a(z)$ has one zero of order 1 at $z = 1$ and a computation reveals that

$$v_{1,1}^0(z) = \frac{1 - \overline{q(1)} q(z)}{1 - z} = \frac{1}{2}.$$

In this case

$$\mathscr{H}(q) = (z - 1) H^2 \oplus \mathbb{C}$$

and so for any positive real number we get $\mathscr{H}(q^r) = \mathscr{H}(q)$ and

$$\mathscr{H}(q^r) = (z - 1) H^2 \dot{+} \mathbb{C} = (z - 1) H^2 \oplus \mathbb{C} \frac{1 - \left(\frac{1 + z}{2}\right)^r}{1 - z}.$$

For more general q we need to review the proof of the Fejer-Riesz theorem which says that if

$$w(e^{i\theta}) = \sum_{j=-n}^{n} c_j e^{ij\theta}$$

is a non-zero trigonometric polynomial which assumes non-negative values for all θ, then there is an analytic polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

so that

$$w(e^{i\theta}) = |p(e^{i\theta})|^2.$$

Since the proof gives us the algorithm for computing p, we give a quick sketch. Indeed, as a function of the complex variable z, we see that if

$$w(z) = \sum_{j=-n}^{n} c_j z^j$$

then $\overline{w(1/\overline{z})} = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$ we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of s occur in pairs $\alpha, 1/\overline{\alpha}$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^{n} (z - \alpha_j)(\frac{1}{z} - \overline{\alpha_j})$$

for some positive constant c and where $\alpha_1, \ldots, \alpha_n$ satisfy $|\alpha_j| \geqslant 1$ for $1 \leqslant j \leqslant n$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^{n} (z - \alpha_j).$$

Note that p is zero free in $\mathbb{D}$ and we can multiply p by a unimodular constant so that $p(0) > 0$.

Recall from the proof of Lemma (3.1) that if $q = p_1/p_2$ is rational then the Pythagorean mate a for q is given by

$$a = \frac{p}{p_2},$$

where p is the analytic polynomial (guaranteed by the Fejer-Riesz theorem) which satisfies $|p(e^{i\theta})|^2 = w(e^{i\theta}) = |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \geqslant 0$, and p is chosen to that $a(0) > 0$.

Example 5.2. Consider the function

$$q(z) = \frac{1}{2}(1-z)(1+z)$$

and note that $q \in \mathbf{b}(H^\infty)$ and is outer. A computation shows that

$$1 - |q(e^{it})|^2 = \frac{1}{4}e^{-2it} + \frac{1}{4}e^{2it} + \frac{1}{2}.$$

Define

$$w(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$

and

$$s(z) = z^2 w(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4}.$$

The polynomial s factors as

$$s(z) = \frac{1}{4}(z-i)^2(z+i)^2.$$

Notice how the zeros occur in pairs $i = 1/\overline{i}$ and $-i = 1/\overline{-i}$ as guaranteed by the above proof of the Fejer-Riesz theorem. Thus the Pythagorean mate a for q is of the form

$$a(z) = c(z-i)(z+i)$$

for some c adjusted so that $a(0) > 0$ and $1 - |q(e^{i\theta})|^2 = |a(e^{i\theta})|^2$. One can check by direct calculation that $c = 1/2$ works and so

$$a(z) = \frac{1}{2}(z-i)(z+i).$$

Of course the exact value of c is not important for our calculations since we only need to identify the zeros of a.

The zeros of a are at $z = i$ and $z = -i$ and have order one and so

$$\mathscr{H}(q) = (z-i)(z+i)H^2 \oplus \bigvee\{v^0_{1,i}, v^0_{1,-i}\},$$

where the kernels can be computed directly as

$$v^0_{1,i}(z) = \frac{1}{2i}(z+i), \qquad v^0_{1,-i}(z) = \frac{1}{2i}(z-i).$$

Again, as in the previous example, $\mathscr{H}(q^r) = \mathscr{H}(q)$ and so

$$\mathscr{H}(q^r) = (z-i)(z+i)H^2 \dot{+} \bigvee\{z+i, z-i\}.$$

Example 5.3. Consider the function

$$q(z) = \frac{1}{4}(z+1)^2$$

and note that q is outer and belongs to $\mathbf{b}(H^\infty)$. Following our Fejer-Riesz computations as in the previous example, note that

$$1 - |q(e^{it})|^2 = -\frac{e^{-it}}{4} - \frac{e^{it}}{4} - \frac{1}{16}e^{-2it} - \frac{1}{16}e^{2it} + \frac{5}{8}.$$

Define

$$w(z) = -\frac{z^2}{16} - \frac{1}{16z^2} - \frac{z}{4} - \frac{1}{4z} + \frac{5}{8}$$

and

$$\begin{aligned} s(z) &= z^2 w(z) \\ &= -\frac{z^4}{16} - \frac{z^3}{4} + \frac{5z^2}{8} - \frac{z}{4} - \frac{1}{16} \\ &= -\frac{1}{16}(-1+z)^2(1+6z+z^2). \end{aligned}$$

The zeros of s are at $z = -1, z = -1, z = -3 - 2\sqrt{2} \approx -5.82843, z = -3 + 2\sqrt{2} \approx -0.171573$. Notice how these roots occur in the pairs $\alpha, 1/\overline{\alpha}$. The function a is then

$$a(z) = c(z-1)(z+3+2\sqrt{2})$$

for some appropriate constant c. There is one zero of a at $z = 1$ of multiplicity one and so

$$\mathscr{H}(q) = (z-1)H^2 \oplus \mathbb{C}v^0_{1,1}(z),$$

where the kernel can be computed to be

$$v^0_{1,1}(z) = \frac{z+3}{4}.$$

As in our previous examples, note that

$$\mathscr{H}(q^r) = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Notice that the q from this example is the square of the q from our first example and thus the corresponding spaces should be the same. Indeed a little algebra will show that

$$(z-1)H^2 \dot{+} \mathbb{C} = (z-1)H^2 \dot{+} \mathbb{C}(z+3).$$

Example 5.4. Reversing the roles of a and q in the preceding example:

$$a(z) = \frac{1}{4}(z+1)^2, \qquad q(z) = c(z-1)(z+3+2\sqrt{2}),$$

with suitable c so that $\|q\|_\infty = 1$ (the maximum modulus on $\mathbb{D}^-$ being attained in -1, one has $c = (4(1+\sqrt{2}))^{-1}$, and $q(-1) = -1$ corresponding to the normalization $q(0) > 0$), we obtain a function a with double zero, and so

$$\mathscr{H}(q) = (z+1)^2 H^2 \oplus \bigvee\{v^0_{1,-1}, v^1_{1,-1}\}$$

where

$$v^0_{1,-1}(z) = \frac{1 - \overline{q(-1)}q(z)}{1 - \overline{(-1)}z} = \frac{1+q(z)}{1+z},$$

and using $q(-1) = -1$ as well as $q'(z) = c(2+2\sqrt{2})$ so that $q'(-1) = -1/2$, we have

$$v^1_{1,-1}(z) = \frac{\frac{1}{2}q(z)(1+z) + z(1+q(z))}{(1+z)^2} = \frac{1}{2}\frac{q(z)(1+3z)+2z}{(1+z)^2}.$$

Example 5.5. Here is an example of a rational function:

$$q(z) = \frac{3(z+1)}{2(z+2)}.$$

Then, as it turns out,

$$a(z) = c\frac{z-1}{z+2}$$

which has a zero of order one at $z = 1$. From here we get

$$\mathscr{H}(q) = (z-1)H^2 \oplus \mathbb{C}v^0_{1,1}(z) = (z-1)H^2 \oplus \mathbb{C}\frac{1}{4+2z}.$$

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