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Solution of a class of reaction-diffusion systems via logarithmic Sobolev inequality

Pierre Fougères*, Ivan Gentil† and Boguslaw Zegarlinski‡

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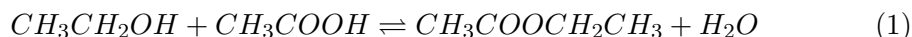
Abstract

We study global existence, uniqueness and positivity of weak solutions of a class of reaction-diffusion systems of chemical kinetics type, under the assumptions of logarithmic Sobolev inequality and appropriate exponential integrability of the initial data.

Keywords: Reaction-diffusion systems, Markov semigroups, logarithmic Sobolev inequality, infinite dimensions.

1 Introduction

A mixture one gets after esterification of one mole of ethyl alcohol by one mole of ethanoic acid contains products (ethyl acetate and water), but also reactants. This is an example of a double displacement reaction



(see [34]).

We consider here chemical reactions between $q \geq 2$ species \mathcal{A}_i , $i = 1, \dots, q$, as follows

$$\sum_{i=1}^q \alpha_i \mathcal{A}_i \rightleftharpoons \sum_{i=1}^q \beta_i \mathcal{A}_i,$$

where $\alpha_i, \beta_i \in \mathbb{N}$. We assume that for any $1 \leq i \leq q$, $\alpha_i - \beta_i \neq 0$ which corresponds to the case of a reaction without a catalyst.

If $\vec{u} = (u_1, \dots, u_q)$ denotes the concentration of the species \mathcal{A}_i then the law of action mass proposed by Waage and Guldberg in 1864 (see again [34]) implies that the concentrations are solutions of the system, for all $i \in \{1, \dots, q\}$,

$$\frac{d}{dt} u_i = (\beta_i - \alpha_i) \left(k \prod_{j=1}^q u_j^{\alpha_j} - l \prod_{j=1}^q u_j^{\beta_j} \right),$$

where $k, l > 0$ are the rate constants of the two reactions.

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When considering substances distributed in space, the concentrations change not only under the influence of the chemical reactions but also due to the diffusion of the species over the space, one gets the following kinetic model for a chemical reaction-diffusion equation

$$\partial_t u_i = L_i u_i + (\beta_i - \alpha_i) \left(k \prod_{j=1}^q u_j^{\alpha_j} - l \prod_{j=1}^q u_j^{\beta_j} \right),$$

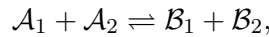
where for all $i = 1, \dots, q$, L_i is an operator which modelizes how the substance diffuses.

We will assume that $L_i = C_i L$ for some $C_i \geq 0$ and some reference operator L . Moreover, by a change of variables, one can assume that there exist (a posteriori two) constants $\lambda_i > 0$ such that the system of reaction-diffusion is given by

$$\partial_t u_i = C_i L u_i + \lambda_i (\beta_i - \alpha_i) \left(\prod_{j=1}^q u_j^{\alpha_j} - \prod_{j=1}^q u_j^{\beta_j} \right), \quad (2)$$

where $\vec{u}(t, x) = (u_1(t, x), \dots, u_q(t, x))$ with $t \geq 0$ and x belongs to the underlying space.

The *two-by-two* system, one of the simplest non trivial example, describes the chemical reaction



and the system of equations can be formulated as follow

$$\begin{cases} \partial_t u_1 = C_1 L u_1 - \lambda (u_1 u_2 - v_1 v_2) \\ \partial_t u_2 = C_2 L u_2 - \lambda (u_1 u_2 - v_1 v_2) \\ \partial_t v_1 = C_3 L v_1 + \tilde{\lambda} (u_1 u_2 - v_1 v_2) \\ \partial_t v_2 = C_4 L v_2 + \tilde{\lambda} (u_1 u_2 - v_1 v_2) \end{cases} \quad (3)$$

where $\lambda, \tilde{\lambda} > 0$ and u_i denotes the concentration of the specie \mathcal{A}_i and v_i the concentration of the specie \mathcal{B}_i for $i = 1, 2$. To make things even simpler, we will assume later that $\lambda = \tilde{\lambda}$.

More general reaction-diffusion systems, of the following form

$$\begin{cases} \partial_t \vec{u} = \mathfrak{C} \Delta_x \vec{u} + F(t, x, \vec{u}), & t > 0, \quad x \in \Omega \\ \vec{u}(0) = \vec{u}_0, \end{cases} \quad (4)$$

with prescribed boundary conditions, were intensively studied in the past. Here, Ω is a (possibly unbounded sufficiently smooth) domain of \mathbb{R}^n , \vec{u} takes values in \mathbb{R}^q , \mathfrak{C} is a usually diagonal $q \times q$ matrix which can be degenerate, and $F(t, x, \cdot)$ is a vector field on \mathbb{R}^q .

Depending on specific choices for \mathfrak{C} and $F(t, x, \cdot)$, such systems can present various behaviours with respect to global existence and asymptotic behaviour of the solution. Paragraph 15.4 in [43] is a nice introduction with a lot of classical references.

In the above setting, local existence follows from general textbooks on parabolic type partial differential equations (see [23], [30], or for fully general boundary value problems [1]).

Global existence question (or how to prevent blow up) gave rise to extensive efforts and to different methods adapted to specific cases (see [2], especially remark 5.4. a), [40], [37] and references therein). Most of these methods consist in deducing \mathbb{L}^∞ bounds on the maximal solution from bounds in weaker norms.

The survey [37] provides a lot of references, positive and negative results, together with a description of open problems. Its first observation is that, for numerous reaction-diffusion systems of interest in applications, the nonlinearity satisfies two general conditions which ensure respectively positivity and a control of the mass (i.e. the \mathbb{L}^1 norm) of a solution. M. Pierre investigates how these \mathbb{L}^1 estimates (as well as \mathbb{L}^1 bounds on the nonlinearity) help to provide global existence.

Further works provide results on asymptotic behaviour. Spectral gap, logarithmic Sobolev inequality and entropy methods are often used to quantify exponential convergence of the solution of an equation to equilibrium, and in the context of reaction-diffusion equations (mostly of type (2)) they were used to study the convergence (to constant steady states) in [16, 17, 15, 25]. Geometric characteristics and approximations of global and exponential attractors of general reaction-diffusion systems may be found in [49, 20, 50] (and references therein) in terms of precise estimates of their Kolmogorov ε -entropy. In these papers, \mathfrak{C} is of positive symmetric part and the nonlinearity must satisfy some moderate growth bound involving the dimension n to ensure global existence. Other cross-diffusion systems are studied by entropy methods in [11].

One way or another, local or global existence results in the above setting rely on regularity theory for the heat semigroup, the maximum principle, and Sobolev inequality through one of its consequences, Gagliardo-Nirenberg inequalities or ultracontractivity of the semigroup (as well as Moser estimates). (Note nevertheless that an approach based on a nonlinear Trotter product formula is proposed in [43], but seems to impose some kind of uniform continuity of the semigroup).

The aim of this article is to prove global existence of a non-negative solution of the reaction-diffusion system (2) with unbounded initial data in a setting where Sobolev inequality possibly does not hold, as e.g. in infinite dimensions or when underlying measure does not satisfy polynomial growth condition. We restrict ourselves to some nonlinearities for which in a finite dimensional setting, \mathbb{L}^∞ bounds of the solution (and so global existence) come for free, [37]. Nevertheless, Sobolev inequality has to be replaced by the weaker logarithmic Sobolev inequality (or other coercive inequalities which survive the infinite dimensional limit; see [8], [6], [39], [7]).

The celebrated paper [26] of L.Gross established equivalence of logarithmic Sobolev inequality and hypercontractivity of the semigroup. No compactness embeddings hold in this context.

For a wide variety of strongly mixing Markov semigroups, logarithmic Sobolev inequality holds for the corresponding Dirichlet form of the generator. For diffusion semigroups on Riemannian manifolds, logarithmic Sobolev inequality follows from positive bound from below of the Ricci curvature (of the generator L), i.e. the so called Bakry-Emery, Γ_2 or $\text{CD}(\rho, \infty)$ criterion (see [4], [5], [44]). For logarithmic Sobolev inequality and discrete state space Markov chains, see [18], [35], [12]. In infinite dimensional spaces, logarithmic Sobolev inequality for spin systems has been extensively studied (see [47], [42], [48], [9], [45], [31], [27], [36]; and in subelliptic setting, [32], [29], [28]). In the present paper, logarithmic Sobolev inequality plays a key role to study existence results in an infinite dimensional setting, by a step-by-step approximation approach.

The paper is organized as follows. In the next section we describe the framework and the main result of the paper: in the two-by-two case, assuming 1) that $C_1 = C_3$ and $C_2 = C_4$, 2) that the linear *diffusion* term satisfies logarithmic Sobolev inequality and 3) that the initial datum \vec{f} is nonnegative and satisfies some exponential

integrability properties (made more precise later), then there exists a unique weak solution of the system of reaction-diffusion equation (3) which is moreover nonnegative. Section 3 presents the iterative procedure we follow to approximate weak solutions of our reaction-diffusion type problem. This is based on some cornerstone linear problem which is stated there. The two following sections are devoted to the details of the proof : section 4 to the convergence of the iterative procedure to the unique nonnegative weak solution of the nonlinear Cauchy problem, whereas section 5 focuses on the cornerstone linear problem.

In Section 6 we extend our result to the general case of system (2), and present how operators $C_i L$ can be modified. (To give a comprehensive proof we focus in the rest of the paper on the two-by-two case which already contains non trivial difficulty).

We recall or detail tools used in the proof in three appendices: the entropic inequality, basics on Orlicz spaces, and finally some further topics on Markov semigroups and Orlicz spaces.

2 Framework and main result

An abstract Reaction-Diffusion equation

In the following we will consider an underlying Polish space \mathbb{M} equipped with a probability measure μ . Let L be a (linear) densely defined selfadjoint Markov operator on $\mathbb{L}^2(\mu) \equiv \mathbb{L}^2(\mathbb{M}, \mu)$, that is the infinitesimal generator of a C_0 Markov semigroup $(P_t)_{t \geq 0}$ symmetric with respect to μ . It is well known that under these assumptions there exist a kernel $p_t(x, dy)$ on $(\mathbb{M}, \mathcal{B}_{\mathbb{M}})$, that is a measurable family of probability measures such that, for any $t \geq 0$, any $f \in \mathbb{L}^1(\mu)$, and for μ almost every $x \in \mathbb{M}$,

$$P_t f(x) = \int_{\mathbb{M}} f(y) p_t(x, dy). \quad (5)$$

Let us consider the following equation

$$\begin{cases} \frac{\partial}{\partial t} \vec{u}(t) &= \mathfrak{C} L \vec{u}(t) + G(\vec{u}(t)) \vec{\lambda}, & t > 0 \\ \vec{u}(0) &= \vec{f} \end{cases} \quad (\text{RDP})$$

where, in the two-by-two case,

- the unknown $\vec{u}(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x), u_4(t, x))$ is a function from $[0, \infty) \times \mathbb{M}$ to \mathbb{R}^4 ; and $L \vec{u} = (Lu_1, Lu_2, Lu_3, Lu_4)$ is defined componentwise.
- $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda(-1, -1, 1, 1) \in \mathbb{R}^4$, with $\lambda \in \mathbb{R}_+$;
- the nonlinearity G is quadratic: $G(\vec{u}) = u_1 u_2 - u_3 u_4$.
- \mathfrak{C} is a diagonal matrix of the following form

$$\mathfrak{C} = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix},$$

where we assume that $C_1 = C_3$ and $C_2 = C_4$.

(This condition is weakened in section 6).

- the initial datum is $\vec{f} = (f_1, f_2, f_3, f_4)$.

Dirichlet form and logarithmic Sobolev inequality

Let $(\mathcal{E}, \mathcal{D})$ be the Dirichlet form associated to (L, μ) (see [14], [24], [33], [10]; or [22] for a minimal introduction). For any $u \in \mathcal{D}(L)$ (the domain of L) and $v \in \mathcal{D}$ (the domain of the Dirichlet form), one has

$$\mathcal{E}(u, v) = -\mu(v Lu).$$

We will denote $\mathcal{E}(u) \equiv \mathcal{E}(u, u)$, for any $u \in \mathcal{D}$. Recall that \mathcal{D} is a real Hilbert space with associated norm

$$\|u\|_{\mathcal{D}} = (\mu(u^2) + \mathcal{E}(u))^{1/2}.$$

We will assume that the Dirichlet structure (\mathcal{E}, μ) satisfies logarithmic Sobolev inequality with constant $C_{LS} \in (0, \infty)$, that is

$$\text{Ent}_{\mu}(u^2) \equiv \mu\left(u^2 \log \frac{u^2}{\mu(u^2)}\right) \leq C_{LS} \mathcal{E}(u), \quad (6)$$

for any $u \in \mathcal{D}$.

Classical function spaces

Let $I = [0, T]$. For any Banach space $(X, \|\cdot\|_X)$, we shall denote by $C(I, X)$ the Banach space of continuous functions from I to X equipped with the supremum norm

$$\sup_{t \in I} \|u(t)\|_X.$$

Let also $\mathbb{L}^2(I, X)$ be the space of (a.e. classes of) Bochner measurable functions from I to X such that $\int_0^T \|u(t)\|_X^2 ds < \infty$. As for vector valued functions, let $\mathbb{L}^2(I, X^4)$ be the space of Bochner measurable functions $t \in I \mapsto (u_1(t), u_2(t), u_3(t), u_4(t)) \in X^4$ such that

$$\int_0^T \sum_{i=1}^4 \|u_i(s)\|_X^4 ds < \infty.$$

All these are Banach spaces.

We'll furthermore consider the space $\mathbb{L}^{\infty}(I, X)$ of Bochner measurable X -valued functions on I such that

$$\text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X < +\infty.$$

The reader may refer to [41] for Bochner measurability, Bochner integration and other Banach space integration topics.

Bochner measurability in an Orlicz space

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ given by $\Phi(x) = \exp(|x|) - 1$ and $\Phi_{\alpha}(x) = \Phi(|x|^{\alpha})$, $\alpha \geq 1$. These are Young functions and the Orlicz space associated to Φ_{α} is denoted by $\mathbb{L}^{\Phi_{\alpha}}(\mu)$. This is the space of measurable functions f such that

$$\mu(\Phi_{\alpha}(\gamma f)) < \infty \quad (7)$$

for *some* $\gamma > 0$ (or functions *whose* α *power is exponentially integrable*).

An important closed subspace $E^{\Phi_{\alpha}}(\mu)$ of $\mathbb{L}^{\Phi_{\alpha}}(\mu)$ consists of those functions such that (7) holds for *any* $\gamma > 0$. This is the closure of the space of simple functions (finitely valued measurable functions) in $\mathbb{L}^{\Phi_{\alpha}}(\mu)$.

A striking property of Markov semigroups is that C^0 property in $\mathbb{L}^2(\mu)$ implies C^0 property in any $\mathbb{L}^p(\mu)$; $1 \leq p < +\infty$ (see [14]). We will need the following weakened result in the context of Orlicz spaces.

Proposition 1 *Let $f \in E^{\Phi_\alpha}(\mu)$, $\alpha \geq 1$. Then the linear semigroup is Bochner measurable in time in $E^{\Phi_\alpha}(\mu)$. More precisely, the mapping $t \in [0, \infty) \mapsto P_t f \in E^{\Phi_\alpha}(\mu)$ belongs to $\mathbb{L}^\infty([0, \infty), E^{\Phi_\alpha}(\mu))$ and*

$$\text{ess sup}_{0 \leq t < \infty} \|P_t f\|_{E^{\Phi_\alpha}(\mu)} \leq \|f\|_{E^{\Phi_\alpha}(\mu)}.$$

The proof is given in appendix C.3.

First regularity result and weak solutions

The following lemma exhibits the main role the entropic inequality (see appendix A) and the logarithmic Sobolev inequality play to deal with the nonlinearity we consider. In short, the multiplication operator by a function in $\mathbb{L}^{\Phi_2}(\mu)$ is a bounded operator, mapping the domain of the Dirichlet form \mathcal{D} to $\mathbb{L}^2(\mu)$.

Lemma 2.1 (Regularity property) *Assume the Dirichlet structure (μ, \mathcal{E}) satisfies logarithmic Sobolev inequality with constant $C_{LS} \in (0, \infty)$. Let $\Phi(x) = \exp(|x|) - 1$ and $\Phi_2(x) = \Phi(x^2)$. Let $u \in \mathbb{L}^2(I, \mathcal{D})$ and $v \in \mathbb{L}^\infty(I, \mathbb{L}^{\Phi_2}(\mu))$. Then $uv \in \mathbb{L}^2(I, \mathbb{L}^2(\mu))$ and the bilinear mapping*

$$(u, v) \in \mathbb{L}^2(I, \mathcal{D}) \times \mathbb{L}^\infty(I, \mathbb{L}^{\Phi_2}(\mu)) \mapsto uv \in \mathbb{L}^2(I, \mathbb{L}^2(\mu)) \quad (8)$$

is continuous. Consequently,

$$(\phi, u, v) \in \mathbb{L}^2(I, \mathbb{L}^2(\mu)) \times \mathbb{L}^2(I, \mathcal{D}) \times \mathbb{L}^\infty(I, \mathbb{L}^{\Phi_2}(\mu)) \mapsto \phi uv \in \mathbb{L}^1(I, \mathbb{L}^1(\mu)) \quad (9)$$

is trilinear continuous.

The reader may note that we will use this lemma to define properly a weak solution of the nonlinear problem below.

Proof

◁ Note that $f \in \mathbb{L}^{\Phi_2}(\mu)$ iff $f^2 \in \mathbb{L}^\Phi(\mu)$ and that

$$\|f^2\|_\Phi = \|f\|_{\Phi_2}^2. \quad (10)$$

First we show that the bilinear mapping

$$(u, v) \in \mathcal{D} \times \mathbb{L}^{\Phi_2}(\mu) \mapsto uv \in \mathbb{L}^2(\mu) \quad (11)$$

is continuous. Fix $0 < \gamma < \|v^2\|_{\mathbb{L}^\Phi(\mu)}^{-1}$. Then, $\mu(\exp(\gamma v^2)) - 1 \leq 1$ and so $\mu(e^{\gamma v^2}) \leq 2$. Hence, using the entropic inequality (36), and then the logarithmic Sobolev inequality, one gets,

$$\begin{aligned} \mu(u^2 v^2) &\leq \frac{1}{\gamma} \mu \left(u^2 \log \left(\frac{u^2}{\mu(u^2)} \right) \right) + \frac{\mu(u^2)}{\gamma} \log \mu(e^{\gamma v^2}) \\ &\leq \frac{1}{\gamma} (C_{LS} \mathcal{E}(u) + \log 2 \mu(u^2)) \leq \frac{\max(\log 2, C_{LS})}{\gamma} \|u\|_{\mathcal{D}}^2. \end{aligned}$$

Letting γ go to $\|v^2\|_{\mathbb{L}^\Phi(\mu)}^{-1}$, using (10) one gets the announced continuity.

If now $u \in \mathbb{L}^2([0, T], \mathcal{D})$ and $v \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$, there exist two sequences of simple functions (see [41] if necessary) $(u_n)_n \subset \mathcal{S}_{I, \mathcal{D}}$ and $(v_n)_n \subset \mathcal{S}_{I, \mathbb{L}^{\Phi_2}}$ converging to u (resp. v) a.e. in \mathcal{D} (resp. \mathbb{L}^{Φ_2}). The continuity of (11) shows that $(u_n v_n)_n$ is a sequence of simple functions with values in $\mathbb{L}^2(\mu)$ which converges a.e. in $\mathbb{L}^2(\mu)$ to uv . Bochner measurability of uv from I to $\mathbb{L}^2(\mu)$ follows.

As for continuity of (8), what precedes shows that, for any t a.e.,

$$\|u(t)v(t)\|_{\mathbb{L}^2(\mu)}^2 \leq \max(\log 2, C_{LS}) \|v\|_{\mathbb{L}^\infty([0,T], \mathbb{L}^{\Phi_2(\mu)})}^2 \|u(t)\|_{\mathcal{D}}^2.$$

Integrating w.r.t. t on $[0, T]$, one gets the result. Finally, continuity of the trilinear mapping follows by Cauchy-Schwarz inequality in \mathbb{L}^2 . \triangleright

Weak solutions. Let $T > 0$. We say that a function

$$\vec{u} \in (\mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu)) \cap \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2(\mu)}))^4 \quad (12)$$

is a *weak solution* of (\mathbb{RDP}) on $[0, T]$ provided, for any $\vec{\phi} \in C^\infty([0, T], \mathcal{D}^4)$ and any $t \in [0, T]$,

$$\begin{aligned} & - \int_0^t \sum_{i=1}^4 \mu(u_i(s) \partial_s \phi_i(s)) ds + \left[\sum_{i=1}^4 \mu(u_i(t) \phi_i(t) - u_i(0) \phi_i(0)) \right] \\ & = - \int_0^t \sum_{i=1}^4 C_i \mathcal{E}(u_i(s), \phi_i(s)) ds + \int_0^t \sum_{i=1}^4 \lambda_i \mu(\phi_i(s) G(\vec{u}(s))) ds. \quad (\text{weak-}\mathbb{RDP}) \end{aligned}$$

When this is satisfied for any $T > 0$, we'll say that \vec{u} is a weak solution on $[0, \infty)$.

Main result

Theorem 2.2 *Let (L, μ) be a selfadjoint Markov generator satisfying logarithmic Sobolev inequality (6) with constant $C_{LS} \in (0, \infty)$.*

Let $\Phi_2(x) = \exp(x^2) - 1$.

Assume $\vec{f} \geq 0$ is a nonnegative initial datum and $\vec{f} \in (E^{\Phi_2(\mu)})^4$.

Then, for any diffusion coefficients $C_1 > 0$ and $C_2 > 0$ and any reaction rate $\lambda > 0$, there exists a unique nonnegative weak solution \vec{u} of (\mathbb{RDP}) on $[0, \infty)$.

Moreover, for any $\alpha \geq 1$, any $\gamma > 0$ and any $i = 1, \dots, 4$, if $\mu(e^{\gamma(f_1+f_3)^\alpha}) < \infty$ and $\mu(e^{\gamma(f_2+f_4)^\alpha}) < \infty$, then

$$\forall t \text{ a.e. in } [0, \infty), \quad \mu(e^{\gamma u_i^\alpha(t)}) \leq \max\left(\mu(e^{\gamma(f_1+f_3)^\alpha}), \mu(e^{\gamma(f_2+f_4)^\alpha})\right).$$

In section 6, we will state the extension of this theorem to the general problem (2).

In short, to prove this theorem, we linearize the system of equations by means of an approximation sequence $(\vec{u}^{(n)})_n$. We show recursively that $\vec{u}^{(n)}(t)$ is nonnegative, belongs to $\mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2(\mu)})$ so that lemma 2.1 guarantees $\vec{u}^{(n+1)}$ is well defined. This propagation is made precise in a lemma studying the linear *cornerstone* problem which underlies the recursive approach.

We will first focus our efforts to prove convergence of the approximation sequence in the space $(\mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu)))^4$. Afterwards, we detail a way to study the cornerstone existence lemma.

Remark 2 *We will exhibit in appendix B a sufficient condition to ensure that $f \in E^{\Phi_\alpha(\mu)}$, namely, that there exist $\beta > \alpha$ and $\gamma > 0$ such that $\mu(e^{\gamma|f|^\beta}) < +\infty$. In particular, it implies that, provided $\vec{f} \geq 0$ belongs to $(E^{\Phi_2(\mu)})^4$, one may choose $\tilde{\gamma} > 0$ large enough such that*

$$\begin{aligned} & \frac{4}{\min(C_1, C_2)} \lambda C_{LS} < \tilde{\gamma}, \text{ and} \\ & \vec{f} \text{ still satisfies } \mu(e^{\tilde{\gamma} f_i}) < \infty, i = 1, \dots, 4 \end{aligned} \quad (13)$$

which will be useful in the proof of existence and uniqueness.

3 Iterative procedure

Let us define the approximation sequence $(\vec{u}^{(n)})_{n \in \mathbb{N}}$ in the following way. (First of all, note the parenthesis in $\vec{u}^{(n)}$ has nothing to do with differentiation, and has been introduced to distinguish the index from powers).

- for all $n \in \mathbb{N}$, $\vec{u}^{(n)}(t=0) = \vec{f} \in (E^{\Phi_2}(\mu))^4$;
- for $n=0$, $\partial_t \vec{u}^{(0)}(t) = \mathfrak{C}L\vec{u}^{(0)}(t)$, $t > 0$;
- for any $n \geq 1$, and $t > 0$,

$$\begin{cases} \partial_t u_1^{(n)}(t) = C_1 L u_1^{(n)}(t) - \lambda \left(u_2^{(n-1)}(t) u_1^{(n)}(t) - u_4^{(n-1)}(t) u_3^{(n)}(t) \right), \\ \partial_t u_3^{(n)}(t) = C_1 L u_3^{(n)}(t) + \lambda \left(u_2^{(n-1)}(t) u_1^{(n)}(t) - u_4^{(n-1)}(t) u_3^{(n)}(t) \right), \end{cases} \quad (\mathbb{RDP}_n)$$

$$\begin{cases} \partial_t u_2^{(n)}(t) = C_2 L u_2^{(n)}(t) - \lambda \left(u_1^{(n-1)}(t) u_2^{(n)}(t) - u_3^{(n-1)}(t) u_4^{(n)}(t) \right), \\ \partial_t u_4^{(n)}(t) = C_2 L u_4^{(n)}(t) + \lambda \left(u_1^{(n-1)}(t) u_2^{(n)}(t) - u_3^{(n-1)}(t) u_4^{(n)}(t) \right). \end{cases}$$

Knowing $\vec{u}^{(n-1)} \in (\mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu)) \cap \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu)))^4$, (which is the case for any $T > 0$ under our hypothesis for $\vec{u}^{(0)}$ by proposition 1), this system may be reduced to the four independant affine scalar equations, with $t > 0$,

$$\begin{cases} \partial_t u_1^{(n)} = C_1 L u_1^{(n)} - \lambda P_{C_2 t}(f_2 + f_4) u_1^{(n)} + \lambda P_{C_1 t}(f_1 + f_3) u_4^{(n-1)}, \\ \partial_t u_3^{(n)} = C_1 L u_3^{(n)} - \lambda P_{C_2 t}(f_2 + f_4) u_3^{(n)} + \lambda P_{C_1 t}(f_1 + f_3) u_2^{(n-1)}, \\ \partial_t u_2^{(n)} = C_2 L u_2^{(n)} - \lambda P_{C_1 t}(f_1 + f_3) u_2^{(n)} + \lambda P_{C_2 t}(f_2 + f_4) u_3^{(n-1)}, \\ \partial_t u_4^{(n)} = C_2 L u_4^{(n)} - \lambda P_{C_1 t}(f_1 + f_3) u_4^{(n)} + \lambda P_{C_2 t}(f_2 + f_4) u_1^{(n-1)}. \end{cases} \quad (14)$$

whose existence, uniqueness and positivity on $[0, T]$ follows from Lemma 3.1 below, with $A(t) = \lambda P_{C_2 t}(f_2 + f_4)$ and $B(t) = \lambda P_{C_1 t}(f_1 + f_3) u_4^{(n-1)}$ (or similarly), using proposition 1 and lemma 2.1.

Lemma 3.1 (Cornerstone existence lemma) *Let L be a Markov generator satisfying logarithmic Sobolev inequality with constant $C_{LS} \in (0, \infty)$. Let $T > 0$ and $A = A(t) \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$ and $B \in \mathbb{L}^2([0, T], \mathbb{L}^2(\mu))$. Then the Cauchy problem*

$$\begin{cases} \partial_t u(t) = Lu(t) - A(t)u(t) + B(t), \\ u(0) = f, f \in \mathbb{L}^2(\mu) \end{cases} \quad (\text{CS})$$

has a unique weak solution on $[0, T]$. Futhermore, provided f , A and B are assumed nonnegative, then the solution u is nonnegative.

Although classical, we recall that $u \in \mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu))$ is a weak solution of (CS) provided, for any $\phi \in C^\infty([0, T], \mathcal{D})$, and any $0 \leq t \leq T$,

$$\begin{aligned} - \int_0^t \mu(u(s) \partial_s \phi(s)) ds + \mu(u(t) \phi(t) - u(0) \phi(0)) &= - \int_0^t \mathcal{E}(u(s), \phi(s)) ds \\ &+ \int_0^t \mu(\phi(s) [-A(s)u(s) + B(s)]) ds. \quad (\text{weak-CS}) \end{aligned}$$

Recursive equivalence of both systems (\mathbb{RDP}_n) and (14) may be seen as follows. Starting from (\mathbb{RDP}_n) , one easily gets

$$\begin{cases} \partial_t (u_1^{(n)} + u_3^{(n)}) = C_1 L (u_1^{(n)} + u_3^{(n)}) \\ \partial_t (u_2^{(n)} + u_4^{(n)}) = C_2 L (u_2^{(n)} + u_4^{(n)}) \end{cases}$$

and writing $u_3^{(n)}(t) = P_{C_1 t}(f_1 + f_3) - u_1^{(n)}(t)$ (and similarly for the other coordinates) gives the announced decoupled system. Conversely, deducing from the decoupled system that $u_1^{(n)} + u_3^{(n)} = P_{C_1 t}(f_1 + f_3)$ (and similarly) follows by induction and uniqueness in lemma 3.1.

To be able to define $\vec{u}^{(n+1)}$, and hence prove that the iterative sequence is well defined, it remains to check that $u_i^{(n)} \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$, for all $i = 1, \dots, 4$. This is based on results stated in appendix C and can be shown as follows.

We may focus on $u_1^{(n)}(t)$ by symmetry. By positivity of the $u_i^{(n)}$'s and constraint $u_1^{(n)} + u_3^{(n)} = P_{C_1 t}(f_1 + f_3)$, the contraction property of the semigroup stated in lemma C.1 implies that, for any $\gamma > 0$, for any t a.e.,

$$\mu\left(e^{\gamma(u_1^{(n)}(t))^2}\right) \leq \mu(e^{\gamma(f_1+f_3)^2}) < +\infty. \quad (15)$$

So that, in particular, for any $t \in [0, T]$, $u_1^{(n)}(t) \in E^{\Phi_2}(\mu)$. Following lemma C.2, what remains to be checked is Bochner measurability of the mapping $t \mapsto u_1^{(n)}(t) \in E^{\Phi_2}(\mu)$.

From the corresponding weak formulation (weak-CS) applied to a constant (in time) test function $\phi(t) \equiv \varphi \in \mathcal{D}$,

$$\begin{aligned} \mu(u_1^{(n)}(t)\varphi) &= \mu(f_1\varphi) - C_1 \int_0^t \mathcal{E}(u_1^{(n)}(s), \varphi) ds \\ &\quad + \int_0^t \mu\left(\varphi(-\lambda P_{C_2 s}(f_2 + f_4) u_1^{(n)}(s) + \lambda P_{C_1 s}(f_1 + f_3) u_4^{(n-1)}(s))\right) ds. \end{aligned}$$

Hence, the function $t \mapsto \mu(u_1^{(n)}(t)\varphi)$ is continuous, for any fixed $\varphi \in \mathcal{D}$. Now, \mathcal{D} is a dense subspace of the dual space $(E^{\Phi_2})' = \mathbb{L}^{\Phi_2^*}(\mu)$ (see appendix C), so that weak measurability of $t \mapsto u_1^{(n)}(t) \in E^{\Phi_2}$ follows. By Pettis measurability theorem* and separability of $E^{\Phi_2}(\mu)$, $t \in [0, T] \mapsto u_1^{(\infty)}(t) \in E^{\Phi_2}(\mu)$ is Bochner measurable.

4 Proof of Theorem 2.2

Convergence of the approximation procedure (\mathbb{RDP}_n)

From now on, we'll use the notation

$$|\vec{u}|^2 = \sum_{i=1}^4 u_i^2 \quad \text{and} \quad \mathcal{E}(\vec{u}) = \sum_{i=1}^4 \mathcal{E}(u_i).$$

The main idea is to show that, with

$$\Sigma_n(t) = \mu(|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2)(t) + 2\kappa \int_0^t \mathcal{E}(\vec{u}^{(n)} - \vec{u}^{(n-1)})(s) ds, \quad (16)$$

for some $\kappa > 0$ (specified later), the supremum $\sup_{t \in [0, T]} \Sigma_n(t)$ goes to 0 exponentially fast as n goes to ∞ provided $T > 0$ is small enough. From lemma 3.1, $\vec{u}^{(n)}$ is defined recursively as a weak solution of the cornerstone linear problem. To make things simpler at this stage, we here perform formal computations to get a priori estimates. Getting the estimates rigorously makes use of Steklov regularisation, which we will illustrate in the proof of the next proposition.

* see [46], [19] or [41] for a proof, [21], appendix E.5, theorem 7, for a statement.

Estimate of the \mathbb{L}^2 -norm derivative

We will focus on the \mathbb{L}^2 -norm of $u_1^{(n)}$.

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu [(u_1^{(n)} - u_1^{(n-1)})^2] &= C_1 \mu [(u_1^{(n)} - u_1^{(n-1)}) L(u_1^{(n)} - u_1^{(n-1)})] \\ &\quad - \lambda \mu [(u_1^{(n)} - u_1^{(n-1)}) (u_1^{(n)} u_2^{(n-1)} - u_3^{(n)} u_4^{(n-1)} - u_1^{(n-1)} u_2^{(n-2)} + u_3^{(n-1)} u_4^{(n-2)})], \end{aligned}$$

and after natural multilinear handlings,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu [(u_1^{(n)} - u_1^{(n-1)})^2] &= -C_1 \mathcal{E} [u_1^{(n)} - u_1^{(n-1)}] - \lambda \mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_2^{(n-1)}] \\ &\quad - \lambda \mu [(u_1^{(n)} - u_1^{(n-1)}) (u_2^{(n-1)} - u_2^{(n-2)}) u_1^{(n-1)}] + \lambda \mu [(u_1^{(n)} - u_1^{(n-1)}) (u_3^{(n)} - u_3^{(n-1)}) u_4^{(n-1)}] \\ &\quad \quad \quad + \lambda \mu [(u_1^{(n)} - u_1^{(n-1)}) (u_4^{(n-1)} - u_4^{(n-2)}) u_3^{(n-1)}]. \end{aligned}$$

Since $\bar{u}^{(n-1)}$ is nonnegative, using the quadratic inequality $ab \leq a^2/2 + b^2/2$, one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu [(u_1^{(n)} - u_1^{(n-1)})^2] &\leq -C_1 \mathcal{E} [u_1^{(n)} - u_1^{(n-1)}] \\ &\quad + \frac{\lambda}{2} \mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_1^{(n-1)}] + \frac{\lambda}{2} \mu [(u_2^{(n-1)} - u_2^{(n-2)})^2 u_1^{(n-1)}] \\ &\quad + \frac{\lambda}{2} \mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_4^{(n-1)}] + \frac{\lambda}{2} \mu [(u_3^{(n)} - u_3^{(n-1)})^2 u_4^{(n-1)}] \\ &\quad + \frac{\lambda}{2} \mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_3^{(n-1)}] + \frac{\lambda}{2} \mu [(u_4^{(n-1)} - u_4^{(n-2)})^2 u_3^{(n-1)}]. \end{aligned}$$

All the similar terms are then estimated thanks to the relative entropy inequality (36). For instance,

$$\begin{aligned} \mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_1^{(n-1)}] &\leq \frac{1}{\gamma} \text{Ent}_\mu [(u_1^{(n)} - u_1^{(n-1)})^2] \\ &\quad + \frac{1}{\gamma} \mu [(u_1^{(n)} - u_1^{(n-1)})^2] \log \mu [e^{\gamma u_1^{(n-1)}}]. \end{aligned}$$

The logarithmic Sobolev inequality (6) and bound (15) give

$$\mu [(u_1^{(n)} - u_1^{(n-1)})^2 u_1^{(n-1)}] \leq \frac{C_{LS}}{\gamma} \mathcal{E} [u_1^{(n)} - u_1^{(n-1)}] + \frac{D}{\gamma} \mu [(u_1^{(n)} - u_1^{(n-1)})^2],$$

where

$$D = \max \left\{ \log \mu (e^{\gamma(f_1+f_3)}), \log \mu (e^{\gamma(f_2+f_4)}) \right\}. \quad (17)$$

Using the same arguments for all the terms leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu [(u_1^{(n)} - u_1^{(n-1)})^2] &\leq -C_1 \mathcal{E} [u_1^{(n)} - u_1^{(n-1)}] \\ &\quad + \frac{\lambda C_{LS}}{2\gamma} \left(3\mathcal{E} [u_1^{(n)} - u_1^{(n-1)}] + \mathcal{E} [u_2^{(n-1)} - u_2^{(n-2)}] \right. \\ &\quad \quad \left. + \mathcal{E} [u_3^{(n)} - u_3^{(n-1)}] + \mathcal{E} [u_4^{(n-1)} - u_4^{(n-2)}] \right) \\ &\quad + D \frac{\lambda}{2\gamma} \left(3\mu [(u_1^{(n)} - u_1^{(n-1)})^2] + \mu [(u_2^{(n-1)} - u_2^{(n-2)})^2] \right. \\ &\quad \quad \left. + \mu [(u_3^{(n)} - u_3^{(n-1)})^2] + \mu [(u_4^{(n-1)} - u_4^{(n-2)})^2] \right). \end{aligned}$$

Completely similar terms are obtained when dealing with the \mathbb{L}^2 -norms of the other components. After summation in all the components, one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu [|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2] &\leq -\min(C_1, C_2) \mathcal{E}[\vec{u}^{(n)} - \vec{u}^{(n-1)}] \\ &\quad + \frac{\lambda C_{LS}}{2\gamma} \left(4\mathcal{E}[\vec{u}^{(n)} - \vec{u}^{(n-1)}] + 2\mathcal{E}[\vec{u}^{(n-1)} - \vec{u}^{(n-2)}] \right) \\ &\quad + \frac{D\lambda}{2\gamma} \left(4\mu [|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2] + 2\mu [|\vec{u}^{(n-1)} - \vec{u}^{(n-2)}|^2] \right). \end{aligned}$$

Let $\kappa \equiv \min(C_1, C_2) - \frac{2\lambda C_{LS}}{\gamma}$ which is positive thanks to the assumed constraint (13).

Use the absolute continuity and the positivity of $\int_0^t \mathcal{E}(\vec{u}^{(n)} - \vec{u}^{(n-1)})(s) ds$, to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\mu [|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2] + 2\kappa \int_0^t \mathcal{E}[\vec{u}^{(n)} - \vec{u}^{(n-1)}](s) ds \right) &\leq \\ D \frac{2\lambda}{\gamma} \left(\mu [|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2] + 2\kappa \int_0^t \mathcal{E}[\vec{u}^{(n)} - \vec{u}^{(n-1)}](s) ds \right) & \\ + D \frac{\lambda}{\gamma} \left(\mu [|\vec{u}^{(n-1)} - \vec{u}^{(n-2)}|^2] + \frac{C_{LS}}{D} \mathcal{E}[\vec{u}^{(n-1)} - \vec{u}^{(n-2)}] \right). & \end{aligned}$$

Reminding the definition (16) of Σ_n and that $\vec{u}^{(n)}(0) = \vec{u}^{(n-1)}(0)$, after integration over $[0, t]$, $t \in [0, T]$, we obtain the following main estimate

$$\begin{aligned} \Sigma_n(t) &\leq D \frac{4\lambda}{\gamma} \int_0^t \Sigma_n(s) ds \\ + D \frac{2\lambda}{\gamma} \left(\int_0^t \mu [|\vec{u}^{(n-1)} - \vec{u}^{(n-2)}|^2](s) ds + \frac{C_{LS}}{D} \int_0^t \mathcal{E}[\vec{u}^{(n-1)} - \vec{u}^{(n-2)}](s) ds \right). \end{aligned} \quad (18)$$

Gronwall argument and convergence

Gronwall type arguments applied to the estimate (18) give for any $t \in [0, T]$,

$$\begin{aligned} \Sigma_n(t) &\leq D \frac{2\lambda}{\gamma} e^{D \frac{4\lambda}{\gamma} t} \left(\int_0^t \mu (|\vec{u}^{(n-1)} - \vec{u}^{(n-2)}|^2)(s) ds \right. \\ &\quad \left. + \frac{C_{LS}}{D} \int_0^t \mathcal{E}(\vec{u}^{(n-1)} - \vec{u}^{(n-2)})(s) ds \right), \end{aligned}$$

It follows that

$$\sup_{t \in [0, T]} \Sigma_n(t) \leq \eta(T) \sup_{t \in [0, T]} \Sigma_{n-1}(t),$$

where $\eta(T) = \frac{2\lambda}{\gamma} e^{D \frac{4\lambda}{\gamma} T} (DT + C_{LS})$.

Condition (13) implies that there exists $T(D) > 0$ such that $\eta(T) < 1$ since $\lim_{T \rightarrow 0} \eta(T) = \frac{4\lambda}{\gamma} C_{LS}$. Therefore, for this choice of $T > 0$, $(\vec{u}^{(n)})_{n \in \mathbb{N}}$ satisfies

$$\begin{aligned} \max \left\{ \int_0^T \mathcal{E}(\vec{u}^{(n)} - \vec{u}^{(n-1)})(s) ds, \sup_{t \in [0, T]} \mu (|\vec{u}^{(n)} - \vec{u}^{(n-1)}|^2)(s) \right\} \\ \leq \eta(T)^{n-1} \sup_{t \in [0, T]} \Sigma_1(t), \end{aligned}$$

Performing a similar estimate for $\frac{1}{2} \frac{d}{dt} \mu ((u_i^{(n)}(t))^2)$, one gets the uniform bound

$$\forall n, \forall t \in [0, T], \Sigma_n(t) \leq e^{\frac{4\lambda D}{\gamma} t} \mu(|\vec{f}|^2). \quad (19)$$

It follows that

$$\|\vec{u}^{(n)} - \vec{u}^{(n-1)}\|_{\mathbb{L}^2([0,T], \mathcal{D}^4) \cap \mathcal{C}([0,T], \mathbb{L}^2(\mu^4))} \leq 2e^{\frac{4\lambda D}{\gamma} T} \mu(|\vec{f}|^2) \eta(T)^{n-1}.$$

Hence, $(\vec{u}^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence: it converges to some function $\vec{u}^{(\infty)} \in \mathbb{L}^2([0, T], \mathcal{D}^4) \cap \mathcal{C}([0, T], \mathbb{L}^2(\mu^4))$.

Global existence of the weak solution

Let $T > 0$ fixed as in the previous computation. We will first prove that the limit $\vec{u}^{(\infty)}$ is a weak solution of (\mathbb{RDP}) in $[0, T]$. Let $\phi \in \mathcal{C}^\infty([0, T], \mathcal{D})$ and use the weak formulation of (\mathbb{RDP}_n) for $\vec{\phi} \equiv (\phi, 0, 0, 0)$. For any $t \in [0, T]$,

$$\begin{aligned} & - \int_0^t \mu(u_1^{(n)}(s) \partial_s \phi) ds + \mu(u_1^{(n)}(t) \phi(t)) - \mu(f_1 \phi(0)) = \\ & - C_1 \int_0^t \mathcal{E}(\phi, u_1^{(n)})(s) ds - \lambda \int_0^t \mu(\phi u_1^{(n)} u_2^{(n-1)})(s) ds + \lambda \int_0^t \mu(\phi u_3^{(n)} u_4^{(n-1)})(s) ds. \end{aligned}$$

We now show we can pass to the limit $n \rightarrow \infty$ in all the terms. (Dealing with other coordinates $u_i^{(n)}$ is similar by symmetry). Thanks to the continuity of the scalar product in $\mathbb{L}^2([0, T], \mathcal{D})$, we have

$$\lim_{n \rightarrow \infty} \int_0^t \mu(u_1^{(n)} \partial_s \phi)(s) ds = \int_0^t \mu(u_1^{(\infty)} \partial_s \phi)(s) ds$$

and

$$\lim_{n \rightarrow \infty} \int_0^t \mathcal{E}(\phi, u_1^{(n)})(s) ds = \int_0^t \mathcal{E}(\phi, u_1^{(\infty)})(s) ds.$$

Moreover, as the convergence also holds in $\mathcal{C}([0, T], \mathbb{L}^2(\mu))$, then $\lim_{n \rightarrow \infty} \mu(u_1^{(n)} \phi)(t) = \mu(u_1^{(\infty)} \phi)(t)$ and $\mu(f_1 \phi(0)) = \lim_{n \rightarrow \infty} \mu(u_1^{(n)} \phi)(0) = \mu(u_1^{(\infty)} \phi)(0)$.

Dealing with the convergence of the term $\int_0^t \mu(\phi u_1^{(n)} u_2^{(n-1)})(s) ds$ (and similarly of $\int_0^t \mu(\phi u_3^{(n)} u_4^{(n-1)})(s) ds$) is more intricate. The difficulty is to show that $u_1^{(\infty)}$ belongs to $\mathbb{L}^\infty([0, T], E^{\Phi_2})$ which will follow indirectly. The details are as follows.

By lemma 2.1, $\tau_n \equiv \tau_n^{(12)} \equiv \mu(\phi u_1^{(n)} u_2^{(n-1)}) \in \mathbb{L}^1([0, T])$. Let us show that this sequence is Cauchy, and so converges to, say, $\tau^{(12)}$ in $\mathbb{L}^1([0, T])$. Indeed,

$$\begin{aligned} \|\tau_n - \tau_m\|_1 & \leq \int_0^t \mu(|\phi(s)| \cdot |u_1^{(n)} u_2^{(n-1)}(s) - u_1^{(m)} u_2^{(m-1)}(s)|) ds \leq \\ & \int_0^t \mu(|\phi| \cdot |u_1^{(n)} - u_1^{(m)}| \cdot |u_2^{(n-1)}|)(s) ds + \int_0^t \mu(|\phi| \cdot |u_2^{(n-1)} - u_2^{(m-1)}| \cdot |u_1^{(m)}|)(s) ds. \end{aligned}$$

But by (15), and again entropic and log-Sobolev inequalities,

$$\begin{aligned} & \int_0^t \mu(|\phi| \cdot |u_1^{(n)} - u_1^{(m)}| \cdot |u_2^{(n-1)}|)(s) ds \\ & \leq \frac{\max(C_{LS}, \log M_\gamma)}{\gamma} \|\phi\|_{\mathbb{L}^2(I, \mathbb{L}^2(\mu))} \|u_1^{(n)} - u_1^{(m)}\|_{\mathbb{L}^2(I, \mathcal{D})} \quad (20) \end{aligned}$$

with $M_\gamma \equiv \max(\mu(e^{\gamma(f_1+f_3)^2}), \mu(e^{\gamma(f_2+f_4)^2}))$. This goes to 0 as $n, m \rightarrow +\infty$.

Now, for any $t \in [0, T]$, $u_1^{(n)}(t) \rightarrow u_1^{(\infty)}(t)$ in $\mathbb{L}^2(\mu)$, so that along a subsequence it converges μ a.s.. Hence first $u_1^{(\infty)}(t)$ is nonnegative (μ a.s.) and secondly by Fatou lemma

$$\mu(e^{\gamma(u_1^{(\infty)}(t))^2}) \leq \liminf_n \mu(e^{\gamma(u_1^{(n)}(t))^2}) \leq M_\gamma < \infty$$

for any t a.e. in $[0, T]$. And this for any $\gamma > 0$. Consequently, for any t a.e., $u^{(\infty)}(t) \in E^{\Phi_2}$.

From lemma C.2, what remains to do is to prove E^{Φ_2} Bochner measurability. Let us summarize what we obtained. One has after taking limit $n \rightarrow +\infty$,

$$\begin{aligned} & - \int_0^t \mu(u_1^{(\infty)}(s) \partial_s \phi) ds + \mu(u_1^{(\infty)}(t) \phi(t)) - \mu(f_1 \phi(0)) = \\ & \quad - C_1 \int_0^t \mathcal{E}(\phi, u_1^{(\infty)})(s) ds - \lambda \int_0^t \tau^{(12)}(s) ds + \lambda \int_0^t \tau^{(34)}(s) ds. \end{aligned}$$

In particular, choosing $\phi(t) = \varphi \in \mathcal{D}$, the mapping $t \in [0, T] \mapsto \mu(\varphi u_1^{(\infty)}(t)) \in \mathbb{R}$ is continuous. Then, arguments detailed on page 9 ensure that $u_1^{(\infty)} \in \mathbb{L}^\infty(I, L^{\Phi_2})$.

Furthermore,

$$\begin{aligned} & \left| \int_0^t \mu(\phi u_1^{(n)} u_2^{(n-1)})(s) ds - \int_0^t \mu(\phi u_1^{(\infty)} u_2^{(\infty)})(s) ds \right| \leq \\ & \int_0^t \mu(|\phi| \cdot |u_1^{(n)} - u_1^{(\infty)}| \cdot |u_2^{(n-1)}|)(s) ds + \int_0^t \mu(|\phi| \cdot |u_2^{(n-1)} - u_2^{(\infty)}| \cdot |u_1^{(\infty)}|)(s) ds. \end{aligned}$$

Letting separately n (resp. m) to $+\infty$ in (20) shows that

$$\int_0^t \mu(\phi u_1^{(n)} u_2^{(n-1)})(s) ds \rightarrow \int_0^t \mu(\phi u_1^{(\infty)} u_2^{(\infty)})(s) ds.$$

All this implies that

$$\vec{u}^{(\infty)} = (u_1^{(\infty)}, u_2^{(\infty)}, u_3^{(\infty)}, u_4^{(\infty)}) \in \left(\mathbb{L}^2(I, \mathcal{D}) \cap C(I, \mathbb{L}^2(\mu)) \cap \mathbb{L}^\infty(I, E^{\Phi_2}) \right)^4$$

is a nonnegative weak solution of (RDIP).

From the local existence in $[0, T]$ to a global existence in $[0, \infty)$ it is enough to prove that we can repeat the method on the interval $[T, 2T]$. This follows from the estimate

$$\mu(e^{\gamma(u_1^{(\infty)} + u_3^{(\infty)})^2}(T)) \leq \mu(e^{\gamma(f_1 + f_3)^2}).$$

See lemma C.1.

Proposition 4.1 (Uniqueness) *Let $\vec{f} \geq 0$ such that, for some $\gamma > 0$,*

$$M \equiv \max \left\{ \mu(e^{\gamma(f_1 + f_3)}), \mu(e^{\gamma(f_2 + f_4)}) \right\} < \infty.$$

Assume the diffusion coefficients C_1 and C_2 , the logarithmic Sobolev constant C_{LS} of L , the reaction rate λ and the exponential integrability parameter γ are linked by the constraint

$$4 \frac{\lambda C_{LS}}{\min(C_1, C_2)} \leq \gamma.$$

Then a weak solution of the Reaction-Diffusion problem (RDIP) with initial datum \vec{f} is unique.

We recall basics on Steklov calculus (see [30] for instance), i.e. appropriate time regularization to deal with weak solutions. For any Banach space X , and any $v \in \mathbb{L}^2([0, T], X)$, the Steklov average, defined by

$$a_h(v)(t) = \begin{cases} \frac{1}{h} \int_t^{t+h} v(\tau) d\tau & , 0 \leq t \leq T - h, \\ 0 & , T - h < t \leq T \end{cases}$$

converges to v in $\mathbb{L}^2([0, T], X)$ when h goes to 0. Moreover, provided $v \in C([0, T], X)$, $a_h(v) \in C^1([0, T - h], X)$, $\frac{d}{dt}a_h(v)(t) = \frac{1}{h}(v(t+h) - v(t))$ in X , and $a_h(v)(t)$ converges to $v(t)$ in X , for every $t \in [0, T]$. The space X will be here $\mathbb{L}^2(\mu)$ or \mathcal{D} depending on the context.

Proof of Proposition 4.1

◁ Let \vec{u} and \vec{v} be two weak solutions of (\mathbb{RDP}) with the same initial datum $\vec{f} \geq 0$. Let $M \in (0, \infty)$ such that, $\forall i = 1, \dots, 4$, $\mu(e^{\gamma|u_i(t)|}) \leq M$, t a.e., (and similarly for \vec{v}). Let $\vec{w} \equiv \vec{u} - \vec{v}$ and $a_h(w^i)(t)$ the Steklov average of the i component of w as defined before. Integrating $\frac{1}{2} \frac{d}{ds} \mu((a_h(w^i)(s))^2) = \mu(a_h(w^i)(s) \partial_s a_h(w^i))$ one gets

$$\mu((a_h(w^i)(t))^2) = \mu((a_h(w^i)(0))^2) + 2 \int_0^t ds \mu(a_h(w^i)(s) \frac{1}{h}(w^i(s+h) - w^i(s))). \quad (21)$$

We then use the definition of a weak solution with the constant test function $a_h(w^i)(s) \in \mathcal{D}$ on the interval $[s, s+h]$ to get

$$\begin{aligned} \mu(a_h(w^i)(s) \frac{1}{h}(w^i(s+h) - w^i(s))) &= -C_i \frac{1}{h} \int_s^{s+h} \mathcal{E}(a_h(w^i)(s), w^i(\tau)) d\tau \\ &\quad + \lambda_i \frac{1}{h} \int_s^{s+h} d\tau \mu(a_h(w^i)(s) \{ (u_1 u_2 - u_3 u_4)(\tau) - (v_1 v_2 - v_3 v_4)(\tau) \}) \end{aligned}$$

Now, first,

$$\frac{1}{h} \int_s^{s+h} \mathcal{E}(a_h(w^i)(s), w^i(\tau)) d\tau = \mathcal{E}(a_h(w^i)(s), a_h(w^i)(s)).$$

And the other term is bounded from above by

$$\begin{aligned} \frac{\lambda}{h^2} \int_{[s, s+h]^2} d\tau d\tau' \mu(|u_i - v_i|(\tau') \{ |u_1 - v_1|(\tau) |u_2|(\tau) + |v_1|(\tau) |u_2 - v_2|(\tau) + \\ |u_3 - v_3|(\tau) |u_4|(\tau) + |v_3|(\tau) |u_4 - v_4|(\tau) \}) \end{aligned}$$

We can deal with the four similar terms by the same way: let us focus on the first one. One first uses

$$\mu(|u_i - v_i|(\tau') |u_1 - v_1|(\tau) |u_2|(\tau)) \leq \frac{1}{2} \mu((u_i - v_i)^2(\tau') |u_2|(\tau) + (u_1 - v_1)^2(\tau) |u_2|(\tau)).$$

Once gain, entropic inequality followed by logarithmic Sobolev inequality give

$$\begin{aligned} \frac{\lambda}{h^2} \int_{[s, s+h]^2} d\tau d\tau' \mu(|u_i - v_i|(\tau') |u_1 - v_1|(\tau) |u_2|(\tau)) &\leq \\ &\leq \frac{\lambda}{2\gamma} \frac{1}{h} \int_s^{s+h} (C_{LS} \mathcal{E}(u_i - v_i)(\tau') + \log M \mu((u_i - v_i)^2(\tau'))) d\tau' \\ &\quad + \frac{\lambda}{2\gamma} \frac{1}{h} \int_s^{s+h} (C_{LS} \mathcal{E}(u_1 - v_1)(\tau) + \log M \mu((u_1 - v_1)^2(\tau))) d\tau. \end{aligned}$$

Note that, up to a constant, the first term of the RHS is the Steklov average of the $\mathbb{L}^1([0, T])$ function $C_{LS} \mathcal{E}(u_i - v_i)(\cdot) + \log M \mu((u_i - v_i)^2(\cdot))$, so that, as $h \rightarrow 0$, it converges in $\mathbb{L}^1([0, T])$ to that function. Going back to (21) and performing all the explained bounds before passing to the limit $h \rightarrow 0$, one gets the estimate (note that $w_i(0) = 0$)

$$\begin{aligned} \mu(w_i^2(t)) \\ \leq 2 \int_0^t ds \left(-C_i \mathcal{E}(w_i)(s) + \frac{\lambda C_{LS}}{2\gamma} [4\mathcal{E}(w_i)(s) + \mathcal{E}(\vec{w})(s)] + \frac{\lambda \log M}{2\gamma} [4\mu(w_i^2)(s) + \mu(\vec{w}^2)(s)] \right) \end{aligned}$$

Summing over all i 's, one gets

$$\begin{aligned} \mu(\bar{w}^2)(t) &\leq 2(-\min(C_1, C_2) + 4\frac{\lambda C_{LS}}{\gamma}) \int_0^t ds \mathcal{E}(\bar{w})(s) + 8\frac{\lambda \log M}{\gamma} \int_0^t ds \mu(\bar{w}^2)(s) \\ &\leq 8\frac{\lambda \log M}{\gamma} \int_0^t ds \mu(\bar{w}^2)(s) \end{aligned}$$

provided the announced constraint $4\frac{\lambda C_{LS}}{\gamma} \leq \min(C_1, C_2)$ is satisfied. Uniqueness follows by Gronwall arguments. \triangleright

5 Proof of Lemma 3.1

Our approach to study the cornerstone linear problem introduced in lemma 3.1 will be as follows. We first complete regularity lemma 2.1 by another preliminary lemma (relative to differentiability) which allow us to perform a recursive approximation of the solution of a mollified problem (with a small action of the semigroup on the extra affine term). On the way, we show a priori estimates which will be useful later to remove the mollification and get a solution of our initial problem. Uniqueness and preservation of positivity are tackled in specific sections.

Such an approach was already proposed in [22], and computations look quite similar. The main difference consists in the fact that, as $A(t) \in \mathbb{L}^{\Phi_2}(\mu)$, then one has $\mu(e^{\gamma|A(t)|}) < \infty$ for any γ (see appendix B), so that, using of the entropic inequality, contribution of the affine extra term may be made small enough to be dominated by the log-Sobolev constant without further constraint.

5.1 Preliminaries

We recall that $\mathbb{L}^2(\mu)$ may be continuously embedded in the dual space \mathcal{D}' of the domain \mathcal{D} . From lemma 2.1, it follows that the multiplication operator by a function $v \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$ is a particular case of a Lipschitz continuous operator from $\mathbb{L}^2([0, T], \mathcal{D})$ to $\mathbb{L}^2([0, T], \mathcal{D}')$. The following lemma may be stated in this more general context (an example of which was studied in [22]).

Lemma 5.1 (Absolute continuity, differentiability a.e. and weak solutions)

Let $z \in \mathbb{L}^2([0, T], \mathcal{D}')$, $f \in \mathbb{L}^2(\mu)$ and $\varepsilon > 0$. Define $u(t) = P_t f + \int_0^t P_{t-s+\varepsilon} z(s) ds$. Then u belongs to $\mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu))$ and is (strongly) absolutely continuous from $[a, T]$ to $\mathbb{L}^2(\mu)$, for any $0 < a < T$. And consequently, the continuous function $t \in [0, T] \mapsto \mu(u^2(t)) \in \mathbb{R}$ is absolutely continuous on $[a, T]$.

Moreover, for all t a.e. in $[0, T]$, $u(t)$ is differentiable w.r.t. t in $\mathbb{L}^2(\mu)$, belongs to the domain of L , and satisfies

$$\begin{cases} \frac{\partial}{\partial t} u(t) = Lu(t) + P_\varepsilon(z(t)), t \text{ a.e.} \\ u(0) = f. \end{cases} \quad (22)$$

As a consequence, $u \in \mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu))$ is a weak solution of (22) i.e., for any $\phi \in C^\infty([0, T], \mathcal{D})$,

$$\begin{aligned} - \int_0^t \mu(u(s) \partial_s \phi(s)) ds + \mu(u(t) \phi(t) - u(0) \phi(0)) &= - \int_0^t \mathcal{E}(u(s), \phi(s)) ds \\ &+ \int_0^t \mu(\phi(s) P_\varepsilon z(s)) ds. \end{aligned} \quad (23)$$

Proof

◁ Let us first note that the Markov semigroup itself satisfies all the announced assertions. We only focus on absolute continuity.

Let $([a_i, b_i])_{i=1, \dots, N}$ be a finite collection of (non empty) non overlapping subintervals of $[a, T]$. Then, $P_{b_i}f - P_{a_i}f = \int_0^{b_i - a_i} P_\tau L P_{a_i} f d\tau$ so that

$$\|P_{b_i}f - P_{a_i}f\|_{\mathbb{L}^2(\mu)} \leq (b_i - a_i) \|LP_{a_i}f\|_{\mathbb{L}^2(\mu)}.$$

Strong absolute continuity follows as, by spectral theory, for any $\alpha > 0$ and any $f \in \mathbb{L}^2(\mu)$,

$$\begin{aligned} \|LP_\alpha f\|_{\mathbb{L}^2(\mu)}^2 &= \mu\left(\left((-L)P_\alpha f\right)^2\right) \\ &= \mu\left(\left((-L)^2 P_{2\alpha} f\right) f\right) = \frac{1}{\alpha^2} \int_0^\infty (\alpha\xi)^2 e^{-2\alpha\xi} \nu_f(d\xi) \leq \frac{C}{\alpha^2} \mu(f^2), \end{aligned} \quad (24)$$

for some constant $C > 0$. Note that, one also has $\mathcal{E}(P_\varepsilon f) \leq \frac{C}{\varepsilon} \mu(f^2)$ for any $f \in \mathbb{L}^2(\mu)$. It follows that $\|P_\varepsilon f\|_{\mathcal{D}} \leq \sqrt{1 + \frac{C}{\varepsilon}} \|f\|_{\mathbb{L}^2(\mu)}$. By duality, $\|f\|_{\mathbb{L}^2(\mu)} \leq \|f\|_{\mathcal{D}'}$, then

$$\|P_\varepsilon z\|_{\mathbb{L}^2(\mu)} \leq \sqrt{1 + \frac{C}{\varepsilon}} \|z\|_{\mathcal{D}'} \in \mathbb{L}^2([0, T]).$$

We'll write $\tilde{z} \equiv P_\varepsilon z \in \mathbb{L}^2([0, T], \mathbb{L}^2(\mu))$ (or even sometime $\tilde{z} \equiv P_{\varepsilon/2} z$).

We now turn our attention to the second term,

$$\Psi_\varepsilon(z)(t) \equiv \int_0^t P_{t-s+\varepsilon} z(s) ds, \quad (\varepsilon > 0).$$

First, we show absolute continuity on $[0, T]$ of $\Psi_\varepsilon(z)$ in $\mathbb{L}^2(\mu)$. With $([a_i, b_i])_{i=1, \dots, N}$ a finite collection of non overlapping subintervals of $[0, T]$,

$$\begin{aligned} \|\Psi_\varepsilon(z)(b_i) - \Psi_\varepsilon(z)(a_i)\|_{\mathbb{L}^2(\mu)} &= \\ &\| \int_{a_i}^{b_i} P_{b_i-s+\varepsilon}(z(s)) ds + \int_0^{a_i} ds [P_{b_i-s} - P_{a_i-s}](P_\varepsilon z(s)) \|_{\mathbb{L}^2(\mu)} \\ &\leq \int_{a_i}^{b_i} \|P_\varepsilon z(s)\|_{\mathbb{L}^2(\mu)} ds + \int_0^{a_i} ds \int_{a_i-s}^{b_i-s} \|P_\tau L(P_\varepsilon z(s))\|_{\mathbb{L}^2(\mu)} \\ &\leq \int_{a_i}^{b_i} \|\tilde{z}(s)\|_{\mathbb{L}^2(\mu)} ds + \frac{C}{\varepsilon} (b_i - a_i) \int_0^T \|z(s)\|_{\mathcal{D}'} ds \end{aligned}$$

by another use of (24). (Strong) Absolute continuity follows.

Continuity of u at $t = 0$ in $\mathbb{L}^2(\mu)$ follows by C^0 property of the semigroup. Indeed, $\|\Psi_\varepsilon(z)(t)\|_{\mathbb{L}^2(\mu)} \leq \int_0^t \|\tilde{z}(s)\|_{\mathbb{L}^2(\mu)} ds$ which goes to 0 as t goes to 0.

To prove that, for any t a.e., $\Psi_\varepsilon(z)(t) \in \mathcal{D}$ and $\int_0^T \mathcal{E}(\Psi_\varepsilon(z)(t)) dt < +\infty$, we will use spectral theory. One has

$$\begin{aligned} \mathcal{E}\left(\int_0^t P_{t-s+\varepsilon}(z(s)) ds, \int_0^t P_{t-\tau+\varepsilon}(z(\tau)) d\tau\right) &\leq \int_{[0, t]^2} ds d\tau \mathcal{E}(P_{t-s+\varepsilon}(z(s)), P_{t-\tau+\varepsilon}(z(\tau))) \\ &\leq t \int_0^t \mathcal{E}(P_{t-s+\varepsilon}(z(s))) ds \end{aligned}$$

thanks to Cauchy-Schwarz inequality. Hence,

$$\int_0^T \mathcal{E}(\Psi_\varepsilon(z)(t)) dt \leq T \int_0^T dt \int_0^t ds \mathcal{E}(P_{t-s}(\tilde{z}(s))) = T \int_0^T ds \int_s^T dt \mathcal{E}(P_{t-s}(\tilde{z}(s)))$$

by a first use of Fubini-Tonelli theorem. Then, noting that

$$\mathcal{E}(P_{t-s}(\tilde{z}(s))) = \int_0^\infty \xi e^{-2(t-s)\xi} \nu_{\tilde{z}(s)}(d\xi)$$

and using once again Fubini-Tonelli theorem for $dt \otimes \nu_{\tilde{z}(s)}(d\xi)$ at fixed s , one gets

$$\begin{aligned} \int_0^T \mathcal{E}(\Psi_\varepsilon(z)(t)) dt &\leq T \int_0^T ds \int_0^\infty \nu_{\tilde{z}(s)}(d\xi) \xi \int_s^T e^{-2(t-s)\xi} dt \\ &\leq T \int_0^T ds \int_0^\infty \nu_{\tilde{z}(s)}(d\xi) \xi \frac{1 - e^{-2(T-s)\xi}}{2\xi} \leq \frac{T}{2} \int_0^T ds \mu(\tilde{z}^2(s)) < \infty. \end{aligned}$$

Now, we show that provided $\varepsilon > 0$, $\Psi_\varepsilon(z)(t)$ is differentiable in $\mathbb{L}^2(\mu)$ for any t a.e. in $[0, T]$ and,

$$\forall t \text{ a.e.}, \frac{\partial}{\partial t} \Psi_\varepsilon(z)(t) = P_\varepsilon(z(t)) + \int_0^t P_{t-s} L P_\varepsilon z(s) ds = P_\varepsilon(z(t)) + L(\Psi_\varepsilon(z)(t)).$$

Let $h > 0$ (the case when $h < 0$ can be dealt with in the same way). Let us consider (in $\mathbb{L}^2(\mu)$) the difference between the associated differential ratio and the expected derivative

$$\frac{1}{h} \left[\int_0^{t+h} P_{t+h-s+\varepsilon} z(s) ds - \int_0^t P_{t-s+\varepsilon} z(s) ds \right] - \int_0^t P_{t-s} L P_\varepsilon z(s) ds - P_\varepsilon z(t).$$

We split it into three terms. First,

$$(I) = \frac{1}{h} \int_t^{t+h} [P_{t+h-s} - Id] P_\varepsilon z(s) ds.$$

Secondly,

$$(II) = \frac{1}{h} \int_t^{t+h} P_\varepsilon z(s) ds - P_\varepsilon z(t).$$

And third,

$$(III) = \int_0^t ds \left[\frac{(P_{t+h-s+\varepsilon} - P_{t-s+\varepsilon})}{h} (z(s)) - P_{t-s} L P_\varepsilon z(s) \right].$$

Now, these three terms all go to 0 in $\mathbb{L}^2(\mu)$ as $0 < h$ goes to 0.

Indeed, we deal with the first term as for absolute continuity of $\Psi_\varepsilon(z)$ above. One has

$$\begin{aligned} \left\| \frac{1}{h} \int_t^{t+h} [P_{t-s+h} - Id] P_\varepsilon z(s) ds \right\|_{\mathbb{L}^2(\mu)} &\leq \frac{1}{h} \int_t^{t+h} ds \int_0^{t+h-s} \left\| P_\tau L P_\varepsilon z(s) \right\|_{\mathbb{L}^2(\mu)} d\tau \\ &\leq \frac{C}{\varepsilon} \int_t^{t+h} \frac{t+h-s}{h} \|\tilde{z}(s)\|_{\mathbb{L}^2(\mu)} ds \leq \frac{C}{\varepsilon} \int_t^{t+h} ds \|\tilde{z}(s)\|_{\mathbb{L}^2(\mu)}, \end{aligned}$$

which goes to 0 as $h \rightarrow 0$.

Convergence of (II) to 0 in $\mathbb{L}^2(\mu)$, and this for any t a.e., follows from the easy part of the fundamental theorem of calculus for Bochner integrable functions with values in $\mathbb{L}^2(\mu)$ (proved via comparison with strongly Henstock-Kurzweil integrable functions and Vitali covering arguments in [41, Theorems 7.4.2 and 5.1.4.] for instance).

Finally, we focus on (III). For any s a.e., as $0 < h$ goes to 0,

$$\frac{(P_{h+\varepsilon} - P_\varepsilon)}{h}(z(s)) \rightarrow LP_\varepsilon z(s)$$

in $\mathbb{L}^2(\mu)$ as $P_\varepsilon z(s) \in \mathcal{D}(L)$. And we can use dominated convergence theorem as, for $g_\varepsilon(\tau, s) \equiv P_\tau(P_\varepsilon z(s))$,

$$\left\| \frac{\partial}{\partial \tau} g_\varepsilon(\tau, s) \right\|_{\mathbb{L}^2(\mu)}^2 = \|P_\tau(-L)P_\varepsilon z(s)\|_{\mathbb{L}^2(\mu)}^2 \leq \frac{C}{\varepsilon^2} \|\tilde{z}(s)\|_{\mathbb{L}^2(\mu)}^2$$

still using (24).

At the end of the day, u is a solution a.e. of (22). Deducing that u is a weak solution is easy. If $\phi \in C^\infty([0, T], \mathbb{L}^2(\mu))$, by bilinearity, $u\phi$ is absolutely continuous in $\mathbb{L}^1(\mu)$ on $[a, T]$, $0 < a < T$, and so is the real valued function $t \mapsto \mu(u(t)\phi(t))$. The weak formulation follows when $a \rightarrow 0$ in the integration by parts formula

$$\int_a^t ds \mu(\partial_s u \phi(s)) = \mu(u(t)\phi(t)) - \mu(u(a)\phi(a)) - \int_a^t ds \mu(u(s) \partial_s \phi).$$

The proof is complete. \triangleright

5.2 A mollified problem

Remark 3 In sections 5.2 to 5.4 below, we use notation introduced in the statement of lemma 3.1. So $T > 0$ is fixed, $A(t) \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$ and $B(t) \in \mathbb{L}^2([0, T], \mathbb{L}^2(\mu))$.

Let us fix $\varepsilon > 0$ and let us consider the following mollified problem

$$\begin{cases} \partial_t u^{(\varepsilon)}(t) = Lu^{(\varepsilon)}(t) + P_\varepsilon(-A(t)u^{(\varepsilon)}(t) + B(t)), \\ u^{(\varepsilon)}(0) = f, f \in \mathbb{L}^2(\mu) \end{cases} \quad (\text{CS}_\varepsilon)$$

We will prove that, for any $\varepsilon > 0$ (and with some more work still at the limit $\varepsilon \rightarrow 0$), the problem (CS_ε) has a weak solution in $[0, T]$ that is $u^{(\varepsilon)} \in \mathbb{L}^2([0, T], \mathcal{D}) \cap C([0, T], \mathbb{L}^2(\mu))$ and, for any $\phi \in C^\infty([0, T], \mathcal{D})$, and any $0 \leq t \leq T$,

$$\begin{aligned} - \int_0^t \mu(u^{(\varepsilon)}(s) \partial_s \phi(s)) ds + \mu(u^{(\varepsilon)}(t)\phi(t) - u^{(\varepsilon)}(0)\phi(0)) &= - \int_0^t \mathcal{E}(u^{(\varepsilon)}(s), \phi(s)) ds \\ &+ \int_0^t \mu(\phi(s) P_\varepsilon[-A(s)u^{(\varepsilon)}(s) + B(s)]) ds. \quad (\text{weak-CS}_\varepsilon) \end{aligned}$$

To handle this problem, let us consider the following iteration scheme which, as we will prove later, converge to the unique weak solution $u^{(\varepsilon)}$ of our problem (CS_ε) . Initially,

$$\begin{cases} \partial_t u_0^{(\varepsilon)} &= Lu_0^{(\varepsilon)} \\ u_0^{(\varepsilon)}|_{t=0} &= f \end{cases}$$

and then define

$$u_{n+1}^{(\varepsilon)}(t) \equiv P_t f + \int_0^t P_{\varepsilon+t-s}(-A(s)u_n^{(\varepsilon)}(s) + B(s)) ds. \quad (25)$$

It follows from Lemmas 2.1 and 5.1 that, for any $f \in \mathbb{L}^2(\mu)$, $u_{n+1}^{(\varepsilon)} \in C([0, T], \mathbb{L}^2(\mu) \cap \mathbb{L}^2([0, T], \mathcal{D}))$, and that for any t a.e. in $[0, T]$, $u_{n+1}^{(\varepsilon)}(t)$ is differentiable in $\mathbb{L}^2(\mu)$ and

$$\begin{cases} \partial_t u_{n+1}^{(\varepsilon)} &= Lu_{n+1}^{(\varepsilon)}(t) + P_\varepsilon \left(-A(t)u_n^{(\varepsilon)}(t) + B(t) \right), \\ u_{n+1}^{(\varepsilon)}|_{t=0} &= f, \end{cases} \quad (26)$$

The convergence scheme we detail below is adapted from the one presented in [22] in another context.

Proposition 4 (Uniform bound) *Fix $\varepsilon > 0$ and $f \in \mathbb{L}^2(\mu)$. Let $u_n^{(\varepsilon)}$ be the recursive solution of the mollified problem introduced above.*

There exists $\beta \in (0, +\infty)$ and $0 < T_0 \leq T$ both independent of ε and of the initial condition f such that for any $n \in \mathbb{N}$,

$$\sup_{0 \leq t \leq T_0} \left(\mu((u_n^{(\varepsilon)})^2(t)) + \int_0^t \mathcal{E}(u_n^{(\varepsilon)})(s) ds \right) \leq \beta (\mu(f^2) + \|B(\cdot)\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2). \quad (27)$$

Proof

◁ We use the notation $\tilde{u}_n^{(\varepsilon)} = P_\varepsilon u_n^{(\varepsilon)}$. For any t a.e.,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu((u_{n+1}^{(\varepsilon)})^2) &= \mu(u_{n+1}^{(\varepsilon)} Lu_{n+1}^{(\varepsilon)}) - \mu(A(t) \tilde{u}_{n+1}^{(\varepsilon)} u_n^{(\varepsilon)}) + \mu(B(t) \tilde{u}_{n+1}^{(\varepsilon)}) \\ &\leq -\mathcal{E}(u_{n+1}^{(\varepsilon)}) + \frac{1}{2} \mu(|A(t)|((\tilde{u}_{n+1}^{(\varepsilon)})^2 + (u_n^{(\varepsilon)})^2) + (\mu(B^2(t)))^{\frac{1}{2}} (\mu(\tilde{u}_{n+1}^{(\varepsilon)})^2)^{\frac{1}{2}} \\ &\leq -\mathcal{E}(u_{n+1}^{(\varepsilon)}) + \frac{1}{2} \mu(|A(t)|((\tilde{u}_{n+1}^{(\varepsilon)})^2 + (u_n^{(\varepsilon)})^2) + \frac{1}{2} \left(\frac{1}{\gamma} \mu(\tilde{u}_{n+1}^{(\varepsilon)})^2 + \gamma \mu(B^2(t)) \right). \end{aligned}$$

Let

$$M_\gamma \equiv \|\mu(e^{\gamma|A(\cdot)|})\|_{\mathbb{L}^\infty([0, T])}. \quad (28)$$

Note that $1 \leq M_\gamma < \infty$ for any $\gamma > 0$ since $A \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$.

By a similar argument, the entropic and the logarithmic Sobolev inequalities give

$$\begin{aligned} \mu(|A(t)|(\tilde{u}_{n+1}^{(\varepsilon)})^2) &\leq \frac{1}{\gamma} \text{Ent}_\mu((\tilde{u}_{n+1}^{(\varepsilon)})^2) + \frac{\mu((\tilde{u}_{n+1}^{(\varepsilon)})^2)}{\gamma} \log \mu(e^{\gamma|A(t)|}) \\ &\leq \frac{CLS}{\gamma} \mathcal{E}(\tilde{u}_{n+1}^{(\varepsilon)}) + \frac{\mu((\tilde{u}_{n+1}^{(\varepsilon)})^2)}{\gamma} \log M_\gamma, \end{aligned}$$

and similarly for the other term. So that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu((u_{n+1}^{(\varepsilon)})^2) &\leq -\mathcal{E}(u_{n+1}^{(\varepsilon)}) + \frac{CLS}{2\gamma} [\mathcal{E}(\tilde{u}_{n+1}^{(\varepsilon)}) + \mathcal{E}(u_n^{(\varepsilon)})] \\ &\quad + \frac{1 + \log M_\gamma}{2\gamma} [\mu((\tilde{u}_{n+1}^{(\varepsilon)})^2) + \mu((u_n^{(\varepsilon)})^2)] + \frac{\gamma}{2} \mu(B^2(t)). \end{aligned}$$

Using $\mathcal{E}(\tilde{u}_{n+1}^{(\varepsilon)}) \leq \mathcal{E}(u_{n+1}^{(\varepsilon)})$ and $\mu((\tilde{u}_{n+1}^{(\varepsilon)})^2) \leq \mu((u_{n+1}^{(\varepsilon)})^2)$ and integrating with respect to t ,

$$\begin{aligned} \mu((u_{n+1}^{(\varepsilon)})^2(t)) &+ 2 \left(1 - \frac{CLS}{2\gamma}\right) \int_0^t \mathcal{E}(u_{n+1}^{(\varepsilon)})(s) ds \leq \mu(f^2) \\ &+ \frac{1 + \log M_\gamma}{\gamma} \int_0^t \mu((u_{n+1}^{(\varepsilon)})^2)(s) ds + \frac{1 + \log M_\gamma}{\gamma} \int_0^t \mu((u_n^{(\varepsilon)})^2)(s) ds \\ &+ \frac{CLS}{\gamma} \int_0^t \mathcal{E}(u_n^{(\varepsilon)})(s) ds + \gamma \|B(\cdot)\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2. \end{aligned}$$

Choosing $\gamma > \frac{C_{LS}}{2}$, $\kappa_\gamma \equiv 1 - \frac{C_{LS}}{2\gamma} > 0$ and setting

$$\theta_n(t) = \mu((u_n^{(\varepsilon)})^2)(t) + 2\kappa_\gamma \int_0^t \mathcal{E}(u_n^{(\varepsilon)})(s)ds,$$

the above inequality implies

$$\begin{aligned} \theta_{n+1}(t) &\leq \mu(f^2) + \gamma \|B(\cdot)\|_{\mathbb{L}^2([0,T],\mathbb{L}^2(\mu))}^2 + \frac{1 + \log M_\gamma}{\gamma} \int_0^t \theta_{n+1}(s)ds \\ &\quad + \frac{1 + \log M_\gamma}{\gamma} \int_0^t \theta_n(s)ds + \frac{C_{LS}}{2\gamma\kappa_\gamma} \theta_n(t). \end{aligned}$$

Hence, by Gronwall type arguments, one gets

$$\theta_{n+1}(t) \leq e^{\frac{1+\log M_\gamma}{\gamma}t} \left[\alpha + \frac{1 + \log M_\gamma}{\gamma} \int_0^t \theta_n(s)ds + \frac{C_{LS}}{2\gamma\kappa_\gamma} \theta_n(t) \right],$$

where

$$\alpha = \mu(f^2) + \gamma \|B(\cdot)\|_{\mathbb{L}^2([0,T],\mathbb{L}^2(\mu))}^2.$$

It gives, for any $0 < T_0 \leq T$,

$$\sup_{t \in [0, T_0]} \theta_{n+1}(t) \leq e^{\frac{1+\log M_\gamma}{\gamma}T_0} \alpha + e^{\frac{1+\log M_\gamma}{\gamma}T_0} \left[\frac{1 + \log M_\gamma}{\gamma} T_0 + \frac{C_{LS}}{2\gamma - C_{LS}} \right] \sup_{t \in [0, T_0]} \theta_n(t)$$

Let us denote $Z_n = \sup_{t \in [0, T_0]} \theta_n(t)$.

Now, provided we choose $\gamma > C_{LS}$, $\frac{C_{LS}}{2\gamma - C_{LS}} < 1$, so that, for $T_0 > 0$ small enough,

$$\eta_{T_0} = e^{\frac{1+\log M_\gamma}{\gamma}T_0} \left[\frac{1 + \log M_\gamma}{\gamma} T_0 + \frac{C_{LS}}{2\gamma - C_{LS}} \right] < 1.$$

we end up with

$$Z_{n+1} \leq e^{\frac{1+\log M_\gamma}{\gamma}T_0} \alpha + \eta_{T_0} Z_n,$$

Hence, by induction,

$$Z_n \leq \alpha e^{\frac{1+\log M_\gamma}{\gamma}T_0} (1 + \dots + \eta_{T_0}^{n-1}) + \eta_{T_0}^n Z_0.$$

Note that

$$\begin{aligned} Z_0 &= \sup_{t \in [0, T_0]} \left\{ \mu(P_t(f)^2) + 2\kappa_\gamma \int_0^t \mathcal{E}(P_s(f))ds \right\} \\ &\leq \sup_{t \in [0, T_0]} \left\{ \mu(P_t(f)^2) + 2 \int_0^t \mathcal{E}(P_s(f))ds \right\} \leq \mu(f^2) \leq \alpha, \end{aligned}$$

since the map $s \mapsto \mu(P_t(f)^2) + 2 \int_0^t \mathcal{E}(P_s(f))ds$ is decreasing. It follows that, for any $n \geq 0$,

$$Z_n \leq \alpha e^{\frac{1+\log M_\gamma}{\gamma}T_0} (1 + \dots + \eta_{T_0}^n) \leq \alpha e^{\frac{1+\log M_\gamma}{\gamma}T_0} \frac{1}{1 - \eta_{T_0}},$$

which is the expected bound. \triangleright

Proposition 5 (Existence for mollified problem; $\varepsilon > 0$) *For any $\varepsilon > 0$ and any initial datum $f \in \mathbb{L}^2(\mu)$, there exists a weak solution $u^{(\varepsilon)}$ on $[0, T]$ of the mollified problem $(\mathbb{CS}_\varepsilon)$ as defined in (weak- \mathbb{CS}_ε).*

Proof

◁ Let $w_{n+1}^{(\varepsilon)} = u_{n+1}^{(\varepsilon)} - u_n^{(\varepsilon)}$ and $\tilde{w}_n^{(\varepsilon)} = P_\varepsilon(w_n^{(\varepsilon)})$. For any $t \geq 0$ a.e.,

$$\frac{1}{2} \frac{d}{dt} \mu((w_{n+1}^{(\varepsilon)})^2) = -\mathcal{E}(w_{n+1}^{(\varepsilon)}) - \mu(\tilde{w}_{n+1}^{(\varepsilon)} A(t) w_n^{(\varepsilon)}).$$

Again thanks to the entropic and the logarithmic Sobolev inequalities,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu((w_{n+1}^{(\varepsilon)})^2) &\leq -\mathcal{E}(w_{n+1}^{(\varepsilon)}) + \frac{C_{LS}}{2\gamma} [\mathcal{E}(\tilde{w}_{n+1}^{(\varepsilon)}) + \mathcal{E}(w_n^{(\varepsilon)})] \\ &\quad + \frac{\log M_\gamma}{2\gamma} [\mu((\tilde{w}_{n+1}^{(\varepsilon)})^2) + \mu((w_n^{(\varepsilon)})^2)], \end{aligned}$$

where M_γ were defined in the proof of Proposition 4. By the same arguments as before,

$$\begin{aligned} \mu((w_{n+1}^{(\varepsilon)})^2(t)) + 2\kappa_\gamma \int_0^t \mathcal{E}(w_{n+1}^{(\varepsilon)})(s) ds &\leq \frac{\log M_\gamma}{\gamma} \int_0^t \mu((w_{n+1}^{(\varepsilon)})^2)(s) ds \\ &\quad + \frac{\log M_\gamma}{\gamma} \int_0^t \mu((w_n^{(\varepsilon)})^2)(s) ds + \frac{C_{LS}}{\gamma} \int_0^t \mathcal{E}(w_n^{(\varepsilon)})(s) ds, \end{aligned}$$

with again $\kappa_\gamma = 1 - \frac{C_{LS}}{2\gamma} > 0$ provided we choose $\gamma > \frac{C_{LS}}{2}$.

Fixing $0 < \tilde{T}_0 \leq T_0$, where T_0 has been defined in the previous proposition, and mimicking what we have done to prove that proposition, this leads to

$$\sup_{t \in [0, \tilde{T}_0]} \{\tilde{\theta}_{n+1}(t)\} \leq \tilde{\eta}_{\tilde{T}_0} \sup_{t \in [0, T]} \{\tilde{\theta}_n(t)\},$$

where $\tilde{\theta}_n(t) = \mu((w_{n+1}^{(\varepsilon)})^2) + 2\kappa_\gamma \int_0^t \mathcal{E}(w_{n+1}^{(\varepsilon)})(s) ds$ and where

$$\tilde{\eta}_{\tilde{T}_0} = (M_\gamma)^{\frac{\tilde{T}_0}{\gamma}} \left[\frac{\log M_\gamma}{\gamma} \tilde{T}_0 + \frac{C_{LS}}{2\gamma - C_{LS}} \right].$$

If we choose $\gamma > C_{LS}$, we may take $0 < \tilde{T}_0 \leq T_0$ small enough (and independent of the initial condition f) so that $\tilde{\eta}_{\tilde{T}_0} < 1$.

Iterating and using uniform bound (27) for $n = 1$ (and $n = 0$), one gets

$$\sup_{t \in [0, \tilde{T}_0]} \{\tilde{\theta}_{n+1}(t)\} \leq \tilde{\beta}(\mu(f^2) + \|B(\cdot)\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2) \tilde{\eta}_{\tilde{T}_0}^n.$$

It follows that $(u_n^{(\varepsilon)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2([0, \tilde{T}_0], \mathcal{D}) \cap C([0, \tilde{T}_0], \mathbb{L}^2(\mu))$.

It converges to some $u^{(\varepsilon)}$ which is a weak solution in $[0, \tilde{T}_0]$ of (CS_ε) (see page 12, but note that things are much simpler here). As \tilde{T}_0 does not depend on f , one easily extends the solution to the entire interval $[0, T]$. ▷

5.3 Uniqueness

We now state uniqueness of a weak solution for both cases : with or without a mollification.

Proposition 6 (Uniqueness) *For any $\varepsilon \geq 0$, a weak solution $u^{(\varepsilon)}$ on $[0, T]$ of the problem (CS_ε) with initial datum $f \in \mathbb{L}^2(\mu)$ is unique.*

We omit the proof which is quite similar to the one of proposition 4.1.

5.4 Existence for the cornerstone linear problem

Recall remark 3:

$T > 0$ is fixed and $A(t) \in \mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$ and $B(t) \in \mathbb{L}^2([0, T], \mathbb{L}^2(\mu))$.

Proposition 7 (Removing the smoothing) *Let $f \in \mathbb{L}^2(\mu)$. There exists $0 < T_0 \leq T$ (independent of f) such that the weak solution $u^{(\varepsilon)}$, $\varepsilon > 0$, of the mollified problem (CS_ε) , (with the same initial datum f) converges as ε goes to 0, to some limit function u in $\mathbb{L}^2([0, T_0], \mathcal{D}) \cap C([0, T_0], \mathbb{L}^2(\mu))$. Moreover, u may be extended to a weak solution of the cornerstone linear problem (CS) , with initial datum f , on $[0, T]$.*

Proof

◁ Let $\varepsilon_1 > \varepsilon_0 > 0$ and let $u_0 = u^{(\varepsilon_0)}$ and $u_1 = u^{(\varepsilon_1)}$ be the associated solutions of the mollified problem (weak- CS_ε). Using Steklov calculus as in the previous proof, we get the same estimate as if we were dealing with strong solutions. Here we avoid such technicalities to focus on the main arguments. Let us denote $w = u_1 - u_0$ and $\tilde{w} = P_{\varepsilon_1} w$. One has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu(w^2) &= -\mathcal{E}(w) + \mu(w[P_{\varepsilon_1}(-A(t)u_1 + B(t)) - P_{\varepsilon_0}(-A(t)u_0 + B(t))]) \\ &= -\mathcal{E}(w) + \underbrace{\mu(wP_{\varepsilon_1}(-A(t)w))}_{(I)} + \mu(w(P_{\varepsilon_1} - P_{\varepsilon_0})(-A(t)u_0 + B(t))). \end{aligned}$$

Term (I) is bounded by $\frac{C_{LS}}{\gamma} \mathcal{E}(w(t)) + \frac{\log(M_\gamma)}{\gamma} \mu(w^2(t))$ as in the previous proof. After integration, using symmetry of the semigroup, one gets

$$\begin{aligned} \mu(w^2(t)) + (1 - \frac{C_{LS}}{\gamma}) \int_0^t \mathcal{E}(w(s)) ds \\ \leq \frac{\log(M_\gamma)}{\gamma} \int_0^t \mu(w^2(s)) ds + \underbrace{\int_0^t \mu((P_{\varepsilon_1} - P_{\varepsilon_0})(w) (-A(s)u_0(s) + B(s))) ds}_{(II)}, \end{aligned}$$

(which is the estimate we would get rigorously after letting $h \rightarrow 0$ in the Steklov regularisation). After using Gronwall type arguments and taking the supremum over $t \in [0, T_0]$, $0 < T_0 \leq T$, we note that, if we prove term (II) goes to 0 as $\varepsilon_1 > \varepsilon_0 > 0$ both go to 0, then $(u^{(\varepsilon)})_{\varepsilon > 0}$ is Cauchy (as ε goes to 0) in the Banach space $\mathbb{L}^2([0, T_0], \mathcal{D}) \cap C([0, T_0], \mathbb{L}^2(\mu))$. Now, by Cauchy-Schwarz inequality,

$$(II) \leq \left(\int_0^t ds \mu[(P_{\varepsilon_1} - P_{\varepsilon_0})(w(s))]^2 \right)^{\frac{1}{2}} \left(\frac{1}{2} \int_0^t ds \mu(A^2(s)u_0^2(s) + B^2(s)) \right)^{\frac{1}{2}}. \quad (29)$$

Following lemma 2.1,

$$\begin{aligned} \int_0^t ds \mu(A^2(s)u_0^2(s) + B^2(s)) &\leq \|Au_0\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2 + \|B\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2 \\ &\leq \max(\log(2), C_{LS}) \|A\|_{\mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))}^2 \|u^{(\varepsilon_0)}\|_{\mathbb{L}^2([0, T], \mathcal{D})}^2 + \|B\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2. \end{aligned}$$

Choosing T_0 as in Proposition 4, one may pass to the limit $n \rightarrow \infty$ in the uniform bound (27) to get that, for any $\varepsilon > 0$,

$$\|u^{(\varepsilon)}\|_{\mathbb{L}^2([0, T], \mathcal{D})}^2 \leq \beta(T_0 + 1) (\mu(f^2) + \|B\|_{\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))}^2). \quad (30)$$

So the second factor of (29) is bounded uniformly in ε_0 . In order to prove convergence to 0 of the other factor $\int_0^t ds \mu [(P_{\varepsilon_1} - P_{\varepsilon_0})(w(s))]^2$ when $\varepsilon_1 > \varepsilon_0 > 0$ both go to 0, one makes use of spectral theory and the above uniform bound (30). Details are given in [22, Theorem 4.10].

Eventually, the limit u of $(u^{(\varepsilon)})_{\varepsilon > 0}$ (as ε goes to 0) in $\mathbb{L}^2([0, T_0], \mathcal{D}) \cap C([0, T_0], \mathbb{L}^2(\mu))$ is a weak solution, which can be extended to a weak solution on the entire interval $[0, T]$ as T_0 doesn't depend on the initial datum f . \triangleright

5.5 Non-negativity

We prove here that, provided A and B are nonnegative, the weak solution u of problem (CS), with a nonnegative initial datum f , is nonnegative.

Let us define $u_- = (-u)_+ = \max(-u, 0)$. Then, formally,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu((u_-(t))^2) &= -\mu(u_-(t) \partial_t u) = -\mu((u_-(t) Lu(t)) + \mu(u_-(t) A(t) u(t)) - \mu(u_-(t) B(t))) \\ &\leq -\mu\left(A(t) \underbrace{(-u)_+(t) (-u)(t)}_{=((-u)_+(t))^2}\right) + \mathcal{E}(u_-(t), u(t)) \leq 0 \end{aligned}$$

using positivity of $A(\cdot)$ and $B(\cdot)$, and $\mathcal{E}(u_-(t), u(t)) \leq -\mathcal{E}(u_-(t), u_-(t)) \leq 0$.

Rigorous arguments to get this are as follows. We consider the Steklov average $a_h(u)(t)$ and its negative part $a_h^-(u)(t) \equiv \max(0, -a_h(u)(t))$. Recall that, as h goes to 0, for any $t \in [0, T]$, $a_h^-(u)(t) \rightarrow u^-(t)$ in $\mathbb{L}^2(\mu)$ and $a_h(u) \rightarrow u$ in $\mathbb{L}^2([0, T], \mathcal{D})$. It follows that $a_h^-(u) \rightarrow u^-$ in $\mathbb{L}^2([0, T], \mathcal{D})$. Namely, from any sequence going to 0, extract a subsequence (h_n) such that, for any t a.e. in $[0, T]$, $a_{h_n}(u)(t) \rightarrow u(t)$ in \mathcal{D} . By continuity of contractions [3], it follows $a_{h_n}^-(u)(t) \rightarrow u^-(t)$, in \mathcal{D} , t a.e. and one may check easily that the sequence $(\|a_{h_n}^-(u)(t) - u^-(t)\|_{\mathcal{D}}^2)_n$ is uniformly integrable in $\mathbb{L}^1([0, T])$.

Moreover, in $W^{1,2}((0, T), \mathbb{L}^2(\mu))$,

$$\partial_s a_h^-(u)(s) = -\partial_s a_h(u)(s) \chi_{\{a_h(u)(s) \leq 0\}} = -\frac{1}{h} (u(s+h) - u(s)) \chi_{\{a_h(u)(s) \leq 0\}}$$

where χ denotes the indicator function. Hence, using the definition of a weak solution (with the constant test function $a_h^-(u)(s) \in \mathcal{D}$), we get

$$\begin{aligned} \frac{1}{2} \mu(a_h^-(u)(t))^2 &= \frac{1}{2} \mu(a_h^-(u)(0))^2 + \frac{1}{2} \int_0^t ds \partial_s \mu(a_h^-(u)(s))^2 \\ &= \frac{1}{2} \mu(a_h^-(u)(0))^2 - \int_0^t ds \mu\left(a_h^-(u)(s) \frac{1}{h} (u(s+h) - u(s))\right) \\ &= \frac{1}{2} \mu(a_h^-(u)(0))^2 + \int_0^t ds \frac{1}{h} \int_s^{s+h} d\tau \left[\mathcal{E}(a_h^-(u)(s), u(\tau)) + \mu\left(a_h^-(u)(s) (A(\tau)u(\tau) - B(\tau))\right) \right] \\ &= \frac{1}{2} \mu(a_h^-(u)(0))^2 + \int_0^t ds \left[\mathcal{E}(a_h^-(u)(s), a_h(u)(s)) + \mu\left(a_h^-(u)(s) a_h(A(\cdot)u(\cdot) - B(\cdot))(s)\right) \right] \end{aligned}$$

We can pass to the limit with $h \rightarrow 0$ which yields (as $\mu((f^-)^2) = 0$)

$$\frac{1}{2} \mu(u^-(t))^2 = \int_0^t ds \mathcal{E}(u^-(s), u(s)) + \mu\left(u^-(s) (A(s)u(s) - B(s))\right) \leq 0,$$

for the same reason as above.

The proof of Lemma 3.1 is complete.

6 Extension to the general case

The chemical reactions we consider here are of the following form

$$\sum_{i \in F} \alpha_i \mathcal{A}_i \rightleftharpoons \sum_{i \in F} \beta_i \mathcal{A}_i,$$

for some given integers $\alpha_i \neq \beta_i$, for any $i \in F$. $F = \{1, \dots, q\}$ is a finite set. The associated reaction-diffusion equation is (after appropriate change of variables)

$$\begin{cases} \partial_t u_i = L_i u_i + \lambda_i (\beta_i - \alpha_i) \left(\prod_{j=1}^q u_j^{\alpha_j} - \prod_{j=1}^q u_j^{\beta_j} \right) \\ u_i|_{t=0} = f_i, \quad i \in F \end{cases} \quad (31)$$

This equation is a particular form of the abstract equation (RDP) on page 4 with constant vector $\lambda_i (\beta_i - \alpha_i)$, $i = 1, \dots, q$ and nonlinearity $G(\vec{u}) = \prod_{j=1}^q u_j^{\alpha_j} - \prod_{j=1}^q u_j^{\beta_j}$. The method we detailed for the two-by-two case may be adapted to this general situation provided the following assumptions hold.

Linearity assumptions

We assume that

- i. F may be partitioned as $F = \sqcup_{k \in K} F_k$ so that, L_i only depends on which F_k , $k \in K$, the index i belongs to. We denote by \tilde{L}_k the common operator for any $i \in F_k$. For any $k \in K$, one has the following:
- ii. \tilde{L}_k is a Markov generator with (selfadjoint in the \mathbb{L}^2 space associated with the) invariant probability measure μ_k on $(\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ (with the same assumptions as in page 4).
- iii. (\tilde{L}_k, μ_k) satisfies logarithmic Sobolev inequality with constant C_k .
- iv. The measures $(\mu_k)_{k \in K}$ are mutually equivalent in the strong sense that there exists a measure μ on $(\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ and $C \in (1, +\infty)$ such that

$$\forall k \in K, \quad \frac{1}{C} \leq \frac{d\mu_k}{d\mu} \leq C.$$

Nonlinearity assumptions

Let $F_k^- = \{i \in F_k, \beta_i - \alpha_i < 0\}$ and $F_k^+ = \{i \in F_k, \beta_i - \alpha_i > 0\}$. We assume that, for any $k \in K$, F_k^- and F_k^+ are not empty.

(Note that this replaces, in the present context, the hypothesis we made in the two-by-two case that $C_1 = C_3$ and $C_2 = C_4$.)

Initial data assumptions

We assume the following common exponential integrability on the initial data.

Common integrability assumption. We assume that, for any $i = 1, \dots, q$, $f_i \in E^{\Phi_{2\theta}}(\mu)$, where $\theta \equiv \max(\sum_{i=1}^q \alpha_i, \sum_{i=1}^q \beta_i) - 1$.

Iterative sequence

We now define an approximation sequence $(\vec{u}^{(n)}(t))_{n \in \mathbb{N}}$ which converges to the solution of problem (31). It is obtained recursively as solutions of the following linear problems.

Let us fix a nonnegative initial datum \vec{f} satisfying the integrability assumptions introduced before.

For any $n \geq 0$, we will impose $\vec{u}^{(n)}(0) = \vec{f}$ and, for $n = 0$, $\partial_t u_i^{(0)} = L_i u_i^{(0)}$, $i = 1, \dots, q$.

Let $N_k = |F_k|$, $N_k^+ = |F_k^+|$ and $N_k^- = |F_k^-|$. Assume $N_k^- \geq N_k^+$ (the other case is similar by symmetry). Let us label elements of F_k^\pm in the following way

$$F_k^- = \{i_1^{k-}, \dots, i_{N_k^-}^{k-}\} \quad \text{and} \quad F_k^+ = \{i_1^{k+}, \dots, i_{N_k^+}^{k+}\}.$$

We consider an onto mapping $\nu_k : F_k^- \rightarrow F_k^+$ defined by

$$\nu_k(i_l^{k-}) = i_m^{k+} \quad \text{provided } l - m \in N_k^+ \mathbb{Z}.$$

Define furthermore, for any $i, j \in F$, $\alpha_j^{(i)} = \begin{cases} \alpha_j & \text{if } j \neq i \\ \alpha_j - 1 & \text{if } j = i \end{cases}$ and similarly for β 's. Let us note here that, for any $i \in F_k^+$ and $j \in F_k^-$, $\beta_i > 0$ and $\alpha_j > 0$. Finally, let $\delta_i = \lambda_i |\beta_i - \alpha_i| > 0$, for any $i \in F$.

The iterated sequence is then defined as follows[†]. In the case $i \in F_k^-$,

$$\partial_t u_i^{(n)} = \tilde{L}_k u_i^{(n)} - \delta_i \left(\prod_{j=1}^q (u_j^{(n-1)})^{\alpha_j^{(i)}} u_i^{(n)} - \prod_{j=1}^q (u_j^{(n-1)})^{\beta_j^{(\nu_k(i))}} u_{\nu_k(i)}^{(n)} \right). \quad (32)$$

And, in the case $i \in F_k^+$,

$$\partial_t u_i^{(n)} = \tilde{L}_k u_i^{(n)} + \frac{\delta_i}{Z_{k,i}} \sum_{r \in \nu_k^{-1}(i)} \delta_r \left(\prod_{j=1}^q (u_j^{(n-1)})^{\alpha_j^{(r)}} u_r^{(n)} - \prod_{j=1}^q (u_j^{(n-1)})^{\beta_j^{(i)}} u_i^{(n)} \right). \quad (33)$$

where $Z_{k,i} = \sum_{r \in \nu_k^{-1}(i)} \delta_r$.

Why the sequence is well defined.

Recall the sequence starts with the heat semigroups associated to the L_i 's

$$u_i^{(0)}(t) \equiv e^{tL_i} f_i, \quad i \in F.$$

It follows from appendix C.3 that under our assumptions on \vec{f} , $u_i^{(0)} \in \mathbb{L}^\infty([0, T], E^{\Phi_{2\theta}}(\mu))$. We hence assume we have proved, $\vec{u}^{(n-1)}$ is well defined, for some $n \geq 1$ and that

$$\vec{u}^{(n-1)} \in (\mathbb{L}^\infty([0, T], E^{\Phi_{2\theta}}(\mu)))^q, \quad (34)$$

for any $T \in (0, +\infty)$. Lemma 8 below ensures that, for any $i \in F$, the mapping $(u_1, \dots, u_q) \in (E^{\Phi_{2\theta}}(\mu))^q \mapsto \prod_{j=1}^q u_j^{\alpha_j^{(i)}} \in E^{\Phi_2}(\mu)$ is continuous, so that

$$\prod_{j=1}^q (u_j^{(n-1)})^{\alpha_j^{(i)}} \in \mathbb{L}^\infty([0, T], E^{\Phi_2}(\mu))$$

(and similarly for β 's).

[†]We recommend to translate at first reading the following general case in the simpler two-by-one case $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3$ with the same diffusion operator.

Lemma 8 Assume $p_1, \dots, p_q \geq r > 0$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_q} = \frac{1}{r}$. Let $\Phi(x) = \exp(|x|) - 1$ and recall $\Phi_\beta(x) = \Phi(|x|^\beta)$, for any $\beta \geq 1$. Then, for any $\alpha > 0$ such that $\alpha r \geq 1$, the q -linear mapping

$$(u_1, \dots, u_q) \in \mathbb{L}^{\Phi_{\alpha p_1}}(\mu) \times \dots \times \mathbb{L}^{\Phi_{\alpha p_q}}(\mu) \mapsto u_1 \dots u_q \in \mathbb{L}^{\Phi_{\alpha r}}(\mu)$$

is continuous.

Proof

◁ Assume $u_i \neq 0$, for all $i = 1, \dots, q$ and denote $\gamma_i \equiv \|u_i\|_{\mathbb{L}^{\Phi_{\alpha p_i}}}^{-1}$. Then one has

$$\forall i = 1, \dots, q, \quad \mu \left(\prod_{j=1}^q u_j^{\alpha_j} - \prod_{j=1}^q u_j^{\beta_j} \exp(|\gamma_i u_i|^{\alpha p_i}) \right) \leq 2.$$

The result will follow if we show that

$$\mu \left(\exp(|\gamma_1 \dots \gamma_q u_1 \dots u_q|^{\alpha r}) \right) \leq 2.$$

Recall Young inequality: for any $a_1, \dots, a_q \geq 0$,

$$\frac{1}{r} a_1^r \dots a_q^r \leq \frac{a_1^{p_1}}{p_1} + \dots + \frac{a_q^{p_q}}{p_q}.$$

Hence, using also Hölder inequality,

$$\begin{aligned} \mu \left(e^{(\gamma_1 |u_1|)^{\alpha r} \dots (\gamma_q |u_q|)^{\alpha r}} \right) &\leq \mu \left(e^{\frac{r}{p_1} (|\gamma_1 u_1|^{\alpha p_1})} \dots e^{\frac{r}{p_q} (|\gamma_q u_q|^{\alpha p_q})} \right) \\ &\leq \mu \left(e^{|\gamma_1 u_1|^{\alpha p_1}} \right)^{\frac{r}{p_1}} \dots \mu \left(e^{|\gamma_q u_q|^{\alpha p_q}} \right)^{\frac{r}{p_q}} \leq 2^{\frac{r}{p_1} + \dots + \frac{r}{p_q}} = 2. \end{aligned}$$

▷

To prove recursively that the sequence $(\vec{u}^{(n)})_n$ is well defined, we have to split the cornerstone existence lemma into the following two lemmas.

Lemma 6.1 (Matrix cornerstone existence lemma) Let (L, μ) be a Markov generator satisfying logarithmic Sobolev inequality with constant $C_{LS} \in (0, \infty)$. Let $T > 0$ and $A = A(t)$ be an $N \times N$ matrix with coefficients in $\mathbb{L}^\infty([0, T], \mathbb{L}^{\Phi_2}(\mu))$ and $\vec{B} \in (\mathbb{L}^2([0, T], \mathbb{L}^2(\mu)))^N$. Then the Cauchy problems

$$\begin{cases} \partial_t \vec{u}(t) = L\vec{u}(t) + A(t) \vec{u}(t) + \vec{B}(t), \\ \vec{u}(0) = \vec{f}, \vec{f} \in (\mathbb{L}^2(\mu))^N \end{cases} \quad (\text{MCS})$$

and

$$\begin{cases} \partial_t \vec{u}(t) = L\vec{u}(t) + A(t) \vec{u}_+(t) + \vec{B}(t), \\ \vec{u}(0) = \vec{f}, \vec{f} \in (\mathbb{L}^2(\mu))^N \end{cases}, \quad (\text{MCS}_+)$$

with $\vec{u}_+ = ((u_1)_+, \dots, (u_q)_+)$, both have a unique weak solution on $[0, \infty)$

Note that we use that $u \mapsto u_+$ is a contraction so that it contracts both the $\mathbb{L}^2(\mu)$ norm and the Dirichlet form \mathcal{E} .

In the system defined by (32) and (33) only blocks made of some $i \in F_k^+$ and j 's in $\nu_k^{-1}(i)$ interact. We now focus on these coordinates. The following lemma ensures that positivity and Bochner measurability (34) propagate along the approximation sequence.

Lemma 6.2 (Positivity and propagation of measurability.) *Let $N \geq 2$ and let $\delta_1, \dots, \delta_{N-1} \geq 0$ such that $Z \equiv \sum_{i=1}^{N-1} \delta_i > 0$. Assume furthermore $\vec{B}(t) = \vec{0}$ and $A(t)$ is of the following form*

$$A(t) = \begin{pmatrix} -a_1(t) & 0 & 0 & \dots & 0 & \delta_1 a_N(t) \\ 0 & -a_2(t) & 0 & \dots & 0 & \delta_2 a_N(t) \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & -a_{N-1}(t) & \delta_{N-1} a_N(t) \\ \frac{1}{Z} a_1(t) & \frac{1}{Z} a_2(t) & \frac{1}{Z} a_3(t) & \dots & \frac{1}{Z} a_{N-1}(t) & -a_N(t) \end{pmatrix} \quad (35)$$

where $a_i \in \mathbb{L}^\infty([0, T], E^{\Phi_2}(\mu))$, $i = 1, \dots, N$, are all nonnegative. Assume the initial datum $\vec{f} \in (\mathbb{L}^2(\mu))^N$ is nonnegative. Then the solution \vec{u} of (MCS) is nonnegative. Moreover, one has

$$\sum_{i=1}^{N-1} u_i(t) + Z u_N(t) = e^{tL} \left(\sum_{i=1}^{N-1} f_i + Z f_N \right)$$

and consequently, provided $\vec{f} \in E^{\Phi_{2\theta}}(\mu)$, then $\vec{u} \in \mathbb{L}^\infty([0, T], E^{\Phi_{2\theta}}(\mu))$.

It is easy to check that $v(t) \equiv \sum_{i=1}^{N-1} u_i(t) + Z u_N(t)$ satisfies $\partial_t v = Lv(t)$. We detail a bit positivity argument (the remaining is similar to the two-by-two case).

Let \vec{v} be the unique weak solution of problem (MCS₊) with initial condition \vec{f} . We now show \vec{v} is nonnegative and so it coincides to the unique solution of (MCS) with initial condition \vec{f} . Thanks to Steklov calculus, the following computation is made rigorous. We focus on the last component (which is the most complicated one). Let $v_N^- \equiv \max(-v_N, 0)$. One has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu((v_N^-)^2) &= -\mu(v_N^- \partial_t v_N) \\ &= -\mu(v_N^- Lv_N) + \mu(a_N v_N^+ v_N^-) - \mu \left(\sum_{i=1}^{N-1} \frac{a_i}{Z} v_i^+ v_N^- \right). \end{aligned}$$

First, $-\mu(v_N^- Lv_N) = -\mathcal{E}((-v_N)^+, -v_N) \leq 0$ as for any $u \in \mathcal{D}$, $0 \leq \mathcal{E}(u_+, u_+) \leq \mathcal{E}(u_+, u)$. Secondly, $\mu(a_N v_N^+ v_N^-) = 0$. And the third term is trivially nonpositive as the a_i 's are assumed nonnegative. Hence, $\mu((v_N^-)^2) \leq \mu((f_N^-)^2) = 0$.

We can state the following theorem.

Theorem 6.3 *Let L_i , $i = 1, \dots, q$, be Markov generators satisfying the linearity assumptions described before. Assume the nonlinearity assumptions are satisfied as well and that $\vec{f} \geq 0$ belongs to $E^{\Phi_{2\theta}}(\mu)$, with θ as in the initial data assumption.*

Then, for any reaction rates $\lambda_i > 0$, there exists a unique nonnegative weak solution \vec{u} of problem (31) on $[0, \infty)$.

Appendix

A The entropic inequality

Let μ be a probability measure. Let $f \geq 0$ be a measurable function s.t. $f \neq 0$ μ -a.e. Then the two following assertions are equivalent:

- i. $f \in \mathbb{L}^1(\mu)$ and $f \log \left(\frac{f}{\mu(f)} \right) \in \mathbb{L}^1(\mu)$,

ii. $f \log_+ f \in \mathbb{L}^1(\mu)$.

Let us extend $\mathbb{L}^1(\mu)$ to the space $\mathbb{L}_{\text{ext}}^{1,-}(\mu)$ of measurable functions f such that $\mu(f_+) < +\infty$ and define $\mu(f) \equiv \mu(f_+) - \mu(f_-) \in \mathbb{R} \cup \{-\infty\}$ if $f \in \mathbb{L}_{\text{ext}}^{1,-}(\mu)$. (Define also symmetrically $\mathbb{L}_{\text{ext}}^{1,+}(\mu)$). Note that $f \in \mathbb{L}_{\text{ext}}^{1,-}(\mu)$ and $g \in \mathbb{L}^1(\mu)$ implies $f+g \in \mathbb{L}_{\text{ext}}^{1,-}(\mu)$ and $\mu(f+g) = \mu(f) + \mu(g)$. Moreover, for any $f, g \in \mathbb{L}_{\text{ext}}^{1,-}(\mu)$, $f \leq g$ implies $\mu(f) \leq \mu(g)$.

Lemma A.1 (Entropic inequality) *Let μ be a probability measure and let f and g be two measurable functions. Assume $f \geq 0$ (excluding $f = 0$ μ -a.e.) such that $f \log_+ f \in \mathbb{L}^1(\mu)$ and $\mu(e^{\gamma g}) < +\infty$ for some $\gamma > 0$. Then $fg \in \mathbb{L}_{\text{ext}}^{1,-}(\mu)$ and*

$$\mu(fg) \leq \frac{1}{\gamma} \mu \left(f \log \frac{f}{\mu(f)} \right) + \frac{\mu(f)}{\gamma} \log \mu(e^{\gamma g}) \quad (36)$$

in $\mathbb{R} \cup \{-\infty\}$.

The proof is based on the following inequality $\forall x \in \mathbb{R}_+, \forall y \in \mathbb{R}, xy \leq x \log x - x + e^y$.

B Basics on Orlicz spaces

Classical properties of Orlicz spaces can be found in [38].

Young functions

Let Φ be a Young function, that is $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ convex, even such that $\Phi(0) = 0$ and Φ is not constant. Note that from this, it follows that $\Phi(x) \geq 0$, that $\Phi(x) \rightarrow +\infty$ when $x \rightarrow \infty$ and that Φ is an increasing function on $[0, +\infty)$.

Associated Orlicz spaces

The space $\mathbb{L}^\Phi(\mu) = \{u \in \mathbb{L}^0(\mu) : \exists \varepsilon > 0 \text{ s.t. } \mu(\Phi(\varepsilon u)) < \infty\}$ is a vector subspace of $\mathbb{L}^0(\mu)$.

Gauge norm

Let $B_\Phi = \{u \in \mathbb{L}^0(\mu) : \mu(\Phi(u)) \leq 1\}$. Then B_Φ is a symmetric ($B_\Phi = -B_\Phi$) convex set in $\mathbb{L}^\Phi(\mu)$ containing 0 and satisfying

$$\mathbb{L}^\Phi(\mu) = \cup_{\lambda > 0} \lambda B_\Phi. \quad (37)$$

From these properties, it follows that the gauge norm

$$\|u\|_\Phi \equiv \inf\{\lambda > 0 : u \in \lambda B_\Phi\}$$

associated to B_Φ is indeed a norm. One has

$$\|u\|_\Phi^{-1} = \sup\{\gamma > 0 : \mu(\Phi(\gamma u)) \leq 1\}. \quad (38)$$

The space $(\mathbb{L}^\Phi(\mu), \|\cdot\|_\Phi)$ is a Banach space.

Comparison of norms

We often have to compare Orlicz norms associated to different Young functions. We already have seen in a footnote that any Young function Φ satisfies $|x| \preceq \Phi(x)$. It leads to the following lemma.

Lemma B.1 *Any Orlicz space may be continuously embedded in $\mathbb{L}_1(\mu)$. More precisely, let M and τ in $(0, \infty)$ such that $|x| \leq \tau \Phi(x)$ for any $|x| \geq M$. Then, for any $f \in \mathbb{L}_\Phi$,*

$$\|f\|_1 \leq (M + \tau) \|f\|_\Phi. \quad (39)$$

Consequently, if Φ and Ψ are two Young functions satisfying, for some constants $A, B \geq 0$, $\Phi(x) \leq A|x| + B\Psi(x)$, then

$$\|f\|_\Phi \leq \max\left(1, A\|Id\|_{\mathbb{L}_\Psi \rightarrow \mathbb{L}_1} + B\right) \|f\|_\Psi. \quad (40)$$

Definition B.2 (Comparison of Young functions) *Let us denote $\Phi(x) \preceq \tilde{\Phi}(x)$ if there exist $x_0 \geq 0$ and $C \in (0, +\infty)$ such that $\forall x \geq x_0$, $\Phi(x) \leq C\tilde{\Phi}(x)$. Furthermore, $\Phi(x) \simeq \tilde{\Phi}(x)$ will mean $\Phi(x) \preceq \tilde{\Phi}(x)$ and $\tilde{\Phi}(x) \preceq \Phi(x)$.*

Remark B.3 *Let Φ and $\tilde{\Phi}$ be two Young functions. The existence of a constant A such that*

$$\forall x \geq 0, \Phi(x) \leq A\left(|x| + \tilde{\Phi}(x)\right)$$

is equivalent to the comparison

$$\Phi(x) \preceq \tilde{\Phi}(x).$$

The previous lemma then claims briefly that comparison of Young functions induces comparison of norms.

Indeed, first assume $\forall x \geq 0$, $\Phi(x) \leq A\left(|x| + \tilde{\Phi}(x)\right)$. As $|x| = O(\tilde{\Phi}(x))$ as x goes to $+\infty$, there exist x_0 and B s.t. $\forall x \geq x_0$, $|x| \leq B\tilde{\Phi}(x)$. So that $\forall x \geq x_0$, $\Phi(x) \leq A(B+1)\tilde{\Phi}(x)$.

Conversely, $\Psi(x) \equiv |x| + \tilde{\Phi}(x)$ is a Young function, so that $\frac{\Psi(x)}{x}$ is non decreasing on $(0, \infty)$ and $\forall x > 0$, $\frac{\Psi(x)}{x} \geq \Psi'(0_+) \geq 1$. Hence, for any $0 < x \leq x_0$,

$$\frac{\Phi(x)}{\Psi(x)} = \frac{\Phi(x)}{x} \underbrace{\frac{x}{\Psi(x)}}_{\leq 1} \leq \frac{\Phi(x)}{x} \leq \frac{\Phi(x_0)}{x_0}.$$

The result follows with $A = \max\left(C, \frac{\Phi(x_0)}{x_0}\right)$.

We will also need to deduce bounds on conjugate functions (as defined in (42)) from bounds on Young functions.

Lemma B.4 ([38], Proposition II.2) *Let Φ and Ψ be Young functions and Φ^* and Ψ^* their conjugate functions. Assume there exists $x_0 \geq 0$ such that*

$$\forall x \geq x_0, \quad \Phi(x) \leq \Psi(x).$$

Then, there exists $y_0 \geq 0$ such that

$$\forall y \geq y_0, \quad \Psi^*(y) \leq \Phi^*(y).$$

Exponential type Young functions and their conjugates

Let us recall we considered Young functions of exponential type

$$\Phi_\alpha(x) = \exp(|x|^\alpha) - 1, \quad \alpha \geq 1.$$

A direct computation shows that, for $y \geq 0$,

$$\Phi_1^*(y) = \begin{cases} 0 & \text{if } y \leq 1 \\ y \log y - y + 1 & \text{if } y \geq 1 \end{cases}$$

As a consequence, $\Phi_1^*(y) \simeq h(y) \equiv y \log_+ y$ and Φ_1^* is Δ_2 . Here $\log_+ y = \max(\log y, 0)$. Using lemmas B.4 and B.1, it follows that, provided $1 \leq \alpha \leq \beta < \infty$

$$\Phi_\beta^* \preceq \Phi_\alpha^* \preceq h \preceq x^2 \text{ so that } \|\cdot\|_{\Phi_\beta^*} \preceq \|\cdot\|_{\Phi_\alpha^*} \preceq \|\cdot\|_h \preceq \|\cdot\|_2 \quad (41)$$

More on $E^{\Phi_\alpha}(\mu)$

One may change parameters in Young inequality to get: for any $\alpha > 1$ and any $\delta, r > 0$, one has $\forall s \geq 0$, $\exp(\delta s) \leq \exp(\frac{\alpha-1}{\alpha}(r\alpha/\delta^\alpha)^{\frac{1}{1-\alpha}}) \exp(rs^\alpha)$. It follows that, for any $\alpha \geq 1$,

$$\cup_{\beta > \alpha} \mathbb{L}^{\Phi_\beta}(\mu) \subset E^{\Phi_\alpha}(\mu).$$

Lemma B.5 (Separability) *Assume \mathbb{M} is a separable metric space. Then, for any Young function Φ , $E^\Phi(\mu)$ is separable.*

(Use that $\mathcal{B}_\mathbb{M}$ is countably generated, monotone class theorem and density of simple functions).

Duality

What follows may be found in [13].

A Young function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy the Δ_2 condition if there exists $K \in (0, \infty)$ and $x_0 \geq 0$ such that, for any $x \geq x_0$, $\Psi(2x) \leq K\Psi(x)$.

In the case of Young functions with rapid growth (as the Φ_α 's introduced before), Δ_2 condition fails. Consequently $E^\Phi(\mu)$ is a proper Banach subspace of $\mathbb{L}^\Phi(\mu)$ (assuming the support of μ is infinite) and $\mathbb{L}^\Phi(\mu)$ is not separable.

Recall that the conjugate function Ψ^* of a Young function Ψ is the Young function defined by

$$\Psi^*(y) \equiv \sup_{x \geq 0} (x|y| - \Psi(x)). \quad (42)$$

The dual space of $E^\Psi(\mu)$ is $E^\Psi(\mu)' = \mathbb{L}^{\Psi^*}(\mu)$.

But when Δ_2 condition fails, the dual space of $\mathbb{L}^\Psi(\mu)$ is more complicated: this is a direct sum of $\mathbb{L}^{\Psi^*}(\mu)$ with some nontrivial subspace made of *singular* linear forms. As a consequence, neither $\mathbb{L}^{\Phi_\alpha}(\mu)$, $E^{\Phi_\alpha}(\mu)$ nor $\mathbb{L}^{\Phi_\alpha^*}(\mu)$ is reflexive.

C Markov Semigroups and Orlicz spaces

C.1 Contraction property

Lemma C.1 *Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a nonnegative convex function. Let $(P_t)_{t \geq 0}$ be a Markov semigroup on $\mathbb{L}^2(\mu)$, for a probability measure μ , as introduced in section 2. Then, for any $f \in \mathbb{L}^1(\mu)$ and any $t \geq 0$,*

$$\mu(\Phi(P_t f)) \leq \mu\Phi(f). \quad (43)$$

In particular, in the case when Φ is a Young function (with domain \mathbb{R}), provided $f \in \mathbb{L}^\Phi(\mu)$, then $P_t f \in \mathbb{L}^\Phi(\mu)$ and $(P_t)_{t \geq 0}$ is a contraction semigroup on $\mathbb{L}^\Phi(\mu)$.

Proof

◁ Let $f \in \mathbb{L}^1(\mu)$, $t \geq 0$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be convex. Nonnegativity of Φ allows to use Jensen inequality for the Markov probability kernels $p_t(x, dy)$. Indeed, for μ almost every $x \in \mathbb{M}$ (such that the representation (5) holds) and any $y \in \mathbb{M}$, by convexity,

$$\Phi(f(y)) \geq \Phi(P_t f(x)) + \Phi'((P_t f(x))_+) (f(y) - P_t f(x)).$$

Integrating w.r.t. $p_t(x, dy)$ leads to

$$P_t(\Phi(f))(x) \geq \Phi(P_t f(x)) \geq 0.$$

Then (43) follows by integration w.r.t. μ and invariance property of P_t .

Let now Φ be a Young function. Assume $f \neq 0$ in $\mathbb{L}^\Phi(\mu) (\subset \mathbb{L}^1(\mu))$. Recall (38) and choose $0 < \gamma \leq \|f\|_{\mathbb{L}^\Phi}^{-1}$. Applying (43) to $\Phi(\gamma \cdot)$ instead of Φ shows that

$$\mu(\Phi(\gamma P_t f)) \leq 1$$

so that $\gamma \leq \|P_t f\|_{\mathbb{L}^\Phi}^{-1}$. And the announced contraction property follows. ▷

C.2 Density of the Dirichlet domain

Using comparison (41), one gets continuous embedding

$$\mathcal{D} \hookrightarrow \mathbb{L}^2(\mu) \hookrightarrow \mathbb{L}^{\Phi_\alpha^*}(\mu), \tag{44}$$

for any $\alpha \geq 1$. As Φ_α^* is Δ_2 , the space of simple functions, and so $\mathbb{L}^2(\mu)$ as well, is dense in $\mathbb{L}^{\Phi_\alpha^*}(\mu)$. Now, \mathcal{D} is dense in $\mathbb{L}^2(\mu)$, and so in $\mathbb{L}^{\Phi_\alpha^*}(\mu)$.

C.3 Bochner measurability

Let X be a Banach space. Recall that an X -valued function $u : I \rightarrow X$ defined on a compact interval I is Bochner measurable provided it is an *a.e.* limit of a sequence of X -valued simple functions on I (see [41] for instance).

The $\mathbb{L}^\infty([0, T], \mathbb{L}^\Phi(\mu))$ space

Lemma C.2 *Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a Young function, $(\mathbb{M}, \mathcal{B}_\mathbb{M}, \mu)$ a probability space and $u \in C([0, T], \mathbb{L}^2(\mu))$. We assume that $x^2 \preceq \Phi(x)$. Then $u \in \mathbb{L}^\infty([0, T], \mathbb{L}^\Phi(\mu))$ iff $u : t \rightarrow u(t, \cdot) \in \mathbb{L}^\Phi(\mu)$ is Bochner measurable and there exist $\gamma, M \in (0, \infty)$ s.t., for any t a.e. in $[0, T]$, $\mu(\Phi(\gamma u(t))) \leq M$. In which case, one has, for any t a.e., $\|u(t)\|_{\mathbb{L}^\Phi(\mu)} \leq \frac{\max(M, 1)}{\gamma}$.*

This is just rewriting the definitions. In particular, provided $M \geq 1$, and $\mu(\Phi(\gamma u(t))) \leq M$, then by convexity, $\mu(\Phi(\frac{\gamma}{M} u(t))) \leq \frac{1}{M} \mu(\Phi(\gamma u(t))) \leq 1$ so that $\|u(t)\|_{\mathbb{L}^\Phi(\mu)} \leq \frac{M}{\gamma}$.

Proof of proposition 1

By density of $\mathbb{L}^2(\mu)$ in $\mathbb{L}^{\Phi_\alpha^*}$ and contraction of P_t in $\mathbb{L}^{\Phi_\alpha^*}$, C_0 property of P_t in $\mathbb{L}^{\Phi_\alpha^*}$ follows from C_0 property in $\mathbb{L}^2(\mu)$. Indeed, let $f \in \mathbb{L}^{\Phi_\alpha^*}$. $\varepsilon > 0$ being fixed, let $g \in \mathbb{L}^2(\mu)$ such that $\|f - g\|_{\Phi_\alpha^*} < \frac{\varepsilon}{3}$. Then

$$\|P_t f - f\|_{\Phi_\alpha^*} \leq 2\|f - g\|_{\Phi_\alpha^*} + \|P_t g - g\|_{\Phi_\alpha^*} \leq \frac{2\varepsilon}{3} + C\|P_t g - g\|_2$$

allows to conclude. As a consequence, provided $f \in E^{\Phi_\alpha}$, $t \mapsto P_t f \in E^{\Phi_\alpha}$ is weakly continuous, and so Bochner measurable as E^{Φ_α} is separable, following Pettis measurability theorem (see page 9 for references).

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