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Abstract
In this paper fault diagnosis is studied for linear time varying descriptor systems, which are the discrete time counterpart of the continuous time dynamic systems described by differential-algebraic equations. This class of systems includes and is broader than the well-known state space systems. The framework of descriptor systems is particularly useful for studying dynamic systems exhibiting time varying singularities. Actuator faults and sensor faults are respectively modeled as parametric changes in the state equation and in the output equation. The main result of this paper consists in extending an adaptive observer, initially designed for state space systems, to descriptor systems. Based on this result, fault diagnosis is performed by estimating the parameters characterizing actuator and sensor faults. Simulations examples are presented to illustrate the proposed method.

Keywords: fault diagnosis, discrete-time descriptor dynamics, time-varying models, adaptive observer.

1. INTRODUCTION

Many modern engineering systems can be modeled by an explicit ordinary differential equation (ODE) of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $x(t)$ and $u(t)$ represent respectively the (vectorial) state and input of the system, $\dot{x}(t)$ denotes the time derivative of $x(t)$, and $f(\cdot, \cdot)$ is some function characterizing the system. Such state space equations have a long-term mathematical history, and a large number of analytical and numerical tools have been developed for their study.

However, in some cases such an explicit state space model for the dynamics of a given system is not available. The system may instead be described by implicit differential equations, known as differential-algebraic equations (DAE), of the form

$$F(\dot{x}(t), x(t), u(t)) = 0 \tag{2}$$

where $F(\cdot, \cdot, \cdot)$ is some vector-valued function. If $\dot{x}(t)$ can be solved for from (2), then the DAE can be converted to an ODE, but this operation is not always possible. It is thus necessary to study DAE systems in some situations.

After linearization around an operating point and discretization in time, the original nonlinear DAE system is approximately described by an implicit discrete time state space equation

$$E_{k+1}x(k+1) = A_k x(k) + B_k u(k) + \mu(k) \tag{3}$$

where $x(k)$, $u(k)$ and $\mu(k)$ are respectively the discrete time state, the input and the modeling errors indexed by $k = 0, 1, 2, ...$, and the matrix E_k may not be of full column rank.

Systems governed by (3) are known as *descriptor systems*. This is a general and convenient framework for studying DAE systems which can be appropriately linearized.

Some descriptor systems can be simply regarded as implicitly written state-space equations, this is the case of systems in which the matrix E_k has full column rank. In principle, the theory developed in the framework of linear state-space equations can be applied in this case. However, even if the matrix E_k is invertible for all k, it is preferable to use the descriptor equation because of the possible ill conditioning of E_k.

Descriptor systems have attracted considerable attentions in recent decades where great efforts where made to investigate descriptor system theory and applications (Nikoukhah et al. (1992), Benveniste et al. (1993), Darouach and Boutayeb (1995), Shields (1997), Polycarpou et al. (1997), Zhang et al. (1998), Vemuri et al. (2001), Koenig and Mammar (2002)).

Fault diagnosis (detection and identification of fault) is rarely tackled in the descriptor case, in contrast to the case of systems with classical state space representations, where the theory is well-established (Ding (2008), Isermann (2006) and references therein). For descriptor system fault diagnosis, most studies are observer-based approaches in the time invariant case (Duan et al. (2002), Marx et al. (2004)).

In this paper, fault diagnosis will be studied for the general class of time-varying discrete time linear descriptor systems in the form of

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$$E_{k+1}x(k+1) = A_k x(k) + B_k u(k) + \Phi_k \theta + \mu(k) \quad (4a)$$
$$y(k+1) = C_{k+1}x(k+1) + \Psi_k \theta + \nu(k) \quad (4b)$$
$$k = 0,\ 1,\ 2,\ \cdots$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input vector (possibly including other known variables), $y(k) \in \mathbb{R}^p$ is the output vector, E_{k+1}, A_k, B_k, C_k are known real matrices of compatible sizes, $\mu(k)$ and $\nu(k)$ represent modeling/measurement errors and are modeled as independent, zero-mean, Gaussian vector sequences with covariance matrices Q_k and R_k respectively, the terms $\Phi_k\theta$ and $\Psi_k\theta$ represent respectively faults possibly affecting the state and the output equation. The matrices $\Phi_k \in \mathbb{R}^{n\times q}$ and $\Psi_k \in \mathbb{R}^{n\times q}$ are assumed known, whereas the parameter vector $\theta \in \mathbb{R}^q$ is unknown and subject to changes caused by faults. The fault-free system is characterized by the nominal value $\bar{\theta}$ of the parameter vector.

The development of this paper is based, on the one hand, on the adaptive estimation techniques for explicit state space systems (Zhang (2002)), and, on the other hand, on the Kalman filter for time-varying discrete linear descriptor systems. The core of the proposed fault diagnosis methods consists of an adaptive observer which is used both to estimate the monitored faults and to improve the robustness against model uncertainty due to parameter changes.

The paper is organized as follows. The first step in the monitoring of a system modeled by (4) is necessarily the state space x reconstruction when the only information available is contained in signals u and y. In Section 2, under some assumptions, a transformation of descriptor systems is introduced which allows us to use a standard Kalman filter for state vector estimation, in the fault-free case. Section 3 is devoted to the outline of the approach discussed in this paper. We detail the motivations of this contribution and explain how adaptive observer based fault diagnosis approach can be extended to general linear descriptor systems. In section 4, a simulation example is used to illustrate the effectiveness of the proposed approach.

2. KALMAN FILTER FOR FAULT-FREE DESCRIPTOR SYSTEMS

Here we recall some basic knowledge about the Kalman filter of discrete linear descriptor systems described by

$$E_{k+1}x(k+1) = A_k x(k) + B_k u(k) + \mu(k) \quad (5a)$$
$$y(k+1) = C_{k+1}x(k+1) + \nu(k) \quad (5b)$$
$$k = 0,\ 1,\ 2,\ \cdots$$

Like the classical Kalman filter for state space systems, the purpose of descriptor system Kalman filter is for state estimation from input and output signals. State reconstruction has been frequently studied in continuous time and usual for state space systems case (when $E = I$) since the early work in (Kalman (1960), Kreindler and Sarachik (1964), Silverman and Meadows (1967)) and in the time invariant descriptor case. Theoretical work on the observability of linear time-varying continuous time descriptor systems can be found in (Campbell and Terrell (1991)).

In (Nikoukhah et al. (1992)), the authors developed a Kalman filter for (5), based on maximum likelihood estimation. The filtered estimate $\hat{x}$ is defined as the maximum likelihood estimate of x based on (5) together with a priori knowledge of the initial condition $x(0)$. Recursively, we obtain, from (5) and the estimate of $x(k)$ denoted by $\hat{x}(k)$, the following equation

$$Y_k = H_{k+1}x(k+1) + V_k, \quad k = 0, 1, \cdots \quad (6)$$

where

$$Y_k = \begin{bmatrix} A_k\hat{x}(k) + B_k u(k) \\ y(k+1) \end{bmatrix},\ H_k = \begin{bmatrix} E_k \\ C_k \end{bmatrix},\ V_k = \begin{bmatrix} w(k) \\ \nu(k) \end{bmatrix}$$

$w(k) = -A_k\tilde{x}(k) - \mu(k)$ and $\tilde{x}(k) = x(k) - \hat{x}(k)$ is the state estimate error.

It has been shown in (Nikoukhah et al. (1992)) (Lemma 2.3 and Lemma 2.4) that the (optimal) estimate $\hat{x}(k+1)$ of $x(k+1)$ based on (5) is the same as its (optimal) estimate based on (6).

Let $\mathcal{R}_k$ be the covariance matrix of V_k. It is shown in (Nikoukhah et al. (1992)) that

$$\mathcal{R}_k = \begin{bmatrix} A_k P_k A_k^T + Q_k & 0 \\ 0 & R_k \end{bmatrix}$$

where P_k is a symmetric positive definite matrix, which can be computed at each iteration of the maximum likelihood estimation algorithm, as recalled later in this section.

We shall make the following assumptions on system (5).

A1 : The initial state $x(0)$ is Gaussian with known mean $\bar{x}_0$ and variance P_0 and independent of μ and ν. This can be represented by the following equation

$$x(0) = \bar{x}_0 + \eta \quad (7)$$

where η is zero-mean, Gaussian and independent of μ and ν, and with covariance P_0.

A2 : The matrix H_k is supposed to be full column rank $\forall\ k \geq 1$

A3 : For all $k \geq 0$, the variance matrices Q_k and R_k are positive definite.

From (6), the maximum likelihood estimate of $x(k+1)$ consists in maximizing the probability density function of V_k parameterized by $x(k+1)$. This estimate is given by

$$\hat{x}(k+1) = \arg\max_{\alpha}\ e^{\{-\frac{1}{2}(Y_k - H_{k+1}\alpha)\mathcal{R}_k^{-1}(Y_k - H_{k+1}\alpha)\}} \quad (8)$$

The solution of the maximization in (8) gives

$$\hat{x}(k+1) = (H_{k+1}^T\mathcal{R}_k^{-1}H_{k+1})^{-1}H_{k+1}^T\mathcal{R}_k^{-1}Y_k \quad (9)$$

The error covariance associated with the estimate $\hat{x}(k+1)$ is

$$P_{k+1} = (H_{k+1}^T\mathcal{R}_k^{-1}H_{k+1})^{-1} \quad (10)$$

Note that, the inverse matrices in (9) and (10) are well defined because of assumptions A2 and A3.

Consider the following partition

$$[L_k \quad K_k] = P_{k+1}H_{k+1}^T\mathcal{R}_k^{-1} \quad (11)$$

The estimate $\hat{x}(k+1)$ can be explicitly expressed from (6), (9) and (11), yielding

$$\hat{x}(k+1) = L_k A_k\hat{x}(k) + K_k y(k+1) + L_k B_k u(k) \quad (12)$$

and the state estimation error is then governed by the equation

$$\tilde{x}(k+1) = L_k A_k \tilde{x}(k) + L_k \mu(k) - K_k \nu(k) \tag{13}$$

Let $\tilde{P}_k$ be the predicted estimate covariance :

$$\tilde{P}_k = A_{k-1} P_k A_{k-1}^T + Q_{k-1} \tag{14}$$

The following Riccati equation is obtained from (10)

$$\tilde{P}_{k+1} = A_k \left(E_{k+1}^T \tilde{P}_k^{-1} E_{k+1} + C_{k+1}^T R_k^{-1} C_{k+1} \right)^{-1} A_k + Q_k \tag{15}$$

We know that the stability of the Kalman filter (the optimal estimator) is generally guaranteed if the covariance matrix $\tilde{P}_k$ is bounded from above and from below by two strictly positive definite matrices for all $k \geq 0$.

In (Nikoukhah et al. (1992)), the asymptotic behavior of the error variance P_k was examined in the time invariant case (with constant matrices A, C, E, Q, R). It can be shown in this case that this error variance $\tilde{P}_k$ converges exponentially fast to the unique positive definite solution of the algebraic descriptor Riccati equation (Nikoukhah et al. (1987)):

$$\tilde{P} = A \left(E^T \tilde{P}^{-1} E + C^T R^{-1} C \right)^{-1} A^T + Q \tag{16}$$

3. ADAPTIVE OBSERVER

Consider the system in (4) subject to actuator and sensor faults. By defining L_k and K_k as in (11), the following explicit state recurrent equation is derived from (4):

$$\begin{aligned} x(k+1) &= L_k A_k x(k) + K_k y(k+1) + L_k B_k u(k) && \text{(17a)} \\ &\quad + [L_k \Phi_k - K_k \Psi_k]\theta + L_k \mu(k) - K_k \nu(k) \\ y(k+1) &= C_{k+1} x(k+1) + \Psi_k \theta + \nu(k) && \text{(17b)} \\ & k = 0,\ 1,\ 2,\ \cdots \end{aligned}$$

We assume, in addition of assumptions A1 - A3, that

A4 : E_k, A_k, B_k, C_k, Φ_k, Ψ_k are all known and bounded
A5 : the parameter vector θ is constant

In fault diagnosis, the terms $\Phi_k\theta$ and $\Psi_k\theta$ can be used to model faults. In particular, they can be used to model actuator or sensor faults according to their location in the system model.

In this section, we aim to estimate the hidden variable vector x and the parameter vector θ based on the knowledge of the input u and the output y signals. One commonly used method to solve this problem is to augment the state $x(k)$ with the parameter vector θ and to implement the Kalman filter (12). While this approach has proved effective in some applications, at least in the case of state space systems (Cox (1964)), it has also some well known drawbacks. In particular,

- treating equally the state vector $x(k)$ and the parameter vector θ as if they had similar dynamics may make the tuning of the Kalman filter delicate.
- the computational cost of the Kalman filter increases mainly because of the larger covariance matrix of the augmented state estimation error.

The approach proposed in this section relies on an adaptive observer. We extend the existing results concerning the adaptive observer for fault diagnosis in linear state space systems (Zhang (2002)) to the descriptor system.

Conceptually similar to the Kalman filter applied to the augmented system, the adaptive observer explicitly takes into account the difference between state variables and unknown parameters.

For the descriptor system described by (4) we propose the following adaptive observer

$$\begin{aligned} \hat{x}(k+1) &= L_k A_k \hat{x}(k) + K_k y(k+1) + L_k B_k u(k) \\ &\quad + [L_k \Phi_k - K_k \Psi_k]\hat{\theta}(k) + \omega_k && \text{(18a)} \\ \hat{y}(k+1) &= C_{k+1}\hat{x}(k+1) + \Psi_k \hat{\theta}(k) && \text{(18b)} \\ \Upsilon_{k+1} &= L_k A_k \Upsilon_k + [L_k \Phi_k - K_k \Psi_k] && \text{(18c)} \\ \Omega_{k+1} &= C_{k+1}\Upsilon_{k+1} + \Psi_k && \text{(18d)} \\ \omega_k &= L_k A_k \Upsilon_k (\hat{\theta}(k) - \hat{\theta}(k-1)) && \text{(18e)} \\ \hat{\theta}(k+1) &= \hat{\theta}(k) + \Theta_{k+1}\left(y(k+1) - \hat{y}(k+1)\right) && \text{(18f)} \\ \Theta_{k+1} &= S_k \Omega_{k+1}^T \left(\lambda R_k + \Omega_{k+1} S_k \Omega_{k+1}^T\right)^{-1} && \text{(18g)} \\ \Gamma_{k+1} &= \left(\lambda R_k + \Omega_{k+1} S_k \Omega_{k+1}^T\right)^{-1} && \text{(18h)} \\ S_{k+1} &= \frac{1}{\lambda}\left(I - \Theta_{k+1}\Omega_{k+1}\right) S_k && \text{(18i)} \\ & k = 0,\ 1,\ 2,\ \cdots \end{aligned}$$

where $\lambda \in (0,1)$ is a forgetting factor, $\hat{\theta}(k)$ is the estimate of the parameter vector θ at time k, matrices gain L_k and K_k are computed as in (11). The term ω_k added in (18a) and defined in (18e) may appear unusual. It will help to establish the proof of the convergence of this adaptive observer later in this section.

This adaptive observer is the discrete time counterpart of the continuous time algorithm presented in (Li et al. (2011)). A simpler discrete time algorithm (no unknown parameter vector in the output equation) has been proposed in (Guyader and Zhang (2003)). Its generalizations to some particular class of nonlinear systems have been considered in (Xu and Zhang (2004), Zhang and Besançon (2008), Farza et al. (2009)).

The convergence of the adaptive observer (18) can be studied both in the noise-free case and in the noise-corrupted case. As the proofs are similar to those of (Guyader and Zhang (2003)) and (Li et al. (2011)), let us formulate here its main lines.

Define the state and parameter estimation errors, respectively given by

$$\tilde{x}(k) = x(k) - \hat{x}(k) \tag{19}$$

$$\tilde{\theta}(k) = \theta - \hat{\theta}(k) \tag{20}$$

In the noise-free case ($\mu(k) = \nu(k) = 0$), the state estimation error is governed by the following equation derived from (17a) and (18a)

$$\tilde{x}(k+1) = L_k A_k \tilde{x}(k) + [L_k \Phi_k - K_k \Psi_k]\tilde{\theta}(k) - \omega_k \tag{21}$$

Define the variable

$$z(k) = \tilde{x}(k) - \Upsilon_k \tilde{\theta}(k-1) \tag{22}$$

From (21), (18c) and (18e), the dynamics of z_k is simplified to an exponentially stable system

$$z(k+1) = L_k A_k z(k) \tag{23}$$

The parameter estimation error is governed by

$$\tilde{\theta}(k+1) = (I - \Theta_{k+1}\Omega_{k+1})\,\tilde{\theta}(k) - \Theta_{k+1}C_{k+1}z(k+1) \tag{24}$$

The key issue is to prove the exponential stability of the homogeneous part of (24) given by

$$\zeta(k+1) = (I - \Theta_{k+1}\Omega_{k+1})\,\zeta(k) \tag{25}$$

Lemma 3.1. Under conditions

- the matrix $L_k A_k$ in (18c) is stable
- there exists an integer N and a real $\alpha > 0$, such that for all $k \geq 0$

$$\alpha I \leq \sum_{i=k}^{k+N-1} \Omega_{i+1}^T R_i^{-1} \Omega_{i+1} \tag{26}$$

- the forgetting factor $\lambda \in (0,\ 1)$
- the initial gain matrix S_0 is symmetric positive definite

then the matrix S_k in (18i) is positive definite and bounded from above and from below by two strictly positive definite matrices for all $k \geq 0$.

A proof of this lemma can be found in Appendix A. The stability of the matrix $L_k A_k$ is directly related to the stability of the Kalman filter (12), which is assumed in this paper. In the literature the condition (26) is referred to as the persistent excitation condition (Narendra and Annaswamy (2005)).

The boundedness of S_k ensured by the above lemma allows us to define the Lyapunov as follows

$$V_k = \zeta(k)^T S_k^{-1} \zeta(k) \tag{27}$$

It follows

$$V_{k+1} = \zeta(k+1)^T S_{k+1}^{-1} \zeta(k+1) \tag{28a}$$

$$= \lambda \zeta(k)^T (I - \Theta_{k+1}\Omega_{k+1})^T S_k^{-1} \zeta(k) \tag{28b}$$

$$= \lambda V_k - \lambda \zeta(k)^T \Omega_{k+1}^T \Theta_{k+1}^T S_k^{-1} \zeta(k) \tag{28c}$$

$$= \lambda V_k - \lambda \left[\Omega_{k+1}\zeta(k)\right]^T \Gamma_{k+1} \left[\Omega_{k+1}\zeta(k)\right] \tag{28d}$$

where Γ_{k+1} is the positive definite matrix given by (18h) i.e $\left[\Omega_{k+1}\zeta(k)\right]^T \Gamma_{k+1} \left[\Omega_{k+1}\zeta(k)\right] \geq 0$.

We then conclude that $V_{k+1} \leq \lambda V_k$. This means that for any $\lambda \in (0\ ,\ 1)$, V_k decreases exponentially and that ζ_k and thus $\tilde{\theta}_k$ also tends exponentially to zero.

It then follows from (22) that, in the noise-free case,

$$\begin{cases} \lim\limits_{k\to+\infty} z_k = 0 \\ \lim\limits_{k\to+\infty} \tilde{\theta}_k = 0 \end{cases} \quad \Rightarrow \quad \lim_{k\to+\infty} \tilde{x}_k = 0$$

In the noise-corrupted case, instead of ordinary convergence, the state and parameter estimation errors converge in the mean to zeros (Guyader and Zhang (2003)).

The proposed adaptive observer has two practical advantages. The gains L_k and K_k for state estimation and Θ_{k+1} for parameter estimation can be tuned in two steps in simulation studies. In the first step the parameter θ is assumed known, hence the tuning of L_k and K_k is like in the case of the classical Kalman filter for state estimation and afterward, in the second step, Θ_{k+1} is tuned while the tuning of L_k and K_k is fixed. Another advantage of the adaptive observer is that the recursive computations of the gains for state and parameter estimations are separated, implying a lower numerical cost, compared to the fully coupled gain matrix computation in the Kalman filter applied to the augmented system.

4. NUMERICAL EXAMPLE

In this example, we use the adaptive observer to monitor faults in a system modeled by (5) with the following data

$$E_{k+1} = \begin{bmatrix} 0 & 0 \\ -1 & \dfrac{k}{k+1} \end{bmatrix}, \quad A_k = \begin{bmatrix} 1 & -\dfrac{k+1}{k+0.5} \\ 0 & \dfrac{1}{k+1} \end{bmatrix}$$

$$B_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(k) = \begin{bmatrix} 1 - \dfrac{e^{-0.018k}}{0.8}\sin(0.03k) \\ 1 + e^{-0.03k}\sin(0.1k) \end{bmatrix}$$

$$C_{k+1} = [0\ 1],\ \bar{\theta} = 1,\ \Phi_k = \begin{bmatrix} 1 - e^{\frac{-k}{30}}\sin(0.1k) \\ 1 - e^{\frac{-k}{15}}\sin(0.5k) \end{bmatrix},\ \Psi_k = 0$$

It is easy to verify that the matrix E_{k+1} is not invertible and that the matrix

$$\begin{bmatrix} E_{k+1} \\ C_{k+1} \end{bmatrix}$$

is full column rank for all $k \geq 0$. The sampling period for the discrete time model is 1s and the simulation is performed during 1000s. We simulate a degradation of the parameter θ starting from time instant $k = 200$. Centered Gaussian white noise is added to the output. The simulated output with noise y is plotted in figure 1. The simulated and estimated parameter (fault) are shown in figure 2, where the dotted lines represent the true simulated parameter values, and the solid lines represent the estimated values. The estimated states are compared with the simulated states in figures 3 and 4, where dotted line represents the true simulated state variables, and the solid line represents the estimated values. These results show that, after the transient time of about 100s corresponding to the transient time (unmodeled transient) of θ, the parameter estimates follow closely the evolution of the simulated parameter. It is then possible to detect the simulated faults and estimate their severity.

5. CONCLUSION

This paper has dealt with fault diagnosis in descriptor systems. We have focused our study on the diagnosis of faults modeled as parameter changes in a class of discrete-time descriptor systems. Under realistic assumptions on the system, a Kalman filtering algorithm is derived. Then, the adaptive observer technique is used to accomplish the fault diagnosis task. We have shown that the adaptive

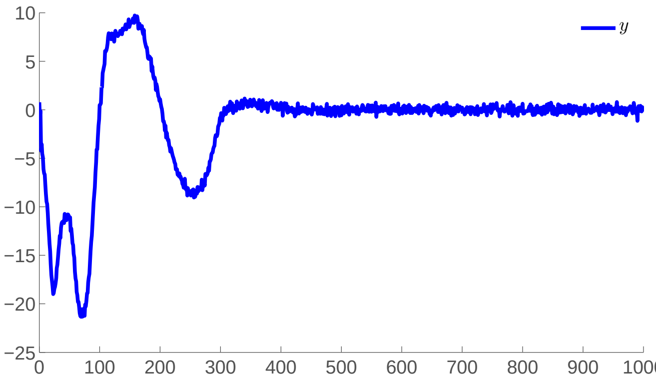

Figure 1. Graphical representations of the simulated noised output y over time (k).

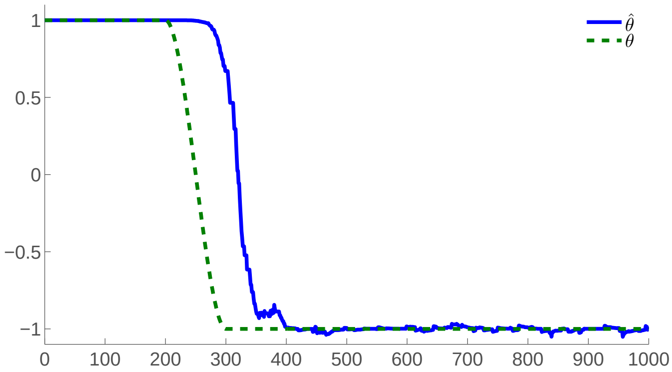

Figure 2. Graphical representations of the simulated parameter θ in dotted line and its estimate $\hat{\theta}$ in solid line, over time (k).

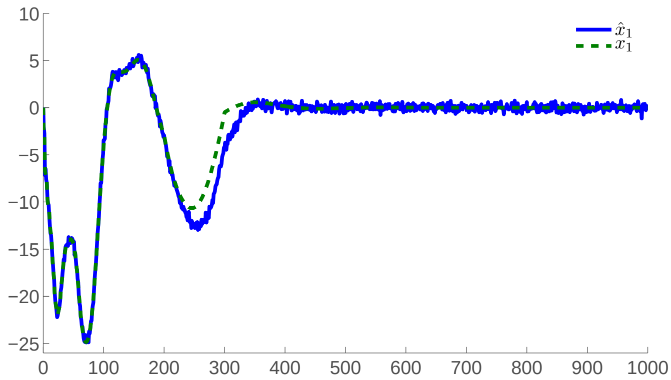

Figure 3. Graphical representations of the simulated state x_1 in dotted line and its estimate in solid line, over time (k).

observer initially developed for explicit linear state space systems can be extended to general linear descriptor systems. The decision for fault diagnosis is based on the time evolution of parameter estimates. Simulation results are produced to illustrate the ability of the proposed approach to detect faults.

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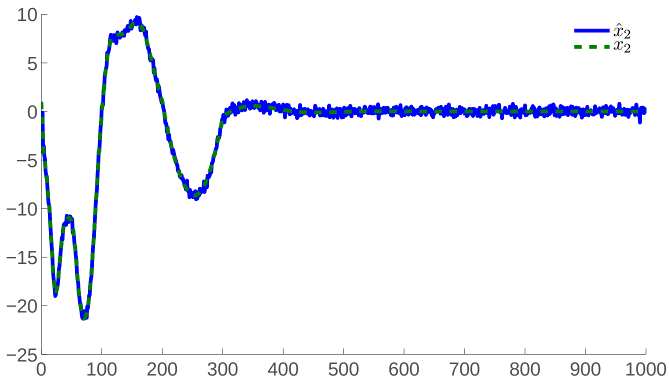

Figure 4. Graphical representations of the simulated state x_2 in dotted line and its estimate in solid line, over time (k).

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Appendix A. PROOF OF LEMMA 3.1

Consider M_k governed by

$$\begin{cases} M_{k+1} = \lambda M_k + \Omega_{k+1}^T R_k^{-1} \Omega_{k+1} \\ M_0 = S_0^{-1} > 0 \end{cases} \tag{A.1}$$

Let us first study the upper bound of M_k. Due to the stability of $L_k A_k$, the matrix Ω_k computed from bounded Φ_k and Ψ_k is also bounded. As $0 < \lambda < 1$, the iterations in (A.1) is exponentially stable. Hence M_k driven by the bounded term $\Omega_{k+1}^T R_k^{-1} \Omega_{k+1}$ is also bounded.

Now consider the lower bound of M_k. A recursive evaluation of (A.1) leads to

$$M_{k+1} = \lambda^{k+1} M_0 + \sum_{j=0}^{k} \lambda^{k-j} \Omega_{j+1}^T R_j^{-1} \Omega_{j+1} \tag{A.2}$$

$$M_{k+1} \geq \lambda^{k+1} M_0 + \sum_{i=1}^{\lfloor \frac{k}{N} \rfloor} \sum_{j=(i-1)N}^{iN-1} \lambda^{k-j} \Omega_{j+1}^T R_j^{-1} \Omega_{j+1}$$

$$\geq \lambda^{k+1} M_0 + \sum_{i=1}^{\lfloor \frac{k}{N} \rfloor} \lambda^{k-iN+1} \sum_{j=(i-1)N}^{iN-1} \Omega_{j+1}^T R_j^{-1} \Omega_{j+1}$$

where $\lfloor \cdot \rfloor$ denotes the floor (greatest integer) function.

It follows from (26) that

$$M_{k+1} \geq \lambda^{k+1} M_0 + \alpha \left(\sum_{i=1}^{\lfloor \frac{k}{N} \rfloor} \lambda^{k-iN+1} \right) I \tag{A.3}$$

Then, we can conclude that M_k is bounded from below by a strictly positive definite matrix. Hence we have proved that M_k has both an upper bound and a strictly positive definite lower bound.

From (A.1) the inverse of the matrix M_{k+1} writes

$$M_{k+1}^{-1} = \frac{1}{\lambda} \left(M_k + \frac{1}{\lambda} \Omega_{k+1}^T R_k^{-1} \Omega_{k+1} \right)^{-1} \tag{A.4}$$

By using the matrix inversion lemma

$$(A + BCD)^{-1} = A^{-1} - A^{-1} B \left(DA^{-1} B + C^{-1} \right)^{-1} DA^{-1}$$

it easy to check that $S_k = M_k^{-1}$ satisfies

$$S_{k+1} = \frac{1}{\lambda} \left(I - \Theta_{k+1} \Omega_{k+1} \right) S_k \tag{A.5}$$

with the initial condition S_0.

Therefore, we conclude that S_k is bounded from above and from below by two strictly positive definite matrices, for all $k \geq 0$. □