A STOCHASTIC APPROXIMATION APPROACH TO QUASI-STATIONARY DISTRIBUTIONS ON FINITE SPACES

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MICHEL BENAÏM AND BERTRAND CLOEZ

ABSTRACT. This work is concerned with the analysis of a stochastic approximation algorithm for the simulation of quasi-stationary distributions on finite state spaces. This is a generalization of a method introduced by Aldous, Flannery and Palacios. It is shown that the asymptotic behavior of the empirical occupation measure of this process is precisely related to the asymptotic behavior of some deterministic dynamical system induced by a vector field on the unit simplex. This approach provides new proof of convergence as well as precise rates for this type of algorithm. We then compare this algorithm with particle system algorithms.

AMS 2010 Mathematical Subject Classification: 65C20, 60B12, 60J10; secondary 34F05, 60J20

Keywords: Quasi-stationary distributions - approximation method - reinforced random walks - random perturbations of dynamical systems.

. CONTENTS

1. INTRODUCTION

Let $(Y_n)_{n\geq 0}$ be a Markov chain on a finite state space F with transition matrix $P = (P_{i,j})_{i,j\in F}$. We assume that this process admits an (attainable) absorbing state, say 0, and that $F^* = F\backslash\{0\}$ is an irreducible class for P; this means that $P_{i,0} > 0$ for some $i \in F^*$, $P_{0,i} = 0$ for all $i \in F^*$ and $\sum_{k\geq 0} P^k_{i,j} > 0$ for all $i, j \in F^*$. For all $i \in F$ and any probability measure μ on F (or F^*), we set

$$\mathbb{P}_i\left(\cdot \right) = \mathbb{P}\left(\cdot \mid Y_0 = i\right),\ \mathbb{P}_\mu = \sum_{i\in F} \mu(i)\mathbb{P}_i,$$

and we let $\mathbb{E}_i, \mathbb{E}_\mu$ denote the corresponding expectations. Classical results [11, 12, 21, 26] imply that Y_n is absorbed by 0 in finite time and admits a unique probability measure ν on F^*, called *quasi-stationary distribution* (QSD), satisfying, for every $k \in F^*$,

$$\nu(k) = \mathbb{P}_\nu(Y_1 = k \mid Y_1 \neq 0) = \frac{\sum_{i\in F^*} \nu(i)P_{i,k}}{\sum_{i,j\in F^*} \nu(i)P_{i,j}} = \frac{\sum_{i\in F^*} \nu(i)P_{i,k}}{1 - \sum_{i\in F^*} \nu(i)P_{i,0}}.$$

Date: May 2, 2014.
Partially supported by ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02.

If we furthermore assume that P is aperiodic, then (see for instance [21, Proposition 1]) for any probability measure μ on F^* and $k \in F^*$,

$$\lim_{n\to+\infty} \mathbb{P}_\mu(Y_n = k \mid Y_n \neq 0) = \nu(k). \quad (1)$$

The existence (and uniqueness) of this measure can be proved through the Perron Frobenius Theorem because a probability measure ν is a QSD if and only if it is a left eigenvector of P (associated to some eigenvalue $\lambda \in (0,1)$); namely

$$\nu P = \lambda\nu \Leftrightarrow \forall k \in F^*, \sum_{i\in F^*} \nu(i)P_{i,k} = \lambda\nu(k). \quad (2)$$

Summing on k the previous expressions gives the following expression of λ:

$$\lambda = 1 - \sum_{i\in F^*} \nu(i)P_{i,0}. \quad (3)$$

Quasi-stationary distributions have many applications as illustrated for instance in [11, 21, 25, 26] and their computation is of prime importance. This can be achieved with deterministic algorithms coming from numerical analysis [26, section 6] based on equation (2), but these type of method fails to be efficient with large state spaces. An alternative approach is to use stochastic algorithms (even if naive Monte-Carlo methods are not well-suited as illustrated in the introduction of [27]). Our main purpose here is to analyze a class of such algorithms based on a method that was introduced by Aldous, Flannery and Palacios [1] and which can be described as follows:
Let Δ be the unit simplex of probabilities over F^*. For $x \in \Delta$, let $K[x]$ be Markov kernel defined by

$$\forall i,j \in F^*,\ K[x]_{i,j} = P_{i,j} + P_{i,0}x(j). \quad (4)$$

and let $(X_n)_{n\geq 0}$ be a process on F^* such that

$$\forall i,j \in F^*,\ \mathbb{P}\left(X_{n+1} = j \mid \mathcal{F}_n\right) = K[x_n]_{i,j},\ \text{on } \{X_n = i\}, \quad (5)$$

where

$$x_n = \frac{1}{n+1}\sum_{k=0}^{n} \delta_{X_k} \quad (6)$$

stands for the *empirical occupation measure* of the process and $\mathcal{F}_n = \sigma\{X_k, k \leq n\}$. In words, the process behaves like $(Y_n)_{n\geq 0}$ until it dies (namely it hits 0) and, when it dies, comes back to life in a state randomly chosen according to it's empirical occupation measure.
Note that, we will use a slight different algorithm which allows us to choose a non-uniform measure on the past. This process is not Markovian and can be understood as an urn process or a reinforced random walk. Using the natural embedding of urn processes into continuous-time multi-type branching processes [2, section V.9], Aldous, Flannery and Palacios prove the convergence of (x_n) to the QSD. As well illustrated in [24], another powerful method for analyzing the behavior of processes with reinforcement is stochastic approximation theory [7, 19] and its dynamical system counterpart [4]. Relying on this approach we recover [1, Theorem 3.8] in a more general context with new rates of convergence. This enables us to compare it with a different algorithm introduced by Del Moral and Guyonnet [14]. We describe it and give a new bound for the convergence based on [6] in section 3. Also note that the process defined in (6) is an instance of the (time) self-interacting Markov chain models studied in [16, 17] and we also extend some of their results in this particular case. Indeed, [17, Theorem 1.2] and [16, Theorem 2.2] gives a L^1-bound for the convergence under a strong mixing assumption which is not always satisfied (a Doeblin type condition). We will prove almost-sure convergence, a central limit theorem and the convergence of $(X_n)_{n\geq 0}$ when $(x_n)_{n\geq 0}$ is a weighted empirical measure.

Outline: the next subsection introduces our main results. The proofs are in section 2. Indeed we study the dynamical system in 2.1, make the link with the sequence $(x_n)_{n\geq 0}$ in 2.2, and end the proof in 2.3. Finally, Section 3 treats the second algorithm based on a particle system.

1.1. **Main results.** Assume that F^* contains $d \geq 2$ elements and let us define the unit simplex of probability measures on F^* by $\Delta = \left\{ x \in \mathbb{R}^d \,\middle|\, x_i \geq 0,\ \sum_{i=1}^d x_i = 1 \right\}$. We embed $\mathbb{R}^d$ with the classical l^1-norm: $\|x\| = \sum_{i \in F^*} |x(i)|$ and Δ with the induced distance (which corresponds, up to a constant, to the total variation distance). Given a law $x \in \Delta$, we denote by $\pi(x)$ the invariant distribution of $K[x]$, defined in (4), and we let $h : \Delta \to T\Delta = \left\{ x \in \mathbb{R}^d \middle| \sum_{i=1}^d x_i = 0 \right\}$ denote the vector field given by $h(x) = \pi(x) - x$. Our aim is to study the *weighted empirical occupation measure* $(x_n)_{n \geq 0}$, defined for every $n \geq 0$ by

$$x_{n+1} = (1 - \gamma_n) x_n + \gamma_n \delta_{X_n} = x_n + \gamma_n (h(x_n) + \epsilon_n), \tag{7}$$

where $\epsilon_n = \delta_{X_{n+1}} - \pi(x_n)$ and $(\gamma_n)_{n \geq 0}$ is a decreasing sequence on $(0,1)$ verifying

$$\sum_{n \geq 0} \gamma_n = +\infty \text{ and } \lim_{n \to +\infty} \gamma_n \ln(n) = 0.$$

The variable X_n is distributed according to the transition (5). Let us set

$$\tau_n = \sum_{k=1}^n \gamma_k, \text{ and } l(\gamma) = \limsup_{n \to +\infty} \frac{\ln(\gamma_n)}{\tau_n}. \tag{8}$$

For instance, if

$$\gamma_n = A n^{-\alpha} \ln(n)^{-\beta}, A > 0, \alpha, \beta \geq 0,$$

then

$$l(\gamma) = \begin{cases} 0, \text{ if } (\alpha, \beta) \in (0,1) \times \mathbb{R}_+, \\ -1/A \text{ if } \alpha = 1, \beta = 0, \\ -\infty \text{ if } (\alpha, \beta) \in \{1\} \times (0,1]. \end{cases}$$

Remark 1.1. *The sequence* (6) *corresponds to the choice* $\gamma_n = \frac{1}{n+1}$. *More generally, let* $(\omega_n)_{n \geq 0}$ *be a sequence of positive number, if*

$$\gamma_n = \frac{\omega_n}{\sum_{i=0}^n \omega_k} \Leftrightarrow \omega_n = \frac{\kappa \gamma_n}{\prod_{k=0}^n (1 - \gamma_i)},$$

for some $\kappa > 0$, *then*

$$x_n = \frac{\sum_{i=0}^n \omega_i \delta_{X_i}}{\sum_{i=0}^n \omega_i}.$$

Notice that with $\omega_n = n^a$ *for* $a > -1, \gamma_n \sim \frac{1+a}{n}$.

The sequence $(x_n)_{n \geq 0}$ is often called a stochastic approximation algorithm with decreasing step [4, 7, 19]. Its long time behavior can be related to the long time behavior of the flow Φ induced by h; namely the solution to

$$\begin{cases} \forall t \geq 0, \forall x \in \Delta,\ \partial_t \Phi(t,x) = h(\Phi(t,x)), \\ \Phi(0,x) = x. \end{cases} \tag{9}$$

In order to state our main result, let us introduce some notation. By Perron-Frobenius Theorem, eigenvalues of P can be ordered as

$$1 > \lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_d| \geq 0,$$

where $\lambda_1 = \lambda$ is given by (3). Set

$$R = 1 - (1 - \lambda) \max_{i \geq 2} \mathrm{Re}\left(\frac{1}{1 - \lambda_i} \right) > 0, \tag{10}$$

where Re is the real part application on $\mathbb{C}$.

Theorem 1.2 (Convergence of $(x_n)_{n\geq 0}$ to the quasi-stationary distribution)**.** *With probability one, x_n tends to ν. If furthermore $l(\gamma) < 0$, then*

$$\limsup_{n\to+\infty} \frac{1}{\tau_n} \ln\left(\|x_n - \nu\|\right) \leq \max\left(-R, \frac{l(\gamma)}{2}\right) \quad a.s.$$

This leads to the following result which generalizes and precises the rates of convergence of [1, Theorem 3.8]

Corollary 1.3. *Suppose $\gamma_n = \frac{A}{n}$ for some $A > 0$ (or, with the notation of remark 1.1, $\omega_n = n^{A-1}$) then for all $\theta < \min(RA, 1/2)$, there exists a random constant $C > 0$ such that*

$$\forall n \geq 0,\ \|x_n - \nu\| \leq C n^{-\theta} \ a.s.$$

Using general results on stochastic approximation, we are also able to quantify more precisely this convergence; we have

Theorem 1.4 (Central limit theorem)**.** *If one of the following conditions is satisfied*

i) $\sum_{k\geq 0} \gamma_k = +\infty, \sum_{k\geq 0} \gamma_k^2 < \infty$ *and* $\lim_{k\to+\infty} \gamma_k^{-1} \ln(\gamma_{k-1}/\gamma_k) = 0$*;*
ii) $\sum_{k\geq 0} \gamma_k = +\infty, \sum_{k\geq 0} \gamma_k^2 < \infty$ *and* $\lim_{k\to+\infty} \gamma_k^{-1} \ln(\gamma_{k-1}/\gamma_k) = \gamma_*^{-1} < 2R$*;*

then there exists a covariance matrix V such that

$$\gamma_n^{-1/2}(x_n - v) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0, V).$$

This gives the following trivial consequence:

Corollary 1.5 (L^p-bound for the convergence of $(x_n)_{n\geq 0}$)**.** *Under the previous assumptions, there exists for all $p \geq 1$ $C_p > 0$ such that for every $n \geq 0$,*

$$\lim_{n\to\infty} \gamma_n^{-1/2} \mathbb{E}\left[\sum_{i\in F^*} |x_n(i) - \nu(i)|^p\right]^{1/p} = C_p$$

Note that this result extends [17, Theorem 1.2] and [16, Theorem 2.2] (at least for this example). Finally, not only the (weighted) empirical occupation measure of $(X_n)_{n\geq 0}$ converges almost surely to ν but (X_n) itself converges in distribution to ν as shown by the next result.

Corollary 1.6 (Convergence in law to ν)**.** *Let $(\mu_n)_{n\geq 0}$ be the sequence of laws of $(X_n)_{n\geq 0}$. Then*

$$\lim_{n\to+\infty} \|\mu_n - \nu\| = 0.$$

If we furthermore assume that the assumptions of Theorem 1.4 hold, there exists $C > 0$ and $0 < \rho < 1$ such that

$$\|\mu_{n+p} - \nu\| \leq C(\rho^p + p\sqrt{\gamma_n}).$$

Proofs of these results are given in section 2 and in particular in 2.2.

2. Study of the flows and proofs of our main results

As explained in the introduction, the proof is based on the ODE method. We study Φ and apply its properties to $(x_n)_{n\geq 0}$ with classical results on perturbed ODE. So we decompose this section into three subsections: the study of the flow Φ, the study of the noise $(\epsilon_n)_{n\geq 0}$ and finally the proof of the main theorems.

2.1. **Analysis of the flow.** For any $x, y \in \Delta$, we will use the following notation:

$$\langle x, y \rangle = \sum_{i \in F^*} x(i)y(i),$$

and $\mathbf{1}$ will denote the unit vector; namely $\mathbf{1}(i) = 1$ for every $i \in F^*$. Let us begin by giving a more tractable expression for π. As $\hat{P} = (P_{i,j})_{i,j \in F^*}$ is sub-stochastic, the matrix $A = \sum_{k \geq 0} \hat{P}^k$ is well defined and is the inverse of $I - \hat{P}$, where I is the identity matrix, and we have

$$\forall x \in \Delta, \ \pi(x) = \frac{xA}{\langle xA, \mathbf{1} \rangle}. \tag{11}$$

Indeed, if $\gamma = \sum_{i \in F^*} \pi(x)(i) P_{i,0}$ then we have

$$\pi(x)K[x] = \pi(x) \Leftrightarrow \pi(x) \cdot (\hat{P} - I) = -\gamma x \Leftrightarrow \pi(x) = \gamma x \cdot (I - \hat{P})^{-1} = \gamma x \cdot A,$$

and as $\pi(x) \in \Delta$, we have

$$1 = \sum_{i \in F^*} \pi(x)(i) = \gamma \sum_{i \in F^*} (x \cdot A)(i) = \gamma \langle xA, \mathbf{1} \rangle.$$

Since A and P have the same eigenvectors and using classical results on linear dynamical system, we deduce the following result

Lemma 2.1 (Long time behavior of Φ)**.** *For all $\alpha \in (0, R)$, there exists $C > 0$ such that for all $x \in \Delta$ and $t \geq 0$, we have*

$$\|\Phi(t, x) - \nu\| \leq Ce^{-\alpha t} \|\Phi(t, x) - \nu\|. \tag{12}$$

Proof. Let us consider $\Phi_1 : (t, x) \mapsto x \cdot e^{tA}$. Writing $x = \nu + (x - \nu)$ and using $\nu A = (1 - \lambda)^{-1} \nu$, it comes

$$\Phi_1(t, x) = e^{(1-\lambda)^{-1} t} \left(\nu + (x - \nu) e^{t(A - (1-\lambda)^{-1} I)} \right). \tag{13}$$

Let

$$\beta < (1 - \lambda)^{-1} - \max_{i \geq 2} \mathrm{Re}((1 - \lambda_i)^{-1}),$$

for t large enough, we have $\|e^{t(A-(1-\lambda)^{-1}I)}\| \leq e^{-\beta t}$. Let now Φ_2 be the semiflow on Δ defined for all $t \geq 0$ and $x \in \Delta$ by

$$\Phi_2(t, x) = \frac{\Phi_1(t, x)}{\langle \Phi_1(t, x), \mathbf{1} \rangle}.$$

It follows from (13) that for some $C > 0$,

$$\forall t \geq 0, \ \|\Phi_2(t, x) - \nu\| \leq Ce^{-\beta t} \|x - \nu\|.$$

Now, note Φ_2 and Φ have the same orbits (up to a time re-parametrization). Indeed, differentiating in t, we find that

$$\begin{cases} \forall t \geq 0, \forall x \in \Delta, \ \partial_t \Phi_2(t, x) = \langle \Phi_2(t, x)A, \mathbf{1} \rangle \left(\dfrac{\Phi_2(t, x)A}{\langle \Phi_2(t, x)A, \mathbf{1} \rangle} - \Phi_2(t, x) \right), \\ \qquad \forall x \in \Delta, \ \Phi_2(0, x) = x. \end{cases}$$

Hence,

$$\forall t \geq 0, \ \forall x \in \Delta, \ \Phi(s(t, x), x) = \Phi_2(t, x), \tag{14}$$

where

$$s(t, x) = \int_0^t \langle \Phi_2(x, s)A, \mathbf{1} \rangle ds.$$

This mapping is strictly increasing because $\Phi_2(x, s)$ belongs to Δ so that $\langle A\Phi_2(x, s), \mathbf{1} \rangle > 0$ for all $s \geq 0$. It follows from (13) that $s(t, x)/t$ tends to $(1 - \lambda)^{-1}$, uniformly in $x \in \Delta$ as t tends to infinity. Thus, fixing $\alpha < \beta(1 - \lambda) < R$, for t large enough, we have $\beta t > \alpha s(t, x)$ and, consequently,

$$\|\Phi(s(t, x), x) - \nu\| \leq Ce^{-\alpha s(t,x)} \|x - \nu\| \Leftrightarrow \|\Phi(s, x) - \nu\| \leq Ce^{-\alpha s} \|x - \nu\|,$$

for s large enough. Replacing C by a sufficiently larger constant, the previous inequality holds for all time and this proves the Lemma. □

Remark 2.2 (Probabilist interpretation of A, Φ_1, Φ_2)**.** *The flow Φ_1 satisfies a linear equation with a positive operator A. If we had $A\mathbf{1} = 0$ then it would be a the semi-group of a continuous-time Markov chain. But the vector $A\mathbf{1}$ has positive coordinates; indeed A represents the Green function, we have*

$$\forall i, j \in F^*,\ A_{i,j} = \mathbb{E}_i \left[\sum_{k \geq 0} \mathbf{1}_{Y_k = j} \right] \text{ and } (A\mathbf{1})_i = \mathbb{E}_i [T_0],$$

where $T_0 = \inf\{n \geq 0 \mid Y_n = 0\}$. However, Φ_1 can be understood as the main measure of a branching particle system; Φ_2 is then the renormalised main measure. See [8] *or* [9, Chapitre 4] *for details.*

Corollary 2.3 (Gradient estimate)**.** *The matrix $D_\nu h$ has all its eigenvalues with real part smaller than $-R$.*

Proof. Let us fix $t \geq 0$ and set $\Phi_t(\cdot) = \Phi(t, \cdot)$. On the first hand, using Lemma 2.1 and $\Phi_t(\nu) = \nu$, we have

$$s^{-1} \|\Phi_t(\nu + su) - \Phi_t(\nu)\| \leq Ce^{-\alpha t} \|u\|$$

for every $s \geq 0$ and $u \in \mathbb{R}^d$; taking the limit $s \to 0$, we find

$$\|D_\nu \Phi_t \cdot u\| \leq Ce^{-\alpha t} \|u\|,$$

for every $\alpha < R$. On the other hand, we have

$$\partial_t \Phi_t(x) = h(\Phi_t(x)) \Rightarrow \partial_t D_\nu \Phi_t = D_\nu \partial_t \Phi_t = D_\nu h(\Phi_t(x)) = D_{\Phi_t(\nu)} h \cdot D_\nu \Phi_t = D_\nu h \cdot D_\nu \Phi_t,$$

and thus $D_\nu \Phi_t = e^{t D_\nu h}$. Finally, if v is an eigenvector, whose eigenvalue is $a + ib, a, b \in \mathbb{R}$, then

$$\|D_\nu \Phi_t \cdot v\| = \|e^{t D_\nu h} \cdot v\| = e^{ta} \|e^{ibt} \cdot v\| \leq Ce^{-\alpha t} \|v\|.$$

This ends the proof. □

2.2. **Links between $(x_n)_{n \geq 0}$ and Φ.** Let us rapidly recall some definitions of [4]. To this end, we define the following continuous time interpolations $X, \bar{X}, \bar{\epsilon}, \bar{\gamma} : \mathbb{R}_+ \to \mathbb{R}^d$ by

$$X(\tau_n + s) = x_n + s \frac{x_{n+1} - x_n}{\tau_{n+1} - \tau_n},\ \bar{X}(\tau_n + s) = x_n,\ \bar{\epsilon}(\tau_n + s) = \epsilon_n \text{ and } \bar{\gamma}(\tau_n + s) = \gamma_n,$$

for every $n \in \mathbb{N}$ and $s \in [0, \gamma_{n+1})$. We also set $m : t \mapsto \sup\{k \geq 0 \mid t \geq \tau_k\}$. A continuous map $Z : \mathbb{R}_+ \mapsto \Delta$ is called an *asymptotic pseudo-trajectory* of Φ if for all $T > 0$,

$$\lim_{t \to +\infty} \sup_{0 \leq h \leq T} \|Z(t + h) - \Phi_h(t)\| = 0.$$

Given $r < 0$, it is called a $r-$pseudo-trajectory of Φ if

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \left(\sup_{0 \leq h \leq T} \|Z(t + h) - \Phi_h(t)\| \right) \leq r,$$

for some (or all) $T > 0$. We have

Lemma 2.4 (Pseudo-trajectory property of X)**.** *With probability one, X is an asymptotic pseudo-trajectory of Φ. If furthermore $l(\gamma) < 0$ then X is almost surely a $l(\gamma)/2$-pseudo-trajectory of Φ.*

Proof. The proof is similar to [3, Section 5] whose some ideas coming from [22]. For $x \in \Delta$, let us denote by $Q[x]$ the solution of the Poisson equation:

$$(I - K[x])Q[x] = Q[x](I - K[x]) = I - \pi(x) \cdot \mathbf{1}^t.$$

Existence, uniqueness and regularity of a solution for this equation is standard [23, Chapter 17] or [18, Theorem 35]; indeed let us recall that the state space is finite. We can write

$$\gamma_n \epsilon_n = \delta_n^1 + \delta_n^2 + \delta_n^3 + \delta_n^4,$$

where, for all $j \in F^*$, we have

$$\delta_n^1(j) = \gamma_n \left(Q[x_n]_{X_{n+1},j} - K[x_n]Q[x_n]_{X_n,j} \right),$$

$$\delta_n^2(j) = \gamma_n K[x_n]Q[x_n]_{X_n,j} - \gamma_{n-1} K[x_n]Q[x_n]_{X_n,j},$$

$$\delta_n^3(j) = \gamma_{n-1} K[x_n]Q[x_n]_{X_n,j} - \gamma_n K[x_{n+1}]Q[x_{n+1}]_{X_{n+1},j},$$

and

$$\delta_n^4(j) = \gamma_n \left(K[x_{n+1}]Q[x_{n+1}]_{X_{n+1},j} - K[x_n]Q[x_n]_{X_{n+1},j} \right).$$

Continuity, smoothness of Q, M and compactness of Δ ensure the existence of $C > 0$ such that

$$\|\delta_n^2\| \leq C(\gamma_{n-1} - \gamma_n), \ \|\sum_{i=n}^{k} \delta_i^3\| \leq C\gamma_n \ \text{ and } \ \|\delta_n^4\| \leq C\gamma_n \|x_{n+1} - x_n\| \leq C\gamma_n^2.$$

Now, if $\mathcal{F}_n = \sigma\{X_k \mid k \leq n\}$, the last term is a $\mathcal{F}_n$-martingale increment and there exists $C_1 > 0$ such that $\|\delta_n^1\|^2 \leq C_1 \gamma_n^2$. From these inequalities, the proof is as [4, Proposition 4.4]. Let us now prove that it is a $l(\gamma)/2-$pseudo-trajectory. Let

$$\Delta(t,T) = \sup_{0 \leq h \leq T} \| \int_t^{t+h} \bar{\epsilon}(s) ds \| \leq \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \gamma_k \epsilon_k \| + C_2,$$

for some $C_2 > 0$. Thanks to Inequality (11) of [4, Proposition 4.1] and the beginning of the proof of [4, Proposition 8.3], it is enough to prove that $\limsup_{t \to \infty} \ln(\Delta(t,T))/t \leq l(\gamma)/2$. From the previous decomposition, we have

$$\begin{aligned}\Delta(t,T) &\leq \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \delta_k^1 \| + \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \delta_k^2 \| + \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \delta_k^3 \| + \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \delta_k^4 \| \\ &\leq \sup_{0 \leq h \leq T} \| \sum_{k=m(t)}^{m(t+h)} \delta_k^1 \| + C\bar{\gamma}(t) + C\bar{\gamma}(t) + CT\bar{\gamma}(t).\end{aligned}$$

Indeed,

$$\| \sum_{k=m(t)}^{m(t+h)} \delta_k^4 \| \leq C \sum_{k=m(t)}^{m(t+T)} \gamma_k^2 = C \int_t^{T+t} \bar{\gamma}(s) ds \leq CT\bar{\gamma}(t)$$

Now the end of the proof is the same as in the Robbins-Monro algorithm situation (see the proof of [4, Proposition 8.3]). □

2.3. **Proof of the main results.**

Proof of Theorem 1.2. By Lemma 2.1, $\{\nu\}$ is a global attractor for Φ. Thus, it contains the limit set of every (bounded) asymptotic pseudo-trajectory (see e.g [4, Theorem 6.9] or [4, Theorem 6.10]). Lemma 2.4 gives the almost-sure convergence. The second part of Theorem 1.2 follows directly from [4, Lemma 8.7] and Lemma 2.4. □

Proof of corollary 1.3. Since the limsup in the definition of $l(\gamma)$ is a limit, the result is a direct consequence of Theorem 1.2. □

Proof of Theorem 1.4. Let us check that our model satisfies the assumptions of [20, Theorem 2.1]. Lemma 2.3 gives that $\mathbf{C1}$ holds. Using the notations of this paper and the one of the proof of Lemma 2.4, we have

$$e_n = \gamma_n^{-1}\delta_n^1 \text{ and } r_n = \gamma_n^{-1}(\delta_n^2 + \delta_n^3 + \delta_n^4).$$

Assumption $\mathbf{C2}(a)$ holds, Assumption $\mathbf{C2}(b)$ holds with $\mathcal{A}_m = \mathcal{A}_{m,k} = \Omega$, where Ω is our probability space. Note that $x_n \to \nu$ with probability one.
Assumption $\mathbf{C2}(c)$ is more tricky but usual. Indeed, one can see that e_n is similar to the one introduced in [20, Section 4] (see the end of page 15), and then, we can use the decomposition developed in page 16 of this article. Using the proof of Lemma 2.4, Assumption $\mathbf{C3}$ is satisfied. Finally, the last assumption is supposed to be true in our setting. □

Proof of Corollary 1.5. The L^p-norm are continuous bounded functions on Δ thus the result is straightforward. □

Proof of Corollary 1.6. By irreducibility of P (and hence $K[\nu]$), $\nu_i > 0$ for all i. Thus, $K[\nu]_{ii} \geq P_{i0}\nu_i > 0$ for all i such that $P_{i0} > 0$. This shows that $K[\nu]$ is aperiodic. Therefore, by the ergodic theorem for finite Markov chains, there exist $C_0 > 0$ and $\rho \in [0, 1)$ such that for all $x \in \Delta$

$$\|xK^n[\nu] - \nu\| \leq C_0\rho^n.$$

In particular, ν is a global attractor for the discrete time dynamical system on Δ induced by the map $x \mapsto xK[\nu]$. To prove that $\mu_n \to \nu$ it then suffices to prove that (μ_n) is an asymptotic pseudo trajectory of this dynamics (that is $\|\mu_n K[\nu] - \mu_{n+1}\| \to 0$) because the limit set of a bounded asymptotic pseudo-trajectory is contained in every global attractor (see e.g [4, Theorem 6.9] or [4, Theorem 6.10]). Now,

$$\|\mu_n K[\nu] - \mu_{n+1}\| = \sum_{j\in F^*} |\mu_n K[\nu](j) - \mu_{n+1}(j)| = \sum_{j\in F^*} |\mathbb{E}\left[K[\nu]_{X_n,j} - K(x_n)_{X_n,j}\right]|$$

$$= \sum_{j\in F^*} |\mathbb{E}\left[P_{X_n,0}(\nu(j) - x_n(j))\right]| \leq \max_{i\in F^*} P_{i,0}\mathbb{E}\left[\|\nu - x_n\|\right]$$

and the proof follows from Theorem 1.2 and dominated convergence.
If one now suppose that assumptions of Corollary 1.5 hold, then, in view of the preceding inequality, there exists $C > 0$ such that

$$\|\mu_n K[\nu] - \mu_{n+1}\| \leq C\sqrt{\gamma_n}.$$

Therefore

$$\|\mu_{n+p} - \mu_n K[\nu]^p\| = \|\sum_{i=0}^{p-1} (\mu_{n+i}K[\nu] - \mu_{n+i+1})\, K[\nu]^{p+i-1}\| \leq C\sum_{i=0}^{p-1} \sqrt{\gamma_{n+i}} \leq pC\sqrt{\gamma_n}$$

and

$$\|\mu_{n+p} - \nu\| \leq \|\mu_{n+p} - \mu_n K[\nu]^p\| + \|\mu_n K[\nu]^p - \nu\| \leq pC\sqrt{\gamma_n} + C_0\rho^p.$$

□

3. A SECOND MODEL BASED ON INTERACTING PARTICLES

A second method to simulate QSD was introduced and well studied by Del Moral and his co-authors in several works on non-linear filtering; see [13]. This one is based on a particle system evolving as follow: at each time, we choose, uniformly at random, a particle i and replace it by another one j; this one is choosen following the probability $P_{i,j}$ or uniformly on the others particles with probability $P_{i,0}$. In this work we will study a slight modification; we allow us the choice to replace the died particle on its previous position. More precisely, let $N \geq 2$ and consider $(X_n^N)_{n\geq 0}$ be the Markov chain on Δ with transition

$$\mathbb{P}\left(X^N(n+1) = x + \frac{1}{N}(\delta_j - \delta_i) \mid X^N(n) = x\right) = p_{i,j}(x), \tag{15}$$

where

$$p_{i,j}(x) = P_{i,j} + P_{i,0}x(j) = K[x]_{i,j}, \tag{16}$$

for every $x \in \Delta$, $n \geq 0$, $i, j \in F^*$. We are interested in the limit of Markov chains X^N, when N is large, and with the time scale $\delta = 1/N$. The key element for such approximation is the vector field $F = (F_j)_{j \in F^*}$, defined by

$$\forall x \in \Delta, \forall j \in F^*,\ F_j(x) = \sum_{i \neq j} (p_{i,j}(x) - p_{j,i}(x)),$$

which, for large N and short time intervals, gives the expected net increase share during the time interval, per time unit. The associated mean-field flow Ψ is the solution to

$$\begin{cases} \forall t \geq 0, \forall x \in \Delta,\ \partial_t \Psi(t,x) = F(\Psi(t,x)), \\ \qquad\quad \forall x \in \Delta,\ \Psi(0,x) = x. \end{cases} \tag{17}$$

Using (16), we have

$$\forall j \in F^*, \forall x \in \Delta,\ F_j(x) = \sum_{i \in F^*} x_i (P_{i,j} + x_j P_{i,0}) - x_j P_{j,0},$$

and Ψ is then the conditioned semi-group of the absorbed Markov process $(U_t)_{t \geq 0}$ generated by $(P - I)$. More precisely, for all $j \in F^*$, $t \geq 0$ and $x \in \Delta$, we have

$$\Psi(t,x) = \frac{\sum_{i \in F^*} \mathbb{P}(U_t = j \mid U_t = i)}{\sum_{i \in F^*} \mathbb{P}(U_t \neq 0 \mid U_t = i)} = \frac{x e^{t(P-I)}}{\langle x e^{t(P-I)}, \mathbf{1} \rangle}.$$

This model was studied in a more general setting in [6]. In particular if we set

$$\forall s \in [0,1),\ \bar{X}^N((n+s)/N) = X_n + s(X_{n+1}^N - X_n^N),$$

then we have

Theorem 3.1 (Deviation inequality)**.** *There exists a (explicit) constant $c > 0$ such that for any $\varepsilon > 0$, $T > 0$, $x \in \Delta$ and N large enough,*

$$\mathbb{P}\left(\max_{0 \leq t \leq T} \|\bar{X}^N(t) - \Psi(t,x)\| \geq \varepsilon \mid X^N(0) = x \right) \leq 2d e^{-c\varepsilon^2 N/T}.$$

In particular, for all $\theta < 1/2$, we have

$$\lim_{N \to +\infty} N^\theta \max_{0 \leq t \leq T} \|\bar{X}^N(t) - \Psi(t,x)\| = 0 \ \text{ a.s.}$$

and

$$\lim_{N \to +\infty} \lim_{n \to +\infty} X_n^N = \nu \ \text{ a.s.}$$

Proof. It comes from [6, Lemma 1], Borel-Cantelli Lemma and [6, Proposition 6] □

In continuous time, we can compare this result with [28, Theorem 1], and [15, Theorem 1.1] which gives a L^1-bound in a more general setting. To our knowledge, it is the first bound almost-sure for this algorithm.
We can also compare our Theorem 1.2 (and Corollary 1.5, more precisely) with [10, Corollary 1.5] (and its proof) and [10, Remark 2.8]. Indeed, using these references, we have that using $t = \gamma \ln(N)$, for some $\gamma > 0$ gives a uniform error term in $N^{-\gamma}$ for the approximation of the QSD, where γ depends on the rate of convergence of the conditioned semi-group to equilibrium (as in our Theorem 1.2).

Remark 3.2 (Others algorithm)**.** *Article* [6] *leads us a new way to develop others methods. Indeed,* [6, Lemma 1] *holds for others choice of F, and thus we have a convergence to the QSD if the flow induced by F converges to the QSD. It is then the case for Φ and Φ_2. For instance, in* (15)*, one can choose*

$$p_{i,j}(x) = A_{i,j} + A_{i,0}x(j).$$

Remark 3.3 (Time versus spatial empirical measure)**.** *In this work, we compare two dynamics based on $K[\mu_r]$ where μ_r is either the time occupation measure or the spatial occupation measure. The analysis of the resultant flows, Φ and Ψ are very similar. This analogy was already observed in others works with the Mc Kean-Vlasov equation; see* [5].

Acknowledgement: We acknowledge financial support from the Swiss National Foundation Grant FN 200020-149871/1. The work of B. Cloez was partially supported by the CIMI (Centre International de Mathématiques et d'Informatique)

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