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Low degree hypersurfaces of projective toric varieties defined over C_1 fields have a rational point

(Long version)

ROBIN GUILBOT

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Introduction

Quasi algebraically closed fields, or C_1 fields, are defined in terms of a low degree condition :

Definition 0.1. *The field K is a C_1 field if every degree d hypersurface of the projective space \mathbb{P}_K^n contains a K -rational point as soon as $d \leq n$.*

J.-L. Colliot-Thélène gives in [CT09] a list of classical results of existence of rational points in various classes of algebraic varieties over given fields. The list ends on the following general question

Question. *For a given type of field, does there exist a natural class of algebraic varieties having automatically a rational point over such a field ?*¹

¹ Sur un type donné de corps, y a-t-il une classe naturelle de variétés algébriques qui sur un tel corps ont automatiquement un point rationnel ?

For C_1 fields, Fano varieties seem to be good candidates for such a natural class, according to conjectures by S. Lang and by Y.I. Manin and J. Kollár ([Lan00] and [Man93, Rem. 2.6]). But during the years 2000, strong results suggested to enlarge a bit the scope of these expectations and give a new formulation of the so-called “ C_1 conjecture” :

Conjecture (J. Kollár, S. Lang, Y.I. Manin). *Every proper smooth separably rationally connected variety over a C_1 field has a rational point.*

A variety X over a field K is rationally connected if every pair of points of X can be joined by a rational curve over any algebraically closed extension of K (see section 7 for more definitions). The notion of rational connectedness was introduced independently by F. Campana and by J. Kollár, Y. Miyaoka and S. Mori in 1992. It arises naturally from Mori’s program of classification of projective algebraic varieties as a good higher-dimensional analogue of rational curves and rational surfaces.

The C_1 conjecture has already been proved for almost every known type of C_1 field :

- For a finite field \mathbb{F}_q , by H. Esnault in [Esn03]. Her result is in fact much stronger since it needs only rational chain-connectedness (see Subsection 7.2 for the definition).
- For a field of functions $\bar{k}(C)$ of a curve C over an algebraically closed field \bar{k} , by T. Graber, J. Harris, A.J. de Jong and J. Starr in [GHS03] and [dJS03]. Their result is more geometric since it is stated in terms of sections of a fibration on the curve C .
- For a field of fractions of formal series with coefficients in an algebraically closed field $\bar{k}((t))$. This is a consequence of the previous case, see [CT09, Th. 7.5].

As there are not many other known C_1 fields (besides the pseudo-algebraically closed ones for which the conjecture is trivial see [Wit10, 1.1.3]), the C_1 conjecture remains open essentially for the maximal unramified extension of the p -adic numbers \mathbb{Q}_p^{nr} .

The aim of this article is to present a new approach to the C_1 conjecture, somewhat orthogonal to the one of the articles cited above : instead of considering *every* separably rationally connected variety over a *given* C_1 field, as is done in those papers, we prove the conjecture *only* for hypersurfaces of a simplicial projective split toric variety endowed with homogeneous coordinates (see 1.1 and 1.2 for precisions) but over an arbitrary C_1 field admitting normic forms of any degree, hence over *every known* C_1 field.

We proceed in two steps. First we define geometrically the notion of low toric degree for a hypersurface D of a simplicial projective split toric variety X_Σ (Theorem 5.1 and Definition 5.2) and show that it implies the existence of a rational point in the hypersurface D when X_Σ and D are defined over a C_1 field whose characteristic is prime to the order of the torsion of the class group $\text{Cl}(X_\Sigma)$ (Corollary 5.4). This takes the major part of the paper. Then we show that any smooth and separably rationally connected hypersurface of X_Σ has low toric degree, yielding the following result :

Theorem (7.1). *Let X_Σ be a simplicial projective toric variety defined over a C_1 field K such that K admits normic forms of arbitrary degree, the dense torus of X_Σ is split over K and the order of the torsion part of $\text{Cl}(X_\Sigma)$ is prime to the characteristic of K . Let D be a hypersurface of X_Σ defined over K . If D is smooth and separably rationally connected then it has low toric degree and in particular admits a rational point over K .*

Along the article we define in fact three different notions of low toric degree. Roughly speaking the first is local, the second is global and the third is a combination of the two others, involving several birational modifications of the ambient variety X_Σ and the toric morphism defined by the linear system of D .

Let us give some details.

Restricted low toric degree

The first step in defining a low degree condition similar to the one of Definition 0.1 for more general ambient varieties was made by J. Kollár in [Kol96]. It draws a notion of low degree for an intersection of hypersurfaces in a weighted projective space by generalizing [Lan52], Th. 4 :

Theorem 0.2 (Th. 6.7 p.232 [Kol96]). *Let K be a C_1 field admitting normic forms of arbitrary degree. Let $K[x_0, \dots, x_n]$ be the polynomial algebra graded by $\deg(x_i) = a_i \in \mathbb{N}^*$. Let $f_1, \dots, f_r \in K[x_0, \dots, x_n]$ be homogeneous polynomials for this graduation. If*

$$\sum_{j=1}^r \deg(f_j) < \sum_{i=0}^n a_i,$$

then the system of equations $f_1 = \dots = f_r = 0$ has a non trivial solution in K^{n+1} .

This result applies also to hypersurfaces in a fake weighted projective space, that is, the quotient of a well formed one by a finite abelian group, provided this ambient space admits homogeneous coordinates (i.e. provided the order of the finite group is prime to the characteristic of K in order for the quotient to be a good geometric one, see 1.2). It thus defines a notion of low weighted degree for (intersections of) hypersurfaces of any complete simplicial toric variety of Picard number 1 with homogeneous coordinates.

We say that an effective divisor D of a (normal separated) toric variety X_Σ has restricted low toric degree if there exists a complete simplicial toric subvariety $W \subseteq X_\Sigma$ of Picard number 1 such that the restriction $D|_W$ has low weighted degree in the sense of Theorem 0.2.

The notion of restricted low toric degree is a local notion since it involves nothing but a specific toric subvariety of the ambient X_Σ . In particular it does not require X_Σ either to be simplicial or to be projective.

One crucial observation is that complete simplicial toric subvarieties $W \subseteq X_\Sigma$ of Picard number 1 appear naturally as fibers of extremal contractions, i.e., contractions of an extremal ray of the Mori cone $\overline{\text{NE}}(X_\Sigma)$ (see 1.5.1). In fact every extremal ray of $\overline{\text{NE}}(X_\Sigma)$ is generated by the class of a line belonging to a family that covers such a fiber W . In particular (see Lemma 2.7) the divisor D has restricted low toric degree as soon as there is an extremal class $[C]$ such that

$$0 < C \cdot D < C \cdot L_W, \tag{1}$$

where L_W is any divisor whose restriction to W is the anticanonical divisor $-K_W$.

An important property of the restricted low toric degree condition for D is that it holds as soon as the same condition holds for one of its pullback ϕ^*D by a toric morphism ϕ (Lemma 2.8).

Global low toric degree

Let X_Σ be a complete simplicial toric variety with homogeneous coordinates (i.e. such that the order of the torsion part of the class group is prime to the characteristic of K , see 1.2). By Cox's construction X_Σ is isomorphic to the good geometric quotient

$$\left(\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \right) // G$$

where $\Sigma(1)$ is the set of 1-dimensional cones of the fan Σ , $G = \text{Hom}(\text{Cl}(X_\Sigma), \mathbb{G}_m)$ is the group whose character group is the class group $\text{Cl}(X_\Sigma)$ and $Z(\Sigma)$ is the exceptional set.

In particular every hypersurface D of X_Σ admits a homogeneous equation : there exists a polynomial f in the $[D]$ -homogeneous piece of the Cox ring $K[x_\rho, \rho \in \Sigma(1)]$ (which is graded by $\text{Cl}(X_\Sigma)$) such that

$$D = \mathbf{V}(f) := \{ \pi(x) \in X_\Sigma \mid f(x) = 0 \},$$

where $\pi : \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \rightarrow X_\Sigma$ is the quotient by G .

With this setting, we can find K -rational points in D as roots $\alpha \in K^{\Sigma(1)}$ of f over K that are not contained in the exceptional set $Z(\Sigma)$, so that $\pi(\alpha) \in D$.

On the one hand, since f is homogeneous for the graduation by the class group $\text{Cl}(X_\Sigma)$, it is also homogeneous for any graduation by \mathbb{Z} induced by the intersection numbers with a given 1-cycle C , i.e. defined by putting $\deg(x_\rho) = mC \cdot D_\rho$ for a well chosen $m \in \mathbb{N}^*$.

By Theorem 0.2, we can hence deduce the existence of a root $\alpha \in K^{\Sigma(1)}$ of f from some low degree condition along a 1-cycle C of X_Σ , that is, from inequalities of the form

$$0 < C \cdot D < \sum_{\rho \in I} C \cdot D_\rho, \quad (2)$$

where $C \cdot D_\rho > 0$ for all $\rho \in I$ (Lemma 3.1).

On the other hand, to get a rational point we have to ensure that such a root is relevant, i.e., does not belong to $Z(\Sigma)$. This is a priori a quite restrictive condition since it requires all the zero coordinates of α to be contained in a single cone of Σ :

$$\exists \sigma \in \Sigma, \{ \rho \in \Sigma(1) \mid \alpha_\rho = 0 \} \subseteq \sigma(1).$$

We then distinguish three cases :

- 1st case The effective divisor D is not Cartier and nef. Then it contains automatically a torus-invariant point, that is a \mathbb{Z} -rational point (Proposition 2.2). We may thus always assume that D is a nef Cartier divisor.
- 2nd case The divisor D is ample. If there is an inequality of the form 2 with C effective, then writing the class $[C]$ as a linear combination of extremal classes with positive coefficient, we can deduce an inequality of the form 1 (Proposition 3.18), which implies that D has restricted low toric degree.
- 3rd case The divisor D is Cartier and nef but not ample. In this case it is quite natural to consider the proper toric morphism $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ associated to the complete linear system of D . It contracts all the curves that have zero intersection with D and presents the class $[D]$ as the pullback of an ample class $[\overline{D}]$ in $X_{\overline{\Sigma}_{[D]}}$. The fan $\overline{\Sigma}_{[D]}$ is obtained from Σ first by removing all the walls $\tau \in \Sigma(n-1)$ such that $C_\tau \cdot D = 0$. This yields a fan $\Sigma_{[D]}$ that is degenerate if D is not big. In this case we quotient by the linear space U_0 contained in every cone of $\Sigma_{[D]}$ to get a genuine fan $\overline{\Sigma}_{[D]}$ of lower dimension.

The key property is that the fan $\Sigma_{[D]}$ is the normal fan of a polytope that can be identified with the Newton polytope of the homogeneous polynomial f defining D (see 1.3.3). The consequence (Proposition 3.6) is that a root $\alpha \in K^{\Sigma(1)}$ of f yields a rational point of D as soon as it does not belong to the exceptional set associated to $\Sigma_{[D]}$, that is as soon as

$$\exists \sigma' \in \Sigma_{[D]}, \{ \rho \in \Sigma(1) \mid \alpha_\rho = 0 \} \subseteq \sigma'. \quad (3)$$

We say that D has global low toric degree if there exist a 1-cycle C on X_Σ and a subset $I \subseteq \Sigma(1)$ such that inequalities 2 are verified and any root of f given by Lemma 3.1 satisfies condition 3.

An important consequence of the appearance of $\Sigma_{[D]}$ in the definition is that the global low toric degree condition can be pulled back by toric birational contractions, i.e. equivariant birational maps surjective in codimension 1 (Proposition 3.15).

Low toric degree

The general notion of low toric degree comes as a combination of the two other notions. It is defined by three conditions whose equivalence constitutes the core result of the paper.

Theorem (5.1). *Let X_Σ be a simplicial projective toric variety.*

Let D be a nef Cartier divisor on X_Σ . Let $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ be the toric morphism of Theorem 1.2, contracting all the rational curves having zero intersection with D and let \overline{D} be an ample divisor on $X_{\overline{\Sigma}_{[D]}}$ such that D is linearly equivalent to the pullback $\phi^\overline{D}$.*

The following conditions are equivalent :

- (i) The ample divisor \overline{D} has restricted low toric degree.*
- (ii) There exist a small modification $\nu : X_\Sigma \dashrightarrow X_{\Sigma'}$ over $X_{\overline{\Sigma}_{[D]}}$ and a subvariety $V' \subseteq X_{\Sigma'}$ such that the restriction $(\nu_*D)|_{V'}$ has global low toric degree.*
- (iii) There exists a sequence of weighted blow-ups $\eta : X_{\tilde{\Sigma}} \rightarrow X_\Sigma$ such that $X_{\tilde{\Sigma}}$ is smooth and the divisor $\tilde{D} = \eta^*D$ has global low toric degree.*

The proof relies entirely on tools from the toric Minimal Model Program. A sketch of the proof is given short after the statement in Section 5.

Here is the outline of the paper :

We begin by reviewing in Section 1 known results of toric geometry, with some explanations on how they apply over an arbitrary field.

In Section 2 we show that a divisor D has automatically rational points over any field if it is not Cartier and nef. Then we define the notion of restricted low toric degree and show that it implies the existence of rational points over C_1 fields. We also point out its relation to low intersection numbers with extremal curves.

We define the notion of global low toric degree in Section 3, giving some of its important properties, notably the fact that it implies the existence of rational points over C_1 fields and its behavior under toric birational contractions.

In Section 4 we prove that any toric rational contraction can be decomposed in a sequence of extremal flips followed by a sequence of extremal divisorial contraction and a sequence of extremal fibering contractions.

Section 5 contains the statement and the proof of the core result of the article yielding the general definition of low toric degree, together with the proof that it ensures the existence of rational points over C_1 fields.

In Section 6 we state some general and easy to handle conditions that are equivalent to the low toric degree condition. Then we discuss briefly the interpretation of low degree conditions in terms of proximity to the boarder of classical cones of divisor classes such as the nef cone or the effective cone.

Finally in Section 7 we prove that hypersurfaces that are smooth and separably rationally connected have low toric degree, proving that the C_1 conjecture is true for hypersurfaces of simplicial projective split toric varieties.

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1 Preliminaries

1.1 General setting : Toric varieties over a C_1 field

In this article, we work over an arbitrary C_1 field K . The only additional hypothesis we make on K is satisfied by every known C_1 field :

$$K \text{ admits normic forms of arbitrary degree.} \quad (\text{H1})$$

A normic form of degree d is a homogeneous polynomial in d variables without non trivial root. The necessity of this hypothesis is clear from the proof of Theorem 0.2 (reproduced in 2.2). Basic facts on C_1 fields, and more generally C_i fields, can be found in [Lan52] and [Wit10, Ch.1].

Many aspects of toric geometry can be studied regardless of the choice a particular ground field. This is what is done for instance by V. Danilov in the classical [Dan78]. However most references among which the (also very classical) [Ful93] and [CLS11] work exclusively over the field of complex numbers \mathbb{C} . In fact, a normal separated toric variety X defined over a field K can be handled almost in the same way as in the complex case as soon as the following hypothesis is satisfied :

$$\text{The dense torus } T \subset X \text{ acting on } X \text{ is split over } K. \quad (\text{H2})$$

This means that we have the isomorphism $T \simeq_K \mathbb{G}_m^n$ over K , where n is the dimension of X . Indeed in this case the usual construction of X as a gluing of toric affine varieties work as well as if K were algebraically closed (see [CLS11, Ch. 3]). More specifically this hypothesis is a necessary and sufficient condition for the character group M of T and its group of 1-parameter subgroups N to be isomorphic to \mathbb{Z}^n . Such a toric variety is said to be split over K . For a discussion of what happens when the torus is not split, see [Hur11] and [ELST14].

Throughout the article, by a toric variety we mean a normal separated split toric variety X_Σ defined by a fan Σ contained in the real span $N_{\mathbb{R}}$ of the lattice N . We fix an isomorphism $T \simeq_K \mathbb{G}_m^n$, inducing another $T_{\overline{K}} \simeq (\overline{K}^*)^n$ over the algebraic closure of K . This in turn induces an isomorphism $M \simeq \mathbb{Z}^n$ given by $m \in \mathbb{Z}^n \leftrightarrow (\chi^m : (t_1, \dots, t_n) \mapsto t_1^{m_1} \dots t_n^{m_n})$. By abuse we identify M to \mathbb{Z}^n and write $m = (m_1, \dots, m_n) \in M$.

The fan Σ is contained in the space $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$. For $k = 1, \dots, n$, we denote by $\Sigma(k)$ the set of k -dimensional cones of Σ . To each cone $\gamma \in \Sigma(k)$ corresponds a toric subvariety $\mathbf{V}(\gamma)$ that is a subvariety of dimension $n - k$ invariant under the action of T . In particular D_ρ denotes the toric divisor associated to $\rho \in \Sigma(1)$ and C_τ denotes the toric curve associated to $\tau \in \Sigma(n - 1)$. The minimal generator of the 1-dimensional cone $\rho \in \Sigma(1)$, is denoted by u_ρ . We say that the fan Σ (or the toric variety X_Σ) is simplicial, resp. complete, if each cone $\gamma \in \Sigma(k)$ is generated by exactly k 1-dimensional cones, resp. if the cones of Σ cover the whole space $N_{\mathbb{R}}$.

The class group of X_Σ is denoted by $\text{Cl}(X_\Sigma) = A^1(X_\Sigma)$ and the space of numerical classes of divisors by $N^1(X_\Sigma) = A^1(X_\Sigma) \otimes \mathbb{R}$. By duality the group of numerical classes of integral curves is denoted by $A_1(X_\Sigma)$ and the corresponding real space is $N_1(X_\Sigma) = A_1(X_\Sigma) \otimes \mathbb{R}$.

1.2 Homogeneous coordinates

Giving an algebro-geometric meaning of earlier analytic results of Audin, Delzant and Kerwan, D. Cox published in 1995 the famous article [Cox95] showing that any (normal separated) complex toric variety is naturally isomorphic to a GIT quotient, this quotient being geometric if and only if the variety is simplicial.

It turns out that this result holds in full generality over an arbitrary field of characteristic zero and also in characteristic p for toric varieties satisfying the following additional hypothesis :

The order of the torsion part of the class group $\text{Cl}(X)$ is prime to p . (H3)

Let us give more details.

D. Cox showed that in the complex case, a simplicial toric variety X_Σ is isomorphic to the good geometric quotient

$$\left(\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \right) // G$$

where

$$Z(\Sigma) = \mathbf{V}(x^{\hat{\sigma}} \mid \sigma \in \Sigma)$$

is the exceptional set, that is the subvariety of $\mathbb{A}^{\Sigma(1)} = \text{Spec}(K[x_\rho, \rho \in \Sigma(1)])$ defined by the vanishing of every monomial of the form $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ for $\sigma \in \Sigma$, and

$$G = \text{Hom}(\text{Cl}(X_\Sigma), \mathbb{G}_m)$$

is the group acting on $\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)$ whose character group is the class group $\text{Cl}(X_\Sigma)$ (see [CLS11, Ch. 5]). In fact, the original proof of the existence of such a quotient (Th. 2.1 of [Cox95]) relies on Theorem 1.1 of [FM82] and its amplification 1.3. These results only depend on the reductivity of the group G by which we quotient. If we denote by F the torsion part of $\text{Cl}(X_\Sigma)$, we have $\text{Cl}(X_\Sigma) \simeq \mathbb{Z}^{s-n} \times F$, where s is the cardinality of $\Sigma(1)$ and n the dimension of X_Σ , so that

$$G \simeq (\mathbb{G}_m)^{s-n} \times \text{Hom}(F, \mathbb{G}_m).$$

The factor on the left is of course a reductive torus. The one on the right is a finite group scheme and hence is reductive as soon as (H3) is verified.

Under the hypotheses (H2) and (H3), subvarieties of X_Σ are thus given by homogeneous ideals of the total coordinate ring (or Cox ring) of X_Σ , which is the polynomial algebra generated over K by one variable for each 1-dimensional cone $\rho \in \Sigma(1)$, and graded by the class group :

$$S = K[x_\rho, \rho \in \Sigma(1)] = \bigoplus_{\beta \in \text{Cl}(X_\Sigma)} S_\beta.$$

In particular, any hypersurface D is the zero locus in X_Σ of a polynomial f belonging to the homogeneous piece $S_{[D]}$:

$$D = \mathbf{V}(f) := \{ \pi(x) \in X_\Sigma \mid f(x) = 0 \},$$

where $\pi : \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \rightarrow X_\Sigma$ is the good geometric quotient.

In this setting, our results give criteria of existence of rational points of the form $\pi(\alpha)$ for $\alpha \in K^{\Sigma(1)}$ a root of f which is not in the exceptional set $Z(\Sigma)$, that is to say such that the set $\{ \rho \in \Sigma(1) \mid \alpha_\rho = 0 \}$ generates a cone of the simplicial fan Σ .

1.3 Projective toric varieties, divisors and polytopes

The fan Σ defining a projective toric variety is always the normal fan of a lattice polytope P . This means that the cones of Σ are generated by the inner normal facets of P . In this case we write $\Sigma = \Sigma_P$. Such fans are said to be strongly polytopal.

In this subsection we review some results from [CLS11, Ch. 2, 4, 5 and 6] linking lattice polytopes with projective toric varieties and their divisors.

1.3.1 Polytopes and projective toric varieties

At a very basic level, projective toric varieties are easy to construct out of lattice polytopes since they appear as Zariski closures of the image of a map defined by such a polytope. Let us recall some of the important results of [CLS11, Ch. 2].

Let $P \subset M_{\mathbb{R}}$ be a full dimensional convex polytope with vertices in the character group M of the n -dimensional torus T , and let $P \cap M = \{m_1, \dots, m_s\}$ be the set of all lattice points contained in P . Suppose, for beginning, that P is normal, which means that for all $k \geq 1$ we have

$$kP \cap M = \underbrace{P \cap M + \dots + P \cap M}_{k \text{ times}}$$

We have a map

$$\Phi : \begin{array}{ccc} T & \rightarrow & \mathbb{P}^{s-1} \\ (t_1, \dots, t_n) & \mapsto & (\chi^{m_1}, \dots, \chi^{m_s}). \end{array}$$

And the Zariski closure $Y_{P \cap M} = \overline{\Phi(T)}$ of the image of Φ is a projective toric variety having $\Phi(T) \simeq T$ as dense open subset.

Now consider the normal fan Σ_P of the polytope P . It is the fan whose maximal cones are the dual cones C_v^{\vee} where

$$C_v = \text{Cone}(P \cap M - v), \quad v \text{ vertex of } P.$$

The toric variety X_{Σ_P} defined from this fan is (isomorphic to) $Y_{P \cap M}$.

In fact, even in the case where the full dimensional polytope P is not a normal polytope, its multiples kP are normal for all $k \geq n - 1$. The toric variety associated to P in this case is therefore

$$X_P := X_{\Sigma_P} \simeq Y_{(n-1)P \cap M}.$$

Note that since, by Sumihiro's Theorem, any (normal separated) toric variety comes from a fan ([CLS11, Cor. 3.1.8]), it follows that any such variety comes from a full dimensional lattice polytope as soon as it is projective. But still, several polytopes may have the same normal fan and hence correspond to the same projective toric variety.

1.3.2 Polytopes and torus-invariant divisors

Let $D = \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}$ be a torus-invariant divisor on a complete simplicial toric variety X_{Σ} . For each 1-dimensional cone $\rho \in \Sigma(1)$, the inequality $\langle m_{\sigma}(D), u_{\rho} \rangle \geq -a_{\rho}$ defines an affine half space of $M_{\mathbb{R}} \simeq \mathbb{R}^n$. The intersection of all these half spaces is the convex polytope

$$P_D := \{m \in M \mid \langle m, u_{\rho} \rangle \geq -a_{\rho} \text{ for all } \rho \in \Sigma(1)\}$$

This polytope characterizes the set of torus-invariant divisors $D' = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$ that are effective and linearly equivalent to D . Indeed for such a divisor we have

$$D' \text{ is effective and } D' \in [D] \quad \Leftrightarrow \quad \exists m \in P_D, D' = D + \text{div}(\chi^m),$$

where $[D]$ denotes the linear equivalence class of D and $\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_{\rho} \rangle D_{\rho}$ is the divisor of the character $\chi^m, m \in M$.

Since Σ is simplicial, for each n -dimensional cone $\sigma \in \Sigma(n)$ the system of linear equations

$$\langle m, u_{\rho} \rangle = -a_{\rho}, \quad \rho \in \sigma(1)$$

has a unique solution denoted $m_{\sigma}(D)$.

The set $\{m_{\sigma}(D) \mid \sigma \in \Sigma(n)\}$ is called the Cartier data of the torus invariant divisor D because D is a Cartier divisor if and only if all the $m_{\sigma}(D)$ are lattice points in M .

Indeed for each $\sigma \in \Sigma(n)$ the fact that $m_\sigma(D)$ is a character of T is equivalent to the existence of a torus-invariant divisor

$$D^\sigma = D + \operatorname{div}(\chi^{m_\sigma}) = \sum_{\rho \in \Sigma(1)} a_\rho^\sigma D_\rho, \quad \text{with } a_\rho^\sigma = a_\rho + \langle m, u_\rho \rangle = 0 \quad \text{for all } \rho \in \sigma(1),$$

which means that the linear equivalence class is trivial on the toric affine open U_σ , and this happens for all $\sigma \in \Sigma(n)$ if and only if D is locally principal. Note that the divisors D^σ are uniquely determined by the class $[D]$, we call them local representatives of $[D]$.

Let us suppose that $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is a Cartier divisor. By definition, each character $m_\sigma(D)$ lie at the intersection of $n = \dim(M_\mathbb{R})$ supporting hyperplanes of the polytope P_D . Hence $m_\sigma(D)$ belongs to P_D if and only if it is one of its vertices, and this happens for all $\sigma \in \Sigma(n)$ if and only if all the local representatives D^σ are effective. Moreover the basepoints of D are necessarily torus-invariant points, that is points of the form $\gamma_\sigma = \bigcap_{\rho \in \sigma(1)} D_\rho$ for some $\sigma \in \Sigma(n)$. Such a point is by definition avoided by the support of the local representative D^σ . It follows that we have

$$P_D = \operatorname{Conv}(m_\sigma(D), \sigma \in \Sigma(n)) \Leftrightarrow \forall \sigma \in \Sigma(n), D^\sigma \geq 0 \Leftrightarrow D \text{ is basepoint free.} \quad (4)$$

And when D is base point free we have (see [CLS11, 6.1] for a proof)

$$m_\sigma(D) \neq m_{\sigma'}(D) \text{ for } \sigma \neq \sigma' \Leftrightarrow D^\sigma \neq D^{\sigma'} \text{ for } \sigma \neq \sigma' \Leftrightarrow D \text{ is ample.} \quad (5)$$

In this case, for $\sigma \in \Sigma(n)$, the character m_σ is the only one verifying for all $\rho \in \Sigma(1)$

$$\langle m, u_\rho \rangle \geq -a_\rho, \quad \text{with equality if and only if } \rho \in \sigma(1).$$

It follows on the one hand that the coefficients of D may be recovered from the polytope P_D by the formula

$$a_\rho = - \min_{m \in P_D} \langle m, u_\rho \rangle = - \min_{m \in P_D \cap M} \langle m, u_\rho \rangle, \quad \text{for all } \rho \in \Sigma(1).$$

On the other hand, the normal fan Σ_{P_D} is in fact the starting fan Σ . Indeed, its maximal cones $C_{m_\sigma}^\vee$ verify for all $\rho \in \Sigma(1)$

$$\begin{aligned} u_\rho \in C_{m_\sigma}^\vee &\Leftrightarrow \forall m \in \operatorname{Cone}(P \cap M - m_\sigma), \langle m, u_\rho \rangle \geq 0 \\ &\Leftrightarrow \forall p \in P \cap M, \langle p, u_\rho \rangle \geq \langle m_\sigma, u_\rho \rangle \\ &\Leftrightarrow \langle m_\sigma, u_\rho \rangle = -a_\rho \Leftrightarrow u_\rho \in \sigma(1). \end{aligned}$$

Finally it turns out that full dimensional lattice polytopes parametrize the set of pairs formed by a projective toric variety and a torus invariant ample divisor on it :

Theorem 1.1 ([CLS11] Th. 6.2.1). *The sets*

$$\{(X_\Sigma, D) \mid \Sigma \text{ is a complete fan in } N_\mathbb{R}, D \text{ is a torus-invariant ample divisor on } X_\Sigma\}$$

and

$$\{P \subset M_\mathbb{R} \mid P \text{ is a full dimensional lattice polytope}\}$$

are in one-to-one correspondence by the inverse maps

$$(X_\Sigma, D) \mapsto P_D \quad \text{and} \quad P \mapsto (X_P, D_P),$$

where D_P is the torus invariant divisor $\sum_{\rho \in \Sigma_P(1)} (-\min_{m \in P} \langle m, u_\rho \rangle) D_\rho$.

1.3.3 Polytopes, morphisms and basepoint free divisors

When we have a couple (X_Σ, D) where $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is a torus-invariant Cartier divisor which is not ample but only nef (which in this setting is equivalent to basepoint free), applying the inverse maps of Theorem 1.1 we get a new couple $(X_{\bar{\Sigma}}, \bar{D})$ where $X_{\bar{\Sigma}}$ is a projective variety and \bar{D} an ample divisor on it, together with a toric morphism $\phi : X_\Sigma \rightarrow X_{\bar{\Sigma}}$ such that D is linearly equivalent to the pullback $\phi^* \bar{D}$. Let us give more details.

A toric morphism $\varphi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$ is a morphism between toric varieties whose restriction to the open tori is a morphism of algebraic groups. In particular to such a morphism corresponds a \mathbb{Z} -linear map $\bar{\varphi} : N_1 \rightarrow N_2$ that is compatible with the fans Σ_1 and Σ_2 , i.e. such that for any cone $\sigma_1 \in \Sigma_1$, there is a cone $\sigma_2 \in \Sigma_2$ containing the image $\bar{\varphi}(\sigma_1)$.

As stated in 4 and 5, when the torus-invariant Cartier divisor D is nef but not ample, the elements of the Cartier data $\{m_\sigma \mid \sigma \in \Sigma(n)\}$ are still the vertices of the polytope P_D but they are not distinct any more. In this case the maximal cones of the normal fan Σ_{P_D} are unions of maximal cones of Σ ([CLS11, Prop. 6.2.5]) :

$$C_v^\vee = \bigcup_{\substack{\sigma \in \Sigma(n) \\ m_\sigma = v}} \sigma.$$

It may happen (when D is not big) that the lattice polytope P_D fail to be full dimensional which means that the C_v^\vee are not strongly convex cones. In this case we say that Σ_{P_D} is a degenerate fan. All the cones of Σ_{P_D} contain a linear space U_0 called the minimal cone of the degenerate fan (it is the orthogonal of the span of P_D).

If we put $N_0 = U_0 \cap N$ then the linear map

$$\bar{\phi} : N \rightarrow \bar{N} = N/N_0$$

is compatible with the fan $\Sigma \subset N_{\mathbb{R}}$ and the nondegenerate fan

$$\bar{\Sigma} = \{\bar{\sigma} := \sigma/U_0 \mid \sigma \in \Sigma_{P_D}\} \subset \bar{N}_{\mathbb{R}} = N_{\mathbb{R}}/U_0. \quad (6)$$

In other words the fan Σ refines the possibly degenerate fan Σ_{P_D} and hence we have a toric morphism between X_Σ and the projective toric variety associated to Σ_{P_D} (after having got rid of the degeneracy).

Here is the precise result :

Theorem 1.2 ([CLS11] Th. 6.2.8). *Let D be a basepoint free Cartier divisor on a complete toric variety X_Σ , and let $X_{\bar{\Sigma}}$ be the toric variety of the fan $\bar{\Sigma} \subset \bar{N} = N/N_0$ defined by 6. Then the linear map $\bar{\phi} : N \rightarrow \bar{N}$ induces a proper toric morphism*

$$\phi : X_\Sigma \rightarrow X_{\bar{\Sigma}}.$$

Furthermore, $X_{\bar{\Sigma}}$ is a projective toric variety and D is linearly equivalent to the pullback of an ample divisor \bar{D} on $X_{\bar{\Sigma}}$.

In other words, if D is basepoint free, the image of the morphism associated to the complete linear system $|D|$ is the toric variety defined by the simplicial fan $\bar{\Sigma}$ in $\bar{N}_{\mathbb{R}}$.

Note that the couple $(X_{\bar{\Sigma}}, \bar{D})$ is the one given by the second map of Theorem 1.1 applied to P_D seen as a full dimensional lattice polytope in its real span.

1.3.4 Polytopes and linear series (Newton polytopes)

Let us first remark that the fans Σ_{P_D} and $\bar{\Sigma}$ do not depend on the precise torus-invariant divisor D , but only on its linear equivalence class $[D]$. Indeed, starting with another torus invariant divisor $D' \in [D]$ in the same class would give a polytope $P_{D'}$ equal to the translate of P_D by the only character $m' \in M$ such that

$$D - D' = \operatorname{div}(m') = \sum_{\rho \in \Sigma(1)} \langle m', u_\rho \rangle D_\rho,$$

and hence give the same normal fan $\Sigma_{P_{D'}} = \Sigma_{P_D}$. The same is true for the nondegenerate fan $\bar{\Sigma}$ and since they do not depend on the choice of a representative of $[D]$, we will denote them by

$$\Sigma_{[D]} = \Sigma_{P_D} \quad \text{and} \quad \bar{\Sigma}_{[D]} = \bar{\Sigma}.$$

This explains why in Theorem 1.2 the divisor D is not supposed to be torus-invariant : we can start with any nef Cartier divisor D up to choose a torus-invariant one $D' \in [D]$ in the same class (which is always possible, see next subsection).

The following result shows that we can even define a “canonical model” of the polytopes $P_{D'}$ for $D' \in [D]$, depending only on the class $[D]$.

Lemma-Definition 1.3. *Let $D \in \operatorname{Div}(X_\Sigma)$ be a Weil divisor and $D' = \sum_{\rho \in \Sigma(1)} a'_\rho D_\rho \in [D]$ be a torus invariant divisor in the class of D . The map*

$$h_{D'} : M_{\mathbb{R}} \longrightarrow \mathbb{R}^{\Sigma(1)}, \quad m \longmapsto (\langle m, u_\rho \rangle + a'_\rho)_{\rho \in \Sigma(1)}$$

embeds $M_{\mathbb{R}}$ as an affine subspace of $\mathbb{R}^{\Sigma(1)}$.

The image $P_{[D]} := h_{D'}(P_{D'})$ of the convex polytope $P_{D'}$ by this map is independent of the choice of $D' \in [D]$ and we have

$$P_{[D]} = h_{D'}(M_{\mathbb{R}}) \cap \mathbb{R}_+^{\Sigma(1)},$$

Proof — It is Exercice 5.4.2 of [CLS11]. The map $h_{D'}$ is injective because Σ is full dimensional, so that

$$\langle m, u_\rho \rangle = \langle m', u_\rho \rangle \text{ for all } \rho \in \Sigma(1) \quad \Rightarrow \quad m = m'.$$

The equality comes from the equivalence

$$m \in P_{D'} \Leftrightarrow \forall \rho \in \Sigma(1), \langle m, u_\rho \rangle \geq -a'_\rho \Leftrightarrow \forall \rho \in \Sigma(1), h_{D'}(m)_\rho \geq 0.$$

In order to prove the independence let us consider another torus invariant divisor $D'' = \sum_{\rho \in \Sigma(1)} a''_\rho D_\rho$ in $[D]$. There exists a character m'' such that

$$D' - D'' = \operatorname{div}(\chi^{m''}) = \sum_{\rho \in \Sigma(1)} \langle m'', u_\rho \rangle D_\rho,$$

that is, $a'_\rho = a''_\rho + \langle m'', u_\rho \rangle$ for all $\rho \in \Sigma(1)$. In particular for all $m \in M$ we have

$$h_{D''}(m) = (\langle m, u_\rho \rangle + a''_\rho)_{\rho \in \Sigma(1)} = (\langle m - m'', u_\rho \rangle + a'_\rho)_{\rho \in \Sigma(1)} = h_{D'}(m - m'')$$

and

$$\begin{aligned} m \in P_{D''} &\Leftrightarrow \forall \rho \in \Sigma(1), \langle m, u_\rho \rangle \geq -a''_\rho \\ &\Leftrightarrow \forall \rho \in \Sigma(1), \langle m - m'', u_\rho \rangle \geq -a'_\rho \Leftrightarrow m - m'' \in P_{D'} \end{aligned}$$

so that

$$\begin{aligned} h_{D''}(P_{D''}) &= \{h_{D''}(m) \mid m \in P_{D''}\} = \{h_{D'}(m - m'') \mid m - m'' \in P_{D'}\} \\ &= \{h_{D'}(m') \mid m' \in P_{D'}\} = h_{D'}(P_{D'}). \end{aligned}$$

□

Let us now make the link with homogeneous coordinates on X_Σ . In [CLS11, 5.4] the authors call D' -homogenization the process associating to a given character χ^m of the torus $T \subset X_\Sigma$, the character of the big torus $T' \subset \mathbb{A}^{\Sigma(1)}$ (or the Laurent monomial)

$$x^{h_{D'}(m)} = \prod_{\rho \in \Sigma(1)} x_\rho^{(m, u_\rho) + a_\rho} \in K(x_\rho, \rho \in \Sigma(1))$$

where $D' = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$.

This provides a nice description of the set of monomials that may appear in the homogeneous polynomial defining an effective divisor on X_Σ . Let us introduce some notations :

Notation 1.4. Let $D \in \text{Div}(X_\Sigma)$ be an effective Weil divisor. We put

$$L_{[D]} = \left\{ (a_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^{\Sigma(1)} \mid \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in [D] \right\} \quad \text{and} \quad \mathcal{A}_{[D]} = \mathbb{N}^{\Sigma(1)} \cap L_{[D]},$$

so that we have

$$S_{[D]} = \bigoplus_{a \in \mathcal{A}_{[D]}} Kx^a,$$

where x^a denotes the monomial $\prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}$ for $a = (a_\rho)_{\rho \in \Sigma(1)} \in \mathbb{N}^{\Sigma(1)}$.

The crucial fact is that the monomials generating the graded piece $S_{[D]}$ are exactly the D' -homogenizations of the characters belonging to $P_{D'}$ for any torus-invariant divisor $D' \in [D]$ in the class $[D]$ (see [CLS11, Prop. 5.4.1]). In other words, for any torus-invariant $D' \in [D]$, the image of the map $h_{D'}$ is the affine hull of the lattice $L_{[D]}$. This means that the elements of $\mathcal{A}_{[D]}$ are exactly the points of $L_{[D]}$ contained in $P_{[D]} : \mathcal{A}_{[D]} = P_{[D]} \cap L_{[D]}$. In particular, for any homogeneous polynomial $f \in S_{[D]}$, the Newton polytope Δ_f of f is contained in the polytope $P_{[D]}$.

Finally let us assume that D is a nef and Cartier divisor. For any torus-invariant $D' \in [D]$ we have $P_{D'} = \text{Conv}(m_\sigma(D'), \sigma \in \Sigma(n))$ in $M_\mathbb{R}$. Hence in $\mathbb{R}^{\Sigma(1)}$ we have

$$P_{[D]} = \text{Conv}(a^\sigma([D]), \sigma \in \Sigma(n)), \quad \text{where } a^\sigma([D]) = h_{D'}(m_\sigma(D')).$$

In particular each $a^\sigma([D])$ belongs to $\mathcal{A}_{[D]}$ and $P_{[D]}$ is a lattice polytope : it is the Newton polytope of the general member of the graded piece $S_{[D]}$. Notice that for each $\sigma \in \Sigma(n)$, the vertex $a^\sigma([D]) = (a_\rho^\sigma)_{\rho \in \Sigma(1)}$ of $P_{[D]}$ is the coefficient vectors of the local representative of $[D]$

$$D^\sigma = \sum_{\rho \in \Sigma(1)} a_\rho^\sigma D_\rho.$$

In particular we have $a_\rho^\sigma = 0$ for all $\rho \in \sigma(1)$ and $a^\sigma([D])$ is the unique element of $\mathcal{A}_{[D]}$ satisfying these vanishings.

1.4 Toric intersection theory

Computing intersections on a normal toric variety X_Σ is made quite easy by the fact that the Chow homology group $A_k(X_\Sigma)$ of classes of k -cycles on X_Σ is generated by the classes $[\mathbf{V}(\gamma)]$ of k -dimensional toric subvarieties ($\gamma \in \Sigma(n-k)$). Most of the computations thus reduce to combinatorics in the fan and linear algebra.

1.4.1 Chow groups

The identification of the Chow cohomology group $A^k(X_\Sigma)$ with $A_{n-k}(X_\Sigma)$, standard in the smooth case, can be extended to the simplicial case up to tensorize by \mathbb{Q} or \mathbb{R} , which kills the torsion and yields vector spaces of numerical classes. In particular we put :

$$N_k(X_\Sigma) := A_k(X_\Sigma) \otimes \mathbb{R}, \quad N^k(X_\Sigma) := A^k(X_\Sigma) \otimes \mathbb{R} = N_{n-k}(X_\Sigma).$$

The class group is computed through the following exact sequence :

$$0 \longrightarrow M \xrightarrow{\text{div}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\text{deg}} A^1(X_\Sigma) = \text{Cl}(X_\Sigma) \longrightarrow 0,$$

where $\text{div}(m) = (\langle m, u_\rho \rangle)_{\rho \in \Sigma(1)}$ and $\text{deg}(D) = [D]$.

Tensoring by \mathbb{R} , this gives

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\text{div}} \mathbb{R}^{\Sigma(1)} \xrightarrow{\text{deg}} N^1(X_\Sigma) \longrightarrow 0,$$

and taking the dual sequence we get

$$0 \longrightarrow N_1(X_\Sigma) \xrightarrow{\text{deg}^*} \mathbb{R}^{\Sigma(1)} \xrightarrow{\text{div}^*} N_{\mathbb{R}} \longrightarrow 0,$$

where $\text{deg}^*([C]) = (C \cdot D_\rho)_{\rho \in \Sigma(1)}$ and $\text{div}^*((b_\rho)_{\rho \in \Sigma(1)}) = \sum_{\rho \in \Sigma(1)} b_\rho u_\rho$.

This last exact sequence identifies the space of numerical classes of 1-cycles $N_1(X_\Sigma)$ with the space of linear relations between minimal generators of the 1-dimensional cones of Σ . We will often consider the numerical class $[C]$ as given by the relation

$$\text{div}^*(\text{deg}^*([C])) = 0, \quad \text{i.e.} \quad \sum_{\rho \in \Sigma(1)} C \cdot D_\rho u_\rho = 0. \quad (7)$$

1.4.2 Intersection numbers

The exact sequences of the previous paragraph show that the numerical classes of divisors (resp. curves) are determined by their intersection numbers with the toric curves (resp. divisors).

In particular for curves we introduce the following notations :

Notation 1.5. $J_C^\pm = \{\rho \in \Sigma(1) \mid C \cdot D_\rho \geq 0\}$ and $J_C = J_C^+ \cup J_C^-$.

The intersection matrix $(C_\tau \cdot D_\rho)_{\tau \in \Sigma(n-1), \rho \in \Sigma(1)}$ is easily computed using the relation 7 and the fact that for a couple $(\tau, \rho) \in \Sigma(n-1) \times \Sigma(1)$ such that $\rho \notin \tau$ we have

$$C_\tau \cdot D_\rho = \begin{cases} \frac{\text{mult}(\tau)}{\text{mult}(\sigma)} & \text{if } \tau + \rho = \sigma \in \Sigma(n) \\ 0 & \text{if } \tau + \rho \notin \Sigma, \end{cases} \quad (8)$$

where the multiplicity of a cone $\alpha = \text{Cone}(u_1, \dots, u_k) \in \Sigma(k)$ is the index of the lattice generated by its minimal generators u_1, \dots, u_k in the lattice N_α equal to the intersection of N with the real span of α :

$$\text{mult}(\alpha) = [N_\alpha : \bigoplus_{i=1}^k \mathbb{Z}u_i].$$

Since we assume Σ to be simplicial and complete, each wall $\tau \in \Sigma(n-1)$ is the intersection of two n -dimensional cones $\sigma, \sigma' \in \Sigma(n)$. Then it follows from 8 that $J_{C_\tau} \subseteq \sigma(1) \cup \sigma'(1)$ and there is a unique relation between the elements of $\{u_\rho \mid \rho \in J_{C_\tau}\}$ up to a real multiple. It follows that, knowing this line of relations, the numerical class $[C_\tau]$ is determined by any of the two numbers $C_\tau \cdot D_\rho$ for $\rho \in \sigma(1) \cup \sigma'(1) \setminus \tau(1)$.

1.4.3 Pushforwards and pullbacks

Pushforwards and pullbacks of cycle classes by toric morphisms can be described explicitly in terms of fans and linear algebra. However the general formulas are rather complicated (see [Par93] and [FS97]).

We focus on cycles of dimension or codimension 1 and describe their behaviour under rational maps. We first fix some terminology.

Definition 1.6. An equivariant (bi-)rational map between toric varieties is called a toric (bi-)rational map. To such a map $\xi : X_\Sigma \dashrightarrow X_{\Sigma'}$ corresponds a linear map $\bar{\xi} : N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ between the latticed spaces containing the fans Σ and Σ' (and $\bar{\xi}$ is compatible with the fans if and only if ξ is a toric morphism).

When ξ is surjective in codimension 1 (i.e. when we have $\Sigma'(1) \subseteq \bar{\xi}(\Sigma(1))$) then it is called a toric (bi-)rational contraction.

Let us describe some properties of pushforwards and pullbacks by a toric birational contraction $\xi : X_\Sigma \dashrightarrow X'_{\Sigma'}$ between simplicial varieties. Let $\Sigma \subset N_{\mathbb{R}}$ and $\Sigma' \subset N'_{\mathbb{R}}$ be complete simplicial fans and $\bar{\xi} : N \rightarrow N'$ the surjective \mathbb{Z} -linear map associated to ξ . By hypothesis we have $\Sigma'(1) \subseteq \bar{\xi}(\Sigma(1))$, and for $\rho \in \Sigma(1)$, we have $\xi_* D_\rho = \begin{cases} D'_{\rho'} & \text{if } \bar{\xi}(\rho) = \rho' \in \Sigma'(1), \\ 0 & \text{otherwise.} \end{cases}$. This

induces a surjective map between the groups of torus-invariant divisors as well as a surjective pushforward map $\xi_* : N^1(X_\Sigma) \rightarrow N^1(X'_{\Sigma'})$ for numerical classes.

In general, the pullback of divisor classes are easy to describe in terms of their associated polytopes or the support functions of some torus-invariant representative, which we do not address here, the interested reader may refer to [CLS11, 6.1 and 6.2]. Let us just mention that in our case the pullback of divisor classes $\xi^* : N^1(X') \rightarrow N^1(X)$ is a section of the pushforward $\xi_* : N^1(X) \rightarrow N^1(X')$. In particular it is injective and the composition $\xi_* \circ \xi^*$ is the identity on $N^1(X')$.

By duality, we can deduce a pullback map for numerical classes of curves. It is called numerical pullback of curves, was introduced by Batyrev in [Bat92] in a more general context, and have been neatly described by C. Araujo in [Ara10].

Definition 1.7. Let $\xi : X_\Sigma \dashrightarrow X'_{\Sigma'}$ be a toric birational contraction. The numerical pullback of curves $\xi_{num}^* : N_1(X') \rightarrow N_1(X)$ is the dual linear map of $\xi_* : N^1(X) \rightarrow N^1(X')$. It is the unique injective linear map verifying the following projection formula :

$$\xi_{num}^*[C'] \cdot [D] = [C'] \cdot \xi_*[D] \quad \text{for all } [C'] \in N_1(X') \text{ and } [D] \in N^1(X).$$

In the toric context, the numerical pullback of curves can be expressed by means of relations between the minimal generators of the fans' rays : $\xi_{num}^*[C']$ is characterized by the equalities

$$\sum_{\rho \in \Sigma(1)} \xi_{num}^*[C'] \cdot [D_\rho] u_\rho = \sum_{\rho' \in \Sigma'(1)} [C'] \cdot [D'_{\rho'}] u_{\rho'} = 0$$

In particular when $[C] = \xi_{num}^*[C']$ we have $J_C^\pm = J_{C'}^\pm$.

Finally the pushforward of curve classes may either be defined directly on invariant curves or by duality via the intersection pairing. In particular it can as well be described in terms of relations between minimal generators, and if ξ is birational, then it admits ξ_{num}^* as a linear section.

Notice that the projection formula always holds :

$$\xi_*[C] \cdot [D'] = [C] \cdot \xi^*[D'] \quad \text{for all } [C] \in N_1(X) \text{ and } [D'] \in N^1(X').$$

1.4.4 Intersections in a toric subvariety

The notions of low toric degree we define in the following sections are based on inequalities between intersection numbers. Since they are designed to ensure the existence of a rational point in a divisor D , one would expect they obey the principle :

For any subvariety $V \subset X_\Sigma$, $D|_V$ has low degree $\Rightarrow D$ has low degree.

It turns out that it is not that simple because for a toric subvariety $V(\gamma) \subset X_\Sigma$, $\gamma \in \Sigma$, the intersection numbers computed in $V(\gamma)$ and the ones computed in X_Σ do not always coincide. Let us explicit the relation between them.

First let us denote by $N(\gamma) = N/N_\gamma$ the quotient of N by $N_\gamma = N \cap \text{Span } \gamma$, and by $\pi : N_{\mathbb{R}} \rightarrow N(\gamma)_{\mathbb{R}} = N(\gamma) \otimes \mathbb{R}$ the surjection between the spanned vector spaces. Then $V(\gamma)$ is the toric variety defined by the fan

$$\text{Star}(\gamma) = \{\pi(\sigma) \mid \gamma \preceq \sigma \in \Sigma\}$$

in $N(\gamma)_{\mathbb{R}}$.

The 1-dimensional cones of $\text{Star}(\gamma)$ are the images through π of the 1-dimensional cones of Σ adjacent to γ :

$$\text{Star}(\gamma)(1) = \{\pi(\rho) \mid \rho \in \text{Adj}(\gamma)\} \text{ where } \text{Adj}(\gamma) = \{\rho \in \Sigma(1) \setminus \gamma(1) \mid \gamma + \rho \in \Sigma\}$$

The tricky point is that, when the variety X_Σ is singular, the minimal generator $u_{\bar{\rho}}$ of $\bar{\rho} = \pi(\rho)$ in the lattice $N(\gamma)$ is not necessarily the image $\pi(u_\rho)$ of the minimal generator of ρ . Instead we have in general

$$\pi(u_\rho) = m_\rho u_{\bar{\rho}} \text{ with } m_\rho = \frac{\text{mult}(\gamma + \rho)}{\text{mult}(\gamma)} \in \mathbb{N}^*.$$

Hence if $j : V(\gamma) \hookrightarrow X_\Sigma$ denotes the inclusion, and $D_{\bar{\rho}}$ denotes the toric divisor in $V(\gamma)$ corresponding to the 1-dimensional cone $\bar{\rho} \in \text{Star}(\gamma)(1)$, then the pullbacks of toric divisors are

$$j^* D_\rho = \begin{cases} \frac{1}{m_\rho} D_{\bar{\rho}} & \text{if } \rho \in \text{Adj}(\gamma), \\ 0 & \text{else.} \end{cases}$$

It follows that for every 1-cycle C on $V(\gamma)$ and every $\rho \in \text{Adj}(\gamma)$ we have

$$C \cdot D_{\bar{\rho}} = C \cdot m_\rho j^* D_\rho = m_\rho j_* C \cdot D_\rho \geq j_* C \cdot D_\rho.$$

and the inequality may be strict when X_Σ is singular.

For later use, notice that we have in particular

$$C \cdot (-K_{V(\gamma)}) \geq j_* C \cdot \sum_{\rho \in \text{Adj}(\gamma)} D_\rho. \quad (9)$$

1.5 Toric Mori theory

In this subsection we recall some of the tools from toric birational geometry that are essential for our results.

1.5.1 Toric Minimal Model Program

The aim of the Minimal Model Program (MMP for short) is the classification of projective algebraic varieties up to birational equivalence, with a special interest given to some special representatives in each class (the minimal models).

This program has been carried out for surfaces by the ‘‘Italian school’’ of Algebraic Geometry during the first half of the 20th century, yielding the Enriques classification of algebraic surfaces.

In higher dimensions it is also called Mori program and is a very active area of research since more than three decades. It relies in an essential manner on the properties of the Mori cone, that is, the cone of numerical classes of effective curves. More specifically, a crucial fact for a \mathbb{Q} -factorial projective variety X , is that the K_X -negative half of the Mori cone $\overline{\text{NE}}(X)$ (i.e. the set of effective classes $[C]$ such that $C \cdot K_X < 0$) is polyhedral. This allows to perform contractions of the rational curves contained in extremal rays of this polyhedral part, with the aim of simplifying the variety towards a minimal model.

Repeating this yields a step by step procedure, again called MMP, consisting in a finite number of divisorial contractions or flips (see below) until the dimension of the variety drops,

or until there is no more curve in the K_X -negative part of $\overline{\text{NE}}(X)$ (that is, the canonical divisor K_X is nef).

An extremal contraction $\psi : X \rightarrow X'$ appearing in the MMP can be of three different types :

- (1) A divisorial contraction. It contracts a divisor to a lower dimensional subvariety, making the Picard number to drop by 1. It is the converse of a weighted blow-up. In this case we replace X by X' and continue the process.
- (2) A flipping (or small) contraction. It contracts a subvariety of codimension ≥ 2 , giving rise to a non \mathbb{Q} -factorial singularity in X' . Since such a singularity is usually considered too bad, we take the flip map $\nu : X \dashrightarrow X^+$ which is an isomorphism in codimension 1 and continue with the \mathbb{Q} -factorial variety X^+ . A typical example of a flip is the composition of a blow-up and a blow-down in “another direction”.
- (3) A fibering contraction. It contracts the whole variety onto a variety of smaller dimension. If ψ is fibering we consider X as minimal and stop the process.

Note that the existence of the flip ν is in general hard to prove (see [BCHM10]) but easy for simplicial toric varieties.

The choice of the extremal rays to contract can obey to various constraints :

- Instead of contracting the rays that are K_X -negative, we can choose a so-called boundary \mathbb{Q} -divisor B and contract the rays that are (K_X+B) -negative. This is called a logarithmic MMP (log-MMP for short) .
- Instead of taking the negative rays in the whole Mori cone $\overline{\text{NE}}(X)$, we can restrict the choices to the rays in a proper face $\overline{\text{NE}}(X/Y)$ of $\overline{\text{NE}}(X)$ consisting of classes of curves contained in fibers of a given morphism $\phi : X \rightarrow Y$. The cone $\overline{\text{NE}}(X/Y)$ is then called the relative Mori cone and we say that we run the MMP relative to Y .

These two restrictions on the choice of rays can be combined to give a relative log-MMP. Such a procedure stops either when it gets to a Mori fiber space or when the image of the divisor $(K_X + B)$ becomes ϕ -nef. An instance of such a procedure is used in the proof of Proposition 5.9.

Note that the canonical class of a toric variety is the class of the torus invariant divisor $K_{X_\Sigma} = -\sum_{\rho \in \Sigma(1)} D_\rho$ which is never effective, and a fortiori never nef. In particular every classical MMP ends on a fibering contraction. The corresponding minimal model is called a Mori fiber space.

In contrast with the general case, which yields extremely difficult issues, the toric MMP is quite simple to describe and it is relatively easy to prove that all its instances described above work for any simplicial projective toric variety (see [Mat02, Ch. 14] for a proof or [CLS11, Ch. 15] for the non-relative case).

Indeed for projective toric varieties, it is not just one half but the whole Mori cone which is polyhedral. This is one of the manifestations of the fact that projective toric varieties are Mori Dream Spaces (see [HK00] for an introduction). In particular the MMP works particularly well for such varieties. Another manifestation of this fact is the existence of homogeneous coordinates, that is, the finite generation of the Cox ring (see paragraph 1.2). The link between the two relies on the theory of variation of geometric invariant theory (VGIT) and can be understood via the study of the secondary fan (see [CLS11, Ch. 14 and 15]). This is a very beautiful subject that provides a convex-geometric insight on the MMP but we do not address it here, for lack of space reasons.

Let us begin with the cone theorem.

Theorem 1.8. *Let X_Σ be a complete toric variety. The Mori cone $\overline{\text{NE}}(X_\Sigma)$ is generated by classes of toric curves :*

$$\overline{\text{NE}}(X_\Sigma) = \text{NE}(X_\Sigma) = \sum_{\tau \in \Sigma(n-1)} \mathbb{R}_+[C_\tau].$$

In particular, it is a rational polyhedral cone in $N_1(X_\Sigma)$.

The fact that extremal rays are generated by classes of toric curves allow a very explicit description of the steps of a toric MMP in terms of the fan of the variety. Let X_Σ be a simplicial projective variety of dimension n and $\psi : X_\Sigma \rightarrow X_{\Sigma'}$ be the contraction of the extremal ray \mathcal{R} in $\overline{\text{NE}}(X_\Sigma)$. The fan defining $X_{\Sigma'}$ is obtained from Σ by removing all the walls $\tau \in \Sigma(n-1)$ such that $[C_\tau] \in \mathcal{R}$. In order to be more precise, we first need to describe some properties of these ‘‘extremal walls’’. For any $\tau \in \Sigma(n-1)$ such that $[C_\tau] \in \mathcal{R}$ let us put

$$\Delta(\tau) := \bigcup_{\substack{\sigma \in \Sigma(n) \\ \tau \prec \sigma}} \sigma.$$

and for any $\rho \in \Sigma(1)$ such that $\rho \subset \Delta(\tau)$ let us put

$$\Delta_\rho(\tau) := \text{Cone}(u_{\rho'} \mid \rho' \subset \Delta(\tau), \rho' \neq \rho).$$

We have the following properties :

Lemma 1.9 ([Rei83]). *Let X_Σ be a simplicial projective toric variety of dimension n and \mathcal{R} be an extremal ray of $\overline{\text{NE}}(X_\Sigma)$. For all $\tau \in \Sigma(n-1)$ such that $[C_\tau] \in \mathcal{R}$ we have*

$$\bigcup_{\rho \in J_{C_\tau}^+} \Delta_\rho(\tau) = \Delta(\tau) = \bigcup_{\rho \in J_{C_\tau}^-} \Delta_\rho(\tau)$$

and

$$\Delta_\rho(\tau) \in \Sigma(n) \quad \text{for all } \rho \in J_{C_\tau}^+.$$

Note that for two toric curve classes $[C_\tau], [C_{\tau'}] \in \mathcal{R}$ the intersection numbers $C_\tau \cdot D_\rho$ and $C_{\tau'} \cdot D_\rho$ have the same sign for all $\rho \in \Sigma(1)$. In particular the sets $J_{C_{\tau'}}^+ = J_{C_\tau}^+$ and $J_{C_{\tau'}}^- = J_{C_\tau}^-$ depend only on the ray \mathcal{R} , hence we can denote them by $J_{\mathcal{R}}^+$ and $J_{\mathcal{R}}^-$.

Here is a combinatorial description of the different steps of the toric MMP :

- (1) If $J_{\mathcal{R}}^- = \{\rho_0\}$ then ψ is a divisorial contraction and D_{ρ_0} is the divisor in X_Σ contracted by ψ . In this case we have $\Sigma'(1) = \Sigma(1) \setminus \{\rho_0\}$ and more generally

$$\Sigma' = \Sigma \setminus \{\sigma \in \Sigma \mid \rho_0 \in \sigma(1)\} \cup \{\Delta(\tau) \mid [C_\tau] \in \mathcal{R}\}.$$

The fan Σ is called the star subdivision of Σ' at ρ_0 .

- (2) If $\text{Card}(J_{\mathcal{R}}^-) \geq 2$ then ψ is a small contraction. If $\nu : X_\Sigma \dashrightarrow X_{\Sigma^+}$ is the corresponding flip, then we have $\Sigma^+(1) = \Sigma'(1) = \Sigma(1)$, and

$$\Sigma^+(n) = \Sigma(n) \setminus \{\Delta_\rho(\tau) \mid [C_\tau] \in \mathcal{R}, \rho \in J_{C_\tau}^+\} \cup \{\Delta_\rho(\tau) \mid [C_\tau] \in \mathcal{R}, \rho \in J_{C_\tau}^-\}.$$

Notice that if we denote by ρ_0 the one dimensional cone of $N_{\mathbb{R}}$ generated by

$$\sum_{\rho \in J_{\mathcal{R}}^+} C_\tau \cdot D_\rho u_\rho = \sum_{\rho \in J_{\mathcal{R}}^-} (-C_\tau \cdot D_\rho) u_\rho$$

then we have

$$\rho_0 = \bigcap_{\substack{\rho \in J_{\mathcal{R}} \\ [C_\tau] \in \mathcal{R}}} \Delta_\rho(\tau)$$

and the fans obtained by star subdividing Σ and Σ' at ρ_0 are the same. In particular the flip ν can be factored as a weighted blow-up followed by a divisorial contraction.

(3) If $J_{\overline{\mathcal{R}}} = \emptyset$ then ψ is a fibering contraction. In this case we have

$$\Sigma' = \{\Delta(\tau) \mid [C_\tau] \in \mathcal{R}\},$$

which is a degenerate fan. Indeed the cones $\Delta(\tau)$ for $[C_\tau] \in \mathcal{R}$ are not strongly convex for they all contain the linear space

$$U_0 = \text{Cone}(u_\rho, \rho \in J_{\overline{\mathcal{R}}}^+) = \text{Span}(u_\rho, \rho \in J_{\overline{\mathcal{R}}}^+)$$

As in paragraph 1.3.3, we get a nondegenerate fan $\overline{\Sigma}'$ by taking the quotient by U_0 . In particular the linear map $\overline{\psi}$ associated to ψ is the quotient of real vector spaces $N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/U_0$ and the variety $X_{\Sigma'} = X_{\overline{\Sigma}'}$ has dimension $n' = n - n_0 < n$ where $n_0 = \dim(U_0)$.

The fiber bundle structure of X_Σ can be read from the fan Σ itself : there exist two subfans $\widehat{\Sigma}$ and Σ_0 of Σ such that $\Sigma_0 = \{\sigma \in \Sigma \mid \sigma \subset U_0\}$ and $\overline{\psi}$ induces a bijection from $\widehat{\Sigma}$ to $\overline{\Sigma}_1$ such that we have

$$\Sigma = \{\sigma_0 + \widehat{\sigma} \mid \sigma_0 \in \Sigma_0, \widehat{\sigma} \in \widehat{\Sigma}\}.$$

In particular it follows from the description of the fan of a toric subvariety given in paragraph 1.4.4 that for every maximal cone $\sigma_0 \in \Sigma_0(n_0)$ the toric subvariety $V(\sigma_0)$ is isomorphic to $X_{\overline{\Sigma}'}$ and the inclusion $V(\sigma_0) \hookrightarrow X_\Sigma$ induces a section $X_{\overline{\Sigma}'} \hookrightarrow X_\Sigma$ of the fibration ψ whose image is $V(\sigma_0)$.

Similarly, for every maximal cone $\widehat{\sigma} \in \widehat{\Sigma}(n')$ the toric subvariety $V(\widehat{\sigma})$ is a fiber of ψ , isomorphic to X_{Σ_0} . It follows from Lemma 1.9 that $\Sigma_0(1) = J_{\overline{\mathcal{R}}}^+$ has cardinality $1 + n_0$, that is, X_{Σ_0} is a complete simplicial toric variety of Picard number 1.

In the divisorial and small cases the contracted locus in X_Σ is the subvariety $V_- := V(\gamma_-)$ where $\gamma_- = \text{Cone}(u_\rho, \rho \in J_{\overline{\mathcal{R}}}) \in \Sigma$ and its image by ψ is $V_+ := V(\gamma'_+)$ where $\gamma'_+ = \text{Cone}(u_\rho, \rho \in J_{\overline{\mathcal{R}}}^+) \in \Sigma'$. The restriction $\psi|_{V_-} : V_- \rightarrow V_+$ is a fibering extremal contraction. In particular there exist a complete simplicial toric subvariety $V(\widehat{\sigma}_-) \subseteq V_- \subset X_\Sigma$ of Picard number 1, that is a fiber of ψ .

As pointed by K. Matsuki in Remark 14-2-3 of [Mat02], these subvarieties need not be weighted projective spaces in general and may instead be quotient of those by a finite abelian group, often called fake weighted projective spaces.

This lead to the following definition :

Definition 1.10. *Well formed weighted projective spaces together with fake weighted projective spaces are exactly the simplicial complete toric varieties of Picard number one. We call them generalized weighted projective spaces.*

The fact that non trivial fibers of toric extremal contractions are generalized weighted projective spaces play a determinant role in the interplay between the different notions of low toric degree.

1.5.2 Cones of positive numerical classes

In addition to the Mori cone $\overline{\text{NE}}(X_\Sigma)$ who plays a central role in the MMP, several cones appear naturally when studying the birational geometry of a projective variety. First the cone of nef divisors, or nef cone, $\text{Nef}(X_\Sigma)$ plays also an important role because it is the dual cone of $\overline{\text{NE}}(X_\Sigma)$: nef divisors are those divisors D having nonnegative intersection number with every effective curve.

It is worth noticing that the contraction of an extremal ray \mathcal{R} of $\overline{\text{NE}}(X_\Sigma)$ is a particular case of the proper toric morphism $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ of Theorem 1.2 : the one associated to any nef divisor D such that the hyperplane $D^\perp := \{[C] \in N_1(X_\Sigma) \mid C \cdot D = 0\}$ intersects $\overline{\text{NE}}(X_\Sigma)$ exactly in \mathcal{R} (such a divisor exists precisely because \mathcal{R} is extremal).

Another interesting couple of dual cones of numerical classes is formed by the effective cone $\text{Eff}(X_\Sigma)$ generated by divisor classes $[D]$ containing at least one effective representative and the cone of mobile curves, or mobile cone, $\text{Mob}(X_\Sigma)$ generated by curve classes $[C]$ whose members move in families covering the whole variety X_Σ .

Since the effective cone is generated by the classes of toric divisors D_ρ , mobile curves are those curves C on X_Σ verifying $J_C^- = \emptyset$ and $J_C = J_C^+$.

The generators of the mobile cone have been characterized by D. Monsôres in his Ph.D thesis [Mon12] : they are the mobile classes $[C]$ such that $\text{Card}(J_C) = \dim(\text{Span}(J_C)) + 1$, and are numerical pullbacks of mobile and extremal curves on a minimal model of X_Σ .

2 First results

In this section we show that a Weil divisor D of a simplicial complete toric variety X_Σ has automatically a rational point over \mathbb{Z} if the Newton polytope of its defining homogeneous polynomial $f \in S_{[D]}$ is too small, which is in particular the case of divisors that are not Cartier and nef. Then we give a first criterion of existence of a rational point based on Theorem 0.2 : the divisor D admits a K -point as soon as its restriction $D|_W$ has lower degree than the anticanonical divisor $-K_W$ of a toric subvariety $W \subseteq X_\Sigma$ isomorphic to a generalized weighted projective space.

2.1 Trivial rational points

It is a trivial fact that a hypersurface of degree d of the projective space \mathbb{P}^n whose homogeneous equation does not feature the monomial x_0^d , automatically contains the \mathbb{Z} -point $(1 : 0 : \dots : 0)$. More generally, as soon as the Newton polytope of a homogeneous polynomial does not contain all the monomials of the form x_i^d , it admits trivial integer solutions. This phenomenon also occurs for a hypersurface D of a complete simplicial toric variety X_Σ endowed with homogeneous coordinate (i.e. satisfying hypothesis (H3), see 1.2) with the role of the special points $(0 : \dots : 0 : 1 : 0 : \dots : 0)$ played by the distinguished points

$$\gamma_\sigma := \bigcap_{\rho \in \sigma(1)} D_\rho \quad \text{for } \sigma \in \Sigma(n).$$

and the role of the extremal monomials x_i^d played by the D' -homogenization of the Cartier data $\{m_\sigma(D') \in M_{\mathbb{R}} \mid \sigma \in \Sigma(n)\}$ for any torus-invariant divisor D' linearly equivalent to D .

Here is the precise result

Lemma 2.1. *Let X_Σ be a simplicial complete toric split variety satisfying hypothesis (H3) and D be an effective divisor on X_Σ , zero locus of the homogeneous polynomial $f \in S_{[D]}$. Let $D' = \sum_{\rho \in \Sigma(1)} a'_\rho D_\rho \in [D]$ be a torus-invariant divisor linearly equivalent to D .*

If the Newton polytope of f does not contain the element

$$a^\sigma([D]) = h_{D'}(m_\sigma(D')) = (\langle m_\sigma(D'), u_\rho \rangle + a'_\rho)_{\rho \in \Sigma(1)} \in \mathbb{R}^{\Sigma(1)}$$

for some $\sigma \in \Sigma(n)$, then D contains the corresponding torus-invariant point $\gamma_\sigma = \bigcap_{\rho \in \sigma(1)} D_\rho$.

Proof — Let us write

$$f = \sum_{a \in \mathcal{A}_{[D]}} \lambda_a \prod_{\rho} x_\rho^{a_\rho} \in S_{[D]},$$

and let us denote by \mathcal{A}_f the set of monomials in $\mathcal{A}_{[D]}$ such that $\lambda_a \neq 0$. By definition the Newton polytope of f is the convex hull Δ_f of \mathcal{A}_f in $\mathbb{R}^{\Sigma(1)}$. Now let us suppose that there exists $\sigma \in \Sigma$ such that $a^\sigma([D])$ does not belong to Δ_f . In particular we have $a^\sigma([D]) \notin \mathcal{A}_f$.

Since $a^\sigma([D])$ is the only element of $\mathcal{A}_{[D]}$ satisfying $a_\rho^\sigma = 0$ for all $\rho \in \sigma(1)$, for all $a \in \mathcal{A}_f$ there exists at least one $\rho \in \sigma(1)$ with $a_\rho \neq 0$.

It follows that putting $\alpha_\rho = 0$ for all $\rho \in \sigma(1)$ and $\alpha_\rho = 1$ for all $\rho \notin \sigma(1)$ gives a root $\alpha \in \{0, 1\}^{\Sigma(1)}$ of f and since $\pi(\alpha) = \gamma_\sigma$ this gives the result. \square

The homogenization of the Cartier data $\{a^\sigma([D]) \mid \sigma \in \Sigma(n)\}$ is a set of lattice points in $\mathbb{Z}^{\Sigma(1)}$ if and only if D is a Cartier divisor and it is contained in the polytope $P_{[D]}$ if and only if D is a nef divisor. The consequence is the following

Proposition 2.2. *Let X_Σ be a simplicial complete split toric variety and D be an effective divisor on X_Σ . If one of the following conditions is satisfied :*

- (i) D is not a nef divisor,
- (ii) X_Σ satisfies hypothesis (H3) and D is not a Cartier divisor,

then D contains a torus invariant point $\gamma_\sigma = \bigcap_{\rho \in \sigma(1)} D_\rho$ for some $\sigma \in \Sigma(n)$, and hence a rational point over \mathbb{Z} .

Proof — If D is not nef then there exists a toric curve C_τ , $\tau \in \Sigma(n-1)$, such that $C_\tau \cdot D < 0$. This implies that C_τ is included in D , and for all $\sigma \in \Sigma(n)$ containing τ we have $\gamma_\sigma \in C_\tau$ which gives the result. If (H3) is verified and D is not Cartier then D is the zero locus of some homogeneous polynomial $f \in S_{[D]}$ and the hypotheses of the previous lemma are verified. The proposition is proved. \square

In the sequel we assume D to be a nef Cartier divisor.

2.2 A simple criterion (Restricted low toric degree)

In order to find rational points in a Cartier divisor D , a first simple idea is to consider the restriction $D|_W$ of D to a weighted projective space W contained in the toric ambient X_Σ and apply Theorem 0.2.

For the convenience of the reader, we reproduce here Theorem 0.2 with its proof.

Theorem (Th. 6.7 p.232 [Kol96]). *Let K be a C_1 field admitting normic forms of arbitrary degree. Let $K[x_0, \dots, x_n]$ be the polynomial algebra graded by $\deg(x_i) = a_i \in \mathbb{N}^*$. Let $f_1, \dots, f_r \in K[x_0, \dots, x_n]$ be homogeneous polynomials for this graduation. If*

$$\sum_{j=1}^r \deg(f_j) < \sum_{i=0}^n a_i,$$

then the system of equations $f_1 = \dots = f_r = 0$ has a non trivial solution in K^{n+1} .

Proof — For $i = 0, \dots, n$ let $g_i(y_{i,1}, \dots, y_{i,a_i})$ be normic forms of degree a_i over K . The g_i 's are homogeneous polynomials of the graded algebra $A = K[y_{i,k_i}, 0 \leq i \leq n, 1 \leq k_i \leq a_i]$ (with the usual grading coming from $\mathbb{P}^{a_0 + \dots + a_n - 1}$).

Now for $j = 1, \dots, r$ let us put

$$F_j := f_j(g_0, \dots, g_n) \in K[y_{i,k_i}, 0 \leq i \leq n, 1 \leq k_i \leq a_i].$$

By construction, F_j is a homogeneous polynomial in A , of degree $\deg(f_j)$ and its number of variables y_{i,k_i} is $a_0 + \dots + a_n$. Since $\sum_{j=1}^r \deg(f_j) < \sum_{i=0}^n a_i$, it follows by Lang's Theorem 4 in [Lan52] that the equation $F_1 = \dots = F_r = 0$ has a solution (b_{i,k_i}) such that $b_{s,t} \neq 0$ for at least one couple (s, t) . As g_s is normic, $c_s := g_s(b_{s,1}, \dots, b_{s,a_s})$ is non zero. Hence putting

$$c_i = g_i(b_{i,1}, \dots, b_{i,a_i}) \quad i = 0, \dots, n$$

yields a non trivial solution of the equation $f_1 = \dots = f_r = 0$. \square

In fact Theorem 0.2 applies as well to generalized weighted projective spaces admitting homogeneous coordinates because in this case any non trivial solution automatically avoids the exceptional set (see 1.2). Here is the precise result :

Corollary 2.3. *Let K be a C_1 field satisfying hypothesis (H1) and W be a generalized weighted projective space over K satisfying hypotheses (H2) and (H3). Let $H_1, \dots, H_k \subset X$ be hypersurfaces in W . For any effective 1-cycle class $[C] \in \overline{NE}(W)$, if*

$$C \cdot (H_1 + \dots + H_k) < C \cdot (-K_W),$$

then the intersection $H_1 \cap \dots \cap H_k$ has a rational point over K .

By focusing on generalized weighted projective spaces that are toric subvarieties of the ambient X_Σ we get an easily handled first notion of low toric degree for divisors that implies the existence of a rational point.

Definition 2.4. *Let X_Σ be a normal toric variety. A Cartier divisor D on X_Σ is said to have restricted low toric degree if there exists a complete simplicial toric subvariety $W \subseteq X_\Sigma$ of Picard number one, such that the restriction $D|_W$ has low degree in the sense of Corollary 2.3 : there exists a 1-cycle C on W such that*

$$C \cdot D|_W < C \cdot (-K_W).$$

Remarks 2.5. (i) This definition does not require the ambient toric variety X_Σ to be simplicial nor projective, and not even to satisfy hypothesis (H3).

(ii) A toric subvariety is an irreducible torus invariant subvariety $\mathbf{V}(\alpha)$, $\alpha \in \Sigma$, we could have considered other subvarieties that are also toric varieties but the $\mathbf{V}(\alpha)$ are sufficient because of their link with the extremal classes (see 1.5.1).

Corollary 2.6. *Let K be a C_1 field satisfying hypothesis (H1) and X_Σ be a simplicial complete toric variety over K satisfying hypothesis (H2) and such that any toric subvariety of Picard number one satisfies hypothesis (H3) (for example X_Σ smooth).*

A restricted low degree divisor on X_Σ has a rational point over K .

Notice that every toric curve C_τ is a normal separated toric variety of dimension 1 and hence is isomorphic to \mathbb{P}^1 . In particular a divisor D has a rational point as soon as its intersection number with one of the toric curves is 1.

Of course this bound can usually be improved for non rigid toric curves, for example by considering the biggest simplicial toric subvariety W of Picard number 1 containing them. In particular one fundamental property of extremal curves is to be mobile in such a generalized weighted projective space W , namely, the general non trivial fiber of the corresponding extremal contraction. It follows that a divisor has restricted low toric degree as soon as it has low degree along an extremal curve, in the precise sense of the following Lemma.

Lemma 2.7. *Let X_Σ be a simplicial projective toric variety and D an effective Cartier divisor on X_Σ . If there exists an extremal curve C on X_Σ such that*

$$0 < C \cdot D < \sum_{\rho \in J_C^+} C \cdot D_\rho$$

then D has restricted low toric degree.

Proof — Since $[C]$ is extremal, it is the class of a toric curve C_τ for some $\tau \in \Sigma(n-1)$. Putting

$$\alpha = \bigcap_{\rho \in J_C^+} \Delta_\rho(\tau),$$

we have $J_C^- \subseteq \alpha(1)$, $J_C^+ = \text{Adj}(\alpha)$ and $W = \mathbf{V}(\alpha)$ is a generalized weighted projective space (fiber of the contraction associated to $[C]$, see 1.5.1). The 1-cycle $C = C_\tau$ is thus the pushforward $j_* \overline{C}$ of a 1-cycle \overline{C} on W and using the projection formula, we have

$$\overline{C} \cdot D|_W = j_* \overline{C} \cdot D = C \cdot D.$$

Since this number is positive, $D|_W$ is a non trivial effective (and hence ample) divisor of W . It is then sufficient to prove that $C \cdot D < \overline{C} \cdot (-K_W)$. We recall from 9 that we have $\overline{C} \cdot (-K_W) \geq j_* C \cdot \sum_{\rho \in \text{Adj}(\alpha)} D_\rho$, so that by hypothesis

$$C \cdot D < \sum_{\rho \in J_C^+} C \cdot D_\rho = \sum_{\rho \in \text{Adj}(\alpha)} C \cdot D_\rho \leq \overline{C} \cdot (-K_W).$$

The lemma is proved. \square

One important property of the restricted low toric degree condition for D is its stability through toric morphisms ϕ such that $\phi^* \phi_* [D] = [D]$. More precisely we have the following result.

Proposition 2.8. *Let X_Σ be a simplicial projective toric variety and $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}}$ be a toric morphism. Let \overline{D} be a Cartier divisor on $X_{\overline{\Sigma}}$. If $D = \phi^* \overline{D}$ has restricted low toric degree then so has \overline{D} .*

Proof — This is a consequence of the effect of toric morphisms on generalized weighted projective spaces : they are either contracted to a point or left unchanged. If D has restricted low toric degree there exist a toric subvariety $W \subset X_\Sigma$ of Picard number 1 and a 1-cycle C on W such that

$$0 < C \cdot D|_W < C \cdot (-K_W).$$

If we denote by $j : W \hookrightarrow X_\Sigma$ the inclusion map, we have

$$j_* C \cdot D = C \cdot j^* D = C \cdot D|_W > 0,$$

and using the projection formula, this implies that

$$\phi_* j_* C \cdot \overline{D} = j_* C \cdot \phi^* \overline{D} > 0.$$

It follows that $j_* C$ is not contracted by ϕ and that the restriction of ϕ to W is an isomorphism

$$\phi|_W : W \xrightarrow{\sim} \overline{W}.$$

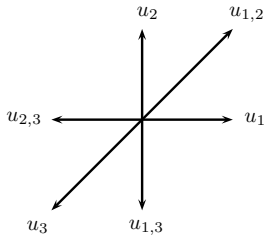
In particular in \overline{W} we again have

$$0 < \overline{C} \cdot \overline{D}|_{\overline{W}} < C \cdot (-K_{\overline{W}})$$

and \overline{D} has restricted low toric degree, which proves the proposition. \square

The main limit of the notion of restricted low toric degree is that it does not account for the specificities of non ample divisors, as illustrated by the following example.

Example 2.9. The Losev-Manin space \overline{L}_3 is a del Pezzo surface of degree 6 obtained from \mathbb{P}^2 by blowing-up the three torus-invariant points (or any three non aligned points up to projectivity). Here are one possible fan for this variety together with the intersection table of its 6 toric divisors :



\cdot	D_1	$D_{1,2}$	D_2	$D_{2,3}$	D_3	$D_{1,3}$
D_1	-1	1	0	0	0	1
$D_{1,3}$	1	-1	1	0	0	0
D_2	0	1	-1	1	0	0
$D_{2,3}$	0	0	1	-1	1	0
D_3	0	0	0	1	-1	1
$D_{1,3}$	1	0	0	0	1	-1

Let $C \subset \mathbb{P}^2$ be a conic that contains none of the three blown-up points, and let $\phi : \overline{L}_3 \rightarrow \mathbb{P}^2$ be the contraction of the three exceptional curves. Since ϕ is an isomorphism outside these three curves, the pullback $D = \phi^*C$ is sent isomorphically to C by ϕ and in particular admits a rational point over any C_1 field. Though, it has intersection number 2 with each strict transform of a toric line of \mathbb{P}^2 , and 0 with each exceptional curve. Indeed if $[L]$ denotes the class of the pullback of any line in \mathbb{P}^2 we have $[D] = 2[L]$ and

$$\begin{aligned} [L] &= [D_{1,3} + D_1 + D_{1,2}] = [D_{1,3} + D_1 + D_{1,2}] = [D_{1,3} + D_1 + D_{1,2}] \\ \Rightarrow D_1 \cdot L &= D_2 \cdot L = D_3 \cdot L = 1 \quad \text{and} \quad D_{1,2} \cdot L = D_{1,3} \cdot L = D_{2,3} \cdot L = 0. \end{aligned}$$

Since every toric subvariety of \overline{L}_3 with Picard number 1 is a \mathbb{P}^1 , this implies that D is not a restricted low toric degree divisor.

Obviously one would expect a good notion of low toric degree to consider this D as a low toric degree divisor. This motivates the search for a notion more stable by birational modifications.

◇

3 Low degree along a 1-cycle (Global low toric degree)

The goal of this section is to give some combinatorial conditions on the numerical class $[D]$ implying the existence of rational points in the divisor D . Contrary to the results of last section, we try here to take the most possible advantage of the specificities of D . In particular we deepen the description of the links between the morphism $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ of Theorem 1.2 and the Newton polytope $P_{[D]}$ of the homogeneous polynomial f defining D , and use it to determine when a root of f in K gives a K -rational point. This yields the notion of global low toric degree.

3.1 Rational points in the total space

In order to generalize Lemma 2.7, one can try to apply Corollary 2.3 from an inequality of the form

$$0 < C \cdot D < \sum_{\rho \in I} C \cdot D_\rho$$

with an arbitrary 1-cycle C . As soon as each D_ρ for $\rho \in I$ has positive degree along C and there are at least two of them, this yields a root of the homogeneous polynomial defining D but not a rational point in general.

Lemma 3.1. *Let D be a nef Cartier divisor on X_Σ and $f \in S_{[D]}$ a homogeneous polynomial such that $D = \mathbf{V}(f)$ as in 1.2. If there exist a 1-cycle C and a subset $I \subseteq J_C^+$ with cardinal at least 2 such that*

$$0 < C \cdot D < \sum_{\rho \in I} C \cdot D_\rho \tag{10}$$

then f has a root $\alpha \in K^{\Sigma(1)}$ such that

$$\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\} \subseteq J_C \setminus \{\rho_1\}$$

for some $\rho_1 \in I$.

Proof — Since f is homogeneous for the graduation by $\text{Cl}(X_\Sigma)$, it is a fortiori homogeneous for the graduation by \mathbb{Z} defined by

$$\deg(x_\rho) = mC \cdot D_\rho \tag{11}$$

where m is any positive integer such that $mC \cdot D_\rho \in \mathbb{Z}$ for all $\rho \in \Sigma(1)$. Let $d \in \mathbb{N}^*$ denote the degree of f for this graduation.

Let us consider the polynomial $g \in S' = K[x_\rho, \rho \in I]$ defined from f by putting $x_\rho = 1$ for all $\rho \in \Sigma(1) \setminus J_C$ (for which $\deg(x_\rho) = 0$) and $x_\rho = 0$ for all $\rho \in J_C \setminus I$.

The polynomial g is again homogeneous of degree d for the graduation of S' induced by the graduation of S defined by 11, which is the graduation of a weighted projective space. Up to multiplying by m , we can thus apply Theorem 0.2 to inequality 10 to get a non trivial zero $\alpha' \in K^I$ of g . This in turn yields a zero $\alpha \in K^{\Sigma(1)}$ of f with $\alpha_\rho = \alpha'_\rho$ for $\rho \in I$, $\alpha_\rho = 0$ for $\rho \in J_C \setminus I$ and $\alpha_\rho = 1$ for $\rho \in \Sigma(1) \setminus J_C$. By construction, we have

$$\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\} \subset J_C$$

and the fact that α' is not trivial exactly means that $\alpha_{\rho_1} \neq 0$ for some $\rho_1 \in I$, which terminates the proof. \square

Lemma 3.1 does not automatically provide a K -rational point in D since for a root $\alpha \in K^{\Sigma(1)}$ of f we have

$$\begin{aligned} \pi(\alpha) \in X_\Sigma &\Leftrightarrow \alpha \notin Z(\Sigma) \\ &\Leftrightarrow \{\rho \in \Sigma(1) \mid \alpha_\rho = 0\} \subseteq \sigma(1) \text{ for some cone } \sigma \in \Sigma. \end{aligned} \quad (12)$$

If we require that for each $\rho \in I$ there exists a cone $\sigma \in \Sigma$ such that $J_C \setminus \{\rho\} \subseteq \sigma(1)$, then obviously 12 is satisfied. But this turns out to be too restrictive a condition and, as shown in next subsection, we only need that each $J_C \setminus \{\rho\}$ is contained in a cone of the (eventually degenerate) fan obtained from Σ by removing all the walls $\tau \in \Sigma(n-1)$ such that $C_\tau \cdot D = 0$.

3.2 The Newton polytope of basepoint free divisors

As we have seen in the preliminaries, to a torus-invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ can be associated a polytope $P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho\}$ in the space of characters $M_{\mathbb{R}}$. To such a polytope are turn associated two objects that only depend on the numerical class $[D]$:

- The normal fan $\Sigma_{P_D} = \Sigma_{[D]}$ whose cones are in duality with the faces of P_D
- The ‘‘canonical’’ polytope $P_{[D]}$, image of P_D through the embedding

$$h_D : M_{\mathbb{R}} \longrightarrow \mathbb{R}^{\Sigma(1)}, \quad m \longmapsto (\langle m, u_\rho \rangle + a_\rho)_{\rho \in \Sigma(1)} \quad (13)$$

If D has no trivial rational point (see Subsection 2.1) then $P_{[D]}$ is naturally identified with the Newton polytope of the homogeneous polynomial defining D . In other words the lattice points of this polytope are the exponent vectors of the monomials belonging to the graded piece $S_{[D]}$ in the total coordinate ring.

Let us describe more precisely the relation between these two objects.

The correspondence between cones of the generalized fan $\Sigma_{P_D} = \Sigma_{[D]}$ and faces of the polytope P_D is one-to-one and dimension reversing :

$$\alpha' \in \Sigma_{P_D} \longleftrightarrow Q_D(\alpha') = \{m \in P_D \mid \langle m, u_\rho \rangle = -a_\rho \text{ for all } \rho \subseteq \alpha'\} \subset P_D$$

Through the embedding h_D of 13, this induces the contravariant correspondence

$$\begin{aligned} \{\text{Cones of } \Sigma_{[D]}\} &\longleftrightarrow \{\text{Faces of } P_{[D]}\} \\ \alpha' &\longmapsto Q_{[D]}(\alpha') \end{aligned} \quad (14)$$

where

$$Q_{[D]}(\alpha') = \{(a_\rho) \in P_{[D]} \mid a_\rho = 0 \text{ for all } \rho \subseteq \alpha'\}.$$

Now let us assume that D is Cartier and nef. In particular the Cartier data $\{m_\sigma = m_\sigma(D) \mid \sigma \in \Sigma(n)\}$ is the set of vertices of the polytope P_D and it follows that the set of vertices of the image $P_{[D]} = h_D(P_D)$ is the set $\{a^\sigma = a^\sigma(D) = h_D(m_\sigma(D)) \in P_{[D]} \mid \sigma \in \Sigma(n)\}$. Moreover, since we have $m_{\sigma_1} = m_{\sigma_2}$ if and only if there exists $\sigma' \in \Sigma_{[D]}$ such that $\sigma_1 \cup \sigma_2 \subseteq \sigma'$, the set of vertices of $P_{[D]}$ is also the set $\{a^{\sigma'} \in P_{[D]} \mid \sigma' \in \Sigma_{[D]}(n)\}$ where

$$\{\rho \in \Sigma(1) \mid a_\rho^{\sigma'} = 0\} = \{\rho \in \Sigma(1) \mid \rho \subseteq \sigma'\}. \quad (15)$$

Since Σ refines $\Sigma_{[D]}$, to each cone α of Σ is naturally associated a cone of $\Sigma_{[D]}$: the smallest that contains α . We generalize this notion with the following definition:

Definition 3.2. Let $\phi : X_\Sigma \rightarrow X_{\Sigma_{[D]}}$ be the toric morphism of Theorem 1.2. Let $c \subset \Sigma(1)$ be a set of 1-cones of Σ all contained in a single cone α of $\Sigma_{[D]}$. We call ϕ -hull of c and denote by \bar{c}^ϕ the smallest cone of $\Sigma_{[D]}$ containing c :

$$\bar{c}^\phi := \bigcap_{\substack{\alpha \in \Sigma_{[D]} \\ c \subset \alpha}} \alpha \subset N_{\mathbb{R}}.$$

For any cone $\alpha \in \Sigma_{[D]}$, the face $Q_{[D]}(\alpha)$ of the polytope $P_{[D]}$ is by definition the intersection of $P_{[D]}$ with the coordinate hyperplanes corresponding to all the 1-dimensional cones $\rho \in \Sigma(1)$ contained in α . The following result shows that we can characterize the face $Q_{[D]}(\alpha)$ by intersections with the coordinate hyperplanes corresponding to any set c of 1-dimensional cones $\rho \in \Sigma(1)$ such that $\bar{c}^\phi = \alpha$:

Lemma-Definition 3.3. Let D be a nef Cartier divisor on X_Σ and $c \subset \Sigma(1)$ be a set of 1-cones of Σ all contained in a single cone of $\Sigma_{[D]}$. Then the set

$$Q_{[D]}(c) := \{(a_\rho) \in P_{[D]} \mid \forall \rho \in c, a_\rho = 0\}$$

is equal to the face $Q_{[D]}(\bar{c}^\phi)$ of $P_{[D]}$.

Remark 3.4. We use the same notation $Q_{[D]}(\cdot)$ for cones in $\Sigma_{[D]}$ and for sets of 1-dimensional cones in Σ . Of course for any $\alpha \in \Sigma_{[D]}$, by putting (with a slight abuse of notation) $\alpha(1) = \{\rho \in \Sigma(1) \mid \rho \subseteq \alpha\}$, we have $\alpha = \text{Cone}(\rho, \rho \in \alpha(1))$ and by the above definition $Q_{[D]}(\alpha(1)) = Q_{[D]}(\alpha)$. What the lemma asserts is that we can in some cases choose a set c strictly smaller than $\alpha(1)$ but still verifying $Q_{[D]}(c) = Q_{[D]}(\alpha)$.

Proof — Since $c \subseteq \bar{c}^\phi$ we clearly have the inclusion

$$Q_{[D]}(\bar{c}^\phi) = \{(a_\rho) \in P_{[D]} \mid a_\rho = 0 \text{ for all } \rho \subseteq \bar{c}^\phi\} \subseteq Q_{[D]}(c).$$

For the reverse inclusion, it follows from the correspondence 14 that the face $Q_{[D]}(\bar{c}^\phi)$ is the convex hull of those vertices $v^{\sigma'}$ of $P_{[D]}$ verifying $\bar{c}^\phi \subset \sigma'$. But for all $\sigma' \in \Sigma_{[D]}$ we have by Definition 3.2:

$$c \subseteq \sigma' \Leftrightarrow \bar{c}^\phi \subseteq \sigma',$$

so that in fact we have

$$Q_{[D]}(\bar{c}^\phi) = \text{Conv}(v^{\sigma'} \mid \sigma' \in \Sigma_{[D]}(n), c \subseteq \sigma'). \quad (16)$$

Now take $a = (a_\rho) \in Q_{[D]}(c) \subset P_{[D]}$. Since $P_{[D]} = \text{Conv}(v^{\sigma'} \mid \sigma' \in \Sigma_{[D]}(n))$ there exist $\sigma'_1, \dots, \sigma'_k \in \Sigma_{[D]}(n)$ such that

$$a = \sum_{i=1}^k \lambda_i v^{\sigma'_i} \quad \text{with } \lambda_i \in \mathbb{Q}_+^* \text{ for all } 1 \leq i \leq k.$$

Then for all $\rho \in c$ we have $0 = a_\rho = \sum_{i=1}^k \lambda_i v_\rho^{\sigma'_i}$ with $v_\rho^{\sigma'_i} \in \mathbb{N}$. Since the λ_i are positive, it follows that for all $\rho \in c$ and all $1 \leq i \leq k$, $v_\rho^{\sigma'_i} = 0$. By 15 this implies that for $i = 1, \dots, k$ every 1-dimensional cone of c is contained in σ'_i and hence by 16 $v^{\sigma'_i}$ is contained in the face $Q_{[D]}(\bar{c}^\phi)$. It follows that $a \in Q_{[D]}(\bar{c}^\phi)$ and the lemma is proved. \square

Corollary 3.5. *Let D be a nef Cartier divisor and C be a 1-cycle on X_Σ . We have*

$$J_C \text{ is contained in a single cone of } \Sigma_{[D]} \Rightarrow C \cdot D = 0,$$

and the implication is an equivalence if $[C]$ is the numerical class of an irreducible curve.

Proof — Suppose there exists a cone $\sigma \in \Sigma_{[D]}(n)$ containing all the 1-dimensional cones of J_C . Then by 15, the corresponding vertex a^σ of the polytope $P_{[D]}$ verifies $a_\rho^\sigma = 0$ for all $\rho \in J_C$. In particular the associated local representative $D^\sigma = \sum_{\rho \in \Sigma(1)} a_\rho^\sigma D_\rho \in [D]$ verifies

$$C \cdot D = C \cdot D^\sigma = C \cdot \sum_{\rho \in J_C} a_\rho^\sigma D_\rho = 0.$$

Now suppose that $C \cdot D = 0$ and that C is irreducible. By [Pay06b, Prop. 1] there exists a cone $\alpha \in \Sigma$ such that $J_C^- \subseteq \alpha(1)$ and $J_C^+ \subseteq \text{Adj}(\alpha)$. Let $\sigma \in \Sigma_{[D]}(n)$ containing α and consider the corresponding local representative $D^\sigma \in [D]$. We have $a_\rho^\sigma = 0$ for all $\rho \in J_C^-$ and the equality

$$0 = C \cdot D^\sigma = \sum_{\rho \in J_C} a_\rho^\sigma C \cdot D_\rho = \sum_{\rho \in J_C^+} a_\rho^\sigma C \cdot D_\rho.$$

implies that $a_\rho^\sigma = 0$ for all $\rho \in J_C^+$ also. Then it follows from 15 that $J_C \subset \sigma$ and the corollary is proved. \square

The main consequence of Lemma 3.3, from the arithmetic point of view is that a root of the homogeneous polynomial f yields a rational point in $D = \mathbf{V}(f)$ as soon as the set $\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\}$ indexing its zero coordinates is contained in a single cone of the (eventually degenerate) fan $\Sigma_{[D]}$.

Proposition 3.6. *Let K be a C_1 field satisfying hypothesis (H1) and X_Σ be a simplicial projective toric variety satisfying hypotheses (H2) and (H3). Let D be a Cartier nef divisor on X_Σ and let $f \in S_{[D]}$ be the homogeneous polynomial such that*

$$D = \mathbf{V}(f) = \{\pi(x) \in X_\Sigma \mid f(x) = 0\}$$

where $\pi : \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \rightarrow X_\Sigma$ is the good geometric quotient. The following conditions are equivalent :

- (1) *There exists a root $\alpha \in K^{\Sigma(1)}$ such that the set $\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\}$ is contained in a single cone of $\Sigma_{[D]}$.*
- (2) *There exists a root $\alpha \in K^{\Sigma(1)}$ such that the set $\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\}$ is contained in a single cone of Σ .*

If these conditions are satisfied then D has a rational point over K .

Proof — The facts that the first condition implies the second and the existence of a rational point are trivial. We have to show that if f has a root α with all zero coordinates indexed by 1-cones of Σ contained in a single cone σ' of $\Sigma_{[D]}$ then it has a root β with all zero coordinates indexed by generators of a cone σ of Σ .

Let us write

$$f = \sum_{a \in \mathcal{A}_{[D]}} \lambda_a x^a \in S_{[D]}.$$

and suppose f has a root α such that the set $c = \{\rho \in \Sigma(1) \mid \alpha_\rho = 0\}$ is contained in some cone of $\Sigma_{[D]}$. Let us consider the polynomial g_1 obtained from f by putting $x_\rho = 0$ for all $\rho \in c$:

$$g_1 = \sum_{a \in Q_{[D]}(c)} \lambda_a x^a \in K[x_\rho, \rho \notin c].$$

Since $\alpha_\rho = 0$ for all $\rho \in c$ the root $\alpha \in K^{\Sigma(1)}$ gives a root $\alpha' \in (K^*)^{\Sigma(1) \setminus c}$ of g_1 .

Since c is contained in a single cone of $\Sigma_{[D]}$, by Definition 3.3 we can put $\sigma' = \bar{c}^\phi \in \Sigma_{[D]}$ and consider the polynomial g_2 obtained from f by putting $x_\rho = 0$ for all $\rho \subseteq \sigma'$:

$$g_2 = \sum_{a \in Q_{[D]}(\sigma')} \lambda_a x^a \in K[x_\rho, \rho \in J_{\sigma'}], \quad J_{\sigma'} := \{\rho \in \Sigma(1) \mid \rho \not\subseteq \sigma'\}$$

By Lemma 3.3 we have $Q_{[D]}(c) = Q_{[D]}(\sigma')$ so that g_1 can be considered as an element of $K[x_\rho, \rho \in J_{\sigma'}]$ equal to g_2 . By forgetting the α_ρ for $\rho \subseteq \sigma'$ we then get a root $\alpha'' \in (K^*)^{J_{\sigma'}}$ of g_2 .

Now let us remark that for every $\sigma \in \Sigma$ of the same dimension as σ' and such that $\sigma \subseteq \sigma'$, we have $\bar{\sigma}^\phi = \sigma'$. Let us fix such a σ and consider the polynomial g_3 obtained from f by putting $x_\rho = 0$ for all $\rho \in \sigma(1)$:

$$g_3 = \sum_{a \in Q_{[D]}(\sigma(1))} \lambda_a x^a \in K[x_\rho, \rho \notin \sigma(1)].$$

Again by Lemma 3.3, g_3 can be considered as an element of $K[x_\rho, \rho \in J_{\sigma'}]$ equal to g_2 and by putting $\beta'_\rho = \alpha''_\rho$ for all $\rho \in J_{\sigma'}$ and $\beta'_\rho = 1$ for all $\rho \in \Sigma(1)$ such that $\rho \subset (\sigma' \setminus \sigma)$, we get a root $\beta' \in (K^*)^{\Sigma(1) \setminus \sigma(1)}$ of g_3 . Finally by putting $\beta_\rho = \beta'_\rho$ for all $\rho \in \Sigma(1) \setminus \sigma(1)$ and $\beta_\rho = 0$ for all $\rho \in \sigma(1)$, we get the desired root $\beta \in K^{\Sigma(1)}$:

$$f(\beta) = g_3(\beta') = g_2(\alpha'') = g_1(\alpha') = f(\alpha) = 0.$$

The proposition is proved. \square

3.3 Global low toric degree

In this subsection we define the notion of global low toric degree, based on Lemma 3.1 and Proposition 3.6, and describe its most important properties. In particular we show that it can be lifted through toric birational contractions (Proposition 3.15).

Definition 3.7. *Let X_Σ be a complete simplicial toric variety. Let D be a nef Cartier divisor on X_Σ , let $\phi : X_\Sigma \rightarrow X_{\Sigma_{[D]}}$ be the toric morphism of Theorem 1.2, contracting all the rational curves having zero intersection with D and $\Sigma_{[D]}$ be the corresponding (eventually degenerate) fan in $N_{\mathbb{R}}$.*

We say that D has global low toric degree if there exist a 1-cycle C on X_Σ and a subset $I \subseteq J_C^+$ with cardinality at least 2 such that the following conditions hold :

$$0 < C \cdot D < \sum_{\rho \in I} C \cdot D_\rho \tag{A}$$

and

$$\forall \rho \in I, \exists \sigma \in \Sigma_{[D]}(n), J_C \setminus \{\rho\} \subset \sigma. \tag{B}$$

Remarks 3.8. (i) The notion of global low toric degree depends only on the numerical class of the divisors. For this reason, we will sometimes tell that the classes itself have (or not) global low toric degree.

(ii) When $C = C_\tau$ is a toric curve there is a minimal subset I of cardinality 2 satisfying condition (B) : the set $\sigma_1(1) \cup \sigma_2(1) \setminus \tau(1)$ where $\sigma_1, \sigma_2 \in \Sigma(n)$ are such that $\tau = \sigma_1 \cap \sigma_2$. Indeed it is a subset of every *primitive collection* contained in $\sigma_1(1) \cup \sigma_2(1)$ (see [CLS11, 5.1] for a definition).

Theorem 3.9. *Let K be a C_1 field satisfying hypothesis (H1) and X_Σ be a complete simplicial toric variety satisfying hypotheses (H2) and (H3). Let D be an effective nef Cartier divisor on X_Σ defined over K . If D has global low toric degree then it has a rational point over K .*

Proof — Let $f \in S_{[D]}$ be a homogeneous polynomial of degree $[D]$ such that

$$D = \mathbf{V}(f) = \{\pi(x) \in X_\Sigma \mid f(x) = 0\}$$

where π is the good geometric quotient. Using A, Lemma 3.1 gives us a root $\alpha \in K^{\Sigma(1)}$ such that

$$\{\rho \in \Sigma(1) \mid \alpha_\rho = 0\} \subseteq J_C \setminus \{\rho\}$$

for some $\rho \in I$. Then by use of B, Proposition 3.6 allows us to construct a rational point of D out of α . This proves the theorem. \square

Proposition 3.10. *In Definition 3.7 we may assume that the class $[C]$ satisfies the following additional properties :*

- (i) *There exist a cone $\alpha \in \Sigma$ such that $J_C^- \subseteq \alpha(1)$. In particular the class $[C]$ is effective.*
- (ii) *The set J_C is minimally linearly dependent, i.e., the space of linear relations between the minimal generators u_ρ for $\rho \in J_C$ has dimension one. In other words we have :*

$$\forall [C_1] \in \overline{\text{NE}}(X_\Sigma), J_{C_1} \subseteq J_C \Rightarrow [C_1] \in \mathbb{R}_+[C].$$

In particular the global low toric degree condition can be checked on a finite number of 1-cycles.

In order to prove Proposition 3.10 we need the following technical lemma.

Lemma 3.11. *Let D be a global low toric degree divisor on X_Σ and let C be a 1-cycle and $I \subseteq J_C^+$ a subset satisfying conditions (A) and (B) of Definition 3.7. Let $\gamma = \overline{J_C \setminus I}^\phi \in \Sigma_{[D]}$ be the minimal cone of $\Sigma_{[D]}$ containing $J_C \setminus I$ and let k be its dimension. For any subset $\mathcal{B} \subset \Sigma(1)$ of cardinality k such that $\{u_\rho \mid \rho \in \mathcal{B}\}$ is linearly independent and included in γ , there exists a 1-cycle C' such that (C', I) satisfies conditions (A) and (B) and $J_{C'} \setminus I \subseteq \mathcal{B}$.*

Proof — By induction on the cardinality of $(J_C \setminus I) \setminus \mathcal{B}$ it is sufficient to prove that for any $\rho_0 \in (J_C \setminus I) \setminus \mathcal{B}$ there exists a 1-cycle C' such that (C', I) satisfies conditions (A) and (B) and $J_{C'} \setminus I \subseteq J_C \setminus \{\rho_0\} \cup \mathcal{B}$. Let $\rho_0 \in (J_C \setminus I) \setminus \mathcal{B}$. Since $\rho_0 \in \gamma$ and the real span of \mathcal{B} is equal to the one of γ , we have a relation

$$\sum_{\rho \in \mathcal{B} \cup \{\rho_0\}} \lambda_\rho u_\rho = 0, \quad \lambda_\rho \in \mathbb{Q}, \lambda_{\rho_0} > 0.$$

Let C_0 be a 1-cycle whose class is given by this relation (i.e. such that $C_0 \cdot D_\rho = \lambda_\rho$ if $\rho \in \mathcal{B} \cup \{\rho_0\}$ and $C_0 \cdot D_\rho = 0$ otherwise). By Corollary 3.5, $J_{C_0} \subseteq \gamma$ implies $C_0 \cdot D = 0$. Similarly $C \cdot D > 0$ implies that J_C is not contained in a single cone of $\Sigma_{[D]}$. In particular for all $\rho \in I$ we have $J_C \not\subseteq \overline{J_C \setminus \{\rho\}}^\phi$ and hence $\rho \notin \overline{J_C \setminus \{\rho\}}^\phi$. A fortiori we have $\rho \not\subseteq \gamma$ and hence $\rho \notin J_{C_0}$, that is $C_0 \cdot D_\rho = 0$.

It follows that putting $C' = \lambda_{\rho_0} C - (C \cdot D_{\rho_0}) C_0$ we have

$$0 < C' \cdot D = \lambda_{\rho_0} C \cdot D < \lambda_{\rho_0} \sum_{\rho \in I} C \cdot D_\rho = \sum_{\rho \in I} C' \cdot D_\rho,$$

which shows that condition (A) is satisfied by (C', I) .

Moreover, by construction we have $J_{C'} \subseteq J_C \cup J_{C_0}$ and

$$C' \cdot D_{\rho_0} = \lambda_{\rho_0} C \cdot D_{\rho_0} - (C \cdot D_{\rho_0}) C_0 \cdot D_{\rho_0} = 0.$$

It follows that $J_{C'} \subseteq J_C \cup J_{C_0} \setminus \{\rho_0\} \subseteq J_C \setminus \{\rho_0\} \cup \mathcal{B}$ as expected.

Finally for all $\rho \in I$ we have $J_{C'} \setminus \{\rho\} \subseteq J_C \setminus \{\rho\} \cup \mathcal{B} \subseteq \overline{J_C \setminus \{\rho\}}^\phi \cup \gamma = \overline{J_C \setminus \{\rho\}}^\phi$ and (C', I) satisfies also condition (B). The lemma is proved. \square

Proof of Proposition 3.10 — Let C be a 1-cycle and $I \subseteq J_C^+$ a subset satisfying conditions (A) and (B). Let $\gamma = \overline{J_C \setminus I}^\phi \in \Sigma_{[D]}$ be the smallest cone of $\Sigma_{[D]}$ containing $J_C \setminus I$ and let k be its dimension. Let $\alpha \in \Sigma(k)$ be a cone of Σ contained in γ . By Lemma 3.11 there exists a 1-cycle C' such that (C', I) satisfies conditions (A) and (B) and $J_{C'} \setminus I \subseteq \alpha(1)$.

In particular $[C'] \in \overline{\text{NE}}(X_\Sigma) = \text{Nef}(X_\Sigma)^\vee$ because any nef class $[D']$ contains an effective divisor whose support does not contain $V(\alpha)$ and hence verifies $[C'] \cdot [D'] \geq 0$. This proves part (i).

It remains to prove that for all $[C_1] \in \overline{\text{NE}}(X_\Sigma)$ such that $J_{C_1} \subseteq J_{C'}$ we have $[C_1] \in \mathbb{R}_+[C']$.

Let $[C_1] \in \overline{\text{NE}}(X_\Sigma)$ be an effective class of curve such that $J_{C_1} \subseteq J_{C'}$ and let us put $\gamma_\rho = \overline{J_{C'} \setminus \{\rho\}}^\phi$ for the smallest cone of $\Sigma_{[D]}$ containing $J_{C'} \setminus \{\rho\}$.

Let us suppose first that there exists $\rho_1 \in I$ such that $\rho_1 \notin J_{C_1}$.

The fact that $J_{C_1} \subseteq J_{C'} \setminus \{\rho\}$ implies that the sets $J_{C_1}^+$ and $J_{C_1}^-$ are each contained in a single cone of the (possibly degenerate) fan $\Sigma_{[D]}$. Let us denote by $\beta_\pm = \overline{J_{C_1}^\pm}^\phi$ the smallest of these cones. Since we have $\sum_{\rho \in J_{C_1}} C_1 \cdot D_\rho u_\rho = 0$, we can consider the vector

$$v = \sum_{\rho \in J_{C_1}^+} \underbrace{C_1 \cdot D_\rho}_{>0} u_\rho = \sum_{\rho \in J_{C_1}^-} \underbrace{(-C_1 \cdot D_\rho)}_{>0} u_\rho$$

By construction, v lies in the relative interior of β_+ and β_- . But even if the complete fan $\Sigma_{[D]}$ is degenerate, the relative interior of its cones form a partition of $N_{\mathbb{R}}$ (with the convention that the minimal cone is its own relative interior). This implies that $\beta_+ = \beta_-$ and we can denote this cone by β . Note that by definition of β_+ and β_- we have $\beta = \overline{J_{C_1}^\pm}^\phi$.

Now let us take $\rho \in I$. We have either $\rho \notin J_{C_1}^+$ or $\rho \notin J_{C_1}^-$, that is either $J_{C_1}^+ \subseteq J_{C'} \setminus \{\rho\}$ or $J_{C_1}^- \subseteq J_{C'} \setminus \{\rho\}$. In either case this implies $\beta \subseteq \gamma_\rho$. As we have seen in the proof of Lemma 3.11, condition (A) implies that $\rho \not\subseteq \gamma_\rho$ because otherwise we would have $J_{C'} \subset \gamma_\rho$ and $C' \cdot D = 0$ by Corollary 3.5. It follows that $\rho \not\subseteq \beta$ and hence $\rho \notin J_{C_1}$. Since this is true for all $\rho \in I$, we have $I \cap J_{C_1} = \emptyset$ or equivalently $J_{C_1} \subseteq J_{C'} \setminus I \subseteq \alpha(1)$, which implies $[C_1] = 0$ since α is a simplicial cone.

This proves that any class $[C_1] \in \overline{\text{NE}}(X_\Sigma)$ such that $J_{C_1} \subseteq J_{C'}$ and $I \not\subseteq J_{C_1}$ is trivial.

Now let us consider $[C_2] \in \overline{\text{NE}}(X_\Sigma)$ with $J_{C_2} \subseteq J_{C'}$ and $I \subseteq J_{C_2}$. If we had $J_{C_2}^+ \subseteq J_{C'} \setminus I \subseteq \alpha(1)$ it would imply $[-C_2] \in \overline{\text{NE}}(X_\Sigma)$ by the argument we used to prove that $[C'] \in \overline{\text{NE}}(X_\Sigma)$. Since we cannot have $[C_2] \in \overline{\text{NE}}(X_\Sigma)$ and $[-C_2] \in \overline{\text{NE}}(X_\Sigma)$ it follows that there exists $\rho_2 \in J_{C_2}^+ \cap I$. Let us put $\lambda_2 = \frac{C_2 \cdot D_{\rho_2}}{C' \cdot D_{\rho_2}} \in \mathbb{Q}_+^*$ and $C_1 = C_2 - \lambda_2 C'$. By construction we have $J_{C_1} \subseteq J_{C'} \cup J_{C_2} = J_{C'}$ and

$$C_1 \cdot D_{\rho_2} = C_2 \cdot D_{\rho_2} - \lambda_2 C' \cdot D_{\rho_2} = 0.$$

It follows that $\rho_2 \notin J_{C_1}$ and in particular $I \not\subseteq J_{C_1}$, which implies $[C_1] = 0$ as above, that is $[C_2] = \lambda_2 [C']$. This proves part (ii).

The final assertion follows from the finiteness of the set of minimal linear relations and the invariance of conditions (A) and (B) by rescaling of C . \square

We now describe the behavior of the global low toric degree property under a toric birational contraction over the variety $X_{\overline{\Sigma}_{[D]}}$ defined out of $[D]$. Let us first precise this notion.

Definition 3.12. Let $X_{\overline{\Sigma}}$ be a projective toric variety. A toric variety X_Σ over $X_{\overline{\Sigma}}$ is a variety such that there exists a proper toric morphism $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}}$. By extension we say that $\xi : X_\Sigma \dashrightarrow X_{\Sigma'}$ is a toric rational map over $X_{\overline{\Sigma}}$ if both X_Σ and $X_{\Sigma'}$ lie over $X_{\overline{\Sigma}}$ and the corresponding toric morphisms ϕ and ϕ' make the following diagram commutative :

$$\begin{array}{ccc} X_\Sigma & \overset{\xi}{\dashrightarrow} & X_{\Sigma'} \\ & \searrow \phi & \swarrow \phi' \\ & X_{\overline{\Sigma}} & \end{array}$$

Lemma 3.13. *Let X_Σ be a complete simplicial toric variety and D be a nef Cartier divisor on X_Σ . Let $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ be the toric morphism of Theorem 1.2, contracting all the rational curves having zero intersection with D and let \overline{D} be the ample divisor on $X_{\overline{\Sigma}_{[D]}}$ such that $[D] = \phi^*[\overline{D}]$.*

Let $X_{\Sigma'}$ be another simplicial projective toric variety over $X_{\overline{\Sigma}_{[D]}}$. Then for all divisor D' on $X_{\Sigma'}$ such that $[D'] = \phi'^[\overline{D}]$ we have*

$$X_{\overline{\Sigma}_{[D']}} = X_{\overline{\Sigma}_{[D]}}.$$

Proof — Let us assume for beginning that there exists a morphism $\psi : X_{\Sigma'} \rightarrow X_\Sigma$. Then the composition $\phi \circ \psi$ is the toric morphism induced by the refinement of the generalized fan $\Sigma_{[D]}$ by the fan Σ' , hence it is the morphism ϕ' . Moreover $\phi \circ \psi$ contracts all the curves having zero intersection with any D' such that $[D'] = \phi'^*[\overline{D}]$. It follows that $X_{\overline{\Sigma}_{[D']}} = X_{\overline{\Sigma}_{[D]}}$ in this case.

Let us now turn to the general case. Let us denote by m and n the respective dimensions of Σ' and Σ . Let us consider a common refinement $\tilde{\Sigma}$ of the fans Σ and Σ' (or the corresponding degenerate fans of dimension $\max(m, n)$). These refinements induce morphisms $\psi : X_{\tilde{\Sigma}} \rightarrow X_\Sigma$ and $\psi' : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma'}$. Moreover, since X_Σ and $X_{\Sigma'}$ are varieties over $X_{\overline{\Sigma}_{[D]}}$, we have a commutative diagram

$$\begin{array}{ccc} & X_{\tilde{\Sigma}} & \\ \psi \swarrow & & \searrow \psi' \\ X_{\Sigma'} & & X_\Sigma \\ \phi \searrow & & \swarrow \phi' \\ & X_{\overline{\Sigma}_{[D]}} & \end{array}$$

If $[D'] = \phi'^*[\overline{D}]$ and $[\tilde{D}] = \psi^*[D]$, we have $[\tilde{D}] = \psi^*\phi^*[\overline{D}] = \psi'^*\phi'^*[\overline{D}] = \psi'^*[D']$ and by the reasoning made above, this implies

$$X_{\overline{\Sigma}_{[D']}} = X_{\overline{\Sigma}_{[\tilde{D}]}} = X_{\overline{\Sigma}_{[D]}}.$$

The lemma is proved. \square

Corollary 3.14. *Let X_Σ be a complete simplicial toric variety and D be a nef Cartier divisor on X_Σ . Let us consider a toric birational map $\xi : X_\Sigma \dashrightarrow X_{\Sigma'}$ over $X_{\overline{\Sigma}_{[D]}}$, and let $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ and $\phi' : X_{\Sigma'} \rightarrow X_{\overline{\Sigma}_{[D]}}$ be the corresponding proper morphisms.*

If D' is a divisor on $X_{\Sigma'}$ such that $[D'] = \xi_[D]$, we have $\Sigma_{[D']} = \Sigma_{[D]}$ and $\phi' : X_{\Sigma'} \rightarrow X_{\overline{\Sigma}_{[D]}}$ is the toric morphism contracting all the curves having zero intersection with D' .*

Proof — Let \overline{D} be the ample divisor on $X_{\overline{\Sigma}_{[D]}}$ such that $[D] = \phi^*[\overline{D}]$. Since $\phi = \phi' \circ \xi$ we have $[D] = \xi^*\phi'^*[\overline{D}]$ and since ξ is birational, this gives $[D'] = \xi_*[D] = \xi_*\xi^*\phi'^*[\overline{D}] = \phi'^*[\overline{D}]$. Then by Lemma 3.13 we have $X_{\overline{\Sigma}_{[D']}} = X_{\overline{\Sigma}_{[D]}}$ and since $[D'] = \phi'^*[\overline{D}]$ with \overline{D} ample, it follows that the curves in $X_{\Sigma'}$ having zero intersection with D' are exactly the ones whose numerical class belongs to the relative Mori cone $\overline{\text{NE}}(X_{\Sigma'}/\Sigma_{[D']})$. The corollary is proved. \square

Proposition 3.15. *Let X_Σ be a complete simplicial toric variety and D be a nef Cartier divisor on X_Σ . Let $\xi : X_\Sigma \dashrightarrow X_{\Sigma'}$ be a toric birational contraction over $X_{\overline{\Sigma}_{[D]}}$ such that $X_{\Sigma'}$ is simplicial. If $\xi_*[D]$ has global low toric degree, then so has $[D]$.*

Proof — We may assume that Σ and Σ' belong to the same real vector space $N_{\mathbb{R}}$ so that every 1-dimensional cone $\rho \in \Sigma'(1)$ can be considered as an element of $\Sigma(1)$. The associated toric divisors of $X_{\Sigma'}$ are denoted D'_ρ while those of X_Σ are denoted D_ρ .

By hypothesis the conditions of Definition 3.7 are satisfied by any divisor D' on $X_{\Sigma'}$ such that $[D'] = \xi_*[D]$: there exists a 1-cycle C' on $X_{\Sigma'}$ and a subset $I \subseteq J_{C'}^+$ with cardinality at least 2 such that

$$\forall \rho \in I, \exists \bar{\sigma} \in \Sigma_{[D']}, J_{C'} \setminus \{\rho\} \subseteq \bar{\sigma}$$

and

$$0 < C' \cdot D' < \sum_{\rho \in I} C' \cdot D'_\rho.$$

But since ξ is birational over $X_{\Sigma_{[D]}}$, it follows from Corollary 3.14 that $\Sigma_{[D']} = \Sigma_{[D]}$. Then we only have to remark that by definition of the numerical pullback of curves, if C is any 1-cycle on X_{Σ} such that $[C] = \xi_{num}^*[C']$, then we have $J_C = J_{C'}$, $[C] \cdot [D] = [C'] \cdot \xi_*[D]$ and $\sum_{\rho \in I} C \cdot D_\rho = \sum_{\rho \in I} C' \cdot D'_\rho$ so that conditions of Definition 3.7 are also satisfied by D . The proposition is proved. \square

3.4 Global vs. Restricted low toric degree

It follows from Proposition 3.15 that in Example 2.9, the pullback of the conic has global low toric degree while it has not restricted low toric degree. In fact, by definition, the restricted low toric degree condition can be viewed as a particular case of the global low toric degree one :

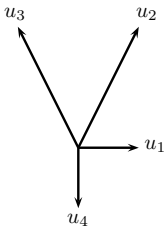
Lemma 3.16. *Let X_{Σ} be a normal toric variety. A Cartier divisor D on X_{Σ} has restricted low toric degree if and only if there exists a complete simplicial toric subvariety $W \subseteq X_{\Sigma}$ of Picard number one, such that the restriction $D|_W$ has global low toric degree.*

It is nevertheless important to observe that a divisor may have restricted low toric degree without having global low toric degree. Indeed the intersection numbers considered in Definition 3.7 are computed in the ambient variety X_{Σ} and can differ from the intersection numbers computed in a proper subvariety if X_{Σ} is singular, as noticed in paragraph 1.4.4. This is where the name global low toric degree comes from. Let us give an example.

Example 3.17. Let Σ be the 2-dimensional fan with the following minimal generators :

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Here is a drawing of Σ together with the intersection matrix of its 4 toric divisors :



\cdot	D_1	D_2	D_3	D_4
D_1	$-\frac{1}{2}$	$\frac{1}{2}$	0	1
D_2	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	0
D_3	0	$\frac{1}{4}$	$\frac{1}{4}$	1
D_4	1	0	1	2

Let D be a divisor linearly equivalent to $4D_3 + 2D_4$. We have

$$D_1 \cdot D = 2, \quad D_2 \cdot D = 1, \quad D_3 \cdot D = 3 \quad \text{and} \quad D_4 \cdot D = 8,$$

which shows that D is an ample divisor with restricted low toric degree since $D_2 \simeq \mathbb{P}^1$ and the degree of the restriction $D|_{D_2}$ is 1. But according to Proposition 3.18 below, D has not global low toric degree since the extremal classes of X_{Σ} are $[D_1]$ and $[D_2]$ and we have

$$D_1 \cdot D = 2 > \frac{3}{2} = D_1 \cdot (D_2 + D_4) \quad \text{and} \quad D_2 \cdot D = 1 > \frac{3}{4} = D_2 \cdot (D_1 + D_4).$$

\diamond

Of course, when X_Σ is smooth, the intersection numbers in the ambient and in its subvarieties coincide such that having restricted low toric degree implies having global low toric degree, the converse being false in general.

The following result shows that for ample divisors, global low toric degree implies restricted low toric degree.

Proposition 3.18. *Let X_Σ be a simplicial projective toric variety and D be an ample divisor on X_Σ . The following conditions are equivalent :*

(1) *There exist an effective 1-cycle C and a subset $I \subseteq \Sigma(1)$ such that*

$$C \cdot D < \sum_{\rho \in I} C \cdot D_\rho.$$

(2) *There exists an extremal class $[C]$ such that*

$$C \cdot D < \sum_{\rho \in J_C^+} C \cdot D_\rho.$$

Proof — The implication (2) \Rightarrow (1) is trivial. Let us show that (1) implies (2).

Let C be an effective 1-cycle and $I \subseteq \Sigma(1)$ a subset such that (1) holds. Since X_Σ is projective there exist extremal classes $[C_1], \dots, [C_m]$ such that

$$[C] = \sum_{i=1}^l \lambda_i [C_i] \text{ with } \lambda_1, \dots, \lambda_l \in \mathbb{Q}_+^*.$$

For all $1 \leq i \leq l$, we have by definition of $J_{C_i}^+$:

$$\sum_{\rho \in J_{C_i}^+} C_i \cdot D_\rho \geq \sum_{\rho \in J_{C_i}^+ \cap I} C_i \cdot D_\rho \geq \sum_{\rho \in I} C_i \cdot D_\rho.$$

It follows that

$$\sum_{i=1}^l \lambda_i C_i \cdot D = C \cdot D < \sum_{\rho \in I} C \cdot D_\rho = \sum_{i=1}^l \lambda_i \sum_{\rho \in I} C_i \cdot D_\rho \leq \sum_{i=1}^l \lambda_i \sum_{\rho \in J_{C_i}^+} C_i \cdot D_\rho,$$

so that we must have

$$C_i \cdot D < \sum_{\rho \in J_{C_i}^+} C_i \cdot D_\rho$$

for at least one $i \in \{1, \dots, l\}$.

This shows that (2) holds and the proposition is proved. \square

The very complementarity of the notions of restricted and global low toric degree comes from their behavior under toric maps (Propositions 2.8 and 3.15) and appears clearly in Theorem 5.1.

4 Decomposition of toric rational contractions

In this section we prove that a toric rational contraction between simplicial projective varieties admits a decomposition as a sequence of extremal flips, divisorial contractions and Mori fibrations in such a way that the flips are performed first and the fibrations last.

From the secondary fan viewpoint this result can be interpreted as a consequence of the convexity of the moving cone and the effective cone, and as such, it should admit a generalization to Mori Dream Spaces in general. However we give here a purely toric version and prove it by considerations on fans.

Proposition 4.1. *Let $X_\Sigma, X_{\overline{\Sigma}}$ be simplicial projective toric varieties and let $\xi : X_\Sigma \dashrightarrow X_{\overline{\Sigma}}$ be a toric rational contraction. There exist simplicial projective toric varieties $X_{\Sigma'}, X_{\Sigma''}$ and a factorization $\xi = \pi \circ \psi \circ \nu$ where*

- $\nu : X_\Sigma \dashrightarrow X_{\Sigma'}$ is a sequence of extremal flips,
- $\psi : X_{\Sigma'} \rightarrow X_{\Sigma''}$ is a sequence of extremal divisorial contractions, and
- $\pi : X_{\Sigma''} \rightarrow X_{\overline{\Sigma}}$ is a sequence of extremal fibering contractions.

$$\begin{array}{ccc} X_\Sigma & \xrightarrow{\nu} & X_{\Sigma'} \\ \xi \downarrow & & \downarrow \psi \\ X_{\overline{\Sigma}} & \xleftarrow{\pi} & X_{\Sigma''} \end{array}$$

We admit the existence of a factorization of ξ as a sequence of elementary steps of a MMP. It is an easy consequence of the fact that X_Σ is a Mori Dream Space (each step corresponding to a wall in the secondary fan) and can also be recovered by running some well chosen MMP without stopping at fibering contractions.

The proof of Proposition 4.1 is then a repeated application of the following two lemmas.

Lemma 4.2. *Let $X_{\Sigma_1}, X_{\Sigma_2}, X_{\Sigma'_2}$ be simplicial projective toric varieties. Suppose that there exist a sequence of extremal divisorial contractions $\psi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$ and a sequence of extremal flips $\nu : X_{\Sigma_2} \dashrightarrow X_{\Sigma'_2}$. Then there exist a simplicial projective toric variety $X_{\Sigma'_1}$, a sequence of extremal flips $\nu' : X_{\Sigma_1} \dashrightarrow X_{\Sigma'_1}$ and a sequence of extremal divisorial contractions $\psi' : X_{\Sigma'_1} \rightarrow X_{\Sigma'_2}$ making the following diagram commutative :*

$$\begin{array}{ccc} X_{\Sigma_1} & \xrightarrow{\nu'} & X_{\Sigma'_1} \\ \psi \downarrow & & \downarrow \psi' \\ X_{\Sigma_2} & \xrightarrow{\nu} & X_{\Sigma'_2} \end{array}$$

Proof — Let us denote by ρ_1, \dots, ρ_k the 1-dimensional cones of Σ_1 corresponding to the toric divisors contracted by ψ . We have $\Sigma'_2(1) = \Sigma_2(1) = \Sigma_1(1) \setminus \{\rho_1, \dots, \rho_k\}$. Let Σ'_1 be the fan obtained from Σ_2 by a sequence of star subdivisions at ρ_1, \dots, ρ_k . The refinement induces a birational toric morphism (i.e. a sequence of extremal divisorial contractions) $\psi' : X_{\Sigma'_1} \rightarrow X_{\Sigma'_2}$. Since $\Sigma'_1(1) = \Sigma_1(1)$, the rational map $\nu' : X_{\Sigma_1} \dashrightarrow X_{\Sigma'_1}$ making the diagram commutative is an isomorphism in codimension 1, i.e. a sequence of extremal flips. The lemma is proved. \square

Lemma 4.3. *Let $X_{\Sigma_1}, X_{\overline{\Sigma}_1}, X_{\overline{\Sigma}_2}$ be simplicial projective toric varieties. Suppose that there exist an extremal fibering contraction $\pi : X_{\Sigma_1} \rightarrow X_{\overline{\Sigma}_1}$ and a toric birational contraction $\xi : X_{\overline{\Sigma}_1} \dashrightarrow X_{\overline{\Sigma}_2}$. Then there exist a simplicial projective toric variety X_{Σ_2} , a toric birational contraction $\xi' : X_{\Sigma_1} \dashrightarrow X_{\Sigma_2}$ and an extremal fibering contraction $\pi' : X_{\Sigma_2} \rightarrow X_{\overline{\Sigma}_2}$ making the following diagram commutative :*

$$\begin{array}{ccc} X_{\Sigma_1} & \xrightarrow{\xi'} & X_{\Sigma_2} \\ \pi \downarrow & & \downarrow \pi' \\ X_{\overline{\Sigma}_1} & \xrightarrow{\xi} & X_{\overline{\Sigma}_2} \end{array}$$

Proof — Let us denote by $N_{\mathbb{R}}$ the space containing the fan Σ_1 , by $\overline{N}_{\mathbb{R}}$ the one containing $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$, and by $\overline{\pi} : N_{\mathbb{R}} \rightarrow \overline{N}_{\mathbb{R}}$ the linear map associated to π . Since π is an extremal fibering contraction, there exist two subfans $\widehat{\Sigma}_1$ and Σ_0 of Σ_1 such that $\Sigma_0 = \{\sigma \in \Sigma_1 \mid \sigma \subseteq \ker \overline{\pi}\}$ is the fan of the general fiber of π and $\overline{\pi}$ induces a bijection from $\widehat{\Sigma}_1$ to $\overline{\Sigma}_1$ such that we have

$$\Sigma_1 = \{\sigma_0 + \widehat{\sigma} \mid \sigma_0 \in \Sigma_0, \widehat{\sigma} \in \widehat{\Sigma}_1\}$$

In particular for each $\bar{\rho} \in \bar{\Sigma}_1(1)$ there is a unique $\rho \in \widehat{\Sigma}_1(1)$ such that $\bar{\pi}(\rho) = \bar{\rho}$. Since by hypothesis we have $\bar{\Sigma}_2(1) \subseteq \bar{\Sigma}_1(1)$ we can define the fan

$$\widehat{\Sigma}_2 = \left\{ \text{Cone}(\rho \in \widehat{\Sigma}_1(1) \mid \bar{\pi}(\rho) \in \bar{\sigma}(1)) \mid \bar{\sigma} \in \bar{\Sigma}_2 \right\}.$$

Then the fan

$$\Sigma_2 = \{ \sigma_0 + \hat{\sigma} \mid \sigma_0 \in \Sigma_0, \hat{\sigma} \in \widehat{\Sigma}_2 \}$$

is a simplicial fan such that $\bar{\pi}$ is compatible with Σ_2 in $N_{\mathbb{R}}$ and $\bar{\Sigma}_2$ in $\bar{N}_{\mathbb{R}}$, inducing a surjective morphism $\pi' : X_{\Sigma_2} \rightarrow X_{\bar{\Sigma}_2}$. The fact that $\Sigma_2(1) \subseteq \Sigma_1(1)$ implies the existence of a toric birational contraction $\xi' : X_{\Sigma_1} \dashrightarrow X_{\Sigma_2}$ making the diagram commutative. The lemma is proved. \square

5 Low toric degree : general definition

This section is devoted to state and prove the core theorem of the article, on which is based our definition of a low toric degree.

Theorem 5.1. *Let X_{Σ} be a simplicial projective toric variety.*

Let D be a nef Cartier divisor on X_{Σ} . Let $\phi : X_{\Sigma} \rightarrow X_{\bar{\Sigma}_{[D]}}$ be the toric morphism of Theorem 1.2, contracting all the rational curves having zero intersection with D and let \bar{D} be an ample divisor on $X_{\bar{\Sigma}_{[D]}}$ such that D is linearly equivalent to the pullback $\phi^\bar{D}$.*

The following conditions are equivalent :

- (i) *The ample divisor \bar{D} has restricted low toric degree.*
- (ii) *There exist a small modification $\nu : X_{\Sigma} \dashrightarrow X_{\Sigma'}$ over $X_{\bar{\Sigma}_{[D]}}$ and a subvariety $V' \subset X_{\Sigma'}$ such that the restriction $(\nu_*D)|_{V'}$ has global low toric degree.*
- (iii) *There exists a sequence of weighted blow-ups $\eta : X_{\bar{\Sigma}} \rightarrow X_{\Sigma}$ such that $X_{\bar{\Sigma}}$ is smooth and the divisor $\tilde{D} = \eta^*D$ has global low toric degree.*

Definition 5.2. *Let X_{Σ} be a simplicial projective toric variety. A Weil divisor D on X_{Σ} is said to have low toric degree if it is not nef and Cartier or if it is nef and Cartier and satisfies the conditions of Theorem 5.1.*

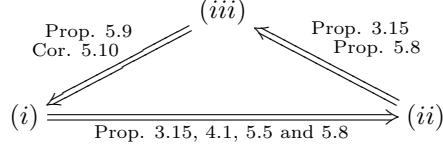
Remark 5.3. It is easily seen that if X_{Σ} is a weighted projective space then the divisor D has low toric degree if and only if it has low weighted degree in the sense of Theorem 0.2. In particular for $X_{\Sigma} = \mathbb{P}_K^n$, D has low toric degree if and only if its degree d verifies $d \leq n$ as in Definition 0.1.

Before proving Theorem 5.1, let us show that the low toric degree condition ensures as expected the existence of rational points over C_1 fields.

Corollary 5.4. *Let K be a C_1 field satisfying hypothesis (H1) and X_{Σ} be a simplicial projective toric variety satisfying hypotheses (H2) and (H3). Every low toric degree divisor on X_{Σ} admits a rational point over K .*

Proof — Let D be an effective divisor on X_{Σ} . If D is not Cartier and nef then it has a K -rational point by Proposition 2.2. If it is Cartier and nef then by hypothesis condition (iii) of Theorem 5.1 is satisfied. In particular the pullback \tilde{D} of D by η admits a rational point $p \in \tilde{D}$ by Theorem 3.9. Then since the morphism η is toric and every toric map is defined over K , the image of p by η is a rational point $\eta(p)$ in D . This proves the corollary. \square

The proof of the equivalences of Theorem 5.1 goes circularly according to the following scheme :

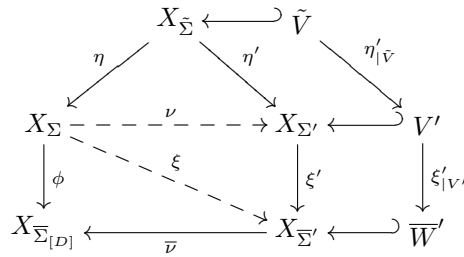


Let us give some details.

Sketch of the proof of Theorem 5.1 — Proving the implication $(i) \Rightarrow (ii)$ amounts to lift the restricted low toric degree condition of the ample divisor \bar{D} in $X_{\bar{\Sigma}_{[D]}}$ to a global low toric degree condition of some restriction $D'_{|V'}$ of its pullback D' to a subvariety V' of a well chosen small modification $X_{\Sigma'}$ of X_{Σ} .

To begin with we recall that by Lemma 3.16 the ample divisor \bar{D} has restricted low toric degree if and only if there exists a complete simplicial toric subvariety $\bar{W} \subset X_{\bar{\Sigma}_{[D]}}$ of Picard number 1, such that the restriction $\bar{D}_{|\bar{W}}$ has global low toric degree. Then we use the fact that the global low toric degree condition can be lifted by toric birational contraction between simplicial varieties (Proposition 3.15). The lifting is done in two steps :

- (1) First we prove the existence of a simplicial refinement $\bar{\Sigma}'$ of $\bar{\Sigma}_{[D]}$ such that $\bar{\Sigma}'(1) = \bar{\Sigma}_{[D]}(1)$ together with a toric subvariety $\bar{W}' \subset X_{\bar{\Sigma}'}$ such that the restriction $\bar{D}'_{|\bar{W}'}$ of the divisor $\bar{D}' = \bar{\nu}^* \bar{D}$ has global low toric degree (Proposition 5.5).
- (2) Then we consider the toric rational map $\xi : X_{\Sigma} \dashrightarrow X_{\bar{\Sigma}'}$ induced by the linear map $\bar{\phi} : N_{\mathbb{R}} \rightarrow \bar{N}_{\mathbb{R}}$. By construction it is surjective in codimension 1. This allows us to use Proposition 4.1 to decompose ξ as a small modification $\nu : X_{\Sigma} \dashrightarrow X_{\Sigma'}$ followed by a sequence of divisorial and fibering extremal contractions $\xi' : X_{\Sigma'} \rightarrow X_{\bar{\Sigma}'}$. Then Proposition 5.8 asserts that for such a sequence of contractions, each subvariety $\bar{W}' \subset X_{\bar{\Sigma}'}$ can be lift to $X_{\Sigma'}$ birationally, in the sense that there exists a $V' \subset X_{\Sigma'}$ such that $\xi'_{|V'} : V' \rightarrow \bar{W}'$ is birational. The implication follows.



The implication $(ii) \Rightarrow (iii)$ is easier. It is enough to choose $\tilde{\Sigma}$ to be a smooth common refinement of Σ and Σ' . The induced morphism $\eta' : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma'}$ is a sequence of weighted blow-ups so that Propositions 3.15 and 5.8 apply again and yield the existence of a subvariety $\tilde{V} \subset X_{\tilde{\Sigma}}$ such that the restriction $\tilde{D}_{|\tilde{V}}$ of the divisor $\tilde{D} = \eta^* D = \eta^* \nu_* D$ to \tilde{V} has global low toric degree. Then the smoothness of $X_{\tilde{\Sigma}}$ implies that \tilde{D} itself has global low toric degree.

Finally $(iii) \Rightarrow (i)$ is a direct application of the toric MMP. Since restricted low toric degrees can be descended by toric morphisms that are compositions of extremal contractions by Proposition 2.8, it is sufficient to prove that some pullback of \bar{D} by such a morphism has restricted low toric degree to get (i) . The idea is to mimic the proof of Proposition 3.18 to

prove that the global low toric degree of \tilde{D} implies the restricted low toric degree of its image by some birational contraction. Namely, starting with the inequalities

$$0 < C \cdot \tilde{D} < \sum_{\rho \in I} C \cdot \tilde{D}_\rho \quad (\text{A})$$

we consider a decomposition $[C] = \sum_{i=1}^l \lambda_i [C'_i]$ of $[C]$ as a positive linear combination of extremal classes and try to show that we can deduce inequalities similar to (A) with some C'_i instead of C . This is in general not possible in $X_{\tilde{\Sigma}}$ since some of the C'_i 's may fail to have positive intersection with \tilde{D} . The solution is therefore to get rid of these superfluous C'_i 's either by contracting them or by flipping them, which can be realized by running a logarithmic MMP with boundary divisor $B = \sum_{\rho \in \tilde{\Sigma} \setminus I} \tilde{D}_\rho$ relatively to the variety $X_{\tilde{\Sigma}_{[D]}}$ (Proposition 5.9). This yields a birational contraction $\mu : X_{\tilde{\Sigma}} \dashrightarrow X_{\tilde{\Sigma}''}$ over $X_{\tilde{\Sigma}_{[D]}} = X_{\tilde{\Sigma}_{[\tilde{D}]}}$ such that the divisor $D'' = \mu_* \tilde{D}$ has restricted low toric degree by Lemma 2.7. Moreover by construction D'' is linearly equivalent to the pullback of \tilde{D} by a proper toric morphism $\phi'' : X_{\tilde{\Sigma}''} \rightarrow X_{\tilde{\Sigma}_{[D'']}} = X_{\tilde{\Sigma}_{[D]}}$, which yields the last implication of Theorem 5.1. \square

Proposition 5.5. *In the conditions of Theorem 5.1, if the ample divisor \bar{D} has restricted low toric degree, then there exist a small surjective toric morphism $\bar{\nu} : X_{\tilde{\Sigma}'} \rightarrow X_{\tilde{\Sigma}_{[D]}}$ with $\tilde{\Sigma}'$ simplicial and a toric subvariety $\bar{W}' \subset X_{\tilde{\Sigma}'}$ such that the restriction $\bar{D}'_{|\bar{W}'}$ of the divisor $\bar{D}' = \bar{\nu}^* \bar{D}$ has global low toric degree.*

The proof of Proposition 5.5 is based upon Proposition 3.15 and the following lemma :

Lemma 5.6. *Let $X_{\tilde{\Sigma}}$ be a projective toric variety and $\bar{W} \subset X_{\tilde{\Sigma}}$ a toric subvariety. For all birational toric morphism $\bar{\nu} : X_{\tilde{\Sigma}'} \rightarrow X_{\tilde{\Sigma}}$ there exists a toric subvariety $\bar{W}' \subset X_{\tilde{\Sigma}'}$ such that the restriction $\bar{\nu}_{|\bar{W}'} : \bar{W}' \rightarrow \bar{W}$ is a birational toric morphism.*

Proof — Let us denote by \bar{N} the lattice of 1-parameter subgroups of the open torus $\bar{T} \subset X_{\tilde{\Sigma}}$, by α the cone of $\tilde{\Sigma}$ such that $\bar{W} = V(\alpha)$ and by $\bar{\pi} : \bar{N}_{\mathbb{R}} \rightarrow \bar{N}(\alpha)_{\mathbb{R}} = (\bar{N}/\bar{N}_\alpha) \otimes \mathbb{R}$ the linear map associated to the inclusion $\bar{W} \subset X_{\tilde{\Sigma}}$ (see 1.4.4). The fan of \bar{W} is $\text{Star}(\alpha) = \{\bar{\pi}(\sigma) \mid \alpha \preceq \sigma \in \tilde{\Sigma}\}$.

Now let $\tilde{\Sigma}'$ be a refinement of $\tilde{\Sigma}$. Choose a cone $\alpha' \in \tilde{\Sigma}'$ such that $\alpha' \subset \alpha$ and $\dim(\alpha') = \dim(\alpha)$, and put $\bar{W}' = V(\alpha')$. Since $N_{\alpha'} = N_\alpha$, the linear map associated to the inclusion $\bar{W}' \subset X_{\tilde{\Sigma}'}$ is again $\bar{\pi}$ and the fan of \bar{W}' is $\text{Star}(\alpha') = \{\bar{\pi}(\sigma') \mid \alpha' \preceq \sigma' \in \tilde{\Sigma}'\}$. But for all $\sigma \in \tilde{\Sigma}$ such that $\alpha \preceq \sigma$ there exists $\sigma' \in \tilde{\Sigma}'$ such that $\sigma' \subset \sigma$ and $\alpha' \preceq \sigma'$. It follows that the fan $\text{Star}(\alpha')$ is a refinement of $\text{Star}(\alpha)$ and the corresponding toric map $\bar{\nu}_{|\bar{W}'} : \bar{W}' \rightarrow \bar{W}$ is a birational toric morphism, which proves the lemma. \square

Proof of Proposition 5.5 — By Lemma 3.16, if the ample divisor \bar{D} has restricted low toric degree, then there exist a complete simplicial toric subvariety $\bar{W} \subset X_{\tilde{\Sigma}_{[D]}}$ of Picard number 1, such that the restriction $\bar{D}_{|\bar{W}}$ has global low toric degree.

If $\tilde{\Sigma}$ is simplicial then we can choose $\bar{\nu}$ to be the identity map of $X_{\tilde{\Sigma}}$ and there is nothing to prove.

If $\tilde{\Sigma}$ is not simplicial then the existence of a simplicial refinement $\tilde{\Sigma}'$ of $\tilde{\Sigma}$ such that the corresponding morphism $\bar{\nu} : X_{\tilde{\Sigma}'} \rightarrow X_{\tilde{\Sigma}}$ is small (that is $\tilde{\Sigma}'(1) = \tilde{\Sigma}(1)$) is proved in [CvR09, Th. 3.1].

Let $\tilde{\Sigma}'$ be such a simplicial refinement. It follows from Lemma 5.6 that there exists a toric subvariety $\bar{W}' \subset X_{\tilde{\Sigma}'}$ such that the restriction $\bar{\nu}_{|\bar{W}'} : \bar{W}' \rightarrow \bar{W}$ is a birational toric morphism. In particular the restriction $\bar{D}'_{|\bar{W}'}$, which is also the pullback $\bar{\nu}_{|\bar{W}'}^* \bar{D}_{|\bar{W}}$, has global low toric degree by Proposition 3.15. The proposition is proved. \square

Remark 5.7. It follows from the proof that in Proposition 5.5, one may replace the quantifier “there exists” by “for all” and that the corresponding subvarieties $\overline{W}' \subset X_{\Sigma'}$ may be chosen all birational to each other.

Proposition 5.8. *Let X_{Σ} and $X_{\Sigma'}$ be simplicial projective toric varieties and $\xi : X_{\Sigma} \rightarrow X_{\Sigma'}$ be a toric morphism, composition of extremal contractions of divisorial and/or fibering type. For any subvariety $V' \subset X_{\Sigma'}$, there exists a subvariety $V \subset X_{\Sigma}$ such that the restriction $\xi|_V : V \rightarrow V'$ is a birational toric morphism.*

Proof — By hypothesis it is sufficient to prove the result in the two special cases where ξ is an extremal contraction either of divisorial or of fibering type.

The fibering type case is the simplest because in this case there exists an equivariant section $s : X_{\Sigma} \rightarrow X_{\Sigma'}$ of ξ , so that taking $V = s(V')$ gives the result.

Let us turn to the divisorial type case. Let D_{ρ_0} denote the divisor of X_{Σ} contracted by ξ and $V(\alpha')$ its image in $X_{\Sigma'}$. We have

$$\Sigma = \{\sigma' \in \Sigma' \mid \alpha' \not\subseteq \sigma'\} \cup \{\text{Cone}(\sigma'(1) \setminus \{\rho\}) \cup \{\rho_0\} \mid \sigma' \in \Sigma', \alpha' \preceq \sigma', \rho \in \alpha'(1)\}$$

Let us denote by β' the cone of Σ' such that $V' = V(\beta')$. We can distinguish four cases :

- 1st case Let us suppose the intersection $V' \cap V(\alpha')$ empty, that is $\alpha' + \beta' \notin \Sigma'$. Then all the cones of Σ' containing β' are also cones of Σ . It follows that $\text{Star}(\beta')$ stays unchanged and by taking $V = V(\beta') \subset X_{\Sigma}$, the restriction $\xi|_V : V \rightarrow V'$ is an isomorphism.
- 2nd case Let us suppose that the intersection $V' \cap V(\alpha')$ has codimension ≥ 2 in V' , that is $\alpha + \beta' \in \Sigma'$ and $\text{Card}(\alpha'(1) \setminus \beta'(1)) \geq 2$. Then β' is again a cone of Σ and the star subdivision of α' in Σ' induces the star subdivision of the image of $\alpha' + \beta'$ in $\text{Star}(\beta')$. In particular by taking $V = V(\beta') \subset X_{\Sigma}$, the morphism $(\xi)|_V$ is the extremal divisorial contraction of $(D_{\rho_0})|_V$.
- 3rd case Let us now suppose that $V' \cap V(\alpha)$ has codimension 1 in V' , that is $\alpha(1) \setminus \beta'(1) = \{\rho_1\}$. Then by taking $V = V(\beta') \subset X_{\Sigma}$ once more, we have an isomorphism $(\xi)|_V : V \rightarrow V'$ because there is a one-to-one correspondence between cones $\sigma \in \Sigma'$ containing β' and cones $\text{Cone}(\sigma(1) \setminus \{\rho_1\}) \cup \{\rho_0\}$ in Σ and the latter have the same image in $\text{Star}(\beta')$ as the former.
- 4th case Finally if V' is a subvariety of $V(\alpha')$, that is $\alpha' \subset \beta'$, then $(\xi)|_{D_{\rho_0}} : D_{\rho_0} \rightarrow V(\alpha')$ is an extremal fibering contraction and hence have an equivariant section $s : V(\alpha) \rightarrow D_{\rho_0}$. In this case also, taking $V = s(V') \subset D_{\rho_0} \subset X_{\Sigma}$ yields an isomorphism $(\xi)|_V : V \rightarrow V'$.

In all the cases the restriction $\xi|_V : V \rightarrow V'$ is a birational toric morphism. The lemma is proved. \square

Proposition 5.9. *Let X_{Σ_1} be a simplicial projective toric variety. Let D_1 be a nef Cartier divisor on X_{Σ_1} , $\phi_1 : X_{\Sigma_1} \rightarrow X_{\overline{\Sigma}_{[D_1]}}$ be the toric morphism of Theorem 1.2 and $\Sigma_{[D_1]}$ be the corresponding (eventually degenerate) fan in $N_{\mathbb{R}}$.*

Suppose that there exist a 1-cycle C_1 on X_{Σ_1} and a subset $I_1 \subset J_{C_1}^+$ such that the following conditions are satisfied :

(a)

$$0 < C_1 \cdot D_1 < \sum_{\rho \in I_1} C_1 \cdot D_{\rho},$$

(b)

$$\exists \sigma \in \Sigma_{[D_1]}, J_{\overline{C}} \subset \sigma \quad \text{and} \quad \forall \rho \in I_1, \rho \not\subset \sigma.$$

Then there exist a toric birational contraction $\mu : X_{\Sigma_1} \dashrightarrow X_{\Sigma_2}$ over $X_{\overline{\Sigma}_{[D_1]}}$ such that $\mu_ D_1$ has restricted low toric degree.*

Proof — It is essentially an application of the relative logarithmic MMP for simplicial projective toric varieties. Let us consider the logarithmic MMP with boundary divisor $B = \sum_{\rho \in \Sigma(1) \setminus I} D_\rho$ relative to the variety $X_{\overline{\Sigma}_{[D_1]}}$. To begin with, we show that conditions (a) and (b) are invariant through each step of the MMP. For this let us describe the full algorithm, keeping track of what happens to $[D_1]$, $[C_1]$ and I_1 .

We start with $X_\Sigma = X_{\Sigma_1}$, $[B] = [B_1]$, $\phi = \phi_1$, $[D] = [D_1]$, $[C] = [C_1]$ and $I = I_1$ and do the following steps :

- (1) If $K_{X_\Sigma} + B$ is ϕ -nef, then stop.
- (2) If $K_{X_\Sigma} + B$ is not ϕ -nef, then there is an extremal ray R in the relative Mori cone $\overline{\text{NE}}(X_\Sigma/X_{\overline{\Sigma}_{[D_1]}})$ with $(K_{X_\Sigma} + B) \cdot R < 0$. Let $\psi : X_\Sigma \rightarrow X_{\Sigma'}$ be the corresponding extremal contraction.
- (3) If ψ is fibering, then stop.
- (4) If ψ is divisorial, then replace $(X_\Sigma, B, \phi, [D], [C], I)$ with $(X_{\Sigma'}, \psi_* B, \phi', \psi_* [D], \psi_* [C], I')$ where $\phi' : X_{\Sigma'} \rightarrow X_{\overline{\Sigma}_{[D_1]}}$ is the unique morphism satisfying $\phi = \phi' \circ \psi$ and $I' = I \cap \Sigma'(1)$. Return to step (1) and continue.
- (5) If ψ is flipping and $\nu : X_\Sigma \dashrightarrow X_\Sigma^+$ is the corresponding flip, replace $(X_\Sigma, B, \phi, [D], [C], I)$ with $(X_\Sigma^+, \nu_* B, \phi^+, \nu_* [D], \nu_* [C], I)$ where $\phi^+ : X_\Sigma^+ \rightarrow X_{\overline{\Sigma}_{[D_1]}}$ is the unique morphism satisfying $\phi = \phi^+ \circ \nu$. Return to step (1) and continue.

Notice that every toric map that appears during the process is a map over $X_{\overline{\Sigma}_{[D_1]}}$, i.e., at each step we have a toric morphism $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D_1]}} = X_{\overline{\Sigma}_{[D_1]}}$ contracting the curves that have zero intersection with D . Moreover since these maps are birational we always have $\Sigma_{[D]} = \Sigma_{[D_1]}$ in $N_{\mathbb{R}}$, therefore what we show is that conditions (a) and (b) are invariant *for a fixed* $\sigma \in \Sigma_{[D_1]}$.

Let us begin by the flipping case. The flip $\nu : X_\Sigma \dashrightarrow X_\Sigma^+$ is an isomorphism in codimension 1 and we have $\Sigma(1) = \Sigma^+(1)$. If we denote by D_ρ and D_ρ^+ the toric divisor corresponding to $\rho \in \Sigma(1)$ respectively in X_Σ and X_Σ^+ , then the pushforward of divisor classes is the isomorphism $\nu_* : N^1(X_\Sigma) \rightarrow N^1(X_\Sigma^+)$ induced by $\nu_* D_\rho = D_\rho^+$. By duality, the numerical pullback of curves is the isomorphism $\nu_{num}^* : N_1(X_{\Sigma^+}) \rightarrow N_1(X_\Sigma)$ defined by

$$\nu_{num}^*[C'] \cdot [D_\rho] = [C'] \cdot [D_\rho^+] \text{ for all } \rho \in \Sigma(1).$$

Its inverse isomorphism is the pushforward of curve classes so that we also have

$$\nu_*[C] \cdot [D_\rho^+] = [C] \cdot [D_\rho] \text{ for all } \rho \in \Sigma(1).$$

It follows that conditions (a) and (b) are satisfied by (D, C, I) if and only if they are satisfied by $(\nu_* D, \nu_* C, I)$.

Then let us turn to the divisorial case. The morphism $\psi : X_\Sigma \rightarrow X_{\Sigma'}$ is surjective and we have $\Sigma'(1) = \Sigma(1) \setminus \{\rho_0\}$ where D_{ρ_0} is the exceptional divisor in X_Σ . If we denote by D_ρ and D'_ρ the toric divisor corresponding to $\rho \in \Sigma'(1)$ respectively in X_Σ and $X_{\Sigma'}$, then the pushforward of divisor classes is the epimorphism $\psi_* : N^1(X_\Sigma) \rightarrow N^1(X_{\Sigma'})$ induced by $\psi_* D_\rho = D'_\rho$. By duality, the numerical pullback of curves is the monomorphism $\psi_{num}^* : N_1(X_{\Sigma'}) \rightarrow N_1(X_\Sigma)$ defined for $[C'] \in N_1(X_{\Sigma'})$ by

$$\psi_{num}^*[C'] \cdot [D_\rho] = \begin{cases} [C'] \cdot [D'_\rho] & \text{if } \rho \in \Sigma'(1) \\ 0 & \text{if } \rho = \rho_0. \end{cases}$$

If $[C_0]$ is a generator of the extremal ray contracted by ψ , then the pushforward of curve classes is the unique epimorphism (with abuse of notation) $\psi_* : N_1(X_\Sigma) \rightarrow N_1(X'_\Sigma)$ sending $[C_0]$ to 0 and admitting ψ_{num}^* as a section. In particular there exists a coefficient $\lambda_{[C]} \in \mathbb{R}$ such that

$$\psi_{num}^* \psi_* [C] = [C] + \lambda_{[C]} [C_0].$$

Since $[D]$ is the pullback of an ample class $[\overline{D}]$ on $X_{\overline{\Sigma}_{[D_1]}}$ and $\psi : X_\Sigma \rightarrow X_{\Sigma'}$ is a toric morphism over $X_{\overline{\Sigma}_{[D_1]}}$ we also have $\psi_* [D] = \phi'^* [\overline{D}]$ and hence $[D] = \psi^* \psi_* [D]$. In particular, by the projection formula we have

$$\psi_* [C] \cdot \psi_* [D] = [C] \cdot \psi^* \psi_* [D] = [C] \cdot [D]. \quad (17)$$

Now let $V' = V(\alpha')$, $\alpha' \in \Sigma'$, be the image of D_{ρ_0} by ψ . We may assume that the class $[C_0]$ is given by the relation

$$-u_{\rho_0} + \sum_{\rho \in \alpha'(1)} c_\rho u_\rho, \quad \text{with } c_\rho \in \mathbb{Q}_+^* \text{ for all } \rho \in \alpha'(1),$$

so that $\psi_{num}^* \psi_* [C] \cdot [D_{\rho_0}] = 0$ implies $\lambda_{[C]} = [C] \cdot [D_{\rho_0}]$.

Note that in this situation, by the definition of ϕ -hull (Definition 3.2) we have

$$\overline{\{\rho_0\}}^\phi = \overline{\alpha'(1)}^\phi = \overline{J_{C_0}}^\phi. \quad (18)$$

To simplify notations let us put $[C'] = \psi_* [C]$ and $[C] \cdot [D_\rho] = b_\rho$ for all $\rho \in J_C$ (in particular $\lambda_{[C]} = b_{\rho_0}$). Since $\psi_{num}^* [C'] = [C] + b_{\rho_0} [C_0]$, for $\rho \in \Sigma'(1)$ we have

$$[C'] \cdot [D'_\rho] = \psi_{num}^* [C'] \cdot [D_\rho] = \begin{cases} b_\rho & \text{if } \rho \in J_C \setminus J_{C_0} \\ b_\rho + b_{\rho_0} c_\rho & \text{if } \rho \in J_C \cap \alpha'(1) \\ b_{\rho_0} c_\rho & \text{if } \rho \in \alpha'(1) \setminus J_C \\ 0 & \text{if } \rho = \rho_0 \text{ or } \rho \notin J_C \cup J_{C_0}. \end{cases} \quad (19)$$

In particular we have $J_{C'} \subset J_C \setminus \{\rho_0\} \cup \alpha'(1)$.

We then distinguish three cases :

1st case If $\rho_0 \notin J_C$ then $[C'] \cdot [D'_\rho] = [C] \cdot [D_\rho]$ for all $\rho \in \Sigma'(1)$ and conditions (a) and (b) are satisfied by $([D], [C], I)$ if and only if they are satisfied by $(\psi_* [D], [C'], I)$.

2nd case If $\rho_0 \in J_C^-$ we claim that $[C_0] \cdot \sum_{\rho \in I} [D_\rho] = 0$, or equivalently that $[C_0] \cdot (K_{X_\Sigma} + B) = 0$, so that this case never occurs.

Indeed, it follows from 18 and condition (b) that

$$\overline{J_{C_0}}^\phi \subset \overline{J_C^-}^\phi \subset \overline{\Sigma}_{[D_1]}.$$

and for all $\rho \in I$, $\overline{\rho}^\phi \notin \overline{\Sigma}_{[D_1]}$. In particular we have $I \cap J_{C_0} = \emptyset$ and $[C_0] \cdot \sum_{\rho \in I} [D_\rho] = 0$ as claimed.

3rd case If $\rho_0 \in J_C^+$ then it follows from 19 that for all $\rho \in J_{C'}$ we have $[C'] \cdot [D'_\rho] \geq [C] \cdot [D_\rho]$. In particular $J_{C'} \subset J_C^- \subset \sigma$ and since $I' = I \cap \Sigma'(1) \subset I$ we have for all $\rho \in I'$, $\rho \notin \sigma$. This shows that condition (b) is satisfied by C' and I' .

For condition (a) we may assume that $\rho_0 \in I$ because otherwise we have $I' = I$ and condition (a) for C' and I' is the same as for C and I . Recall that with our notations we have

$$[C_0] \cdot [D_\rho] = \begin{cases} c_\rho & \text{if } \rho \in \alpha'(1) \\ -1 & \text{if } \rho = \rho_0 \\ 0 & \text{else.} \end{cases}$$

The condition $[C_0] \cdot [K_{X_\Sigma} + B] < 0$ coming from step (2) is equivalent to $[C_0] \cdot \sum_{\rho \in I} [D_\rho] > 0$, that is

$$\sum_{\rho \in \alpha'(1) \cap I} c_\rho > 1.$$

Since $[C'] \cdot [D'_{\rho_0}] = 0$ and $I \setminus J_{C_0} \subset J_C \setminus J_{C_0}$, we have by 19 :

$$\begin{aligned} \sum_{\rho \in I} [C'] \cdot [D'_\rho] &= \sum_{\rho \in I \setminus \{\rho_0\}} [C'] \cdot [D'_\rho] \\ &= \sum_{\rho \in \alpha'(1) \cap I} [C'] \cdot [D'_\rho] + \sum_{\rho \in I \setminus J_{C_0}} [C'] \cdot [D'_\rho] \\ &= \sum_{\rho \in \alpha'(1) \cap I} b_\rho + b_{\rho_0} c_\rho + \sum_{\rho \in I \setminus J_{C_0}} b_\rho \\ &= \sum_{\rho \in I \setminus \{\rho_0\}} b_\rho + \sum_{\rho \in \alpha'(1) \cap I} b_{\rho_0} c_\rho \\ &> \sum_{\rho \in I \setminus \{\rho_0\}} b_\rho + b_{\rho_0} = \sum_{\rho \in I} b_\rho = \sum_{\rho \in I} [C] \cdot [D_\rho]. \end{aligned}$$

Now with 17 this gives

$$0 < [C'] \cdot \psi_*[D] < \sum_{\rho \in I} [C'] \cdot [D'_\rho],$$

which shows that condition (a) is satisfied in this case too.

Finally, since steps (4) and (5) preserve conditions (a) and (b), we claim that step (3) cannot occur because every extremal class of fibering type in the relative Mori cone $\overline{\text{NE}}(X_\Sigma / X_{\overline{\Sigma}_{[D_1]}})$ have zero intersection with $(K_{X_\Sigma} + B)$. Indeed let $[C_0]$ be such a class. We have $J_{C_0} = J_{C_0}^+ \subset \ker \bar{\phi}$ so that by condition (b), $I \cap J_{C_0} = \emptyset$. In particular we have $[C_0] \cdot \sum_{\rho \in I} [D_\rho] = 0$, that is $[C_0] \cdot [K_{X_\Sigma} + B] = 0$ as claimed.

It follows that when the algorithm finishes, we get a variety X_{Σ_2} , a divisor D_2 , a morphism $\phi_2 : X_{\Sigma_2} \rightarrow X_{\overline{\Sigma}_{[D_1]}}$, a 1-cycle C_2 a set $I_2 \subset J_{C_2}^+$ and a boundary divisor B_2 such that conditions (a) and (b) are satisfied by (D_2, C_2, I_2) and the divisor $-\sum_{\rho \in I_2} D_\rho = K_{X_2} + B_2$ is ϕ_2 -nef. By construction, the composition μ of all the birational maps appearing at each step of the MMP is a birational toric contraction over $X_{\overline{\Sigma}_{[D_1]}}$ and we have $D_1 = \mu^* D_2$.

It remains to prove that $D_2 = \mu_* D_1$ has restricted low toric degree.

Since $J_{C_2}^-$ is contained in a single cone of Σ_2 by condition (b), we know that the class $[C_2]$ belongs to the Mori cone $\overline{\text{NE}}(X_{\Sigma_2})$. Let us consider its decomposition as a linear combination of extremal classes with positive rational coefficients :

$$[C_2] = \sum_{i=1}^l \lambda_i [C'_i] \quad \lambda_1, \dots, \lambda_l \in \mathbb{Q}_+^*.$$

Up to a change of numbering we may assume that there exists $k \in \{1, \dots, l\}$ such that $C'_i \cdot D_2 > 0$ for $i = 1, \dots, k$ and $C'_i \cdot D_2 = 0$ for $i = k+1, \dots, l$, that is, the extremal classes $[C'_{k+1}], \dots, [C'_l]$ are contracted by ϕ_2 . Since $-\sum_{\rho \in I_2} D_\rho$ is ϕ_2 -nef, we have

$$C'_i \cdot \sum_{\rho \in I_2} D_\rho \leq 0 \quad \text{for } i = k+1, \dots, l.$$

By condition (a) for (C_2, I_2) this implies

$$\sum_{i=1}^k \lambda_i C'_i \cdot \sum_{\rho \in I_2} D_\rho \geq \sum_{i=1}^l \lambda_i C'_i \cdot \sum_{\rho \in I_2} D_\rho = C_2 \cdot \sum_{\rho \in I_2} D_\rho > C_2 \cdot D_2 = \sum_{i=1}^k \lambda_i C'_i \cdot D_2.$$

Since all the λ_i are positive, it follows that there exists necessarily at least one index $i \in \{1, \dots, k\}$ such that

$$C'_i \cdot \sum_{\rho \in I_2} D_\rho > C'_i \cdot D_2.$$

Moreover, by definition of $J_{C'_i}^+$ we have

$$C'_i \cdot \sum_{\rho \in J_{C'_i}^+} D'_\rho \geq C'_i \cdot \sum_{\rho \in I' \cap J_{C'_i}^+} D'_\rho \geq C'_i \cdot \sum_{\rho \in I'} D'_\rho.$$

Finally, since $i \leq k$ we also have $C'_i \cdot D_2 > 0$ so that

$$0 < C'_i \cdot D_2 < \sum_{\rho \in I'} C'_i \cdot D'_\rho$$

and since C'_i is extremal, this implies that D_2 has restricted low toric degree by Lemma 2.7. This achieves the proof of the proposition. \square

Corollary 5.10. *Let X_Σ be a simplicial projective toric variety and D be a nef Cartier divisor on X_Σ . If D has global low toric degree then there exist a simplicial projective toric variety $X_{\Sigma''}$, a birational map $\xi : X_\Sigma \dashrightarrow X_{\Sigma''}$ surjective in codimension 1 and a nef Cartier divisor D' on $X_{\Sigma''}$ such that $[D] = \xi^*[D']$ and D' has restricted low toric degree.*

Proof — It is sufficient to show that whenever D is a nef Cartier divisor having global low toric degree on X_Σ , there exists a 1-cycle C and a subset $I \subseteq J_C^+$ satisfying conditions (a) and (b) of Proposition 5.9.

It is clear that condition (a) is just condition (A) of Definition 3.7, minus the condition on the cardinality of the indices set I . Let us suppose that condition (B) of the definition is verified and prove that so is condition (b) of the proposition.

For $\rho \in I$, let us denote by $\gamma_\rho = \overline{J_C \setminus \{\rho\}}^\phi$ the smallest cone of $\Sigma_{[D]}$ containing $J_C \setminus \{\rho\}$. Since the set J_C^- is contained in $J_C \setminus \{\rho\}$ for all $\rho \in I$, we have $J_C^- \subseteq \gamma = \bigcap_{\rho \in I} \gamma_\rho \in \Sigma_{[D]}$. Moreover, as shown in the proof of Lemma 3.11, for $\rho \in I$, we cannot have $\rho \subseteq \gamma$ because this would imply $J_C \subseteq \gamma_\rho$ and hence $C \cdot D = 0$ by Corollary 3.5. This shows that (b) is verified and the corollary is proved. \square

Proof of Theorem 5.1 — We prove the implications circularly : $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ Suppose that the ample divisor \overline{D} has restricted low toric degree. By Proposition 5.5 there exist a small surjective toric morphism $\overline{\nu} : X_{\overline{\Sigma}'} \rightarrow X_{\overline{\Sigma}_{[D]}}$ with $\overline{\Sigma}'$ simplicial and a toric subvariety $\overline{W}' \subset X_{\overline{\Sigma}'}$ such that the restriction $\overline{D}'_{|\overline{W}'}$ of the divisor $\overline{D}' = \overline{\nu}^* \overline{D}$ has global low toric degree.

Now let $\xi : X_\Sigma \dashrightarrow X_{\overline{\Sigma}'}$ be the equivariant rational map such that $\phi = \overline{\nu} \circ \xi$. It is a map over $X_{\overline{\Sigma}_{[D]}}$ by definition and it is surjective in codimension 1 since $\overline{\Sigma}'(1) = \overline{\Sigma}(1)$. Then by Proposition 4.1, there exist a small modification $\nu : X_\Sigma \dashrightarrow X_{\Sigma'}$ and a toric morphism $\xi' : X_{\Sigma'} \rightarrow X_{\overline{\Sigma}'}$ ($\xi' = \pi \circ \psi$), such that $\xi = \xi' \circ \nu$. Moreover by Proposition 5.8 there exists a subvariety $V' \subseteq X_{\Sigma'}$ such that the restriction $\xi'_{|V'} : V' \rightarrow \overline{W}'$ is a birational toric morphism.

$$\begin{array}{ccccc} X_\Sigma & \xrightarrow{\nu} & X_{\Sigma'} & \longleftarrow & V' \\ \phi \downarrow & \searrow \xi & \downarrow \xi' & & \downarrow \xi'_{|V'} \\ X_{\overline{\Sigma}_{[D]}} & \xleftarrow{\overline{\nu}} & X_{\overline{\Sigma}'} & \longleftarrow & \overline{W}' \end{array}$$

Since $\xi : X_\Sigma \dashrightarrow X_{\bar{\Sigma}}$ is a toric rational contraction over $X_{\bar{\Sigma}_{[D]}}$, so are ν and ξ' . In particular there is a morphism $\phi' : X_{\Sigma'} \rightarrow X_{\bar{\Sigma}_{[D]}}$ contracting all the curves that have zero intersection with ν_*D . If we denote by $\phi_{V'} : V' \rightarrow \bar{V}'$ the morphism of Theorem 1.2 contracting all the curves that have zero intersection with $(\nu_*D)|_{V'}$, we have $\phi_{V'} = \phi'_{|V'}$ and $\xi'_{|V'} : V' \rightarrow \bar{W}'$ is a birational morphism over \bar{V}' . Then by Proposition 3.15 the fact that $\bar{D}'_{|\bar{W}'}$ has global low toric degree implies that so has $(\nu_*D)|_{V'}$, which is (ii).

(ii) \Rightarrow (iii) Suppose that (ii) is verified. Consider the coarsest common refinement $\tilde{\Sigma}'$ of Σ and Σ' . It can be obtained by a sequence of weighted blow-ups corresponding to the star subdivisions at $\rho_0 = \bigcap_{\rho \in J_{\mathcal{R}}, [C_\tau] \in \mathcal{R}} \Delta_\rho(\tau)$ curves C_τ with $\tau \in \Sigma(1) \setminus \Sigma'(1)$ (see 1.5.1).

By [CLS11, Th. 11.1.9] there exists a smooth refinement $\tilde{\Sigma}$ of $\tilde{\Sigma}'$ that can also be obtained by a sequence of star subdivisions. It follows that the morphisms $\eta : X_{\tilde{\Sigma}} \rightarrow X_\Sigma$ and $\eta' : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma'}$ induced by the refinements of Σ and Σ' by $\tilde{\Sigma}$ are sequences of weighted blow-ups. In particular, by Proposition 5.8 there exists a subvariety $\tilde{V} \subseteq X_{\tilde{\Sigma}}$ such that the restriction $\eta'_{|\tilde{V}} : \tilde{V} \rightarrow V'$ is a birational toric morphism. Then it follows from (ii) and Proposition 3.15 that the restriction $\tilde{D}_{|\tilde{V}}$ of the divisor $\tilde{D} = \eta^*D = \eta'^*\nu_*D$ to \tilde{V} has global low toric degree. Finally since $X_{\tilde{\Sigma}}$ is smooth this implies that \tilde{D} itself has global low toric degree, which is (iii).

(iii) \Rightarrow (i) Suppose that (iii) is verified. First notice that since $[\tilde{D}] = \eta^*[D] = \eta^*\phi^*[\bar{D}]$, the morphism $\tilde{\phi} : X_{\tilde{\Sigma}} \rightarrow X_{\bar{\Sigma}_{[\tilde{D}]}}$ of Theorem 1.2, contracting all the curves having zero intersection with \tilde{D} is just $\tilde{\phi} \circ \eta$. In particular we have $X_{\bar{\Sigma}_{[\tilde{D}]}} = X_{\bar{\Sigma}_{[D]}}$.

Now by Corollary 5.10, there exist a simplicial projective toric variety $X_{\Sigma''}$, a birational map $\mu : X_{\tilde{\Sigma}} \dashrightarrow X_{\Sigma''}$ surjective in codimension 1 and a nef Cartier divisor D'' on $X_{\Sigma''}$ such that $\tilde{D} = \mu^*D''$ and D'' has restricted low toric degree.

It follows that $X_{\bar{\Sigma}_{[D'']}} = X_{\bar{\Sigma}_{[\tilde{D}]}} = X_{\bar{\Sigma}_{[D]}}$ and $[D'']$ is the pullback of $[\bar{D}]$ by the morphism $\phi'' : X_{\Sigma''} \rightarrow X_{\bar{\Sigma}_{[D]}}$ contracting all the curves that have zero intersection with D'' . In particular, by Proposition 2.8, the fact that D'' has restricted low toric degree implies that so has \bar{D} , which is (i).

This concludes the proof of Theorem 5.1. \square

6 More criteria

In this section we give the weakest and the most general criteria we could find to determine whether a given nef Cartier divisor has low toric degree. Then we give some more natural criteria, linking the notion of low toric degree to different notions of positivity for divisor classes.

6.1 Most general criterion

Let us first remark that along the proof we found a numerical condition weaker to the global low toric degree condition that implies the low degree one :

Lemma 6.1. *X_Σ be a simplicial projective toric variety and D be a nef Cartier divisor on X_Σ . Let $\Sigma_{[D]}$ be the (possibly degenerate) fan obtained from Σ by removing the codimension 1 cones $\tau \in \Sigma(n-1)$ such that $C_\tau \cdot D = 0$.*

Then D has low toric degree as soon as there exist a 1-cycle C on X_Σ and a subset $I \subseteq J_C^+$ verifying

$$0 < C \cdot D < \sum_{\rho \in I} C \cdot D_\rho, \quad (\text{a})$$

and

$$\exists \sigma \in \Sigma_{[D]}, J_C^- \subseteq \sigma \quad \text{and} \quad \forall \rho \in I, \rho \not\subseteq \sigma. \quad (\text{b})$$

Proof — It is sufficient to remark that conditions (a) and (b) allow to apply Proposition 5.9. Then it follows from Proposition 2.8 that \overline{D} has restricted low toric degree (see the proof of the implication (iii) \Rightarrow (i) of Theorem 5.1). \square

Then it follows from Theorem 5.1 and Definition 5.2 that the low toric degree property of the nef Cartier divisor D on X_Σ is essentially a property of its “ample model” \overline{D} on $X_{\overline{\Sigma}_{[D]}}$. In particular we have the following consequence

Theorem 6.2. *X_Σ be a simplicial projective toric variety and D be a nef Cartier divisor on X_Σ . Let $\phi : X_\Sigma \rightarrow X_{\overline{\Sigma}_{[D]}}$ be the toric morphism of Theorem 1.2, contracting all the rational curves having zero intersection with D and let \overline{D} be an ample divisor on $X_{\overline{\Sigma}_{[D]}}$ such that D is linearly equivalent to the pullback $\phi^*\overline{D}$.*

The divisor D has low toric degree if and only if one of the following equivalent conditions is satisfied :

- (1) *There exists a toric subvariety $V \subseteq X_\Sigma$ such that the restriction $D|_V$ has low toric degree.*
- (2) *There exists a simplicial toric variety $X_{\Sigma'}$ over $X_{\overline{\Sigma}_{[D]}}$ such that the pullback of \overline{D} in $X_{\Sigma'}$ has low toric degree.*
- (3) *There exist a simplicial toric variety $X_{\Sigma'}$ over $X_{\overline{\Sigma}_{[D]}}$, a subvariety $V' \subseteq X_{\Sigma'}$, a 1-cycle C' on V' and a subset $I' \subseteq J_{C'}^+$, verifying conditions (a) and (b) of Lemma 6.1.*

Proof — The three conditions are clearly necessary so we just have to see that they are sufficient :

- (1) If we denote by $\phi_V : V \rightarrow \overline{V}$ the morphism contracting all the curves that have zero intersection with $D|_V$ then we have $\phi_V = \phi|_V$ and $D|_V = \phi_V^*\overline{D}|_{\overline{V}}$. It follows that $D|_V$ has low toric degree if and only if $\overline{D}|_{\overline{V}}$ has restricted low toric degree which implies that \overline{D} has restricted low toric degree.
- (2) By hypothesis there exists a proper toric morphism $\phi' : X_{\Sigma'} \rightarrow X_{\overline{\Sigma}_{[D]}}$. Let us put $D' = \phi'^*\overline{D}$. It follows from Lemma 3.13 that $X_{\overline{\Sigma}_{[D']}} = X_{\overline{\Sigma}_{[D]}}$ and hence D' has low toric degree if and only if \overline{D} has restricted low toric degree.
- (3) This one is just a combination of the other two and Lemma 6.1.

The theorem is proved. \square

6.2 Criteria in terms of positivity cones

The inequalities characterizing a low toric degree, of the form

$$0 < C \cdot D < C \cdot \sum_{\rho \in I} D_\rho,$$

can be interpreted in terms of the proximity of $[D]$ to the border of a cone of “positive divisor classes”. Indeed if the hyperplane $H_{[C]} \subset N^1(X_\Sigma)$ of classes having zero intersection with $[C]$ is a supporting hyperplane of a given polyhedral cone \mathcal{K} of divisor classes, then the first

inequality holds as soon as $[D]$ belongs to the interior of \mathcal{K} and the second exactly means that subtracting a divisor of the form $[\sum_{\rho \in I} D_\rho]$ is enough to get out of \mathcal{K} . We detail two important examples when $\mathcal{K} = \text{Nef}(X_\Sigma)$ and $\mathcal{K} = \text{Eff}(X_\Sigma)$.

Proposition 6.3. *Let X_Σ be a simplicial projective toric variety and D be an ample divisor on X_Σ . If there exists a subset $I \subseteq \Sigma(1)$ such that*

$$\left[D - \sum_{\rho \in I} D_\rho \right] \notin \text{Nef}(X_\Sigma) \quad (20)$$

then D has restricted low toric degree.

Proof — It is a direct consequence of the fact that the Mori cone $\overline{\text{NE}}(X_\Sigma)$ is the dual of the Nef cone $\text{Nef}(X_\Sigma)$ and is generated by extremal classes. Indeed the fact that $D - \sum_{\rho \in I} D_\rho$ is not nef means that there exists an extremal curve C such that $C \cdot (D - \sum_{\rho \in I} D_\rho) < 0$. By Lemma 2.7 and Proposition 3.18 this is sufficient to prove that D has restricted low toric degree. \square

Remarks 6.4. (i) When X_Σ is smooth, then condition 20 is in fact necessary and sufficient. In the general case, as explained in paragraph 1.4.4, we have to consider every toric subvariety.

(ii) The same arguments work with the sum $\sum_{\rho \in I} D_\rho$ replaced by any \mathbb{Q} divisor D' satisfying $0 \leq D' \leq -K_{X_\Sigma}$. Moreover, it follows from 8 that in any generalized weighted projective space W of dimension k different from \mathbb{P}^k , there exists a toric curve C such that $C \cdot (-K_W) \leq k$. Using inequality 9, this shows that if D has restricted low toric degree then there exists an extremal curve C_τ in X_Σ such that $C_\tau \cdot D \leq \dim(W)$ where W is a general fiber of the contraction of $[C_\tau]$. In particular $C_\tau \cdot D \leq n$ and we recover the simplicial case of Fujita's freeness conjecture for toric varieties (see [Pay06a]).

(iii) As Theorem 0.2 applies to any intersection of hypersurfaces, Proposition 6.3 extends immediately to any intersection of ample divisors. Such an intersection contains a K -rational point as soon as the sum of the divisors has restricted low toric degree.

Proposition 6.5. *Let X_Σ be a simplicial projective toric variety and D be a Cartier divisor on X_Σ that is nef and big. If there exists a subset $I \subseteq \Sigma(1)$ such that*

$$\left[D - \sum_{\rho \in I} D_\rho \right] \notin \text{Eff}(X_\Sigma)$$

then D has low toric degree.

Proof — Since the dual cone of the effective cone is the cone of classes of mobile curves, by reasoning like in the proof of the previous proposition we prove that there exists a mobile curve C satisfying

$$0 < C \cdot D < C \cdot \sum_{\rho \in I} D_\rho.$$

This is condition (a) of Lemma 6.1 (1), it remains to prove condition (b). Since D is big, the fan $\Sigma_{[D]}$ is not degenerate, that is, its minimal cone is the trivial cone $\{0\}$. Hence we can take $\sigma = \{0\}$ and since C is mobile, we have $J_{\overline{C}} = \emptyset \subset \sigma$ so that (b) is satisfied and the proposition is proved. \square

7 Relation to Rational Connectedness

7.1 Separably rationally connected hypersurfaces have low toric degree

Let us recall some definitions.

A variety Y of dimension n over an algebraically closed field \overline{K} is said to be uniruled (resp. separably uniruled) if there exists a projective variety Z of dimension $n - 1$ and a dominant (resp. dominant and separable) rational map $\varphi : \mathbb{P}^1 \times Z \dashrightarrow Y$. Similarly a variety Y over \overline{K} is said to be rationally connected (resp. separably rationally connected) if there exists a quasi-projective variety Z and a morphism $\varphi : \mathbb{P}^1 \times Z \rightarrow Y$ such that the morphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \times Z \rightarrow Y \times Y, \quad (t, t', z) \mapsto (\varphi(t, z), \varphi(t', z))$$

is dominant (resp. a dominant and separable).

A variety Y over an arbitrary field K is said to be uniruled, separably uniruled, rationally connected or separably rationally connected, if the variety $Y \otimes_K \overline{K}$ is, where \overline{K} is an uncountable algebraically closed extension of K .

It is clear from the definitions that any rationally connected (resp. separably rationally connected) variety is also uniruled (resp. separably uniruled). It is also clear that the four notions are stable by any birational modification that is a separable map, i.e. generically smooth, in particular by any divisorial contraction. Similarly it follows from the definition that (separable) rational connectedness is stable by any surjective morphism.

A variety Y is separably uniruled (resp. separably rationally connected) if and only if it contains a free curve (resp. a very free curve), that is roughly speaking, a curve that can be deformed to pass by any point of Y (resp. by any two points of Y). For precise definitions and proofs see [Bon10, Cours 2].

For our arithmetic purpose, the key property of free curves on a variety Y is that they have negative intersection with the canonical divisor. When Y is a smooth hypersurface of a smooth ambient X , this gives a low degree to Y through the adjunction formula.

Before stating the result let us recall the general hypotheses defining our framework (see 1.1 and 1.2) : we consider a simplicial projective toric variety X_Σ defined over a C_1 field K such that

- (H1) K admits normic forms of arbitrary degree.
- (H2) The torus T acting on X_Σ is split over K .
- (H3) The order of the torsion part of the class group $\text{Cl}(X_\Sigma)$ is prime to the characteristic of K .

Then we have the following result :

Theorem 7.1. *Let X_Σ be a simplicial projective toric variety defined over a C_1 field K such that hypotheses (H1), (H2) and (H3) are verified. Let D be a hypersurface of X_Σ defined over K . If D is smooth and separably rationally connected then it has low toric degree and in particular admits a rational point over K .*

The proof of the theorem is splitted in two parts stated as propositions.

Proposition 7.2. *Let K and X_Σ be as in Theorem 7.1. Let D be a big divisor of X_Σ defined over K . If D is separably uniruled and smooth, then it has low toric degree.*

Proof — First recall that we may assume that D is Cartier and nef because otherwise it has low degree by Definition 5.2.

Then let us consider a desingularization $\psi : X_{\tilde{\Sigma}} \rightarrow X_\Sigma$ and the pullback $\tilde{D} = \psi^* D$ of D by ψ . We may assume \tilde{D} is also the strict transform of D because otherwise this would mean that

D contains some component of the blown-up locus, which consist only of toric subvarieties and hence D would again contain torus-invariant points.

Since ψ is birational it follows that \tilde{D} is also separably uniruled and smooth. Moreover it has low toric degree if and only if D has by Theorem 6.2 (2), so that up to replacing X_Σ by $X_{\tilde{\Sigma}}$ and D by \tilde{D} , we may assume that X_Σ is smooth.

Now since D is separably uniruled, it contains a free curve C .

The key point is that the free curve C have negative intersection with the canonical divisor of D :

$$C \cdot K_D < 0.$$

Let us consider the pullback of divisors j^* and the pushforward of curves j_* associated to the inclusion $j : D \hookrightarrow X_\Sigma$. Since X_Σ and D are smooth, we have by the adjunction formula

$$K_D = (K_{X_\Sigma} + D)|_D = j^*(K_{X_\Sigma} + D).$$

On the other hand, by the projection formula we have :

$$j_*C \cdot (K_{X_\Sigma} + D) = C \cdot j^*(K_{X_\Sigma} + D) = C \cdot K_D < 0.$$

Using the fact that $K_{X_\Sigma} = -\sum_{\rho \in \Sigma(1)} D_\rho$ this gives :

$$j_*C \cdot \left(D - \sum_{\rho \in \Sigma(1)} D_\rho \right) < 0.$$

Finally, since C is a mobile curve on D , which is nef, j_*C is also mobile on X_Σ . Since D is a big this gives $j_*C \cdot D > 0$ and we have

$$0 < j_*C \cdot D < j_*C \cdot \sum_{\rho \in \Sigma(1)} D_\rho = j_*C \cdot \sum_{\rho \in J_C} D_\rho,$$

which is condition (a) of Lemma 6.1 with $I = J_C = J_C^+$. It remains to prove condition (b), that is, there exists a cone σ in the fan $\Sigma_{[D]}$ containing all the set J_C^- but no element of J_C^+ . Since C is mobile we have $J_C^- = \emptyset$, hence we can choose $\sigma = \{0\}$ because $\Sigma_{[D]}$ since D is big. As $\{0\}$ contains no 1-cone of Σ condition (b) is satisfied and D has low degree. The lemma is proved. \square

The big hypothesis is necessary here because if the divisor D is not big it may happen that for all free curve C in D we have $j_*(C) \cdot D = 0$ For example if E is a non singular elliptic curve in \mathbb{P}^2 without rational point over K , then the nef but not big hypersurface

$$Y = E \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$$

has no rational point over the C_1 field K , though it is smooth and separably uniruled in the smooth complete toric variety $\mathbb{P}^2 \times \mathbb{P}^1$.

Proposition 7.3. *Let K, X_Σ be as in Theorem 7.1. Let D be a non big divisor of X_Σ defined over K . If D is separably rationally connected and smooth, then it has low toric degree.*

Proof — We are reasoning like in the proof of Theorem 5.1. First let us consider a simplicial refinement $\bar{\Sigma}'$ of $\bar{\Sigma}_{[D]}$ and the induced small toric morphism $\bar{\nu} : X_{\bar{\Sigma}'} \rightarrow X_{\bar{\Sigma}_{[D]}}$. Then let us denote by $\xi : X_\Sigma \dashrightarrow X_{\bar{\Sigma}'}$ the equivariant rational map over $X_{\bar{\Sigma}_{[D]}}$ such that $\phi = \bar{\nu} \circ \xi$. It is surjective in codimension 1 because $\bar{\Sigma}'(1) = \bar{\Sigma}(1)$. By Proposition 4.1 there exist a simplicial projective variety $X_{\Sigma'}$, a small modification $\nu : X_\Sigma \dashrightarrow X_{\Sigma'}$ and a toric morphism $\xi' : X_{\Sigma'} \rightarrow X_{\bar{\Sigma}'}$ such that $\xi = \xi' \circ \nu$. By Proposition 5.8 there also exists a subvariety $V' \subseteq X_{\Sigma'}$ such that the restriction $\xi'_{|V'} : V' \rightarrow X_{\bar{\Sigma}'}$ is birational. Finally by considering a common

smooth refinement $\tilde{\Sigma}$ of Σ and Σ' , we have sequences of weighted blow-ups $\eta : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma}$ and $\eta' : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma'}$ and again there exists a subvariety $\tilde{V} \subseteq X_{\tilde{\Sigma}}$ such that the restriction $\eta'_{|\tilde{V}} : \tilde{V} \rightarrow V'$ is birational. The situation is described by the following commutative diagram :

$$\begin{array}{ccccc}
& & X_{\tilde{\Sigma}} & \longleftarrow & \tilde{V} \\
& \eta \swarrow & & \searrow \eta' & \eta'_{|\tilde{V}} \searrow \\
X_{\Sigma} & \overset{\nu}{\dashrightarrow} & X_{\Sigma'} & \longleftarrow & V' \\
\downarrow \phi & \overset{\xi}{\dashrightarrow} & \downarrow \xi' & \swarrow \xi'_{|V'} & \\
X_{\tilde{\Sigma}_{[D]}} & \overset{\bar{\nu}}{\longleftarrow} & X_{\tilde{\Sigma}'_{[D]}} & &
\end{array}$$

Now we use the fact that separable rational-connectedness is stable both by surjective morphisms and by birational maps. Since D is separably rationally connected, so is its pullback \tilde{D} by η (which is also its strict transform by the reasoning made in the proof of last proposition). Hence so is also the image $\bar{D}' \subset X_{\tilde{\Sigma}'}$ of \tilde{D} by the surjective morphism $\xi' \circ \eta' : X_{\tilde{\Sigma}} \rightarrow X_{\tilde{\Sigma}'}$.

Since the morphism $\xi'_{|V'} \circ \eta'_{|\tilde{V}} : \tilde{V} \rightarrow X_{\tilde{\Sigma}'}$ is birational, it induces also a birational map from $\tilde{D}_{|\tilde{V}}$ to \bar{D}' . It follows that $\tilde{D}_{|\tilde{V}}$ is separably rationally connected and in particular separably uniruled. Finally, the morphism $\bar{\nu} \circ \xi'_{|V'} \circ \eta'_{|\tilde{V}} : \tilde{V} \rightarrow X_{\tilde{\Sigma}_{[D]}}$ is equal to the restriction $\phi'_{|\tilde{V}}$ of the morphism $\phi' = \phi \circ \eta : X_{\tilde{\Sigma}} \rightarrow X_{\tilde{\Sigma}_{[D]}}$ contracting all the curves that have zero intersection with \tilde{D} . Hence it is the morphism contracting all the curves that have zero intersection with $\tilde{D}_{|\tilde{V}}$ and since it is birational, it follows that $\tilde{D}_{|\tilde{V}}$ is a big divisor in \tilde{V} . We may thus apply Proposition 7.2 to prove that $\tilde{D}_{|\tilde{V}}$ has low toric degree. This implies that D has low toric degree by Theorem 6.2 which proves the proposition and achieves the proof of Theorem 7.1. \square

7.2 A low degree condition implying Rational Chain-Connectedness

The question of the converse implication, from low toric degree to rational connectedness seems difficult to treat in full generality by our method but as suggested by C. Araujo, it is quite easy to find conditions “of low toric degree type” implying a weak version of rational connectedness : rational chain-connectedness.

While any two points of a rationally connected variety Y over a field K can be joined by a rational curve over the algebraically closed extensions of K , two points of a rationally chain-connected variety can be joined only by a chain of such rational curves. The condition is weaker and implies less good properties. For instance rational chain-connectedness is stable by fibration but not by birational modification. Indeed for any variety Y , the cone over Y is rationally chain-connected because any two points can be joined to the node by a line and hence joined together by a length 2 chain of rational curves. But if Y is not rationally chain-connected then the blow-up of the cone at the node is not either rationally chain-connected.

The following result gives a condition that imply rational chain-connectedness. It is easy to see that it implies also (global *and* restricted) low toric degree. Notice that in this case no requirement is done about the singularities of the hypersurface.

Proposition 7.4. *Let K and X_{Σ} be as in Theorem 7.1. Let D be a hypersurface of X_{Σ} defined over K . If there exists a mobile extremal curve C such that*

$$0 < C \cdot D < C \cdot (-K_{X_{\Sigma}})$$

then D is rationally chain-connected.

Proof — Since the class $[C]$ is mobile and extremal its contraction is a Mori fibration

$$\pi : X_{\Sigma} \rightarrow X_{\tilde{\Sigma}}$$

with general fiber F isomorphic to a generalized weighted projective space. The condition $C \cdot D < C \cdot (-K_{X_\Sigma})$ means that the restriction $D|_F$ of D to such a fiber is \mathbb{Q} -Fano and hence rationally chain connected.

On the other hand, the condition $0 < C \cdot D$ implies that D is linearly equivalent to a torus invariant effective divisor $\sum a_\rho D_\rho$ with at least one $\rho \in \Sigma(1)$ such that $a_\rho > 0$ and $C \cdot D_\rho > 0$. Since such a D_ρ is a section of ϕ it follows that D dominates $X_{\overline{\Sigma}}$.

Putting this together we have that the restriction of ϕ to \overline{D} is a fibration onto the toric (hence rational) variety $X_{\overline{\Sigma}}$ with rationally chain connected fibers $D|_F$. Hence D is rationally chain connected and the proposition is proved. \square

If the hypersurface is smooth and the characteristic of K is zero, then we can make two improvements. First, rational connectedness and rational chain-connectedness are equivalent for smooth projective varieties (see [Bon10, Cours 4]). Second, in characteristic zero every map is separable so that rational connectedness is also equivalent to separable rational connectedness. It follows that in this case we have

$$\text{Low degree along a mobile extremal curve} \Rightarrow \text{Rationally connected} \Rightarrow \text{Low toric degree.}$$

Let us remark that having low degree along a mobile extremal curve is quite a strong condition for D since it implies that D is a fiber bundle with toric base whose fibers are \mathbb{Q} -Fano hypersurfaces of a generalized weighted projective space.

Here is an example

Example 7.5. Let Y be a hypersurface of \mathbb{P}^n of degree d containing a projective subspace Z of dimension k with multiplicity m . Let $\eta : X_\Sigma \rightarrow \mathbb{P}^n$ denote the blowing-up of \mathbb{P}^n along Z and let D be the strict transform of Y in X_Σ . Notice that D may be singular.

Let us number the toric divisors from 1 to $n+2$ with D_{n+2} the exceptional divisor of the blowing-up and denote by $D'_i = \eta(D_i)$ the image of D_i in \mathbb{P}^n . Up to projectivity we may assume that $Y = D_1 \cdot D_2 \cdots D_{n-k}$ is a torus-invariant subvariety of \mathbb{P}^n . There are three different classes of toric curves in X_Σ : the class $[C_1]$ of a line contracted by η , the class $[C_2]$ of the strict transform of a line in \mathbb{P}^n containing a point of Z and the class $[C_3] = [C_1] + [C_2]$ of the strict transform of a line disjoint from Z .

The corresponding intersection table is given by

\cdot	D_1	\cdots	D_{n-k}	D_{n-k+1}	\cdots	D_{n+1}	D_{n+2}
C_1	1	\cdots	1	0	\cdots	0	-1
C_2	0	\cdots	0	1	\cdots	1	1
C_3	1	\cdots	1	1	\cdots	1	0
	$\underbrace{\hspace{10em}}_{n-k}$			$\underbrace{\hspace{10em}}_{k+1}$			

The classes $[C_1]$ and $[C_2]$ are extremal and $[C_2]$ is moreover a mobile class, whose contraction is the fibration of X_Σ onto \mathbb{P}^{n-k-1} .

The hypersurface D dominates \mathbb{P}^{n-k-1} . Indeed by construction it is a fiber bundle with base \mathbb{P}^{n-k-1} and fibers isomorphic to hypersurfaces of \mathbb{P}^{k+1} of degree $d-m$. In particular we have

$$C_1 \cdot D = m, \quad C_2 \cdot D = d - m \quad \text{and} \quad C_3 \cdot D = d.$$

It follows that whenever the multiplicity m is greater than $d - k - 2$ we have

$$0 < C_2 \cdot D = d - m < k + 2 = C_2 \cdot (-K_{X_\Sigma}),$$

and proposition 7.4 applies: D has low toric degree and is rationally chain-connected for the same reason.

\diamond

References

- [Ara10] Carolina Araujo. The cone of pseudo-effective divisors of log Fano varieties after Batyrev. *Math. Z.*, 264(1):179–193, 2010.
- [Bat92] Victor V. Batyrev. The cone of effective divisors of threefolds. In *Proceedings of the International Conference on Algebra*, volume 131, pages 337–352, Contemp. Math., AMS, Novosibirsk, 1992.
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James M^cKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23:405–468, 2010.
- [Bon10] Laurent Bonavero. Variétés rationnellement connexes sur un corps algébriquement clos. In *Variétés rationnellement connexes : aspects géométriques et arithmétiques*, pages 9–60, Panor. Synthèses, 31, Soc. Math. France, Paris, 2010.
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric Varieties*. AMS, 2011.
- [Cox95] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4:15–50, 1995.
- [CT09] Jean-Louis Colliot-Thélène. Variétés presque rationnelles, leurs points rationnels et leurs dégénérescences. In *Arithmetic geometry, Lecture Notes in Math.*, pages 1–44, Springer, Berlin, 2009.
- [CvR09] David A. Cox and Christine von Renesse. Primitive collections and toric varieties. *Tohoku Math. J.*, 61:309–332, 2009.
- [Dan78] Vladimir I. Danilov. The geometry of toric varieties. *Russian Math. Surveys*, 33:97–154, 1978.
- [dJS03] Aise Johan de Jong and Jason Starr. Every rationally connected variety over the function field of a curve has a rational point. *Amer. J. Math.*, 125(3):567–580, 2003.
- [ELST14] E. Javier Elizondo, Paulo Lima-Filho, Frank Sottile, and Zach Teitler. Arithmetic toric varieties. *Math. Nachr.*, 287(2-3):216–241, 2014.
- [Esn03] Hélène Esnault. Varieties over a finite field with trivial chow group of 0-cycles have a rational point. *Invent. Math.*, 151(1):187–191, 2003.
- [FM82] John Fogarty and David Mumford. *Geometric Invariant Theory*. Springer-Verlag, 2nd edition, 1982.
- [FS97] William Fulton and Bernd Sturmfels. Intersection theory on toric varieties. *Topology*, 36(2):335–353, 1997.
- [Ful93] William Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.
- [GHS03] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67, 2003.
- [HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000.
- [Hur11] Mathieu Huruguen. Toric varieties and spherical embeddings over an arbitrary field. *J. Algebra*, 342:212–234, 2011.
- [Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. Springer, 1996.

- [Lan52] Serge Lang. On quasi algebraic closure. *Annals of math.*, 55(2):373–390, 1952.
- [Lan00] Serge Lang. Cyclotomic points, very anti-canonical varieties, and quasi-algebraic closure. *Preprint*, 2000.
- [Man93] Yuri Ivanovitch Manin. Notes on the arithmetic of fano threefolds. *Compositio math.*, 85(1):37–55, 1993.
- [Mat02] Kenji Matsuki. *Introduction to the Mori Program*. Springer, 2002.
- [Mon12] Douglas Monsôres. *Toric Birational Geometry and Applications to Lattice Polytopes*. PhD thesis, IMPA, 2012. arXiv:1307.1449 [math.AG].
- [Par93] Hye Sook Park. The Chow rings and GKZ-decompositions for \mathbb{Q} -factorial toric varieties. *Tohoku Math. J.*, 45:109–145, 1993.
- [Pay06a] Sam Payne. Fujita’s very ampleness conjecture for singular toric varieties. *Tohoku Math. J.*, 58(3):447–459, 2006.
- [Pay06b] Sam Payne. Stable base loci, movable curves, and small modifications, for toric varieties. *Math. Z.*, 253(2):421–431, 2006.
- [Rei83] Miles Reid. Decomposition of toric morphisms. In *Arithmetic and geometry*, volume II, pages 395–418, Progr. Math. 36, Birkhauser Boston, Boston, MA, 1983.
- [Wit10] Olivier Wittenberg. La connexité rationnelle en arithmétique. In *Variétés rationnellement connexes: aspects géométriques et arithmétiques*, pages 61–114, Panor. Synthèses, 31, Soc. Math. France, Paris, 2010.