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CONTINUOUS FIELDS OF PROPERLY INFINITE C*-ALGEBRAS

ETIENNE BLANCHARD

ABSTRACT. The only separable unital continuous $C([0, 1])$ -algebra with fibres isomorphic to the Cuntz algebra \mathcal{O}_∞ is the trivial continuous field $\mathcal{O}_\infty \otimes C([0, 1])$. But there exist non properly infinite separable unital continuous $C([0, 1])$ -algebras with properly infinite fibres.

1. INTRODUCTION

One of the basic C*-algebras studied in the classification programme of nuclear C*-algebras through K-theoretical invariants launched by G. Elliott ([Ell94]) is the Cuntz C*-algebra \mathcal{O}_∞ generated by infinitely many isometries with pairwise orthogonal ranges ([Cun77]). This C*-algebra is pretty rigid in so far as it is a strongly self-absorbing C*-algebra ([TW07]): Any separable unital continuous $C(X)$ -algebra A the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial $C(X)$ -algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra \mathcal{Z} ([Win09]). Hence, this C*-algebra D is K_1 -injective ([Ror04]) and the $C(X)$ -algebra A satisfies $A \cong D \otimes C(X)$ ([DW08].) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product $\prod_{n=0}^\infty S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2^∞ ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dad09, §3] for all pair (Γ_0, Γ_1) of discrete countable torsion groups a unital separable continuous $C(X)$ -algebra A such that:

- the base space X is the compact Hilbert cube $X = \mathfrak{X}$ of infinite dimension,
- all the fibres A_x ($x \in \mathfrak{X}$) are isomorphic to the strongly self-absorbing Cuntz C*-algebra \mathcal{O}_2 generated by two isometries s_1, s_2 satisfying $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$,
- $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0, 1]$ is the canonical Cantor set.

These K-theoretical conditions imply that the $C(\mathfrak{X})$ -algebra A is not a trivial one. These arguments does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra \mathcal{O}_∞ ([Cun77]), in so far as $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ is a torsion free group.

We show directly in this article that the Pimsner-Toeplitz algebra ([Pim95]) of the nontrivial Dixmier-Douady Hilbert $C(X)$ -module E_{DD} ([DD63]) is a nontrivial unital $C(X)$ -algebra with fibres \mathcal{O}_∞ . This notably implies that there exists a properly infinite C*-algebra A which is not K_1 -injective, *i.e.* the mapping $\mathcal{U}(A)/\mathcal{U}_0(A) \rightarrow K_1(A)$ is not

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injective, and there exist separable unital continuous $C([0, 1])$ -algebras with properly infinite fibres which are not properly infinite C^* -algebras.

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2. NOTATIONS

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of any subset S in a Banach space.

Definition 2.1. ([Dix69], [Kas88], [Blan97]) *Let X be a compact Hausdorff space and let $C(X)$ be the C^* -algebra of continuous function on X .*

- *A unital $C(X)$ -algebra is a unital C^* -algebra A endowed with a unital morphism of C^* -algebra from $C(X)$ to the centre of A .*
- *For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a|_F$ the image of a in the quotient $A|_F := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in X , one calls fibre at x the quotient $A_x := A|_{\{x\}}$ and one write a_x for $a|_{\{x\}}$.*
- *The $C(X)$ -algebra A is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.*

Remarks 2.2. a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the C^* -algebra $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$ is the universal unital C^* -algebra generated by n isometries s_1, \dots, s_n satisfying the relation

$$s_1 s_1^* + \dots + s_n s_n^* \leq 1. \quad (2.1)$$

b) A unital C^* -algebra A is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Ror03, Proposition 2.1]):

- the C^* -algebra A contains two isometries with mutually orthogonal range projections, *i.e.* A unittally contains a copy of \mathcal{T}_2 ,
- the C^* -algebra A contains a unital copy of the simple Cuntz C^* -algebra \mathcal{O}_∞ generated by infinitely many isometries with pairwise orthogonal ranges.

3. GLOBAL PROPER INFINITENESS

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following stable proper infiniteness for continuous $C(X)$ -algebras with properly infinite fibres.

Proposition 3.1. *Let X be a second countable perfect compact Hausdorff space and let A be a separable unital continuous $C(X)$ -algebra with properly infinite fibres.*

1) *There exist:*

- (a) *a finite integer $n \geq 1$,*
- (b) *a covering $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$ by the interiors of closed balls F_1, \dots, F_n ,*
- (c) *unital embeddings of C^* -algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A|_{F_k}$ ($1 \leq k \leq n$).*

2) *The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integers p .*

Proof. 1) For all point $x \in X$, the semiprojectivity of the C^* -subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) entails that there are a closed neighbourhood $F \subset X$ of the point x and a unital embedding $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A|_F$ of $C(F)$ -algebra. The compactness of the topological space X enables to conclude.

2) Proposition [BRR08, Proposition 2.7] entails that $M_{2^{n-1}}(A)$ is properly infinite and [Ror97, Proposition 2.1] implies that $M_p(A)$ for all integer $p \geq 2^{n-1}$. \square

Remark 3.2. If X is an ordinary second countable compact Hausdorff space and A is a separable unital continuous $C(X)$ -algebra, then $\tilde{X} := X \times [0, 1]$ is a perfect space, $\tilde{A} := A \otimes C([0, 1])$ is a unital continuous $C(\tilde{X})$ -algebra and any morphism $\mathcal{O}_\infty \rightarrow \tilde{A}$ induces a morphism $\mathcal{O}_\infty \rightarrow A$ by composition with the projection map $\tilde{A} \rightarrow A$ coming from the injection $x \in X \mapsto (x, 0) \in \tilde{X}$.

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the C^* -algebra A is properly infinite ([HR98]). Indeed, there exists a unital C^* -algebra A which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is properly infinite ([Ror03, Proposition 4.5]). We nevertheless have the following corollary.

Corollary 3.3. *Let j_0, j_1 denote the two canonical unital embeddings of the C^* -algebra \mathcal{T}_2 in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ and let $\tilde{u} \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a K_1 -trivial unitary satisfying $j_1(s_1) = \tilde{u} \cdot j_0(s_1)$ ([BRR08, Lemma 2.4]).*

Then the following conditions are equivalent:

- (a) *The full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1 -injective.*
- (b) *The unitary \tilde{u} belongs to the connected component $\mathcal{U}_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ of $1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2}$.*
- (c) *Every separable unital continuous $C(X)$ -algebra A with properly infinite fibres is a properly infinite C^* -algebra.*

Proof. (a) \Rightarrow (b) A unital C^* -algebra A is called K_1 -injective if and only if every unitary $v \in \mathcal{U}(A)$ is homotopic to the unit 1_A in $\mathcal{U}(A)$ (see e.g. [Roh09]). Thus, (b) is a special case of (a).

(b) \Rightarrow (c) Let A be a separable unital continuous $C(X)$ -algebra with properly infinite fibres. Take a finite covering such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \rightarrow A|_{F_k}$ ($1 \leq k \leq n$). Set $G_k := F_1 \cup \dots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A|_{G_k}$ such that the two projections $w_k w_k^*$ and $1|_{G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A|_{G_k}$:

– If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

– If $k \in \{1, \dots, n-1\}$ and the isometry $w_k \in A|_{G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital C^* -algebra $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \rightarrow A|_{G_k \cap F_{k+1}}$ and a K_1 -trivial unitary $u_{k+1} \in \mathcal{U}(A|_{G_k \cap F_{k+1}})$ satisfying:

$$\begin{aligned} & - \pi_k(j_0(s_1)) = w_k|_{G_k \cap F_{k+1}} \quad \text{and} \\ & - \pi_k(j_1(s_1)) = \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = u_{k+1} \cdot w_k|_{G_k \cap F_{k+1}}. \end{aligned} \tag{3.1}$$

If the unitary \tilde{u} belongs to $\mathcal{U}_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$, then the unitary $u_{k+1} = \pi_k(\tilde{u})$ is homotopic to $1_{A|_{G_k \cap F_{k+1}}} = \pi_k(1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2})$ in $\mathcal{U}(A|_{G_k \cap F_{k+1}})$, so that u_{k+1} admits a unitary lifting z_{k+1}

in $\mathcal{U}_0(A|_{F_{k+1}})$ (see *e.g.* [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A|_{G_{k+1}}$ satisfying the two constraints:

$$w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1) \quad (3.2)$$

verifies that the two projections $w_{k+1}w_{k+1}^*$ and $1|_{G_{k+1}} - w_{k+1}w_{k+1}^*$ are properly infinite and full in $A|_{G_{k+1}}$.

The proper infiniteness of the projection $w_n w_n^* \in A|_{G_n} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$ is also a properly infinite projection in A , *i.e.* the C^* -algebra A is properly infinite.

(c) \Rightarrow (a) The C^* -algebra $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2); f(0) \in \mathcal{J}_0(\mathcal{T}_2) \text{ and } f(1) \in \mathcal{J}_1(\mathcal{T}_2)\}$ is a unital continuous $C([0, 1])$ -algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C^* -algebra \mathcal{D} is properly infinite, a statement which is equivalent to the K_1 -injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([Blan10, Proposition 4.2]). \square

4. THE PIMSNER-TOEPLITZ ALGEBRA OF A HILBERT $C(X)$ -MODULE

Definition 4.1. ([Pim95]) *Let X be a compact Hausdorff space and E a full Hilbert $C(X)$ -module E , *i.e.* without any zero fibre.*

a) *The full Fock Hilbert $C(X)$ -module $\mathcal{F}(E)$ of E is the direct sum of Hilbert $C(X)$ -module*

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)} m)}, \quad (4.1)$$

where $E^{(\otimes_{C(X)} m)} := \begin{cases} C(X) & \text{if } m = 0, \\ E \otimes_{C(X)} \dots \otimes_{C(X)} E \text{ (} m \text{ terms)} & \text{if } m \geq 1. \end{cases}$

b) *The Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ of E is the unital subalgebra of the $C(X)$ -algebra $\mathcal{L}_{C(X)}(\mathcal{F}(E))$ of adjointable $C(X)$ -linear operator acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(\zeta)$ ($\zeta \in E$), where:*

$$\begin{aligned} - \ell(\zeta) (f \cdot \hat{1}_{C(X)}) &:= f \cdot \zeta = \zeta \cdot f & \text{for } f \in C(X) & \text{and} \\ - \ell(\zeta) (\zeta_1 \otimes \dots \otimes \zeta_k) &:= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_k & \text{for } \zeta_1, \dots, \zeta_k \in E & \text{if } k \geq 1. \end{aligned} \quad (4.2)$$

c) *There is a unique coaction α of the group C^* -algebra $(C^*(\mathbb{Z}), \Delta)$ on the Pimsner-Toeplitz $C(X)$ -algebra $\mathcal{T}(E)$ such that $\alpha(\ell(\zeta)) = \ell(\zeta) \otimes \mathbf{u}$ for all $\zeta \in E$, *i.e.**

$$\begin{aligned} \alpha : \mathcal{T}(E) &\rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(\mathbb{T}, \mathcal{T}(E)) \\ \ell(\zeta) &\mapsto \ell(\zeta) \otimes \mathbf{u} = (z \mapsto \ell(z\zeta)) \end{aligned} \quad (4.3)$$

The fixed point $C(X)$ -subalgebra $\mathcal{T}(E)^\alpha = \{a \in \mathcal{T}(E); \alpha(a) = a \otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^\alpha = [C(X) \cdot 1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^*]. \quad (4.4)$$

Besides, the projection $P \in \mathcal{L}(\mathcal{F}(E))$ onto the submodule E induces a quotient morphism of $C(X)$ -algebra $a \in \mathcal{E}^\alpha \mapsto \bar{\mathbf{q}}(a) := P \cdot a \cdot P \in \mathcal{K}(E) + C(x) \cdot 1 \subset \mathcal{L}(E)$.

Proposition 4.2. *Let X be a second countable compact Hausdorff perfect space, i.e. without any isolated point, and let E be a separable Hilbert $C(X)$ -module with infinite dimensional fibres.*

1) *There exist a covering $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_m$ by the interiors of closed subsets F_1, \dots, F_m and m sections ζ_1, \dots, ζ_m in E such that $\mathcal{T}(E) = C^* \langle \mathcal{T}(E)^\alpha, \ell(\zeta_1), \dots, \ell(\zeta_m) \rangle$ and $\|(\zeta_k)_x\| = 1$ for all $k \in \{1, \dots, m\}$ and all $x \in F_k$.*

2) *If the full unital free product $\mathcal{T}_2 *_C \mathcal{T}_2$ is K_1 -injective, then there exists a section $\xi \in E$ such that $\|\xi_x\| = 1$ for all $x \in X$.*

Proof. 1) For all point $x \in X$, there exists a section $\zeta \in E$ such that $\|\zeta_x\| = 1$, whence an isomorphism of C^* -algebra $\mathcal{T}(E)_x \cong \mathcal{T}(E_x) = C^* \langle \mathcal{T}(E_x)^\alpha, \ell(\zeta_x) \rangle$. The semiprojectivity of the C^* -algebra \mathcal{O}_∞ and the compactness of the space X then imply that there exist a finite covering $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_m$ by the interiors of closed subsets F_1, \dots, F_m and m contractions ζ_1, \dots, ζ_m in E such that $\|(\zeta_k)_x\| = 1$ for all index $k \in \{1, \dots, m\}$ and all point $x \in F_k$.

2) Set $G_k := F_1 \cup \dots \cup F_k$ for all $k \in \{1, \dots, m\}$ (as in Corollary 3.3) and let us construct inductively sections $\xi(k) \in E|_{G_k}$ such that $\|\xi(k)_x\| = 1$ for all $x \in G_k$.

– If $k = 1$, the section $\xi(1) := (\zeta_1)|_{F_1}$ has the requested properties.

– Suppose now that $k \in \{1, \dots, m-1\}$ and a convenient section $\xi(k)$ in $E|_{G_k}$ has already been constructed. There exists by Corollary 4.4 a morphism of unital C^* -algebra $\pi_k : \mathcal{T}_2 *_C \mathcal{T}_2 \rightarrow \mathcal{T}(E)|_{G_k \cap F_{k+1}}$ such that $\pi_1(j_0(s_1)) = \ell(\xi(k))|_{G_k \cap F_{k+1}}$ and $\pi_1(j_1(s_1)) = \ell(\zeta_{k+1})|_{G_k \cap F_{k+1}} = u_k \cdot \ell(\xi(k))|_{G_k \cap F_{k+1}}$ for some K_1 -trivial unitary $u_k \in (\mathcal{U}(\mathcal{T}(E)|_{G_k \cap F_{k+1}})^\alpha) \cong (\mathcal{U}(\mathcal{T}(E)^\alpha)|_{G_k \cap F_{k+1}})$. If $\mathcal{T}_2 *_C \mathcal{T}_2$ is K_1 -injective, this unitary u_k belongs to $\mathcal{U}_0(\mathcal{T}(E)^\alpha)|_{G_k \cap F_{k+1}}$ and it admits a unitary lifting $z_{k+1} \in \mathcal{U}_0(\mathcal{T}(E)^\alpha)$ (see e.g. [LLR00, Lemma 2.1.7]). The only section $\xi(k+1) \in E|_{G_{k+1}}$ such that $\xi(k+1)|_{G_k} = \xi(k)$ and $\xi(k+1)|_{F_{k+1}} = \overline{\mathbf{q}}(z_{k+1})^* \cdot \xi_{k+1}|_{F_{k+1}}$ satisfies the desired conditions. \square

Theorem 4.3. *The full unital free product $\mathcal{T}_2 *_C \mathcal{T}_2$ is properly infinite, but not K_1 -injective.*

Proof. Let \mathfrak{X} be the complex Hilbert cube $\mathfrak{X} := \{z \in \mathbb{C}; |z| \leq 1\}^{\mathbb{N}}$. It is a compact space when equipped with the distance $d(x, y) = \sum_p 2^{-p-2} |x_p - y_p|$. Let also E_{DD} be the non-trivial separable Hilbert $C(\mathfrak{X})$ -module contracted by J. Dixmier and A. Douady ([DD63]): It has infinite dimensional fibres and every section $\zeta \in E_{DD}$ satisfies $\zeta_x = 0$ for at least one $x \in \mathfrak{X}$. Thus proposition 4.2 implies that $\mathcal{T}_2 *_C \mathcal{T}_2$ is not K_1 -injective. \square

The $C([0, 1])$ -algebra $\{f \in C([0, 1], \mathcal{T}_2 *_C \mathcal{T}_2); f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2)\}$ introduced in Corollary 3.3 is an example for the following corollary:

Corollary 4.4. *There exists a separable unital continuous $C([0, 1])$ -algebra with properly infinite fibres which is not properly infinite.*

Question 4.5. The Pimsner-Toeplitz algebra $\mathcal{T}(E_{DD})$ is locally purely infinite ([BK04b, Definition 1.3]) since all its simple quotients are isomorphic to the Cuntz algebra \mathcal{O}_∞ ([BK04b, Proposition 5.1]). But is $\mathcal{T}(E_{DD})$ properly infinite?

5. A K_1 -INJECTIVE SUPQUOTIENT

We show in this section that some *reduced* quotient of the full unital free product $\mathcal{T}_2 *_C \mathcal{T}_3$ is a simple K_1 -injective C^* -algebra which contains the algebraic unital free product $\mathcal{T}_2 \otimes_C \mathcal{T}_3$.

Let $\ell^2(\mathbb{N})$ be the Hilbert space of all the sequences $\xi = (\xi_k) \in \mathbb{C}^{\mathbb{N}}$ which satisfy $\|\xi\|_2^2 = \sum_k |\xi_k|^2 < \infty$ and take an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ in $\ell^2(\mathbb{N})$. Define for all integer $n \geq 2$:

- the n isometries $s_{n,1}, \dots, s_{n,n}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$ satisfying $s_{n,i} \cdot e_k = e_{nk+i}$ ($k \in \mathbb{N}$),
- the Cuntz extension $\mathcal{T}_n := C^* \langle s_{n,1}, \dots, s_{n,n} \rangle \subset \mathbb{B}(\ell^2(\mathbb{N}))$ ([Cun77]),
- the state $\psi_n(a) = \frac{1}{n} \sum_{i=1}^n \langle e_i, a \cdot e_i \rangle$ on the C^* -algebra $\mathcal{T}_n \subset \mathbb{B}(\ell^2(\mathbb{N}))$,
- the unitary $u_n := s_{n,1} s_{n,2}^* + \dots + s_{n,n} s_{n,1}^* + p \in \mathcal{T}_n$, where p is the rank 1 projection $p = 1 - \sum_i s_{n,i} s_{n,i}^* = \theta_{e_0, e_0}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$.

Lemma 5.1. 1) One has $\psi_n(a) = \psi_n(u_n^* a u_n)$ for all $a \in \mathcal{T}_n$ and $\psi_n((u_n)^j) = 0$ for all $j \in \{1, \dots, n-1\}$,

2) The state ψ_n on \mathcal{T}_n has a faithful GNS representation.

3) The restriction of the state ψ_n to the C^* -subalgebra $C^*(u_n) \subset \mathcal{T}_n$ is faithful.

Proof. 1) The unitary u_n satisfies $(u_n)^n = 1$ and $(u_n)^j \cdot e_1 = e_{n-j+1}$ for all $1 \leq j \leq n$. Thus, there exists an isomorphism of C^* -algebra $\alpha_n : C^*(u_n) \rightarrow C^*(\mathbb{Z}/n\mathbb{Z})$ and

$$\begin{aligned} \psi_n(a) &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^i \cdot e_1, a (u_n)^i \cdot e_1 \rangle \\ &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^{i+1} \cdot e_1, a (u_n)^{i+1} \cdot e_1 \rangle = \psi_n(u_n^* a u_n). \end{aligned}$$

We also have $\psi_n((u_n)^j) = \langle e_1, (u_n)^j \cdot e_1 \rangle = \langle e_1, e_{n-j+1} \rangle = 0$ for all $j \in \{1, \dots, n-1\}$.

2) The C^* -algebra \mathcal{T}_n is contained in $\mathbb{B}(\ell^2(\mathbb{N}))$ and the Hilbert space $\ell^2(\mathbb{N})$ is the closure of $\mathcal{T}_n \cdot e_0$. Thus, if $a \in \mathcal{T}_n \setminus \{0\}$ is non-zero, there exists an element $b \in \mathcal{T}_n$ such that

$$\begin{aligned} 0 < \|ab \cdot e_0\|^2 &= \|ab s_{n,1}^* \cdot e_1\|^2 = \langle e_1, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_1 \rangle \\ &= \sum_{i=1}^n \langle e_i, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_i \rangle \\ &= n \cdot \psi_n(s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^*). \end{aligned}$$

3) The composition $\psi_n \circ \alpha_n^{-1}$ equals the only Haar state h_n on the finite dimensional Hopf C^* -algebra $(C^*(\mathbb{Z}/n\mathbb{Z}), \Delta)$, which is faithful on $C^*(\mathbb{Z}/n\mathbb{Z})$. \square

As a consequence, the following holds:

Proposition 5.2. Let $(\mathfrak{B}, \psi) := (\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$ be the reduced unital free product defined by D . Voiculescu ([Voi85]).

1) The C^* -algebra \mathfrak{B} contains the algebraic unital free product generated by \mathcal{T}_2 and \mathcal{T}_3

2) The C^* -algebra \mathfrak{B} is simple and has stable rank 1.

3) The C^* -algebra \mathfrak{B} is K_1 -injective.

Proof. 1) This embedding result is essentially contained in Proposition 4.2 of [DS01]. But we include here the main steps of that proof for completeness. Define:

- the tensor product C^* -algebra $B := \mathcal{T}_2 \otimes \mathcal{T}_3$,
- the state $\phi := \psi_2 \otimes \psi_3$ on the C^* -algebra B ,
- the full countably generated Hilbert B -bimodule $E := L^2(B, \phi) \otimes B$,
- the full Fock Hilbert B -bimodule $\mathcal{F}(E) = B \oplus E \oplus (E \otimes E) \oplus \dots$,
- the Pimsner-Toeplitz C^* -algebra $\mathcal{T}(E)$ generated in $\mathcal{L}(\mathcal{F}(E))$ by the *creation* operators $\ell(\xi)$, $\xi \in E$ ([Pim95]).
- the conditional expectation $\mathfrak{E} : \mathcal{T}(E) \rightarrow B$ given by compression with the orthogonal projection from the full Fock Hilbert B -module $\mathcal{F}(E)$ onto the first summand $B \subset \mathcal{F}(E)$ ([DS01]).

The C^* -algebra B unitaly embeds in $\mathcal{T}(E) \subset \mathcal{L}(\mathcal{F}(E))$ by

$$b \cdot \ell(\Lambda_\phi b_1 \otimes b_2) \cdot b' = \ell(\Lambda_\phi b b_1 \otimes b_2 b').$$

Take a unitary v in the C^* -subalgebra $C^* \langle \ell(\Lambda_\phi 1 \otimes 1) \rangle \subset \mathcal{T}(E)$ such that $\mathfrak{E}(v^j) = 0$ for all non-zero integer $j \in \mathbb{Z}$ and let $\pi_2 : \mathcal{T}_2 \rightarrow \mathcal{T}(E)$, $\pi_3 : \mathcal{T}_3 \rightarrow \mathcal{T}(E)$ be the two unital $*$ -morphisms $\pi_2(a) = v^2 \cdot (a \otimes 1_{\mathcal{T}_3}) \cdot v^{-2}$ and $\pi_3(a) = v^3 \cdot (1_{\mathcal{T}_2} \otimes a) \cdot v^{-3}$. Then, the algebraic amalgamated free product $\mathcal{T}_2 \otimes_{\mathbb{C}} \mathcal{T}_3$ studied by B. Blackadar ([Blac78]) and G. Pedersen ([Ped94]) is (isomorphic to) the unital $*$ -algebra generated by $\pi_2(\mathcal{T}_2)$ and $\pi_3(\mathcal{T}_3)$ in $\mathcal{T}(E)$, whereas Voiculescu's reduced unital free product $(\mathcal{T}_2, \psi_2) *_\mathbb{C} (\mathcal{T}_3, \psi_3)$ ([Voi85]) is (isomorphic to) its closure (see *e.g.* [Blan09, Theorem 4.1]).

2) The previous Lemma 5.1 and [BD01, Theorem 1.3] imply that there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_\mathbb{C} (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \hookrightarrow (\mathfrak{B}, \psi) = (\mathcal{T}_2, \psi_2) *_\mathbb{C} (\mathcal{T}_3, \psi_3)$, if h_2 and h_3 denote the Haar states on the compact groups $(C^*(\mathbb{Z}/2\mathbb{Z}), \Delta_2)$ and $(C^*(\mathbb{Z}/3\mathbb{Z}), \Delta_3)$. Hence, the reduced free product \mathfrak{B} is simple ([Av82, Proposition 3.1+its corollary]) and this unital C^* -algebra has stable rank 1 ([DHR97, Theorem 3.8]), *i.e.* the open subset of invertible elements is dense in \mathfrak{B} .

3) With the notations of [Rief83], the set $Lg_1(\mathfrak{B})$ of invertible elements is dense in the stable rank 1 C^* -algebra \mathfrak{B} , so that this C^* -algebra \mathfrak{B} is K_1 -injective by [Rief83, Proposition 1.6] and [Rief87, Theorem 2.10]. (A self-contained proof is also available in Theorem 3.2.11 of [Roh09].) \square

Remark 5.3. For all strictly positive integer n , the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are normal subgroups in the cyclic groups $\mathbb{Z}/2n\mathbb{Z}$ and $\mathbb{Z}/3n\mathbb{Z}$. Thus, there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_\mathbb{C} (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \subset (C^*(\mathbb{Z}/2n_1 \cdot \mathbb{Z}), h_{2n_1}) *_\mathbb{C} (C^*(\mathbb{Z}/3n_2 \cdot \mathbb{Z}), h_{3n_2})$ ([BD01]) and the reduced unital free product $(\mathcal{T}_{2n_1}, \psi_{2n_1}) *_\mathbb{C} (\mathcal{T}_{3n_2}, \psi_{3n_2})$ is both simple and K_1 -injective for all strictly positive integers n_1, n_2 .

Question 5.4. Is the reduced unital free product $(\mathcal{T}_2, \psi_2) *_\mathbb{C} (\mathcal{T}_2, \psi_2)$ also K_1 -injective?

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