A Piecewise Deterministic Markov Toy Model for Traffic/Maintenance and Associated Hamilton-Jacobi Integrodifferential Systems on Networks

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A Piecewise Deterministic Markov Toy Model for Traffic/Maintenance and Associated Hamilton-Jacobi Integrodifferential Systems on Networks

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Abstract

We study optimal control problems in infinite horizon when the dynamics belong to a specific class of piecewise deterministic Markov processes constrained to star-shaped networks (corresponding to a toy traffic model). We adapt the results in [35] to prove the regularity of the value function and the dynamic programming principle. Extending the networks and Krylov's "shaking the coefficients" method, we prove that the value function can be seen as the solution to a linearized optimization problem set on a convenient set of probability measures. The approach relies entirely on viscosity arguments. As a by-product, the dual formulation guarantees that the value function is the pointwise supremum over regular subsolutions of the associated Hamilton-Jacobi integrodifferential system. This ensures that the value function satisfies Perron's preconization for the (unique) candidate to viscosity solution.

1 Introduction

This paper aims at the study of optimal control problems in infinite horizon when the dynamics belong to a specific class of piecewise deterministic Markov processes constrained to networks. The starting point is a toy model inspired by traffic. Our point of view is the one of a traffic regulator who observes the generic traffic $X_{\cdot}$ and has the possibility to intervene in the regulation by imposing speed limits via some (external) control. In this basic model, the generic vehicle should remain on some star-shaped network containing several edges bound to a common intersection. At the same time as the traffic, the regulator should ensure the maintenance of the network by observing a second (pure jump) component $\Gamma_{\cdot}$ (known as mode). The functionality of the network evolves stochastically and damage to a specific edge occurs exponentially distributed with a parameter $\lambda(X, \Gamma, \alpha)$ depending on the traffic, on the previous state of the network and on regulator's control policy $\alpha_{\cdot}$. In this context of controlled switched Piecewise Deterministic Markov Processes (PDMP), the regulator seeks to minimize its (discounted) operating cost

$$v^{\delta}(x, \gamma) := \inf_{\alpha, X_{\cdot}^{x, \gamma, \alpha} \in network} \mathbb{E}\left[\int_0^{\infty} e^{-\delta t} l_{\Gamma_t^{x, \gamma, \alpha}}\left(X_t^{x, \gamma, \alpha}, \alpha_t\right) dt\right].$$

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In this paper, we study the Hamilton-Jacobi integrodifferential systems on networks associated to the previous control problem.

To our best knowledge, for deterministic dynamics, the constrained optimal control problem with continuous cost was studied for the first time in [34] (see also [35] for a stochastic framework). The value function of an infinite horizon control problem with space constraints was characterized as a continuous solution to a corresponding Hamilton–Jacobi–Bellman equation. For discontinuous cost functionals, the deterministic control problem with state constraints was studied in [21], [22], [30] using viability theory tools. However, the results of these papers do not directly apply to (deterministic) control problems on star-shaped networks. Several very recent results are available on this subject when dealing with deterministic systems (cf. [33], [1], [28], [11], [2], [27]). The cited papers rely either on Bellman's approach or on Perron's method for the existence of solutions of the associated Hamilton-Jacobi equation and propose several methods for the uniqueness part. There is also an increasing literature on problems inspired by stratified domains or interfaces and discontinuities that partly share the same difficulties (e.g. [10], [7], [31], [4], [5]).

Our control problem is governed by a switch PDMP with characteristic triple (f, λ, Q) (cf. [17], see also Section 2 for the explicit construction). A switch process is often used to model various aspects in biology (see [12], [14], [36], [13], [23]), reliability or storage (in [9], [18]), finance (in [32]), communication networks ([26], [3]). We proceed as follows. In the first part, we prove that v^δ satisfies, in some generalized viscosity sense the associated Hamilton-Jacobi integrodifferential equation. As in the deterministic counterpart, we use Bellman's approach. We begin Section 4 with proving the regularity of the deterministic value function and the dynamic programming principle (DPP) for this case. For available (active) roads, the controllability assumptions are the same as those in [1]. However, entering inactive roads from intersection should be prohibited and other assumptions must be made for this case in order to guarantee the uniform continuity of the value function. Next, we iterate the value functions and the DPP between jumps to prove the uniform continuity of the (stochastic) value function and the DPP. As a by-product, we prove that the value function satisfies in a (relaxed) viscosity sense the associated Hamilton-Jacobi integrodifferential system (in Section 5).

We then focus on a different notion of uniqueness (in Section 6): The well-known method of Perron consists in proposing the supremum over regular subsolution as candidate to the viscosity solution. Using this intuition, we proceed backward and prove that the value function given in the previous section is the pointwise supremum over such regular subsolutions (with a slightly modified notion). The major argument in proving this result is to extend the intersection with some additional directions and impose convenient extensions of the dynamics. Then, we adapt Krylov's "shaking the coefficients" method (cf. [29], [6]) to exhibit a sequence of regular subsolutions of our Hamilton-Jacobi system converging to the initial control problem. These arguments allow the linearization of the value function. It is shown (in Theorem 27) that the value function can be interpreted in connection to an optimization problem set on a family of convenient probability measures. This family is completely described by the Dynkin operator of our process. Moreover, the dual value allows one to state that the initial value function is, indeed, the pointwise supremum over regular subsolutions.

The paper is organized as follows. In Section 2, we recall the basic construction of piecewise deterministic Markov switch processes and give the main assumptions on the dynamics. We present our traffic model and introduce the different types of admissible controls and the controllability assumptions in Section 3. Section 4 is dedicated to the study of regularity of the value function and the dynamic programming principles. The basic ingredient is the technical projection Lemma 6 allowing to prove the uniform continuity of the value function in the deterministic setting (in Theorem 8). We proceed as in [35] by iterating the value function and the dynamic programming principle. In Section 5, we introduce a sequential relaxation of the dynamics and prove that the regular value function exhibited before satisfies, in some generalized viscosity sense, the associated

Hamilton-Jacobi intergrodifferential system. Section 6 is dedicated to the linearization of our value function. We begin with extending the graph and the dynamics by mirroring the trajectories in the inactive case and using the inertia otherwise. We briefly present the adaptation of Krylov's "shaking the coefficients" method and exhibit a family of regular subsolutions converging to the initial value function (in Theorem 25). The main ingredients in proving the convergence are successive projection arguments given by Lemmas 23 and 24 (whose proofs are postponed to the Appendix). The main result (Theorem 27) shows that the value function can be interpreted in connection to an optimization problem set on a family of convenient probability measures. Moreover, the dual of this problem allows one to characterize the value as the pointwise supremum over regular subsolutions (as predicted by Perron's method).

2   Standard construction of controlled switched PDMPs

We consider A (the control space) to be a compact subspace of a metric space $\mathbb{R}^d$ and $\mathbb{R}^m$ be the state space, for some $d, m \geq 1$. Moreover, we consider a finite set E.

We summarize the construction of controlled piecewise deterministic Markov processes (PDMP) of switch type (cf. [15], [16], [17]) having as characteristic triple $f_\gamma : \mathbb{R}^m \times A \longrightarrow \mathbb{R}^m$, for all $\gamma \in E$, $\lambda : \mathbb{R}^m \times E \times A \longrightarrow \mathbb{R}_+$ and $Q : \mathbb{R}^m \times E^2 \times A \longrightarrow [0, 1]$. These functions are assumed to satisfy some usual continuity conditions (to be made precise at the end of the section). The switch PDMP is constructed on a space $(\Omega, \mathcal{F}, \mathbb{P})$ allowing to consider a sequence of independent, $[0, 1]$ uniformly distributed random variables (e.g. the Hilbert cube starting from $[0, 1]$ endowed with its Lebesgue measurable sets and the Lebesgue measure for coordinate, see [17, Section 23]). We let $\mathbb{L}^0 (\mathbb{R}_+ \times \mathbb{R}^m \times E; A)$ denote the space of A-valued Borel measurable functions defined on $\mathbb{R}^m \times E \times \mathbb{R}_+$. Whenever $\alpha_1 \in \mathbb{L}^0 (\mathbb{R}_+ \times \mathbb{R}^m \times E; A)$ and $(t_0, x_0, \gamma_0) \in \mathbb{R}_+ \times \mathbb{R}^m \times E$, we consider the ordinary differential equation

$$\begin{cases} dy_{\gamma_0} (t; t_0, x_0, \alpha_1) = f_{\gamma_0} (y_{\gamma_0} (t; t_0, x_0, \alpha_1), \alpha_1 (t - t_0; x_0, \gamma_0)) \, dt, \ t \geq t_0, \\ y_{\gamma_0} (t_0; t_0, x_0; \alpha_1) = x_0. \end{cases}$$

For the sake of simplicity, whenever $t_0 = 0$, we denote by $y_{\gamma_0} (t; x_0, \alpha_1)$ the solution of the previous ordinary differential equation such that $y_{\gamma_0} (0; x_0, \alpha_1) = x_0$.

We pick the first jump time τ_1 such that the jump rate is $\lambda (y_{\gamma_0} (t; x_0, \alpha_1), \gamma_0, \alpha_1 (t; x_0, \gamma_0))$ i.e.

$$\mathbb{P} (\tau_1 \geq t \ / \ y_{\gamma_0} (0; x_0; \alpha_1) = x_0) = \exp \left(- \int_0^t \lambda (y_{\gamma_0} (s; x_0, \alpha_1), \gamma_0, \alpha_1 (s; x_0, \gamma_0)) \, ds \right).$$

The controlled piecewise deterministic Markov processes (PDMP) is defined by

$$(X_t^{x_0, \gamma_0, \alpha}, \Gamma_t^{x_0, \gamma_0, \alpha}) = (y_{\gamma_0} (t; x_0, \alpha_1), \gamma_0), \ \text{if } t \in [0, \tau_1).$$

The post-jump location is denoted by (Y_1, Υ_1). Since we deal with continuous switching, $Y_1 = y_{\gamma_0} (\tau_1; x_0, \alpha_1)$ and Υ_1 is a random variable who has $Q (y_{\gamma_0} (\tau; x_0, \alpha), \gamma_0, \alpha_1 (\tau, x_0, \gamma_0), \cdot)$ as conditional distribution given $\tau_1 = \tau$. Starting from (Y_1, Υ_1) at time τ_1, we select the inter-jump time $\tau_2 - \tau_1$ such that

$$\mathbb{P} (\tau_2 - \tau_1 \geq t \ / \ \tau_1, (Y_1, \Upsilon_1)) = \exp \left(- \int_{\tau_1}^{\tau_1 + t} \lambda (y_{\Upsilon_1} (s; \tau_1, Y_1, \alpha_2), \Upsilon_1, \alpha_2 (s - \tau_1; Y_1, \Upsilon_1)) \, ds \right),$$

where $\alpha_2 \in \mathbb{L}^0 (\mathbb{R}_+ \times \mathbb{R}^m \times E; A)$. We set

$$(X_t^{x_0, \gamma_0, \alpha}, \Gamma_t^{x_0, \gamma_0, \alpha}) = (y_{\Upsilon_1} (t; \tau_1, Y_1, \alpha_2), \Upsilon_1), \ \text{if } t \in [\tau_1, \tau_2).$$

The post-jump location (Y_2, Υ_2) satisfies

$$\mathbb{P} ((Y_2, \Upsilon_2) \in \mathcal{Y} \times \mathcal{E} \ / \ \tau_2, \tau_1, Y_1, \Upsilon_1) = \mathbf{1}_{y_{\Upsilon_1} (\tau_2; \tau_1, Y_1, \alpha_2) \in \mathcal{Y}} Q (y_{\Upsilon_1} (\tau_2; \tau_1, Y_1, \alpha_2), \Upsilon_1, \mathcal{E}, \alpha_2 (\tau_2 - \tau_1; Y_1, \Upsilon_1)),$$

for all Borel sets $\mathcal{Y} \subset \mathbb{R}^m$ and $\mathcal{E} \subset E$. (Of course, the set E is endowed with the discrete topology.) And so on.

Throughout the paper, unless stated otherwise, we assume the following:

(A1) The functions $f_\gamma : \mathbb{R}^m \times A \longrightarrow \mathbb{R}^m$ are uniformly continuous on $\mathbb{R}^m \times A$ and there exists a positive real constant $C > 0$ such that

$$(A1) \qquad \langle f_\gamma(x,a) - f_\gamma(y,a), x-y \rangle \leq C|x-y|^2, \text{ and } |f_\gamma(x,a)| \leq C,$$

for all $x, y \in \mathbb{R}^m$ and all $a \in A$.

(A2) The function $\lambda : \mathbb{R}^m \times E \times A \longrightarrow \mathbb{R}_+$ is uniformly continuous on $\mathbb{R}^m \times \{\gamma\} \times A$ and there exists a positive real constant $C > 0$ such that

$$(A2) \qquad |\lambda(x,\gamma,a) - \lambda(y,\gamma,a)| \leq C|x-y|, \text{ and } \lambda(x,\gamma,a) \leq C,$$

for all $x, y \in \mathbb{R}^m$, all $\gamma \in E$ and all $a \in A$.

(A3) The function $Q : \mathbb{R}^m \times E^2 \times A \longrightarrow [0,1]$ is a stochastic matrix : i.e. $\sum_{\gamma' \in E} Q(x,\gamma,\gamma',a) = 1$, for all $\gamma \in E$ and all $(x,a) \in \mathbb{R}^m \times A$. Moreover, we assume that $Q(x,\gamma,\gamma,a) = 0$, for all $\gamma \in E$ and that there exists some positive real constant $C > 0$ such that

$$(A3) \qquad \sup_{\substack{a \in A \\ \gamma,\gamma' \in E}} |Q(x,\gamma,\gamma',a) - Q(y,\gamma,\gamma',a)| \leq C|x-y|.$$

(A4) The cost functions $l_\gamma : \mathbb{R}^m \times A \longrightarrow \mathbb{R}$ are uniformly continuous on $\mathbb{R}^m \times A$ and there exists a positive real constant $C > 0$ such that

$$(A4) \qquad |l_\gamma(x,a) - l_\gamma(y,a)| \leq C|x-y|, \text{ and } |l_\gamma(x,a)| \leq C,$$

for all $x, y \in \mathbb{R}^m$ and all $a \in A$.

Remark 1 *(i) The assumptions (**A1-A4**) are quite standard when dealing with viscosity theory in PDMP. They appear under this form in [35] and are needed to infer the uniform continuity of the value function.*

(ii) We have chosen this presentation in order to emphasize the continuity of the X component (continuous switch). Readers who are familiar with the construction in [35], may skip this subsection and just think of a characteristic triple

$$\overline{f} : \mathbb{R}^{m+d} \times A \longrightarrow \mathbb{R}^{m+d}, \ \overline{f}((x,\gamma),a) = (f_\gamma(x,a), 0_{\mathbb{R}^d}), \ \overline{\lambda} = \lambda$$

$$\overline{Q} : \mathbb{R}^{m+d} \times A \to \mathcal{P}\left(\mathbb{R}^{m+d}\right), \ \overline{Q}((x,\gamma),a,dy,d\theta) = \delta_x(dy) Q(x,\gamma,d\theta).$$

Here, $\mathcal{P}\left(\mathbb{R}^{m+d}\right)$ stands for the family of probability measures on $\mathbb{R}^{m+d}$.

3 A traffic problem

We consider a traffic problem on a network given by :
- a family of vertices $(e_j)_{j=1,2,\dots,N}$, for some $N \in \mathbb{N}^* \smallsetminus \{1\}$,
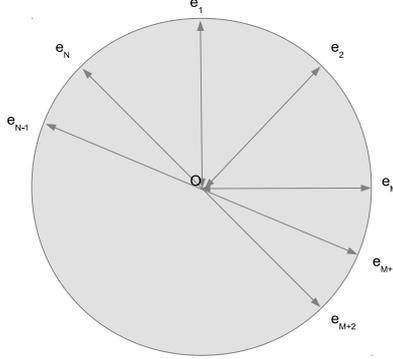- a central intersection denoted by O.

Fig 1. Original intersection

We let $J_j := (0,1)\,e_j$, for all $j = 1, 2, ..., N$, $\mathcal{G} := \underset{j=1,2,...,N}{\cup} [0,1)\,e_j$ and $\overline{\mathcal{G}} := \underset{j=1,2,...,N}{\cup} [0,1]\,e_j$.

Our point of view is the one of a traffic regulator who observes the generic traffic and has the possibility to intervene in the regulation by imposing speed limits via some (external control). Given an initial point $x \in \overline{\mathcal{G}}$, the generic vehicle will move (in a continuous trajectory X_t) on $\overline{\mathcal{G}}$. At the same time as the actual traffic, the regulator observes the quality of the road (Γ_t) and distinguishes between roads which are functional (active) and those which need repairing (inactive). For functional roads, speeding up the traffic at the intersection in both directions is possible. In the inactive case, the road needs repairing and the vehicles that have just entered the road are directed to the junction. We emphasize that we have a simplified toy model in which the x component stands for the position of a generic vehicle (in opposition with the usual density component in traffic models).

This leads to controlled switch PDMP dynamics $(X_t^{x,\gamma,\alpha}, \Gamma_t^{x,\gamma,\alpha})$ governed by the speed of the vehicle f, a jump parameter λ depending on both the traffic and the quality of the road λ and a postjump transition Q specifying functionality of the network. We denote by E the family of all possible functionality variables (e.g. $\{0,1\}^N$) and introduce, for all $j = 1, 2, ..., N$ a partition of $E = E_j^{active} \cup E_j^{inactive}$.

Given an initial couple describing the position and configuration $(x, \gamma) \in \overline{\mathcal{G}} \times E$, we introduce the set of feasible (network-constrained) controls for the deterministic framework by setting

$$\mathcal{A}_{\gamma,x} := \left\{ \alpha : \mathbb{R}_+ \longrightarrow A : \alpha \text{ is Borel measurable}, \, y_\gamma(t; x, \alpha) \in \overline{\mathcal{G}}, \text{ for all } t \geq 0, \, \right\},$$

for all $(x, \gamma) \in \overline{\mathcal{G}} \times E$. In general, without further assumptions, these sets might be empty. We will specify hereafter some controllability conditions that guarantee consistence of these sets. We also introduce the set of constant, locally-admissible controls for the deterministic problem by setting

$$A_{\gamma,x} = \left\{ a \in A : y_\gamma(t; x, a) \in \overline{\mathcal{G}}, \text{ for some } \theta > 0 \text{ and all } t \in [0, \theta] \right\},$$

for all $(x, \gamma) \in \overline{\mathcal{G}} \times E$.

Unless stated otherwise, throughout the paper, we will use the following assumptions.

(Aa) There exist nonempty subsets $A^{\gamma,j} \subset A$ such that

$$A_{\gamma,x} = A^{\gamma,j}, \text{ if } x \in J_j,$$
$$A_{\gamma,O} = \underset{j=1,2,...,N}{\cup} \left\{ a \in A^{\gamma,j} : f(O, a) \in \mathbb{R}_+ e_j \right\},$$
$$A_{\gamma,e_j} = \left\{ a \in A^{\gamma,j} : \langle f_\gamma(e_j, a), e_j \rangle \leq 0 \right\} \neq \emptyset,$$

for all $\gamma \in E$ and all $j = 1, 2, ..., N$. Moreover, we assume that, for every $\gamma \in E$ and every $j = 1, 2, ..., N$, either $A_{\gamma, e_j} = A^{\gamma, j}$ or, otherwise, there exists some $\beta > 0$ and some $a_{\gamma, j} \in A^{\gamma, j}$ satisfying

$$\langle f_\gamma (e_j, a_{\gamma, j}), e_j \rangle < -\beta.$$

(Ab)
The active case :
For all $\gamma \in E_j^{active}$, there exist some $\beta > 0$ and some $a_{\gamma, j}^+, a_{\gamma, j}^- \in A^{\gamma, j}$ such that

$$\left\langle f_\gamma \left(O, a_{\gamma, j}^+\right), e_j \right\rangle > \beta \text{ and } \left\langle f_\gamma \left(O, a_{\gamma, j}^-\right), e_j \right\rangle < -\beta.$$

The inactive case :
For $\gamma \in E_j^{inactive}$, there exist some $\beta > 0$, $1 > \eta > 0$, $\kappa \in [0, 1)$ and $a_{\gamma, j}^-, a_{\gamma, j}^0 \in A^{\gamma, j}$ such that

$$\left\langle f_\gamma \left(x, a_{\gamma, j}^-\right), e_j \right\rangle \le -\beta \langle x, e_j \rangle^\kappa,$$

for all $x \in J_j$, $|x| \le \eta$ and $f_\gamma \left(O, a_{\gamma, j}^0\right) = 0$. Moreover,

$$\langle f_\gamma (x, a), e_j \rangle \le 0,$$

for all $a \in A^{\gamma, j}$ and all $x \in J_j$, $|x| \le \eta$.
  (Ac) Whenever $\gamma \in E_j^{inactive}$, $l_\gamma (O, a) = l_\gamma (O)$, for all $a \in A^{\gamma, j}$.

Remark 2 *(i) The condition **(Ab)** states that if the road is functional (active), then one has a behavior similar to the one introduced in [1] (speeding up the traffic at the intersection in both directions is possible).*

*If the road is inactive, then, again according to **(Ab)**, for the cars that have "just" entered the road, the only possibility is to move back into the intersection (the road needs clearing up for repairing). A measure $(a_{\gamma, j}^-)$ is possible to get them off this inactive road within a controlled time and, eventually, they are allowed to stay in O (due to the control $a_{\gamma, j}^0$) until the road is repaired.*

*The condition **(Ac)** is intended for technical reasons. It can be interpreted as : if the road is inactive, the presence of vehicles at the entrance of the road prevents the authority to intervene and repair the road and thus involves a certain cost. For vehicles that intend to get to e_j, there is a global "waiting" cost at junction. However, if $\{a \in A^{\gamma, j} : f(O, a) \in \mathbb{R}_+ e_j\} = A^{\gamma, j}$, then **(Ac)** is no longer necessary.*

*(ii) Under the assumption **(Aa)**, if $A_{\gamma, e_j} \ne A^{\gamma, j}$, then there exists $\frac{1}{2} > \eta > 0$ such that*

$$\langle f_\gamma (x, a_{\gamma, j}), e_j \rangle < -\beta,$$

*whenever $|x - e_j| \le \eta$. Similarly, under the assumption **(Ab)**, for every $\gamma \in E_j^{active}$ and some $\eta > 0$,*

$$\left\langle f_\gamma \left(x, a_{\gamma, j}^-\right), e_j \right\rangle < -\beta, \ \left\langle f_\gamma \left(x, a_{\gamma, j}^+\right), e_j \right\rangle > \beta,$$

whenever $|x| \le \eta$.

Our assumptions guarantee the following.

Proposition 3 *Under the assumptions **(Aa)** and **(Ab)**, the set $\mathcal{A}_{\gamma, x}$ is nonempty for all $(x, \gamma) \in \overline{\mathcal{G}} \times E$.*

Proof. If $\gamma \in E_j^{active}$ and $x \in [0, 0.5)\, e_j$, we define

$$t_{x,\gamma,e_j}^+ := \inf\left\{ t > 0 : y_\gamma\left(t; x, a_{\gamma,j}^+\right) = e_j \right\},$$

If $x \in [0.5, 1]\, e_j$, we let

$$t_{x,\gamma,O}^- := \inf\left\{ t > 0 : y_\gamma\left(t; x, a\right) = O \right\},$$

where a is any point of A_{γ,e_j}. One notices that $t_{x,\gamma,e_j}^+ \geq \frac{0.5}{\max(|f|_0, 1)}$ and $t_{x,\gamma,O}^- \geq \frac{0.5}{\max(|f|_0, 1)}$, where $|f|_0 = \max\limits_{\gamma \in E,\ x \in \overline{\mathcal{G}},\ a \in A} |f_\gamma(x, a)|$. For $x \in [0, 0.5)\, e_j$, we set

$$\alpha_{x,\gamma}^0(t) := \left\{ \begin{array}{c} a_{\gamma,j}^+,\ \text{if } t \in \left[0, t_{x,\gamma,e_j}^+\right) \cup \left[t_{x,\gamma,e_j}^+ + t_{e_j,\gamma,O}^-, t_{x,\gamma,e_j}^+ + t_{e_j,\gamma,O}^- + t_{O,\gamma,e_j}^+\right) \cup ..., \\ a,\ \text{otherwise.} \end{array} \right.$$

The estimates on $t^{+,-}$ imply that $\alpha_{x,\gamma}^0$ is defined on $\mathbb{R}_+$. Moreover, it is clear that $\alpha_{x,\gamma}^0 \in A_{\gamma,x}$. Similar construction holds true for $x \in [0.5, 1]\, e_j$. If $\gamma \in E_j^{inactive}$, one gets similar results by replacing $a_{\gamma,j}^+$ with $a_{\gamma,j}^0$. (In fact, in this case, if $t_{x,\gamma,O}^-$ is finite, then the solution stays at O after the time $t_{x,\gamma,O}^-$). This concludes the proof of our assertion. ∎

We introduce the set $\mathcal{A}_{ad}$ given by

$$(1) \qquad \mathcal{A}_{ad} := \left\{ \begin{array}{c} \alpha : \mathbb{R}_+ \times \overline{\mathcal{G}} \times E \longrightarrow A : \alpha \text{ is Borel measurable,} \\ X_t^{x_0, \gamma_0, \alpha} \in \overline{\mathcal{G}},\ \text{for all } t \geq 0, \mathbb{P}-a.s.,\ \text{for all } (x_0, \gamma_0) \in \overline{\mathcal{G}} \times E \end{array} \right\}.$$

Here, $X_t^{x_0, \gamma_0, \alpha}$ is the continuous component of our PDMP constructed as in Section 2 by using $\alpha_i = \alpha$, for all $i \geq 1$.

Remark 4 *(a) Under the assumptions (Aa, Ab) it is clear that $\mathcal{A}_{ad}$ is nonempty. In fact, it suffices to note that all the times t^+, t^- in the previous proposition are measurable functions of (x, γ).*

(b) The set $\mathcal{A}_{\gamma,x}$ can be seen as a subset of $\mathcal{A}_{ad}$ by choosing some $\overline{\alpha}_0 \in \mathcal{A}_{ad}$ and setting

$$\overline{\alpha}(t, y, \eta) := \left\{ \begin{array}{l} \alpha(t),\ \text{if } (y, \eta) = (x, \gamma), \\ \overline{\alpha}_0(t; y, \eta),\ \text{otherwise,} \end{array} \right.$$

for all $\alpha \in \mathcal{A}_{\gamma,x}$.

Example 5 *Let us exhibit a simple example for which the previous assumptions (particularly (A1), (Aa-Ab)) are satisfied. We consider $N = 3$ and $e_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $e_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: -e_3$, $A := [-1, 1]\, e_1 \cup [-1, 1]\, e_2$, $E := \left\{ \begin{array}{c} (0,0,0), (0,1,1), \\ (1,0,0), (1,1,1) \end{array} \right\} \subset \{0,1\}^3$,*

$$f_\gamma(x, a) := \gamma_1 \langle a, e_1 \rangle e_1 + \gamma_2 \langle a, e_2 \rangle e_2 - |a| \left[\begin{array}{c} (1 - \gamma_1) \langle x, e_1 \rangle^{\frac{1}{2}} e_1 + (1 - \gamma_2) \left(\langle x, e_2 \rangle^+ \right)^{\frac{1}{2}} e_2 \\ + (1 - \gamma_2) \left(\langle x, e_3 \rangle^+ \right)^{\frac{1}{2}} e_3 \end{array} \right],$$

for $x \in \mathbb{R} \times \mathbb{R}_+$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in E$. Here, $z^+ = \max(z, 0)$, for $z \in \mathbb{R}$. Then $E_1^{inactive} := \{\gamma \in E : \gamma_1 = 0\}$ and $E_2^{inactive} = E_3^{inactive} := \{\gamma \in E : \gamma_2 = 0\}$. The reader is invited to note that f_γ is Lipschitz-continuous for active configurations. Also, we wish to note that, for this particular case, whenever J_2 is inactive (i.e. $\gamma \in E_2^{inactive}$), $f_\gamma(re_2, a) = -f_\gamma(-re_2, a)$, for all $r \in \mathbb{R}$. The intersection acts as a mirror in the inactive case.

The cost l can be chosen increasing with the speed, very high as one reaches the intersection and null at the destination vertex. Moreover, it can be chosen decreasing with respect to the number of available/ active roads

$$l_\gamma \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, a \right) := l_0 + \frac{1}{\gamma_1 + \gamma_2 + 1} \left(1 - |x_1| \right)^2 + |a| \left(|x_1| - |x_1|^2 \right) \text{ and symmetrically for } \begin{pmatrix} 0 \\ x_2 \end{pmatrix}.$$

Here, $l_0 > 0$ is some minimal cost.

The rate λ can be chosen in a similar way as a propensity function : we define $\widetilde{\lambda}_\gamma(x, a) := \lambda_0 l_\gamma(x, a)$ for some $\lambda_0 > 0$, then $\lambda_\gamma(x, a) := \sum_{\gamma' \in E \smallsetminus \{\gamma\}} \widetilde{\lambda}_{\gamma'}(x, a)$. The jump measure Q can be chosen proportional to the relative contribution to the propensity function

$$Q(x, \gamma, \gamma', a) := \begin{cases} \frac{\widetilde{\lambda}_{\gamma'}(x, a)}{\lambda_\gamma(x, a)}, & \text{if } \gamma' \in E \smallsetminus \{\gamma\}, \\ 0, & \text{if } \gamma' = \gamma. \end{cases}$$

4 The dynamic programming principle and the regularity of the value function(s)

The aim of the traffic regulator will be to minimize the expectation of the (infinite horizon, discounted) operating cost l satisfying (for the time being and unless stated otherwise), the assumption **(A4)**

$$\inf_\alpha \mathbb{E} \left[\int_0^\infty e^{-\delta t} l_{\Gamma_t^{x, \gamma, \alpha}} \left(X_t^{x, \gamma, \alpha}, \alpha_t \right) dt \right].$$

The discount parameter $\delta > 0$ will be fixed throughout the paper. The set of control policies (keeping the vehicle on the network) as well as the meaning of α_t will be given later on.

The program of this first part relies on the paper [35] : we study the regularity properties in the deterministic setting via some projection argument, then define some iterated value functions. Next, we prove the uniform continuity of these iterates and the dynamic programming principles (DPP). This leads to a regular limit function satisfying a DPP.

Throughout the paper, if ϕ is a bounded real-valued function on some set $\mathcal{X} \times F$, where $\mathcal{X} \subset \mathbb{R}^M$ and F is compact such that $\phi(\cdot, \varsigma)$ is Lipschitz-continuous for all $\varsigma \in F$, we set

$$|\phi|_0 := \sup_{(y, \varsigma) \in \mathcal{X} \times F} |\phi(y, \varsigma)| \text{ and } Lip(\phi) := \sup_{\varsigma \in F} \sup_{\substack{y, y' \in \mathcal{X} \\ y \neq y'}} \frac{|\phi(y, \varsigma) - \phi(y', \varsigma)|}{|y - y'|}.$$

Whenever f is not Lipschitz continuous (recall that (A1) is weaker than Lipschitz-continuity), by abuse of notation, we let

$$Lip(f) := \sup_{(\gamma, a) \in E \times A} \sup_{\substack{y, y' \in \mathbb{R}^m \\ y \neq y'}} \frac{\langle f_\gamma(y, a) - f_\gamma(y', a), y - y' \rangle}{|y - y'|^2}.$$

Of course, whenever the function f is only defined and satisfies the regularity assumptions on $\overline{\mathcal{G}}$, the supremum can be taken over $j = 1, 2, ..., N$ and y, y' which are colinear with e_j and $a \in A^{\gamma, j}$.

4.1 A projection argument

Whenever $\varepsilon > 0$ is small enough, we let

$$t_\varepsilon := -\frac{1}{\delta} \ln \left(\frac{\varepsilon \delta}{2 |f|_0} \right), \rho_\varepsilon := \frac{\eta}{4} e^{-Lip(f) t_\varepsilon}.$$

We will make extensive use of the following result.

Lemma 6 *We assume* (***Aa-Ac***) *and* (***A1-A4***) *to hold true.*

(i) There exists some $C > 0$ such that, for every $\varepsilon > 0$, every $\gamma \in E$, $x, y \in J_1 \cup \{O, e_1\}$ satisfying $|x - y| \leq \rho_\varepsilon^{\frac{2}{1-\kappa}}$ and every $\alpha \in \mathcal{A}_{\gamma,x}$, there exists $\mathcal{P}_{x,y}(\alpha) \in \mathcal{A}_{\gamma,y}$ such that

$$
(2) \qquad |y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)) - y_\gamma(t; x, \alpha)| \leq C |x - y|^{\frac{1-\kappa}{2}},
$$

and
$$
(3)
$$
$$
\left| \int_0^t e^{-\delta s} l_\gamma(y_\gamma(s; y, \mathcal{P}_{x,y}(\alpha)), \mathcal{P}_{x,y}(\alpha)(s)) \, ds - \int_0^t e^{-\delta s} l_\gamma(y_\gamma(s; x, \alpha), \alpha(s)) \, ds \right| \leq C |x - y|^{\frac{1-\kappa}{2}},
$$

for all $t \leq t_\varepsilon$.

(ii) Moreover, if $\alpha \in \mathcal{A}_{ad}$, then, for every $\varepsilon > 0$ and every $(\gamma, x) \in E \times (J_1 \cup \{O, e_1\})$, there exists $\mathcal{P}_{(x,\gamma)}(\alpha) \in \mathcal{A}_{ad}$ such that the previous inequalities are satisfied with $\mathcal{P}_{(x,\gamma)}(\alpha)(\cdot; y, \gamma)$ replacing $\mathcal{P}_{x,y}(\alpha)(\cdot)$, for all $y \in J_1 \cup \{O, e_1\}$ satisfying $|x - y| \leq \rho_\varepsilon^{\frac{2}{1-\kappa}}$.

Remark 7 *(i) A brief look at the proof shows that the constant C in the previous lemma only depends on $Lip(l), |l|_0, Lip(f), |f|_0$ and β but not of the actual coefficient f nor of the actual cost function l.*

(ii) The assumption (***Ac***) *is only needed if $\{a \in A^{\gamma,1} : f(O, a) \in \mathbb{R}_+ e_1\} \neq A^{\gamma,1}$. Otherwise, both the cases (b1) and the analogous (c3.2) in the proof need not being treated as special cases.*

At this point, we introduce the value function for the deterministic case ($\lambda = 0$, or, equivalently, the road functionality γ is immutable) by setting

$$
v_0^\delta(x, \gamma) = \inf_{\alpha \in \mathcal{A}_{\gamma,x}} \int_0^\infty e^{-\delta t} l_\gamma(y_\gamma(t; x, \alpha), \alpha(t)) \, dt,
$$

for all $x \in \overline{\mathcal{G}}$ and all $\gamma \in E$.

As a consequence of our projection lemma, we get the following continuity result :

Theorem 8 *The deterministic value functions $v_0^\delta(\cdot, \gamma)$ are bounded and uniformly continuous on $\overline{\mathcal{G}}$.*

Proof. Since the domain $\overline{\mathcal{G}}$ is compact, it suffices to prove that $v^\delta(\cdot, \gamma)$ is continuous. Let us fix $x \in \overline{\mathcal{G}} \setminus \{O\}$ and consider $\varepsilon > 0$. Without loss of generality, we assume that $x \in J_1 \cup \{e_1\}$. Then, there exists some $\alpha \in \mathcal{A}_{\gamma,x}$ such that

$$
v_0^\delta(x, \gamma) + \varepsilon \geq \int_0^{t_\varepsilon} e^{-\delta t} l_\gamma(y_\gamma(t; x, \alpha), \alpha(t)) \, dt - \frac{1}{\delta} e^{-\delta t_\varepsilon} |l|_0.
$$

Hence, for every $y \in J_1 \cup \{e_1, O\}$ such that $|x - y| \leq \rho_\varepsilon^{\frac{2}{1-\kappa}}$, using the previous lemma, there exists $\mathcal{P}_{x,y}(\alpha) \in \mathcal{A}_{\gamma,y}$ such that

$$
\begin{aligned}
v_0^\delta(x, \gamma) + \varepsilon &\geq \int_0^{t_\varepsilon} e^{-\delta t} l_\gamma(y(t; x, \mathcal{P}_{x,y}(\alpha)), \mathcal{P}_{x,y}(\alpha)(t)) \, dt - C\rho_\varepsilon - \frac{1}{\delta} e^{-\delta t_\varepsilon} |l|_0 \\
&\geq \int_0^\infty e^{-\delta t} l_\gamma(y(t; x, \mathcal{P}_{x,y}(\alpha)), \mathcal{P}_{x,y}(\alpha)(t)) \, dt - C\rho_\varepsilon - \frac{2}{\delta} e^{-\delta t_\varepsilon} |l|_0 \\
&\geq v_0^\delta(y, \gamma) - C\rho_\varepsilon - \frac{|l|_0}{|f|_0} \varepsilon.
\end{aligned}
$$

9

The continuity property follows by recalling that $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon \to 0} \rho_\varepsilon = 0$. In the case when $x = O$, the same arguments yield

$$\lim_{\substack{y \to O \\ y \in J_j}} v_0^\delta(y, \gamma) = v_0^\delta(O, \gamma),$$

for every $j = 1, 2, ..., N$. The proof of our theorem is now complete. ∎

Remark 9 *The reader is invited to note that the continuity modulus of v_0^δ depends only on $Lip(l)$, $|l|_0$, $Lip(f)$, $|f|_0$ and β but not of the actual coefficient f nor of the actual cost function l.*

4.2 Iterated value function

Following the ideas of [35], we introduce the iterated value functions v_m^δ defined by

$$v_m^\delta(x, \gamma) := \inf_{\alpha \in \mathcal{A}_{ad}} J_m(x, \gamma, \alpha),$$

where

$$J_m(x, \gamma, \alpha) := \mathbb{E}\left[\int_0^{\tau_1} e^{-\delta t} l_\gamma(X_t^{x,\gamma,\alpha}, \alpha(t; x, \gamma)) \, dt + e^{-\delta \tau_1} v_{m-1}^\delta(Y_1, \Upsilon_1)\right].$$

We recall that (Y_1, Υ_1) are the post-jump locations at the first jump time τ_1 depending on x, γ, α, (cf. Section 2). Hence, we have $(Y_1, \Upsilon_1) = (X_{\tau_1}^{x,\gamma,\alpha}, \Gamma_{\tau_1}^{x,\gamma,\alpha})$ and $\tau_1 = \tau_1^{x,\gamma,\alpha}$. The process is constructed as in section 2 using $\alpha_i = \alpha \in \mathcal{A}_{ad}$, for all $i \geq 1$. The reader is invited to note that a simple recurrence argument yields

(4) $$\left|v_m^\delta(x, \gamma)\right| \leq \frac{|l|_0}{\delta}, \text{ for all } (x, \gamma) \in \overline{\mathcal{G}} \times E.$$

Throughout the section, unless stated otherwise, we assume (**Aa-Ac**) and (**A1-A4**) to hold true. In order to simplify our presentation, we assume that λ and Q are independent of the control parameter a. The general case follows from similar arguments as those of Lemma 6 (the estimates on l) if one assumes

(**Ac'**) Whenever $\gamma \in E_j^{inactive}$, $Q(O, \gamma, \gamma', a) = Q(O, \gamma, \gamma')$ and $\lambda(O, \gamma, a) = \lambda(O, \gamma)$.

Again, (**Ac'**) is only needed for those j such that $\gamma \in E_j^{inactive}$ and $\{a \in A^{\gamma,j} : f(O, a) \in \mathbb{R}_+ e_j\} \neq A^{\gamma,j}$.

The same arguments as those employed in [35, Lemma 3.1] yield

Lemma 10 *Let us assume that $v_{m-1}^\delta(\cdot, \gamma)$ is continuous on $\overline{\mathcal{G}}$. Then, for every $T > 0$, one has*

$$v_m^\delta(x, \gamma) = \inf_{\alpha \in \mathcal{A}_{ad}} \mathbb{E}\left[\begin{array}{c} \int_0^{\tau_1 \wedge T} e^{-\delta t} l_\gamma(y_\gamma(t; x, \alpha), \alpha(t; x, \gamma)) \, dt \\ + e^{-\delta \tau_1} v_{m-1}^\delta(Y_1, \Upsilon_1) \mathbf{1}_{\tau_1 \leq T} + e^{-\delta T} v_m^\delta(y_\gamma(T; x, \alpha), \gamma) \mathbf{1}_{\tau_1 > T} \end{array}\right],$$

for all $(x, \gamma) \in \overline{\mathcal{G}} \times E$.

The proof is identical (no changes needed) to the one in [35, Lemma 3.1] and will be omitted from our (already long enough) presentation.

Theorem 11 *The functions $v_m^\delta(\cdot, \gamma)$ are uniformly continuous on $\overline{\mathcal{G}}$, for all $m \geq 0$ and uniformly with respect to $\gamma \in E$.*

Proof. We prove our theorem by recurrence over m. For $m = 0$, we invoke theorem 8. Let us assume that $v_{m-1}^\delta (\cdot, \gamma')$ is continuous for all $\gamma' \in E$. We let ω_{m-1} be the continuity modulus

$$\omega_{m-1}(r) := \sup \left\{ \left| v_{m-1}^\delta (x, \gamma') - v_{m-1}^\delta (y, \gamma') \right| : |x - y| \leq r, \ \gamma' \in E \right\}.$$

We also introduce

$$\omega_m (\gamma, r) := \sup \left\{ \left| v_m^\delta (x, \gamma) - v_m^\delta (y, \gamma) \right| : |x - y| \leq r \right\},$$

for all $r > 0$. Obviously, $\omega_m(r) = \sup_{\gamma \in E} \omega_m(\gamma, r)$. It is straightforward that $\omega_m(r) \leq 2 \frac{|l|_0}{\delta}$. Let us fix, for the time being, $(\gamma, x, y) \in E \times \overline{\mathcal{G}}^2$, $\varepsilon > 0$ and assume that $|x - y| \leq \rho_\varepsilon^{\frac{2}{1-\kappa}}$. Then, due to the previous lemma, there exists some admissible control process $\alpha \in \mathcal{A}_{ad}$ such that

$$v_m^\delta (x, \gamma) \geq -\varepsilon + \mathbb{E} \left[\begin{array}{c} \int_0^{\tau_1 \wedge t_\varepsilon} e^{-\delta t} l_\gamma (y_\gamma (t; x, \alpha), \alpha (t; x, \gamma)) \, dt \\ + e^{-\delta \tau_1} v_{m-1}^\delta (Y_1, \Upsilon_1) \mathbf{1}_{\tau_1 \leq t_\varepsilon} + e^{-\delta t_\varepsilon} v_m^\delta (y_\gamma (t_\varepsilon; x, \alpha), \gamma) \mathbf{1}_{\tau_1 > t_\varepsilon}. \end{array} \right]$$

We denote by $\widetilde{\alpha}$ the admissible control process $\mathcal{P}_{(x,\gamma)} (\alpha) \in \mathcal{A}_{ad}$ given by the assertion (ii) in Lemma 6. Moreover, we let $\widetilde{\tau}_1$ be the first jump time starting from (y, γ) and using the control $\widetilde{\alpha}$. We introduce the following notations :

$$y(t) := y_\gamma (t; x, \alpha), \ \alpha(t) := \alpha (t; x, \gamma), \ \lambda(t) := \lambda (y(t), \gamma), \ \Lambda(t) = \exp \left(-\int_0^t \lambda(s) \, ds \right),$$

$$\widetilde{y}(t) := y_\gamma (t; y, \widetilde{\alpha}), \ \widetilde{\alpha}(t) := \widetilde{\alpha} (t; y, \gamma), \ \widetilde{\lambda}(t) := \lambda (\widetilde{y}(t), \gamma), \ \widetilde{\Lambda}(t) = \exp \left(-\int_0^t \widetilde{\lambda}(s) \, ds \right).$$

Then

$$v_m^\delta (y, \gamma) \leq \begin{array}{c} \mathbb{E} \left[\int_0^{\widetilde{\tau}_1 \wedge t_\varepsilon} e^{-\delta t} l_\gamma (\widetilde{y}(t), \widetilde{\alpha}(t)) \, dt \right] \\ + \mathbb{E} \left[e^{-\delta \widetilde{\tau}_1} v_{m-1}^\delta \left(\widetilde{Y}_1, \widetilde{\Upsilon}_1 \right) \mathbf{1}_{\tau_1 \leq t_\varepsilon} + e^{-\delta t_\varepsilon} v_m^\delta (\widetilde{y}(t_\varepsilon), \gamma) \mathbf{1}_{\widetilde{\tau}_1 > t_\varepsilon}. \right] \end{array}$$

The right-hand member can be written as

$$(5) \qquad I_m (y, \gamma, \widetilde{\alpha}) = \int_0^{t_\varepsilon} \widetilde{\lambda}(t) \widetilde{\Lambda}(t) \int_0^t e^{-\delta s} l_\gamma (\widetilde{y}(s), \widetilde{\alpha}(s)) \, ds \, dt$$

$$+ \int_0^{t_\varepsilon} \widetilde{\lambda}(t) \widetilde{\Lambda}(t) e^{-\delta t} \sum_{\gamma' \in E \setminus \{\gamma\}} v_{m-1}^\delta (\widetilde{y}(t), \gamma') Q (\widetilde{y}(t), \gamma, \gamma') \, dt$$

$$+ \widetilde{\Lambda}(t_\varepsilon) \int_0^{t_\varepsilon} e^{-\delta t} l_\gamma (\widetilde{y}(t), \widetilde{\alpha}(t)) \, dt + \widetilde{\Lambda}(t_\varepsilon) e^{-\delta t_\varepsilon} v_m^\delta (\widetilde{y}(t_\varepsilon), \gamma).$$

Then, using the estimates (2) in Lemma 6 and recalling that (**A2**) holds true, one has

$$(6) \qquad I_m (y, \gamma, \widetilde{\alpha}) \leq C |x - y|^{\frac{1-\kappa}{2}} + \int_0^{t_\varepsilon} \lambda(t) \Lambda(t) \int_0^t e^{-\delta s} l_\gamma (\widetilde{y}(s), \widetilde{\alpha}(s)) \, ds \, dt$$

$$+ \int_0^{t_\varepsilon} \lambda(t) \Lambda(t) e^{-\delta t} \sum_{\gamma' \in E \setminus \{\gamma\}} v_{m-1}^\delta (\widetilde{y}(t), \gamma') Q (\widetilde{y}(t), \gamma, \gamma') \, dt$$

$$+ \Lambda(t_\varepsilon) \int_0^{t_\varepsilon} e^{-\delta t} l_\gamma (\widetilde{y}(t), \widetilde{\alpha}(t)) \, dt + \Lambda(t_\varepsilon) e^{-\delta t_\varepsilon} v_m^\delta (\widetilde{y}(t_\varepsilon), \gamma),$$

for some generic constant $C > 0$ independent of ε, γ, y, x, α which may change from one line to another. This constant only depends on the supremum norm and the Lipschitz constants of λ, Q, f

and l. Again by (2) and using the assumption (**A3**), we get

$$(7) \qquad \sum_{\gamma' \in E \backslash \{\gamma\}} v_{m-1}^{\delta} \left(\widetilde{y}(t), \gamma' \right) Q \left(\widetilde{y}(t), \gamma, \gamma' \right) - \sum_{\gamma' \in E \backslash \{\gamma\}} v_{m-1}^{\delta} \left(y(t), \gamma' \right) Q \left(y(t), \gamma, \gamma' \right)$$

$$\leq \omega_{m-1} \left(C |x - y|^{\frac{1-\kappa}{2}} \right) + \sum_{\gamma' \in E \backslash \{\gamma\}} \left| v_{m-1}^{\delta} \left(y(t), \gamma' \right) \right| \left| Q \left(\widetilde{y}(t), \gamma, \gamma' \right) - Q \left(y(t), \gamma, \gamma' \right) \right|$$

$$\leq \omega_{m-1} \left(C |x - y|^{\frac{1-\kappa}{2}} \right) + C |x - y|^{\frac{1-\kappa}{2}},$$

for all $t \leq t_{\varepsilon}$. Moreover

$$e^{-\delta t_{\varepsilon}} v_m^{\delta} \left(\widetilde{y}(t_{\varepsilon}), \gamma' \right) \leq e^{-\delta t_{\varepsilon}} v_m^{\delta} \left(y(t_{\varepsilon}), \gamma' \right) + e^{-\delta t_{\varepsilon}} \omega_m \left(C |x - y|^{\frac{1-\kappa}{2}} \right).$$

Returning to (6) and using (7) and the previous relation, we get

$$v_m^{\delta}(y, \gamma) \leq v_m^{\delta}(x, \gamma) + \varepsilon + C |x - y|^{\frac{1-\kappa}{2}} + \omega_{m-1} \left(C |x - y|^{\frac{1-\kappa}{2}} \right) + e^{-\delta t_{\varepsilon}} \omega_m \left(C |x - y|^{\frac{1-\kappa}{2}} \right).$$

Hence, whenever $|x - y| \leq r \leq \rho_{\varepsilon}^{\frac{2}{1-\kappa}}$,

$$\omega_m(r, \gamma) \leq \varepsilon + C r^{\frac{1-\kappa}{2}} + \omega_{m-1} \left(C r^{\frac{1-\kappa}{2}} \right) + e^{-\delta t_{\varepsilon}} \omega_m \left(C r^{\frac{1-\kappa}{2}} \right).$$

Taking the supremum over $\gamma \in E$, we can replace $\omega_m(r, \gamma)$ with $\omega_m(r)$. We can assume, without loss of generality, that $C > 1$ and the conclusion follows (similar to Lemma 3.3 in [35]). Indeed, one considers $r = C^{-\frac{2}{1+\kappa} \left[\left(\frac{1-\kappa}{2} \right)^{-n} - 1 \right]}$ and iterates in the previous inequality to get

$$\omega_m \left(C^{-\frac{2}{1+\kappa} \left[\left(\frac{1-\kappa}{2} \right)^{-n} - 1 \right]} \right) = \varepsilon \frac{1}{1 - e^{-\delta t_{\varepsilon}}} + e^{-\delta t_{\varepsilon}(n-1)} \sum_{k=0}^{n-1} \omega_{m-1} \left(C^{-\frac{2}{1+\kappa} \left[\left(\frac{1-\kappa}{2} \right)^{-k} - 1 \right]} \right) e^{\delta k t_{\varepsilon}}$$

$$+ e^{-\delta t_{\varepsilon}(n-1)} \sum_{k=0}^{n-1} \left(C^{-\frac{2}{1+\kappa} \left[\left(\frac{1-\kappa}{2} \right)^{-k} - 1 \right]} \right) e^{\delta k t_{\varepsilon}} + 2 e^{-\delta t_{\varepsilon} n} \frac{|l|_0}{\delta},$$

for n large enough and recall that $\varepsilon > 0$ is arbitrary. Then, by the recurrence assumption and allowing $n \to \infty$, one gets

$$\omega_m(0) \leq \varepsilon \frac{1}{1 - e^{-\delta t_{\varepsilon}}} = \frac{\varepsilon}{1 - \frac{\varepsilon \delta}{2|f|_0}}.$$

To complete the proof, one only needs to recall that this inequality holds true for arbitrary $\varepsilon > 0$. ∎

Remark 12 *In fact, all these continuity moduli depend only the supremum norm and the Lipschitz constants of λ, Q, f and l but the particular choice of the coefficients is irrelevant (see also Remark 9).*

As a corollary, using the same proof as in the first part of Theorem 3.4 in [35], we get

Corollary 13 *Under our assumptions (**A1-A4, Aa-Ac, Ac'**), the value function $v^{\delta}(\gamma, \cdot)$ given by*

$$v^{\delta}(x, \gamma) := \inf_{\alpha \in \mathcal{A}_{ad}^{\mathbb{N}}} \mathbb{E} \left[\sum_{n \geq 0} \int_{\tau_n}^{\tau_{n+1}} e^{-\delta t} l_{\Gamma_{\tau_n}^{x, \gamma, \alpha}} \left(y_{\Gamma_{\tau_n}^{\gamma, x, \alpha}} \left(t; X_{\tau_n}^{x, \gamma, \alpha}, \alpha_{n+1} \right), \alpha_{n+1} \left(t - \Gamma_{\tau_n}^{x, \gamma, \alpha}; X_{\tau_n}^{x, \gamma, \alpha}, \Gamma_{\tau_n}^{x, \gamma, \alpha} \right) \right) \right]$$

is bounded and uniformly continuous on $\overline{\mathcal{G}}$, for all $\gamma \in E$. Moreover, it satisfies the following Dynamic Programming Principle :

$$v^{\delta}(x, \gamma) = \inf_{\alpha \in \mathcal{A}_{ad}} \mathbb{E} \left[\begin{array}{l} \int_0^{T \wedge \tau_1} e^{-\delta t} l_{\gamma} \left(y_{\gamma}(t; x, \alpha), \alpha(t; x, \gamma) \right) dt \\ + e^{-\delta(T \wedge \tau_1)} v^{\delta} \left(y_{\gamma}(T \wedge \tau_1; x, \alpha), \Gamma_{T \wedge \tau_1}^{x, \gamma, \alpha} \right) \end{array} \right],$$

for all $T > 0$ and all $(\gamma, x) \in E \times \overline{\mathcal{G}}$.

Again, once we have established the ingredients of uniform continuity in the previous theorem, the proof is identical with the first part of Theorem 3.4 in [35] and will be omitted from our (long enough) paper. One iterates Lemma 10 to get v_m^δ and recalls that λ is bounded and, thus, the jumping times cannot accumulate.

5   Existence of the viscosity solution

At this point, we introduce the following Hamilton-Jacobi integrodifferential system

$$(8) \qquad \delta v^\delta (x, \gamma) + \sup_{a \in A_{\gamma,x}} \left\{ \begin{array}{c} - \left\langle f_\gamma (x, a), Dv^\delta (x, \gamma) \right\rangle - l_\gamma (x, a) \\ - \lambda (x, \gamma, a) \sum_{\gamma' \in E} Q (x, \gamma, \gamma', a) \left(v^\delta (x, \gamma') - v^\delta (x, \gamma) \right) \end{array} \right\} = 0.$$

5.1   Relaxing the dynamics

In addition to the standard assumptions **(Aa-Ac)**, we will need the following.

(Ad) For every $1 \leq j \leq N$, every $\gamma \in E$ and every $x \in J_j$, there exists $\theta > 0$ such that, whenever $\alpha \in \mathcal{A}_{ad}$, one has $\alpha (t; x, \gamma) \in A^{\gamma,j}$ for almost all $t \in [0, \theta]$.

For every $x \in \overline{\mathcal{G}}$, we let $\mathcal{T}_x (\overline{\mathcal{G}})$ denote the set of tangent directions to $\overline{\mathcal{G}}$ at x : $\mathcal{T}_x (\overline{\mathcal{G}}) = \mathbb{R}e_j$ if $x \in J_j$, $\mathcal{T}_{e_j} (\overline{\mathcal{G}}) = \mathbb{R}_- e_j$ and $\mathcal{T}_O (\overline{\mathcal{G}}) = \underset{1 \leq j \leq N}{\cup} \mathbb{R}_+ e_j$. The set $\mathcal{M}_+ (E)$ denotes the family of (positive) measures $\zeta = (\zeta (\gamma))_{\gamma \in E} \in \mathbb{R}_+^E$. The following standard notations will be employed throughout the section.

$$\overline{FL} (x, \gamma) := \left\{ \begin{array}{c} (\xi, \zeta, \eta) \in \mathcal{T}_x (\overline{\mathcal{G}}) \times \mathcal{M}_+ (E) \times \mathbb{R} : \exists (\alpha_n)_n \subset \mathcal{A}_{ad}, \ (t_n)_n \subset \mathbb{R}_+, \ \text{s.t.} \\ \lim_{n \to \infty} t_n = 0, \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f_\gamma (x, \alpha_n (s; x, \gamma)) \, ds = \xi, \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \lambda (x, \gamma, \alpha_n (s; x, \gamma)) Q (x, \gamma, \alpha_n (s; x, \gamma)) \, ds = \zeta, \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} l_\gamma (x, \alpha_n (s; x, \gamma)) \, ds = \eta \end{array} \right\},$$

$$\overline{F} (x, \gamma) := \left\{ \begin{array}{c} (\xi, \zeta) \in \mathcal{T}_x (\mathcal{G}) \times \mathcal{M}_+ (E) : \exists (\alpha_n)_n \subset \mathcal{A}_{ad}, \ (t_n)_n \subset \mathbb{R}_+, \ \text{s.t.} \\ \lim_{n \to \infty} t_n = 0, \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f_\gamma (x, \alpha_n (s; x, \gamma)) \, ds = \xi, \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \lambda (x, \gamma, \alpha_n (s; x, \gamma)) Q (x, \gamma, \alpha_n (s; x, \gamma)) \, ds = \zeta \end{array} \right\},$$

$$\overline{fl} (x, \gamma, a) := (f_\gamma (x, a), \lambda (x, \gamma, a) Q (x, \gamma, a), l_\gamma (x, a)).$$

Remark 14 *(a) The reader is invited to note that, in the previous notations, "$(\alpha_n)_n \subset \mathcal{A}_{ad}$" (resp. "$\alpha (s; x, \gamma)$") and can be replaced by "$(\alpha_n)_n \subset \mathcal{A}_{\gamma,x}$" (resp. "$\alpha (s)$", see also the second part of Remark 4).*

(b) Also, the assumptions on the coefficients imply that

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f_\gamma (x, \alpha_n (s; x, \gamma)) \, ds = \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f_\gamma (y_\gamma (s; x, \alpha_n (s; x, \gamma)), \alpha_n (s; x, \gamma)) \, ds,$$

and similar assertions hold true in the definition of η and ζ.

(c) Finally, we have <u>dropped the dependency on λQ</u> in these terms for the sake of simplicity. One should have written $\overline{f (\lambda Q) l}$, etc.

We begin with the following technical result.

Lemma 15 *We assume (**Aa-Ad**) and (**A1-A4**) to hold true. For every $x \in \mathcal{G} \setminus \{O\}$, the following equality holds true*

$$\overline{FL} (x, \gamma) = \overline{co} \overline{fl} (x, \gamma) := \overline{co} \left\{ \overline{fl} (x, \gamma, a) : a \in A_{\gamma,x} \right\}.$$

Moreover, for every $j \leq N$,

$$\overline{FL} (e_j, \gamma) \subset \overline{co} \overline{fl} (e_j, \gamma) := \overline{co} \left\{ \overline{fl} (e_j, \gamma, a) : a \in A^{\gamma,j} \right\} \cap (\mathbb{R}_- e_j \times \mathcal{M}_+ (E) \times \mathbb{R}).$$

Proof. Without loss of generality, we first assume that $x \in J_1$. It is clear that

$$\overline{FL}(x, \gamma) \subset \overline{co} \left\{ \overline{fl}(x, \gamma, a) : a \in A_{\gamma, x} \right\}.$$

Indeed, it suffices to use the assumption **(Ad)** to get the existence of some $\theta > 0$ such that whenever $\alpha \in \mathcal{A}_{ad}$, one has $\alpha(t; x, \gamma) \in A^{\gamma, 1}$ for almost all $t \in [0, \theta]$. Then, for every $(\alpha_n)_n \subset \mathcal{A}_{ad}$, and every sequence $(t_n)_n \subset \mathbb{R}_+$ such that $t_n \leq \theta$, one has

$$\begin{pmatrix} \frac{1}{t_n} \int_0^{t_n} f_\gamma(x, \alpha_n(s; x, \gamma)) \, ds \\ \frac{1}{t_n} \int_0^{t_n} \lambda(x, \gamma, \alpha_n(s; x, \gamma)) Q(x, \gamma, \alpha_n(s; x, \gamma)) \\ \frac{1}{t_n} \int_0^{t_n} l_\gamma(x, \alpha_n(s; x, \gamma)) \end{pmatrix} \in \overline{co} \left\{ \overline{fl}(x, \gamma, a) : a \in A_{\gamma, x} \right\}.$$

If $x = e_1$, then

$$\frac{1}{t_n} \int_0^{t_n} f_\gamma(y_\gamma(s; e_1, \alpha_n), \alpha_n(s; e_1, \gamma)) \, ds = \frac{y_\gamma(t_n; e_1, \alpha_n) - e_1}{t_n} \in \mathbb{R}_- e_1.$$

Hence, invoking part (b) of the Remark 14, it follows that

$$\overline{FL}(e_1, \gamma) \subset \overline{co} \left\{ \overline{fl}(e_1, \gamma, a) : a \in A^{\gamma, 1} \right\} \cap (\mathbb{R}_- e_1 \times \mathcal{M}_+(E) \times \mathbb{R}).$$

For the converse inclusion, we fix $x \in \mathcal{G} \setminus \{O\}$. One begins by noticing that $\overline{FL}(x, \gamma)$ is closed. Hence, it suffices to prove that

$$co \left\{ \overline{fl}(x, \gamma, a) : a \in A^{\gamma, 1} \right\} \subset \overline{FL}(x, \gamma).$$

We consider $\lambda_i \geq 0, i \in \{1, .., K\}$ such that $\sum_{i=1}^{K} \lambda_i = 1$ and $a_i \in A^{\gamma, 1}$, pour tout $i \in \{1, .., K\}$. Since $x \in J_1$, whenever $t_n < \frac{\min(|x|, |x - e_1|)}{\max(|f|_0, 1)}$, an admissible control $\alpha \in \mathcal{A}_{\gamma, x}$ is obtained by setting $\alpha_n(t) = \sum_{i=1}^{K} a_i 1 \left[\left(\sum_{j=1}^{i-1} \lambda_j \right) t_n, \left(\sum_{j=1}^{i} \lambda_j \right) t_n \right)$ (t) and the conclusion follows. $\blacksquare$

The family of admissible test functions will be given as in [1] by $\varphi \in C_b(\overline{\mathcal{G}})$ for which $\varphi \mid_{\overline{J_j}} \in C_b^1(\overline{J_j})$, for all $j = 1, 2, ..., N$. If $x \in J_j$, we recall that

$$D\varphi(x; \xi) := \lim_{t \to 0} \frac{\varphi(x + t\xi) - \varphi(x)}{t}, \text{ for all } \xi \in \mathbb{R}e_j.$$

We also recall that

$$D\varphi(e_j; \xi) := \lim_{t \to 0+} \frac{\varphi(e_j + t\xi) - \varphi(x)}{t}, \text{ for all } \xi \in \mathbb{R}_- e_j$$

and

$$D\varphi(O; \xi) := \lim_{t \to 0+} \frac{\varphi(t\xi) - \varphi(O)}{t}, \text{ whenever } \xi \in \mathbb{R}_+ e_j.$$

If $\varkappa : [0, 1] \longrightarrow \mathcal{G}$ is continuous and $(t_n)_n \subset (0, 1]$ is such that $\lim_{n \to \infty} t_n = 0$ and

$$\lim_{n \to \infty} \frac{\varkappa(t_n)}{t_n} = \xi,$$

we have

$$D\varphi(O; \xi) := \lim_{n \to \infty} \frac{\varphi(\varkappa(t_n)) - \varphi(O)}{t_n}$$

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and one notes that this limit does not depend on the choice of $\varkappa$. To simplify the notations, we will also write $\langle \xi, D\varphi(x) \rangle$ instead of $D\varphi(x; \xi)$. One notices easily that the choice of test functions is equivalent to taking a family of test functions $\varphi_j \in C_b^1\left(\overline{J_j}\right)$ such that $\varphi_j(O) = \varphi_{j'}(O)$, for all $1 \leq j, j' \leq N$. For further details of this family of test functions, the reader is referred to [1, Subsection 3.1].

We now introduce the definition of the generalized solution of the system (8).

Definition 16 *A bounded, upper semicontinuous function V is said to be a generalized viscosity subsolution of (8) if, for every $(\gamma_0, x_0) \in E \times \mathcal{G}$ whenever $\varphi \in C_b\left(\overline{\mathcal{G}}\right)$ for which $\varphi\mid_{\overline{J_j}} \in C_b^1\left(\overline{J_j}\right)$, for all $j = 1, 2, ..., N$ is a test function such that $x_0 \in Argmax\left(V\left(\cdot, \gamma_0\right) - \varphi\left(\cdot\right)\right)$, one has*

$$\delta V\left(x_0, \gamma_0\right) + \sup_{(\xi, \zeta, \eta) \in \overline{FL}(x_0, \gamma_0)} \left\{ \begin{array}{c} -\langle D\varphi\left(x_0; \xi\right)\rangle - \eta \\ -\sum_{\gamma' \in E} \zeta\left(\gamma'\right)\left(V\left(x_0, \gamma'\right) - V\left(x_0, \gamma_0\right)\right) \end{array} \right\} \leq 0.$$

A bounded, lower semicontinuous function V is said to be a generalized viscosity supersolution of (8) if, for every $(\gamma_0, x_0) \in E \times \overline{\mathcal{G}}$ whenever $\varphi \in C_b\left(\overline{\mathcal{G}}\right)$ for which $\varphi\mid_{\overline{J_j}} \in C_b^1\left(\overline{J_j}\right)$, for all $j = 1, 2, ..., N$ is a test function such that $x_0 \in Arg\min\left(V\left(\cdot, \gamma_0\right) - \varphi\left(\cdot\right)\right)$, one has

$$\delta V\left(x_0, \gamma_0\right) + \sup_{(\xi, \zeta, \eta) \in \overline{FL}(x_0, \gamma_0)} \left\{ \begin{array}{c} -\langle D\varphi\left(x; \xi\right)\rangle - \eta \\ -\sum_{\gamma' \in E} \zeta\left(\gamma'\right)\left(V\left(x_0, \gamma'\right) - V\left(x_0, \gamma_0\right)\right) \end{array} \right\} \geq 0.$$

5.2 (A) Viscosity solution

We are now able to state and proof the main result of the section.

Theorem 17 *We assume (**Aa-Ad, Ac'**) and (**A1-A4**) to hold true. Then, the value function v^δ is a bounded uniformly continuous generalized solution of (8).*

Proof. We begin with the proof of the subsolution condition. Let us fix $(\gamma_0, x_0) \in E \times (\mathcal{G} \setminus \{O\})$ and consider a regular test function φ such that $x_0 \in Argmax\left(v^\delta\left(\cdot, \gamma_0\right) - \varphi\left(\cdot\right)\right)$. Then

$$\varphi\left(x_0\right) - \varphi\left(x\right) \leq v^\delta\left(x_0, \gamma_0\right) - v^\delta\left(x, \gamma_0\right),$$

for all $x \in \overline{\mathcal{G}}$. We can assume, without loss of generality, that $\varphi\left(x_0\right) = v^\delta\left(x_0, \gamma_0\right)$. Let us consider $(\xi, \zeta, \eta) \in \overline{FL}\left(x_0, \gamma_0\right)$. Then, there exist $(\alpha_n)_n \subset \mathcal{A}_{ad}$, $(t_n)_n \subset \mathbb{R}_+$, s.t. $\lim\limits_{n \to \infty} t_n = 0$ and

$$\begin{cases} \lim\limits_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f_{\gamma_0}\left(x_0, \alpha_n\left(s; x_0, \gamma_0\right)\right) ds = \xi, \\ \lim\limits_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \lambda\left(x_0, \gamma_0, \alpha_n\left(s; x_0, \gamma_0\right)\right) Q\left(x_0, \gamma_0, \alpha_n\left(s; x_0, \gamma_0\right)\right) ds = \zeta, \\ \lim\limits_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} l_{\gamma_0}\left(x_0, \alpha_n\left(s; x_0, \gamma_0\right)\right) ds = \eta. \end{cases}$$

We fix, for the time being, $n \in \mathbb{N}$. We let τ_1^n be the first jumping time associated to $\alpha_n\left(\cdot; x, \gamma\right)$.

Using the dynamic programming principle, one gets

$$
0 = v^\delta(x_0, \gamma_0) - \varphi(x_0) \leq \mathbb{E}\left[\begin{array}{l} \int_0^{t_n \wedge \tau_1^n} e^{-\delta s} l_{\gamma_0}(y_{\gamma_0}(s; x_0, \alpha_n), \alpha_n(s; x_0, \gamma_0))\, ds \\ + e^{-\delta(t_n \wedge \tau_1^n)} v^\delta\left(y_{\gamma_0}(t_n \wedge \tau_1^n; x_0, \alpha_n), \Gamma_{t_n \wedge \tau_1^n}^{x_0, \gamma_0, \alpha_n}\right) \end{array}\right] - \varphi(x_0)
$$

$$
\leq \mathbb{E}\left[\begin{array}{l} \int_0^{t_n \wedge \tau_1^n} e^{-\delta s} l_{\gamma_0}(x_0, \alpha_n(s; x_0, \gamma_0))\, ds + \int_0^{t_n \wedge \tau_1^n} e^{-\delta s} Lip(l)\, |f|_0\, s\, ds \\ + e^{-\delta \tau_1^n} v^\delta\left(y_{\gamma_0}(\tau_1^n; x_0, \alpha_n), \Gamma_{\tau_1^n}^{x_0, \gamma_0, \alpha_n}\right) \mathbf{1}_{\tau_1^n < t_n} + e^{-\delta t_n} \varphi(y_{\gamma_0}(t_n; x_0, \alpha_n)) \mathbf{1}_{\tau_1^n \geq t_n} \end{array}\right]
$$
$$
- \varphi(x_0)
$$

$$
\leq |f|_0\, Lip(l)\, t_n \mathbb{E}[t_n \wedge \tau_1^n] + |l|_0\left(\int_0^{t_n}\left(1 - e^{-\delta s}\right) ds + t_n \mathbb{P}(\tau_1^n < t_n)\right) + \delta |\varphi|_0\, t_n \mathbb{P}(\tau_1^n < t_n)
$$

$$
\mathbb{E}\left[\int_0^{t_n} l_{\gamma_0}(x_0, \alpha_n(s; x_0, \gamma_0))\, ds\right] + e^{-\delta t_n} \varphi(y_\gamma(t_n; x_0, \alpha_n)) - \varphi(x_0)
$$

$$
+ \mathbb{E}\left[e^{-\delta \tau_1^n}\left(v^\delta\left(y_{\gamma_0}(\tau_1^n; x_0, \alpha_n), \Gamma_{\tau_1^n}^{x_0, \gamma_0, \alpha_n}\right) - \varphi(y_{\gamma_0}(t_n; x_0, \alpha_n))\right) \mathbf{1}_{\tau_1^n < t_n}\right].
$$

We set

$$
\lambda(s) := \lambda(y_{\gamma_0}(s; x_0, \alpha_n), \gamma_0, \alpha_n(s; x_0, \gamma_0)) \text{ and } \Lambda(s) := \exp\left(-\int_0^s \lambda(r)\, dr\right)
$$

and one gets

$$
0 = v^\delta(x_0, \gamma_0) - \varphi(x_0)
$$
$$
\leq |f|_0\, Lip(l)\, t_n \mathbb{E}[t_n \wedge \tau_1^n] + Lip(\varphi)\, |f|_0\, t_n \mathbb{P}(\tau_1^n < t_n) + \delta |\varphi|_0\, t_n \mathbb{P}(\tau_1^n < t_n)
$$
$$
+ |l|_0\left(\int_0^{t_n}\left(1 - e^{-\delta s}\right) ds + t_n \mathbb{P}(\tau_1^n < t_n)\right)
$$
$$
+ e^{-\delta t_n}\left(\varphi(y_\gamma(t_n; x_0, \alpha_n)) - \varphi(x_0)\right) + \left(e^{-\delta t_n} - 1\right)\varphi(x_0) + \int_0^{t_n} l_{\gamma_0}(x_0, \alpha_n(s; x_0, \gamma_0))\, ds
$$

(9)
$$
+ \int_0^{t_n} e^{-\delta s} \lambda(s) \Lambda(s)\left(\sum_{\gamma' \neq \gamma_0} Q(y_{\gamma_0}(s; x_0, \alpha_n), \gamma_0, \gamma', \alpha_n(x_0, \gamma_0, s))\left(v^\delta(y_{\gamma_0}(s; x_0, a), \gamma') - \varphi(x_0)\right)\right) ds.
$$

The reader is invited to note that

$$
\left\{\begin{array}{l} \left|e^{-\delta s} \lambda(s) \Lambda(s) - \lambda(x_0, \alpha_n(s; x_0, \gamma_0))\right| \leq (|f|_0\, Lip(\lambda) + |\lambda|_0(\delta + |\lambda|_0))\, t_n, \\ |Q(y_{\gamma_0}(s; x_0, \alpha_n), \gamma_0, \gamma', a) - Q(x_0, \gamma_0, \gamma', a)| \leq |f|_0\, Lip(Q)\, t_n, \\ \left|v^\delta(y_{\gamma_0}(s; x_0, \alpha_n), \gamma') - v^\delta(x_0, \gamma')\right| = \omega^\delta(|f|_0\, t_n), \end{array}\right.
$$

whenever $s \leq t_n$, where ω^δ denotes the continuity modulus of v^δ. Also,

$$
\frac{y_\gamma(t_n; x_0, \alpha_n) - x_0}{t_n} = \frac{\int_0^{t_n} f_\gamma(y_\gamma(s; x_0, \alpha_n), \alpha_n(s; x_0, \gamma_0))\, ds}{t_n} = \frac{\int_0^{t_n} f_\gamma(x_0, \alpha_n(s; x_0, \gamma_0))\, ds}{t_n} + \omega(t_n),
$$

(where $\lim_{\varepsilon \to 0} \omega(\varepsilon) = 0$). We divide (9) by t_n and allow $n \to \infty$ to get

$$
0 \leq \eta - \delta \varphi(x_0) + D\varphi(x_0; \xi) + \sum_{\gamma' \neq \gamma} \zeta(\gamma')\left(v^\delta(x_0, \gamma') - v^\delta(x_0, \gamma)\right).
$$

The conclusion follows by recalling that $(\xi, \zeta, \eta) \in \overline{FL}(x_0, \gamma_0)$ is arbitrary.

To prove that v^δ is a viscosity supersolution of the associated Hamilton-Jacobi integrodifferential equation, let us fix, for the time being, $\varepsilon > 0$. We equally fix $(\gamma_0, x_0) \in E \times \mathcal{G}$ and consider a test function φ such that $x_0 \in Argmin\left(v^\delta\left(\cdot, \gamma_0\right) - \varphi\left(\cdot\right)\right)$. Then

$$\varphi\left(x_0\right) - \varphi\left(x\right) \geq v^\delta\left(x_0, \gamma_0\right) - v^\delta\left(x, \gamma_0\right),$$

for all $x \in \overline{\mathcal{G}}$. We can assume, without loss of generality, that $\varphi\left(x_0\right) = v^\delta\left(x_0, \gamma_0\right)$. There exists an admissible control α^ε such that

$$v^\delta\left(x_0, \gamma_0\right) + \varepsilon \geq \mathbb{E}\left[\begin{array}{c} \int_0^{\sqrt{\varepsilon} \wedge \tau_1} e^{-\delta s} l_{\gamma_0}\left(y_{\gamma_0}\left(s; x_0, \alpha^\varepsilon\right), \alpha^\varepsilon\left(s\right)\right) ds \\ + e^{-\delta\left(\sqrt{\varepsilon} \wedge \tau_1\right)} v^\delta\left(y_{\gamma_0}\left(\sqrt{\varepsilon} \wedge \tau_1; x_0, \alpha^\varepsilon\right), \Gamma_{\sqrt{\varepsilon} \wedge \tau_1}^{x_0, \gamma_0, \alpha^\varepsilon}\right) \end{array} \right].$$

(For notation purposes, we have dropped the dependency of γ_0, x_0 in α^ε). As in the first part of our proof, τ_1 denotes the first jumping time associated to the admissible control process α^ε. Using similar estimates to the first part, one gets

$$
\begin{aligned}
0 &= v^\delta\left(x_0, \gamma_0\right) - \varphi\left(x_0\right) \\
&\geq -\varepsilon - |f|_0\, Lip\left(l\right) \sqrt{\varepsilon} \mathbb{E}\left[\sqrt{\varepsilon} \wedge \tau_1\right] - Lip\left(\varphi\right) |f|_0 \sqrt{\varepsilon} \mathbb{P}\left(\tau_1 < \sqrt{\varepsilon}\right) - \delta\,|\varphi|_0 \sqrt{\varepsilon} \mathbb{P}\left(\tau_1 < \sqrt{\varepsilon}\right) \\
&\quad - |l|_0 \left(\int_0^{\sqrt{\varepsilon}} \left(1 - e^{-\delta s}\right) ds + \sqrt{\varepsilon}\mathbb{P}\left(\tau_1 < \sqrt{\varepsilon}\right)\right) \\
&\quad + e^{-\delta\sqrt{\varepsilon}}\left(\varphi\left(y_\gamma\left(\sqrt{\varepsilon}; x_0, \alpha^\varepsilon\right)\right) - \varphi\left(x_0\right)\right) + \left(e^{-\delta\sqrt{\varepsilon}} - 1\right) \varphi\left(x_0\right) \\
&\quad + \int_0^{\sqrt{\varepsilon}} e^{-\delta s} \lambda\left(s\right) \Lambda\left(s\right) \left(\sum_{\gamma' \neq \gamma_0} Q\left(y_{\gamma_0}\left(s; x_0, \alpha^\varepsilon\right), \gamma_0, \gamma', \alpha^\varepsilon\left(s\right)\right) \left(v^\delta\left(y_{\gamma_0}\left(s; x_0, \alpha^\varepsilon\right), \gamma'\right) - \varphi\left(x_0\right)\right)\right) ds,
\end{aligned}
$$

where $\lambda\left(s\right) := \lambda\left(y_{\gamma_0}\left(s; x_0, \alpha^\varepsilon\right), \gamma_0, \alpha^\varepsilon\left(s\right)\right)$ and $\Lambda\left(s\right) := \exp\left(-\int_0^s \lambda\left(r\right) dr\right)$. We recall that f, λ and Q are Lipschitz-continuous and bounded and v^δ is uniformly continuous and bounded. The conclusion follows similarly to the subsolution case by dividing the inequality by $\sqrt{\varepsilon}$, recalling the definition of $\overline{FL}\left(x_0, \gamma_0\right)$ and allowing ε (or some subsequence) to go to 0. ∎

6   Extending the intersection and linearizing the value function

6.1   Additional directions

Without loss of generality, we assume that $-e_j \notin \overline{\mathcal{G}}$, for all $j \leq M \leq N$ and $-e_j \in \overline{\mathcal{G}}$, for all $M < j \leq N$. We define

$$e_j := -e_{j-N}, E_j^{active} := E_{j-N}^{active}, E_j^{inactive} := E_{j-N}^{inactive},$$

whenever $N < j \leq M + N$. For every $\varepsilon > 0$, we complete $\mathcal{G}$ into $\mathcal{G}^{+,\varepsilon}$ by adding $[0, \varepsilon e_j)$ for $N < j \leq M + N$ and $(1, 1 + \varepsilon) e_j$, for $j \leq N$.
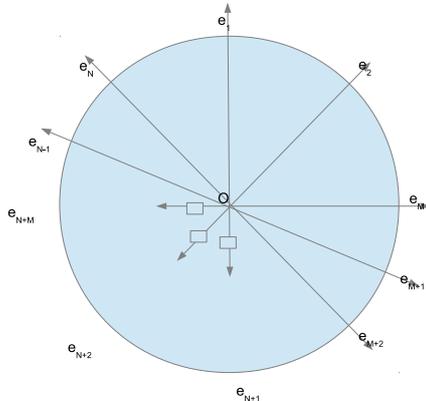

Fig 2. The complete intersection

Throughout the remaining of the paper we make the following assumption.

(B) Whenever $M < j, j' \leq N$ are such that $e_{j'} = -e_j$, then $A^{\gamma,j} = A^{\gamma,j'}$, for all $\gamma \in E$.

Remark 18 *Roughly speaking, on the roads that cross the intersection (of type $(-1, 1) e_j$), the same family of (piecewise constant) controls can be used both at the entrance and at the exit of the intersection.*

6.1.1 Inactive roads

The reader is invited to notice that, if $e_j, e_{j'} = -e_j \in \overline{\mathcal{G}}$ then, for every $\gamma \in E_j^{inactive} \cap E_{j'}^{inactive}$, $f_\gamma (O, a) = 0$, for all $a \in A^{\gamma,j} \cap A^{\gamma,j'}$. This is a simple consequence of the assumption **(Ab)** which implies that $\langle f_\gamma (O, a), e_j \rangle \leq 0$ and $\langle f_\gamma (O, a), e_{j'} \rangle \leq 0$, for all $a \in A^{\gamma,j} \cap A^{\gamma,j'}$. In particular, if **(B)** holds true, then $f_\gamma (O, a) = 0$, for all $a \in A^{\gamma,j} (= A^{\gamma,j'})$ whenever $\gamma \in E_j^{inactive} \cap E_{j'}^{inactive}$.

Hence, in order to obtain a similar behavior for the completed intersection, it is natural to strengthen the assumption **(Ab)**. We will assume that,

(Ab') Whenever $\gamma \in E_j^{inactive}$ for some $j \leq M$, then $f_\gamma (O, a) = 0$, for all $a \in A^{\gamma,j}$.

Remark 19 *This is, of course, less general than the existence of one $a_{\gamma,j}^0 \in A^{\gamma,j}$ guaranteed by **(Ab)**. The assumption states that, whenever the road j is inactive, a vehicle that needs to go on this road should wait until it is repaired.*

6.2 Extending the dynamics

Unless stated otherwise, we assume the (pseudo-)controllability conditions **(Aa, Ab, Ad)**, the compatibility at the intersection **(Ab', Ac'),** the regularity of the coefficients and cost functions **(A1-A4)** and the compatibility condition **(B)** to hold true.

We are now able to extend f (and λ, Q) to $\left(\bigcup_{j=1,2,\ldots,N} \mathbb{R}e_j \right) \times A$ by setting

$$f_\gamma (x, a) = \begin{cases} f_\gamma (x, a), & \text{if } x \in (0, 1) e_j, \ j \leq N, \\ f_\gamma (e_j, a), & \text{if } x \in [1, \infty) e_j, \ j \leq N, \\ -f_\gamma (-x, a), & \text{if } \gamma \in E_j^{inactive}, x \in \mathbb{R}_- e_j, \ j \leq M, \\ f_\gamma (O, a), & \text{if } \gamma \in E_j^{active}, x \in \mathbb{R}_- e_j, \ j \leq M. \end{cases}$$

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For the other elements ($\varphi \in \{\lambda, l, Q\}$), we set

$$\varphi(x, \gamma, a) = \begin{cases} \varphi(x, \gamma, a), & \text{if } x \in (0, 1) e_j, \ j \le N, \\ \varphi(e_j, \gamma, a), & \text{if } x \in [1, \infty) e_j, \ j \le N, \\ \varphi(O, \gamma, a), & \text{otherwise.} \end{cases} \quad .$$

(by abuse of notation, $l(x, \gamma, a) = l_\gamma(x, a)$).

This particular construction for f is needed in order to guarantee that the assumptions (**Aa**) and (**Ab**) hold true for the new system on $\mathcal{G}^{+,\varepsilon}$. It basically suggests that in the active case, the vehicle will continue its road on the extension of the road with the same speed as in O. In the inactive case, the extension of the road is obtained by looking at the road j using a mirror.

6.3 Krylov's "shaking the coefficients" method

We wish to construct a family of regular functions satisfying a suitable subsolution condition and converging, as $\varepsilon \to 0$ to our value function. Regularization can be achieved by classical convolution. However, because of the convolution, the subsolution condition should not only concern a point x but some neighborhood. This is the reason why, one needs to introduce a perturbation in the Hamiltonian or, equivalently, in the coefficients. The method is known as "shaking the coefficients" and has been introduced, in the framework of Brownian diffusions, in [29].

For $r > 0$, we let B_r denote the r-radius closed ball $B_r = \{y \in \mathbb{R}^m : |y| \le r\}$. We set

$$\begin{cases} f_\gamma^\rho(x, a, b) = f_\gamma(x + \rho b, a), \\ \varphi^\rho(x, \gamma, a, b) = \varphi(x + \rho b, \gamma, a), & \text{if } \varphi \in \{\lambda, Q, l\}, \end{cases}$$

for all $(x, a, b) \in \underset{j=1,2,\ldots,N}{\cup} ([-\varepsilon, 1 + \varepsilon] e_j \times A \times [-1, 1] e_j)$, and all $|b| \le 1$. Let us fix, for the time being, $\varepsilon \ge \rho > 0$ and consider the control problem on $\mathcal{G}^{+,\varepsilon}$. We denote by

$$J_j^{\varepsilon,+} := (0, 1 + \varepsilon) e_j, \text{ for all } j = 1, 2, \ldots, N, \quad J_j^{\varepsilon,-} := \begin{cases} (-\varepsilon, 0) e_j, & \text{for } j \le M, \\ (-1 - \varepsilon, 0) e_j, & \text{for } M < j \le N, \end{cases} \quad , J_j^\varepsilon := J_j^{\varepsilon,+} \cup J_j^{\varepsilon,-}.$$

for all $j = 1, 2, \ldots, N$.

We set

$$\overline{A} := \underset{j=1,2,\ldots,N}{\cup} \left(A^{\gamma,j} \times [-1, 1] e_j \right), \quad \overline{A}^{\gamma,j} := A^{\gamma,j} \times [-1, 1] e_j.$$

For this extended system, we will check that our controllability assumptions (**Aa**) and (**Ab**) hold true for explicit sets of controls.

The reader is invited to notice that the following hold true :

$$(\overline{Aa}) \qquad \overline{A}_{\gamma,x} = \overline{A}^{\gamma,j}, \text{ if } x \in J_j^{\varepsilon,+}, \ \overline{A}_{\gamma,O} = \underset{j=1,2,\ldots,N}{\cup} \overline{A}^{\gamma,j}, \ \overline{A}_{\gamma,(1+\varepsilon)e_j} = A_{\gamma,e_j} \times [-1, 1] e_j,$$

for all $j = 1, 2, \ldots, N$. Let us fix $j \le M$.

(i) If $\gamma \in E_j^{active}$, then

$$\overline{A}_{\gamma,-\varepsilon e_j} = \left\{ (a, b) \in \overline{A}^{\gamma,j} : f_\gamma(O, a) \in \mathbb{R}_+ e_j \right\}.$$

The set $\overline{A}_{\gamma,-\varepsilon e_j}$ is nonempty. Indeed, the control $\left(a_{\gamma,j}^+, b \right)$ ($a_{\gamma,j}^+$ given by the assumption (**Ab**) and $b \in [-1, 1] e_j$ arbitrary) belongs to $\overline{A}_{\gamma,-\varepsilon e_j}$ and

$$\left\langle f_\gamma^\rho \left(-\varepsilon e_j, a_{\gamma,j}^+, b \right), (-e_j) \right\rangle = \left\langle f_\gamma \left(O, a_{\gamma,j}^+ \right), (-e_j) \right\rangle < -\beta,$$

for all $b \in [-1, 1] e_j$.

(ii) If $\gamma \in E_j^{inactive}$, then

$$\overline{A}_{\gamma,-\varepsilon e_j} = \overline{A}^{\gamma,j}.$$

Indeed,

$$\left\langle f_\gamma^\rho\left(-\varepsilon e_j, a, b\right), \left(-e_j\right)\right\rangle = \left\langle -f_\gamma\left(\varepsilon e_j - \rho b, a\right), \left(-e_j\right)\right\rangle = \left\langle f_\gamma\left(\varepsilon e_j - \rho b, a\right), e_j\right\rangle \leq 0,$$

for ε small enough and all $(a,b) \in \overline{A}^{\gamma,j}$.

Thus, (**Aa**) holds true for the system driven by $(f^\rho, \lambda^\rho, Q^\rho)$.

Concerning the assumption (**Ab**), for the already existing branches, it suffices to take $b = 0$ and the controls $a_{\gamma,j}^+, a_{\gamma,j}^-, a_{\gamma,j}^0$. Let us now fix $j \leq M$.

(i) If $\gamma \in E_j^{active}$, then $\gamma \in E_{j+N}^{active}$, by construction. We recall that $e_{j+N} = -e_j$. Moreover we have

$$\left\langle f_\gamma^\rho\left(O, \left(a_{\gamma,j}^+, 0\right)\right), -e_j\right\rangle < -\beta \text{ and } \left\langle f_\gamma^\rho\left(O, \left(a_{\gamma,j}^-, 0\right)\right), e_j\right\rangle > \beta.$$

(ii) For $\gamma \in E_j^{inactive} = E_{j+N}^{active}$,

$$\left\langle f_\gamma^\rho\left(x, \left(a_{\gamma,j}^-, 0\right)\right), -e_j\right\rangle = \left\langle -f_\gamma\left(-x, a_{\gamma,j}^-\right), -e_j\right\rangle \leq -\beta\left\langle -x, e_j\right\rangle^\kappa,$$

for all $x \in [-\varepsilon, 0] e_j$ and $f_\gamma^\rho\left(O, \left(a_{\gamma,j}^0, 0\right)\right) = 0$.

We cannot have

$$\left\langle f_\gamma^\rho\left(x, (a,b)\right), -e_j\right\rangle \leq 0,$$

for all $(a,b) \in \overline{A}^{\gamma,j}$ and all $x \in J_j$, $|x| \leq \eta$ (close enough to O). Nevertheless, as we have already hinted before (see Remark 7 (ii)), this condition and the one in (**Ac**) are no longer necessary since every control is (locally) admissible at O. Thus, the conclusion of Lemma 6 holds true and so do all the assertions on the value functions in this framework.

At this point, we consider the process $\left(X_t^{\rho,x_0,\gamma_0,\overline{\alpha}}, \Gamma_t^{\rho,x_0,\gamma_0,\overline{\alpha}}\right)$ constructed as in Section 2 using $(f^\rho, \lambda^\rho, Q^\rho)$ and controls $\overline{\alpha}$ with values in $\overline{A}$. We also let y^ρ denote the solution of the ordinary differential equation driven by f^ρ.

Then, the value functions

$$v^{\delta,\varepsilon,\rho}(x,\gamma)$$

$$:= \inf_{\overline{\alpha} \in \overline{\mathcal{A}}_{ad}^{\mathbb{N}}} \mathbb{E}\left[\sum_{n \geq 0} \int_{\tau_n}^{\tau_{n+1}} e^{-\delta t} l_{\Gamma_{\tau_n}^{\rho,x,\gamma,\overline{\alpha}}}^\rho \left(y_{\Gamma_{\tau_n}^{\rho,\gamma,x,\overline{\alpha}}}^\rho\left(t; X_{\tau_n}^{\rho,x,\gamma,\overline{\alpha}}, \overline{\alpha}_{n+1}\right), \overline{\alpha}_{n+1}\left(t - \Gamma_{\tau_n}^{\rho,x,\gamma,\overline{\alpha}}; X_{\tau_n}^{\rho,x,\gamma,\overline{\alpha}}, \Gamma_{\tau_n}^{\rho,x,\gamma,\overline{\alpha}}\right)\right)\right]$$

are bounded, uniformly continuous and satisfy, in the generalized sense given by Definition 16 and Theorem 17 the Hamilton-Jacobi integrodifferential system

(10)
$$\delta v^{\delta,\varepsilon,\rho}(x,\gamma) + \sup_{(a,b)\in\overline{A}_{\gamma,x}} \left\{ \begin{array}{c} -\left\langle f_\gamma\left(x+\rho b, a\right), Dv^{\delta,\varepsilon,\rho}(x,\gamma)\right\rangle - l_\gamma\left(x+\rho b, a\right) \\ -\lambda\left(x+\rho b, \gamma, a\right) \sum_{\gamma'\in E} Q\left(x+\rho b, \gamma, \gamma', a\right)\left(v^{\delta,\varepsilon,\rho}(x,\gamma') - v^{\delta,\varepsilon,\rho}(x,\gamma)\right) \end{array} \right\} \leq 0,$$

for all $(x,\gamma) \in \mathcal{G}^{+,\varepsilon} \times E$.

6.4 Another definition for solutions in the extended intersection

We define

$$\overline{co}\overline{f}l^\rho(O,\gamma) := \bigcup_{j=1,2,\ldots,N} \overline{co}\left\{\overline{f}l^\rho\left(O,\gamma,(a,b)\right) : (a,b) \in \overline{A}^{\gamma,j}\right\}$$

and recall that
$$\overline{FL}^{\rho}(x,\gamma) = \overline{co}\,\overline{fl}^{\rho}(x,\gamma)\left(:= \overline{co}\left\{\overline{fl}^{\rho}(x,\gamma,a) : a \in A_{\gamma,x}\right\}\right),$$

for all $x \in \mathcal{G}^{+,\varepsilon} \smallsetminus \{O\}$ and, for every $j \leq N$,
$$\overline{FL}^{\rho}((1+\varepsilon)\,e_j,\gamma) \subset \overline{co}\,\overline{fl}^{\rho}((1+\varepsilon)\,e_j,\gamma)$$
$$\left(:= \overline{co}\left\{\overline{fl}^{\rho}((1+\varepsilon)\,e_j,\gamma,a) : a \in A^{\gamma,e_j}\right\} \cap \left(\mathbb{R}_- e_j \times \mathcal{M}_+(E) \times \mathbb{R}\right)\right).$$

Also, for every $j \leq M$,
$$\overline{FL}^{\rho}(-\varepsilon e_j,\gamma) \subset \overline{co}\,\overline{fl}^{\rho}(-\varepsilon e_j,\gamma)$$
$$\left(:= \overline{co}\left\{\overline{fl}^{\rho}(-\varepsilon e_j,\gamma,a) : a \in A^{\gamma,e_j}\right\} \cap \left(\mathbb{R}_+ e_j \times \mathcal{M}_+(E) \times \mathbb{R}\right)\right)$$

We consider another definition for viscosity subsolutions by taking more regular test functions.

Definition 20 *A bounded, upper (resp. lower) semicontinuous function V is said to be a classical constrained viscosity subsolution (resp subsolution of (10) if, for every $(\gamma_0, x_0) \in E \times \mathcal{G}^{+,\varepsilon}$ (resp. $E \times \overline{\mathcal{G}}^{+,\varepsilon}$), whenever $\varphi \in C_b\left(\overline{\mathcal{G}}^{+,\varepsilon}\right)$ for which $\varphi\mid_{\overline{J_j^{\varepsilon}}} \in C_b^1\left(\overline{J_j^{\varepsilon}}\right)$, for all $j = 1,2,...,N$ is a test function such that $x_0 \in Argmax\left(V\left(\cdot,\gamma_0\right) - \varphi\left(\cdot\right)\right)$, one has*

$$\delta V(x_0,\gamma_0) + \sup_{(\xi,\zeta,\eta)\in\overline{co}\,\overline{fl}^{\rho}(x_0,\gamma_0)}\left\{ \begin{array}{c} -\langle D\varphi(x_0;\xi)\rangle - \eta \\ -\sum_{\gamma'\in E}\zeta(\gamma')\left(V(x_0,\gamma') - V(x_0,\gamma_0)\right) \end{array} \right\} \leq 0,$$

(resp. ≥ 0).

We get the following characterization of $v^{\delta,\varepsilon,\rho}$.

Theorem 21 *The bounded uniformly continuous function $v^{\delta,\varepsilon,\rho}$ is a classical constrained viscosity subsolution of (10). Moreover, it satisfies the supersolution condition on $E \times \left(\overline{\mathcal{G}}^{+,\varepsilon} \smallsetminus \{O\}\right)$.*

Proof. The reader is invited to note that the test functions in this case are more regular than in Definition 16. Thus, the equality $\overline{FL}^{\rho}(x,\gamma) = \overline{co}\,\overline{fl}^{\rho}(x,\gamma)$ implies the viscosity sub/super condition at every point $x \in \mathcal{G}^{+,\varepsilon} \smallsetminus \{O\}$. The supersolution condition at $(1+\varepsilon)\,e_j$ (resp. $-\varepsilon e_j$) follows from the inclusion $\overline{FL}^{\rho}((1+\varepsilon)\,e_j,\gamma) \subset \overline{co}\,\overline{fl}^{\rho}((1+\varepsilon)\,e_j,\gamma)$ (resp. $\overline{FL}^{\rho}(-\varepsilon e_j,\gamma) \subset \overline{co}\,\overline{fl}^{\rho}(-\varepsilon e_j,\gamma)$).

The constant control $(a,b) \in \overline{A}^{\gamma,1}$ is locally admissible at O (on the extended graph $\overline{\mathcal{G}}^{+,\varepsilon}$). Hence, reasoning as in the subsolution part of theorem 17 (for constant $\overline{\alpha_n} = (a,b)$), one proves that if φ is a regular test function such that $O \in Argmax\left(v^{\delta}\left(\gamma,\cdot\right) - \varphi\left(\cdot\right)\right)$, then

$$0 \leq l_{\gamma}^{\rho}(O,(a,b)) - \delta\varphi(O) + \left\langle D\left(\varphi\mid_{\overline{J_1^{\varepsilon}}}\right)(O), f_{\gamma}(x,(a,b))\right\rangle +$$
$$\lambda^{\rho}(O,\gamma,(a,b)) \sum_{\gamma'\neq\gamma} Q^{\rho}(O,\gamma,\gamma',(a,b))\left(v^{\delta,\varepsilon,\rho}(O,\gamma') - v^{\delta,\varepsilon}(O,\gamma)\right).$$

Thus, continuity and convexity arguments imply that

$$\delta\varphi(O) + \sup_{(\xi,\zeta,\eta)\in\overline{co}\left\{\overline{fl}^{\rho}(O,\gamma,(a,b)):(a,b)\in\overline{A}^{\gamma,1}\right\}}\left\{ \begin{array}{c} -\langle D\varphi(O;\xi)\rangle - \eta \\ -\sum_{\gamma'\in E}\zeta(\gamma')\left(v^{\delta,\varepsilon,\rho}(O,\gamma') - v^{\delta,\varepsilon,\rho}(O,\gamma)\right) \end{array} \right\} \leq 0$$

and the subsolution condition follows. ∎

Remark 22 *In order to have (classical) uniqueness, one has to impose further conditions at the junction O. For example, in the case when $l_\gamma(O, a)$ does not depend on a for all $\gamma \in \bigcup_{j \leq N} E_j^{active}$, one reasons in the same way as in Section 5.2 of [1]. The arguments are quasi-identical and we prefer to concentrate on a different approach to uniqueness. Alternatively, one can impose the analog of the Assumption 2.3 in [1], i.e.*

$$\left(\{0\} \times \mathcal{M}_+(E) \times \left\{ \inf_{a \in A} l_\gamma(O, a) \right\} \right) \cap \overline{co} \left\{ \overline{fl}^0(O, \gamma, (a, b)) : (a, b) \in \overline{A}^{\gamma, j} \right\} \neq \emptyset,$$

for all j such that $\gamma \in E_j^{active}$.

6.5 Convergence to the initial value function

Unless stated otherwise, we assume the controllability conditions (**Aa, Ab, Ad**), the compatibility at the intersection (**Ab', Ac'**), the regularity of the coefficients and cost functions (**A1-A4**) and the compatibility condition (**B**) to hold true.

(**C**) Throughout the subsection, we also assume that l does not depend on the control at O and the nodes e_j.

This "projection long-run compatibility condition" will allow to change the control process around the "critical" points in order to obtain, from admissible controls on $\overline{\mathcal{G}}^{+, \varepsilon}$ an admissible control keeping the trajectory in $\overline{\mathcal{G}}$. This assumption (**C**) is only needed to prove Lemma 23 in its full generality. We have chosen to give a deeper result in Lemma 23 for further developments on the subject.

Let us fix $\varepsilon > 0$ small enough. We introduce the following notations:

$$t_\varepsilon := -\frac{1}{\delta} \ln \left(\frac{\varepsilon \delta}{2 |f|_0} \right), \quad \rho_\varepsilon := -\frac{\varepsilon^{1 + \frac{2 Lip(f)}{(1-\kappa)\delta}}}{\ln(\varepsilon)}, r'_\varepsilon \leq \frac{\rho_\varepsilon}{2},$$

$$\omega_\varepsilon(t; r) := e^{Lip(f)t} \left(r + (2\rho_\varepsilon \vee 4r'_\varepsilon) Lip(f) t \right), \ t \geq 0, r \geq 0, \ \Phi(\varepsilon) := \left(\frac{|f|_0}{(1-\kappa)\beta} + 1 \right) \left(\omega_\varepsilon \left(t_\varepsilon; r'_\varepsilon \right) \right)^{1-\kappa}.$$

The reader is invited to note that

$$\omega_\varepsilon(t; \omega_\varepsilon(t^*; r)) \leq \omega_\varepsilon(t^* + t; r),$$

for all $t, t^*, r \geq 0$. To get the best approximation and simplify the proof of Lemma 23, we also strengthen (**A1**) and ask that the restriction of f_γ to $[0, 1] e_j$ be Lipschitz-continuous for $\gamma \in E_j^{active}$. We emphasize that this only affects the definition of ρ_ε in Lemma 23 but not Theorem 25.

With these notations, we establish.

Lemma 23 *Whenever $\gamma \in E$, $x \in J_1^\varepsilon$ and $\overline{\alpha} = (\alpha, \beta) \in \overline{\mathcal{A}}_{\gamma, x}$, there exists $\mathcal{P}_x^\varepsilon(\alpha)$ (also depending on γ) such that $(\mathcal{P}_x^\varepsilon(\alpha), 0) \in \overline{\mathcal{A}}_{\gamma, x}$ such that*

$$(11) \qquad \left| y_\gamma^{\rho_\varepsilon}(t; x, (\mathcal{P}_x(\overline{\alpha}), 0)) - y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha}) \right| \leq \omega_\varepsilon(t_\varepsilon; \Phi(\varepsilon)),$$

*for $t \leq t_\varepsilon$. Moreover, when (**C**) holds true,*

$$(12) \qquad \limsup_{\varepsilon \to 0} \sup_{t \leq t_\varepsilon} \left| \begin{array}{l} \int_0^t e^{-\delta s} l_\gamma^{\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t; x, (\mathcal{P}_x^\varepsilon(\alpha), 0)), (\mathcal{P}_x^\varepsilon(\alpha)(s), 0) \right) ds \\ - \int_0^t e^{-\delta s} l_\gamma \left(y_\gamma(s; x, \overline{\alpha}), \overline{\alpha}(s) \right) ds \end{array} \right| = 0.$$

(ii) Moreover, if $\overline{\alpha} = (\alpha, \beta) \in \overline{\mathcal{A}}_{ad}$, then, for every $\varepsilon > 0$ there exists $(\mathcal{P}^\varepsilon(\overline{\alpha}), 0) \in \overline{\mathcal{A}}_{ad}$ such that the previous inequalities are satisfied with $\mathcal{P}^\varepsilon(\overline{\alpha})(\cdot, x, \gamma)$ replacing $\mathcal{P}_x^\varepsilon(\overline{\alpha})$.

We postpone the proof of this Lemma to the Appendix. We emphasize that whenever $(\alpha, 0) \in \overline{\mathcal{A}}_{ad}$, one has $y_\gamma^{\rho_\varepsilon}(t; x, (\alpha, 0)) = y_\gamma(t; x, \alpha)$ (and similar for $l_\gamma^{\rho_\varepsilon}, Q_\gamma^{\rho_\varepsilon}, \lambda_\gamma^{\rho_\varepsilon}$), even though α may not belong to $\mathcal{A}_{\gamma, x}$. The second argument takes care of this later issue.

Lemma 24 *Let us consider $T > 0$. Then, there exists a decreasing function $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such $\omega(0) = \omega(0+) = 0$ and whenever $\gamma \in E$, $x \in \overline{\mathcal{G}}$, and $(\alpha, 0) \in \overline{\mathcal{A}}_{\gamma, x}$, there exists $\mathcal{P}_{\gamma, x}(\alpha) \in \mathcal{A}_{\gamma, x}$ such that*

$$|y_\gamma(t; x, \alpha) - y_\gamma(t; x, \mathcal{P}_{\gamma, x}(\alpha))| \leq \omega(\varepsilon),$$

$$\sup_{t \leq T} \begin{array}{c} \int_0^t e^{-\delta s} l_\gamma(y_\gamma(t; x, \mathcal{P}_{\gamma, x}(\alpha)), \mathcal{P}_{\gamma, x}(\alpha)(s)) \, ds \\ - \int_0^t e^{-\delta s} l_\gamma(y_\gamma(t; x, \alpha), \alpha(s)) \, ds \end{array} \leq \omega(\varepsilon),$$

and

$$\sup_{s \leq T} \begin{array}{c} |Q(y_\gamma(s; x, \mathcal{P}_{\gamma, x}(\alpha)), \gamma, \gamma', \mathcal{P}_{\gamma, x}(\alpha)(s)) - Q(y_\gamma(s; x, \alpha), \gamma, \gamma', \alpha(s))| \\ + |\lambda(y_\gamma(s; x, \mathcal{P}_{\gamma, x}(\alpha)), \gamma, \mathcal{P}_{\gamma, x}(\alpha)(s)) - \lambda(y_\gamma(s; x, \alpha), \gamma, \alpha(s))| \end{array} \leq \omega(\varepsilon),$$

for all $\gamma' \in E$.

(ii) Moreover, if $(\alpha, 0) \in \overline{\mathcal{A}}_{ad}$, then, for every $\varepsilon > 0$ there exists $\mathcal{P}(\alpha) \in \mathcal{A}_{ad}$ such that the previous inequalities are satisfied with $\mathcal{P}(\alpha)(\cdot, x, \gamma)$ replacing $\mathcal{P}_{\gamma, x}(\alpha)(\cdot)$.

Although the approach is rather obvious (when looking at the proofs of Lemmas 6 or 23), hints on the proof are given in the Appendix. We wish to emphasize that, although the trajectories can be kept close up to a fixed T due to the proximity of $\overline{\mathcal{G}}^{+,\varepsilon}$ and $\overline{\mathcal{G}}$, we cannot do better then ε. Thus, we are unable to give the same kind of estimates up to t_ε.

The main result of the subsection is the following convergence theorem.

Theorem 25 *Under the assumption (C), the following convergence holds true*

$$\lim_{\varepsilon \to 0} \sup_{x \in \overline{\mathcal{G}}, \gamma \in E} \left| v^{\delta, \varepsilon, \rho_\varepsilon}(x, \gamma) - v^\delta(x, \gamma) \right| = 0.$$

Proof. The definition of our value functions yields $v^{\delta, \varepsilon, \rho_\varepsilon} \leq v^\delta$ on $\overline{\mathcal{G}} \times E$. Hence, we only need to prove the converse inequality. The proof is very similar to that of Theorem 15 in [23]. Let us fix $(x, \gamma) \in \overline{\mathcal{G}} \times E$, $T > 0$ and (for the time being,) $\varepsilon > 0$. Then using the dynamic programming principle for $v^{\delta, \varepsilon, \rho_\varepsilon}$ one gets the existence of some admissible control process $\overline{\alpha}$ such that

$$(13) \qquad v^{\delta, \varepsilon, \rho_\varepsilon}(x, \gamma) \geq \mathbb{E}\left[\begin{array}{c} \int_0^{T \wedge \tau_1} e^{-\delta t} l_\gamma^{\rho_\varepsilon}(y_\gamma^{\rho_\varepsilon}(t; x, \alpha), \overline{\alpha}(t; x, \gamma)) \, dt \\ + e^{-\delta(T \wedge \tau_1)} v^\delta\left(y_\gamma^{\rho_\varepsilon}(T \wedge \tau_1; x, \alpha), \Gamma_{T \wedge \tau_1}^{\rho_\varepsilon, x, \gamma, \overline{\alpha}}\right) \end{array} \right] - \varepsilon.$$

For simplicity, we let $\mathcal{P}$ and $\mathcal{P}^\varepsilon$ denote the two projectors of the previous lemmas and introduce the following notations:

$$\overline{\alpha}_t = \overline{\alpha}(t; x, \gamma), \quad \alpha_t = \mathcal{P}(\mathcal{P}^\varepsilon(\overline{\alpha}))(t; x, \gamma),$$

$$\overline{\lambda}(t) = \lambda\left(y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha}), \gamma, \overline{\alpha}_t\right), \quad \overline{\Lambda}(t) = \exp\left(-\int_0^t \overline{\lambda}(s) \, ds\right)$$

$$\lambda(t) = \lambda\left(y_\gamma(t; x, \alpha), \gamma, \alpha_t\right), \quad \Lambda(t) = \exp\left(-\int_0^t \lambda(s) \, ds\right),$$

for all $t \geq 0$. We denote the right-hand member of the inequality (13) by I. Then, I is explicitly

given by

$$I = \int_0^T \overline{\lambda}(t)\overline{\Lambda}(t) \int_0^t e^{-\delta s} l_\gamma^{\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(s;x,\overline{\alpha}), \overline{\alpha}_s \right) ds dt$$

$$+ \int_0^T \overline{\lambda}(t)\overline{\Lambda}(t) e^{-\delta t} \sum_{\gamma' \in E} v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \gamma' \right) Q^{\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \gamma, \gamma', \overline{\alpha}_t \right) dt$$

$$+ \overline{\Lambda}(T) \int_0^T e^{-\delta t} l_\gamma^{\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \overline{\alpha}_t \right) dt + \overline{\Lambda}(T) e^{-\delta T} v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(T;x,\overline{\alpha}), \gamma \right)$$

$$= I_1 + I_2 + I_3 + I_4.$$

The conclusion follows using the Lemmas 23 and 24. These estimates are tailor-made to allow substituting $\overline{\lambda}$, $\overline{\Lambda}$, $l_\gamma^{\rho_\varepsilon}$ and $y_\gamma^{\rho_\varepsilon}$ with $\lambda, \Lambda, l_\gamma$ and y_γ and the error is some (generic) $\omega(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$ (the reader may also want to take a glance at the proof of Theorem 15 in [23]). In the following, this function ω may change from one line to another. Let us recall (see Remark 12) that $v^{\delta,\varepsilon,\rho_\varepsilon}$ have the same continuity modulus (denoted ω^δ and independent of ε). Then, $v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \gamma' \right)$ can be replaced by $v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma(t;x,\overline{\alpha}), \gamma' \right)$ with an error $\omega^\delta \left(|y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}) - y_\gamma(t;x,\alpha)| \right)$, hence, again some $\omega(\varepsilon)$. The only interesting terms in I are I_2 and I_4. For the term I_2, one writes

$$I_2 \geq \int_0^T \lambda(t)\Lambda(t) e^{-\delta t} \sum_{\gamma' \in E} v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \gamma' \right) Q^{\rho_\varepsilon} \left(y_\gamma^{\rho_\varepsilon}(t;x,\overline{\alpha}), \gamma, \gamma', \overline{\alpha}_t \right) dt + \omega(\varepsilon)$$

$$\geq \int_0^T \lambda(t)\Lambda(t) e^{-\delta t} \sum_{\gamma' \in E} v^{\delta,\varepsilon,\rho_\varepsilon} \left(y_\gamma(t;x,\alpha), \gamma' \right) Q \left(y_\gamma(t;x,\alpha), \gamma, \gamma', \alpha_t \right) dt + \omega(\varepsilon)$$

$$\geq \int_0^T \lambda(t)\Lambda(t) e^{-\delta t} \sum_{\gamma' \in E} v^\delta \left(y_\gamma(t;x,\alpha), \gamma' \right) Q \left(y_\gamma(t;x,\alpha), \gamma, \gamma', \alpha_t \right) dt$$

$$(14) \qquad - \int_0^T \lambda(t)\Lambda(t) e^{-\delta t} dt \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z,\gamma') - v^\delta(z,\gamma') \right| + \omega(\varepsilon)$$

Similar,

$$(15) \qquad I_4 \geq \Lambda(T) e^{-\delta T} v^\delta \left(y_\gamma(T;x,\alpha) \right) - \Lambda(T) e^{-\delta T} \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z,\gamma') - v^\delta(z,\gamma') \right| + \omega(\varepsilon).$$

Hence, using (14, 15), one gets

$$I \geq \int_0^T \lambda(t)\Lambda(t) \int_0^t e^{-\delta s} l_\gamma \left(y_\gamma(s;x,\alpha), \alpha_s \right) ds dt$$

$$+ \int_0^T \lambda(t)\Lambda(t) e^{-\delta t} \sum_{\gamma' \in E} v^\delta \left(y_\gamma(t;x,\alpha), \gamma' \right) Q \left(y_\gamma(t;x,\alpha), \gamma, \gamma', \alpha_t \right) dt$$

$$+ \Lambda(T) \int_0^T e^{-\delta t} l_\gamma \left(y_\gamma(t;x,\alpha), \alpha_t \right) dt + \Lambda(T) e^{-\delta T} v^\delta \left(y_\gamma(T;x,\alpha), \gamma \right)$$

$$- \left[\int_0^T \lambda(t)\Lambda(t) e^{-\delta t} dt + \Lambda(T) e^{-\delta T} \right] \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z,\gamma') - v^\delta(z,\gamma') \right| + \omega(\varepsilon).$$

Then, using the dynamic programming principle for v^δ and (13), one gets

$$v^{\delta,\varepsilon,\rho_\varepsilon}(x,\gamma) \geq v^\delta(x,\gamma) - \left[1 - \delta \int_0^T \Lambda(t) e^{-\delta t} dt \right] \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z,\gamma') - v^\delta(z,\gamma') \right| + \omega(\varepsilon)$$

$$\geq v^\delta(x,\gamma) - \left[1 - \delta \int_0^T e^{-(\delta+|\lambda|_0)t} dt \right] \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z,\gamma') - v^\delta(z,\gamma') \right| + \omega(\varepsilon)$$

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Thus,

$$(0 \leq) v^\delta(x, \gamma) - v^{\delta,\varepsilon,\rho_\varepsilon}(x, \gamma) \leq \left[1 - \delta \int_0^T e^{-(\delta + |\lambda|_0)t} dt \right] \sup_{\gamma' \in E, z \in \overline{\mathcal{G}}} \left| v^{\delta,\varepsilon,\rho_\varepsilon}(z, \gamma') - v^\delta(z, \gamma') \right| + \omega(\varepsilon).$$

The conclusion follows by taking the supremum over $x \in \overline{\mathcal{G}}$ and $\gamma \in E$ and allowing $\varepsilon \to 0$. ∎

Remark 26 *We recall (cf. Remark 12) that $v^{\delta,\varepsilon,\rho_\varepsilon}$ have the same continuity modulus (independent of ε). Moreover, $v^{\delta,\varepsilon,\rho_\varepsilon}(\cdot) \leq \frac{|l|_0}{\delta}$. Therefore, applying Arzela-Ascoli Theorem, there exists $\lim_{\varepsilon \to 0} \left(v^{\delta,\varepsilon,\rho_\varepsilon} \big|_{\overline{\mathcal{G}}} \right)$ and this limit is uniformly continuous. It would have sufficed, therefore, to prove that $\lim_{\varepsilon \to 0} v^{\delta,\varepsilon,\rho_\varepsilon}(x; \gamma) = v^\delta(x, \gamma)$ for all $x \in \underset{i=1,2,...,N}{\cup} (0, 1) e_i$.*

6.6 Linearizing the problem

We assume the (pseudo-)controllability conditions (**Aa, Ab, Ad**), the compatibility at the intersection (**Ab', Ac'),** the regularity of the coefficients and cost functions (**A1-A4**), the compatibility condition (**B**) and the projection compatibility condition (**C**) to hold true.

6.6.1 Smooth subsolutions

Starting from $v^{\delta,\varepsilon,\rho_\varepsilon}$, we will construct a family of smooth subsolutions of (10) (with $\varepsilon = \rho = 0$) that converge to the value function v^δ. To this purpose, we regularize the functions $v^{\delta,\varepsilon,\rho_\varepsilon}$ in each direction given by e_j, for $j = 1, 2, ..., N$. Finally, we conveniently modify the value at the junction point O.

We begin by picking $(\psi_\epsilon)_\epsilon$ to be a sequence of standard mollifiers $\psi_\epsilon(y) = \frac{1}{\epsilon} \psi\left(\frac{y}{\epsilon}\right)$, $y \in \mathbb{R}$, $\epsilon > 0$, where $\psi \in C^\infty(\mathbb{R})$ is a positive function such that

$$Supp(\psi) \subset [-1, 1] \text{ and } \int_{\mathbb{R}} \psi(y) dy = 1.$$

For every $\varepsilon > 0$ and every $0 < \epsilon \leq \rho_\varepsilon$, one can define regular functions $v_{\varepsilon,\epsilon}^{\delta,j}$ by setting

$$v_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma) = \int_{-\varepsilon}^{\varepsilon} v^{\delta,\varepsilon,\rho_\varepsilon}(x - y e_j, \gamma) \psi_\epsilon(y) \, dy,$$

for all $x \in (-\varepsilon, 1 + \varepsilon) e_j$, $j = 1, M$, or $x \in (-1 - \varepsilon, 1 + \varepsilon) e_j$, if $M < j \leq N$. Using the same methods as those employed in [24], Appendix (see also [23], Appendix A2 or [29] or [6], Lemma 2.7), it is easy to prove that

$$(16) \qquad \delta v_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma) + \left\{ \begin{array}{c} -\left\langle f_\gamma(x, a), Dv_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma) \right\rangle - l_\gamma(x, a) \\ -\lambda(x, \gamma, a) \sum_{\gamma' \in E} Q(x, \gamma, \gamma', a) \left(v_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma') - v_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma) \right) \end{array} \right\} \leq 0,$$

for all $x \in [0, 1] e_j$, $j \leq N$ and all $a \in A^{\gamma,j}$. Also, we note that

$$\left| v_{\varepsilon,\epsilon}^{\delta,j}(x, \gamma) - v^\delta(x, \gamma) \right| \leq \left| v^{\delta,\varepsilon,\rho_\varepsilon} - v^\delta \right|_0 + \omega^\delta(\epsilon) =: \omega(\varepsilon, \epsilon),$$

for all $x, \gamma \in \overline{\mathcal{G}} \times E$, where ω^δ is the continuity modulus of v^δ (with respect to the space component). Theorem 25 yields

$$\lim_{\varepsilon,\epsilon \to 0} \omega(\varepsilon, \epsilon) = 0.$$

We define an admissible test function by setting

$$v_\varepsilon^\delta(x,\gamma) = v_{\varepsilon,\rho_\varepsilon}^{\delta,j}(x,\gamma) - v_{\varepsilon,\rho_\varepsilon}^{\delta,j}(O,\gamma) + \min_{j'=1,2,\dots,N} v_{\varepsilon,\rho_\varepsilon}^{\delta,j'}(O,\gamma) - 4\frac{|\lambda|_0}{\delta}\omega(\varepsilon,\rho_\varepsilon),$$

for $x \in [0,1]\,e_j$, $1 \le j \le N$ and $\gamma \in E$. Then v_ε^δ is a regular test function (continuous at O) which satisfies

$$(17) \qquad \begin{cases} \left(\begin{array}{c} \delta v_\varepsilon^\delta(x,\gamma) - \langle f_\gamma(x,a), Dv_\varepsilon^\delta(x,\gamma)\rangle - l_\gamma(x,a) \\ -\lambda(x,\gamma,a) \sum\limits_{\gamma' \in E} Q(x,\gamma,\gamma',a)\left(v_\varepsilon^\delta(x,\gamma') - v_\varepsilon^\delta(x,\gamma)\right) \end{array} \right) \le 0, \text{ and} \\ \lim_{\varepsilon \to 0} \left|v_\varepsilon^\delta - v^\delta\right|_0 = 0, \end{cases}$$

for all (x,γ,a) such that $x \in [0,1]\,e_j$, $\gamma \in E$, $a \in A^{\gamma,j}$, $j \le N$. These functions are Lipschitz continuous on $\overline{\mathcal{G}}$. (In fact, the reader can check rather easily that the Lipschitz constant of v_ε^δ does

not exceed $\sqrt{2}\dfrac{\max\limits_{1 \le j \le N}\left|D\left(v_\varepsilon^\delta|_{[-1,1]e_j}\right)\right|_0}{\sqrt{1-\max\limits_{i',j' \in \left\{1,\dots,M+\frac{N-M}{2}\right\},\ i' \ne j'}\cos\left(e_{i'},e_{j'}\right)}}$). Hence, (using Kirszbraun's Theorem,) one can find an extension (explicitly given by

$$\widetilde{v}_\varepsilon^\delta(x,\gamma) := \inf_{y \in \overline{\mathcal{G}}}\left(v_\varepsilon^\delta(y,\gamma) + Lip\left(v_\varepsilon^\delta\right)|x-y|\right)$$

which is Lipschitz continuous on $\mathbb{R}^m$. As a by-product, this function (identified with $v_\varepsilon^\delta(\cdot,\gamma)$ whenever no confusion is at risk) is absolutely continuous on $\mathbb{R}^m$ $(AC(\mathbb{R}^m))$.

6.6.2 Occupation measures and embedding

To every admissible control $\alpha \in \mathcal{A}_{ad}^{\mathbb{N}}$ and $\gamma \in E, x \in \overline{\mathcal{G}}$, we can associate a probability measure $\mu^{x,\gamma,\alpha} \in \mathcal{P}(\mathbb{R}^m \times E \times A)$ by setting

$$\mu^{x,\gamma,\alpha}(A \times B \times C) = \delta\mathbb{E}\left[\int_0^\infty e^{-\delta t}1_{A \times B \times C}\left(X_t^{x,\gamma,\alpha}, \Gamma_t^{x,\gamma,\alpha}, \alpha_t\right)\right],$$

for all Borel sets $A \times B \times C \subset \mathbb{R}^m \times E \times A$. As before, if $(\tau_i)_{i \ge 0}$ denote the switch times, then $\alpha_t = \alpha_{i+1}(t - \tau_i, X_{\tau_i}^{x,\gamma,\alpha}, \Gamma_{\tau_i}^{x,\gamma,\alpha})$ on $t \in [\tau_i, \tau_{i+1})$. Obviously, the choice of admissible controls (under constraints) yields

$$Supp\left(\mu^{x,\gamma,\alpha}\right) \subset \widehat{\overline{\mathcal{G}} \times E} \times A := \left\{(y,\gamma',a) \in \overline{\mathcal{G}} \times E \times A : a \in A^{\gamma',j} \text{ whenever } y \in \overline{J_j}\right\}.$$

We note that the set $\widehat{\overline{\mathcal{G}} \times E} \times A$ is compact.

We denote by $BAC(\mathbb{R}^m \times E; \mathbb{R})$ the set of all bounded functions $\varphi : \mathbb{R}^m \times E \longrightarrow \mathbb{R}$ such that $\varphi(\cdot,\gamma') \in AC(\mathbb{R}^m)$ for all $\gamma' \in E$. Then, Itô's formula (see [17, Theorem 31.3]) yields

$$(18) \qquad \begin{aligned} &\delta e^{-\delta T}\mathbb{E}\left[\varphi\left(X_T^{x,\gamma,\alpha}, \Gamma_T^{x,\gamma,\alpha}\right)\right] \\ &= \delta\varphi(x,\gamma) + \mathbb{E}\int_0^T \delta e^{-\delta s}\left[-\delta\varphi\left(X_t^{x,\gamma,\alpha}, \Gamma_t^{x,\gamma,\alpha}\right) + \mathcal{U}^{\alpha_t}\varphi\left(X_t^{x,\gamma,\alpha}, \Gamma_t^{x,\gamma,\alpha}\right)\right]dt. \end{aligned}$$

Here,

$$\mathcal{U}^a\varphi(y,\gamma') = \langle f_\gamma(y,a), D\varphi(y,\gamma')\rangle + \lambda(y,\gamma',a)\sum_{\gamma'' \in E}Q(y,\gamma',\gamma'',a)\left(\varphi(\gamma'',x) - \varphi(y,\gamma')\right),$$

for regular $\varphi\left(\cdot,\gamma\right) \in C_b^1\left(\mathbb{R}^m\right)$ is the classical generator of the PDMP. We recall that the extended domain of $\mathcal{U}^a$ includes functions such that $\varphi\left(\cdot,\gamma'\right) \in AC\left(\mathbb{R}^m\right)$ (cf. Theorem 31.3 in [17]). Hence, passing to the limit as $T \to \infty$ in (18) (and recalling that φ is bounded), one gets

$$\int_{\mathbb{R}^m \times E \times A} \left[\mathcal{U}^a\varphi\left(y,\gamma'\right) - \delta\left[\varphi\left(y,\gamma'\right) - \varphi\left(x,\gamma\right)\right]\right] \mu^{x,\gamma,\alpha}\left(dy d\gamma' da\right) = 0.$$

We set

$$(19) \quad \begin{aligned} &\Theta_{\overline{\mathcal{G}}}^0\left(x,\gamma\right) := \left\{\mu^{x,\gamma,\alpha} : \alpha \in \mathcal{A}_{ad}^{\mathbb{N}}\right\} \text{ and} \\ &\Theta_{\overline{\mathcal{G}}}\left(x,\gamma\right) := \left\{ \begin{array}{c} \mu^{x,\gamma,\alpha} \in \mathcal{P}\left(\overline{\mathcal{G} \times E \times A}\right) : \forall \varphi \in BAC\left(\mathbb{R}^m \times E; \mathbb{R}\right) \\ \int_{\mathbb{R}^m \times E \times A}\left[-\mathcal{U}^a\varphi\left(y,\gamma'\right) + \delta\left[\varphi\left(y,\gamma'\right) - \varphi\left(x,\gamma\right)\right]\right] \mu\left(dy d\gamma' da\right) = 0. \end{array} \right\} \end{aligned}$$

We are now able to state (and prove) the main linearization result.

Theorem 27 *The following equalities hold true*

$$\begin{aligned} &\delta v^\delta\left(x,\gamma\right) \\ &= \Lambda^\delta\left(x,\gamma\right) := \inf_{\mu \in \Theta_{\overline{\mathcal{G}}}(x,\gamma)} \int_{\mathbb{R}^m \times E \times A} l_{\gamma'}\left(y,a\right)\mu\left(dy d\gamma' da\right) \\ &= \Lambda^{\delta,*}\left(x,\gamma\right) := \sup\left\{ \begin{array}{c} \eta \in \mathbb{R} : \exists \varphi \in BAC\left(\mathbb{R}^m \times E; \mathbb{R}\right), \; \textit{for all } \left(y,\gamma',a\right) \in \overline{\mathcal{G} \times E \times A}, \\ \eta \le \mathcal{U}^a\varphi\left(y,\gamma'\right) + l_{\gamma'}\left(y,a\right) - \delta\left[\varphi\left(y,\gamma'\right) - \varphi\left(x,\gamma\right)\right]. \end{array} \right\}, \end{aligned}$$

for all $\left(x,\gamma\right) \in \overline{\mathcal{G}} \times E$.

Proof. Let us fix $\left(x,\gamma\right) \in \overline{\mathcal{G}} \times E$. It is clear that

$$\delta v^\delta\left(x,\gamma\right) \ge \inf_{\mu \in \Theta_{\overline{\mathcal{G}}}(x,\gamma)} \int_{\mathbb{R}^m \times E \times A} l_{\gamma'}\left(y,a\right)\mu\left(dy d\gamma' da\right)$$

since $\Theta_{\overline{\mathcal{G}}}^0\left(x,\gamma\right) \subset \Theta_{\overline{\mathcal{G}}}\left(x,\gamma\right)$. Next, if $\eta \le \mathcal{U}^a\varphi\left(y,\gamma'\right) + l_{\gamma'}\left(y,a\right) - \delta\left[\varphi\left(y,\gamma'\right) - \varphi\left(x,\gamma\right)\right]$, for all $\left(y,\gamma',a\right) \in \overline{\mathcal{G} \times E \times A}$, then, due to the definition of $\Theta_{\overline{\mathcal{G}}}\left(x,\gamma\right)$, if $\mu \in \Theta_{\overline{\mathcal{G}}}\left(x,\gamma\right)$, by integrating the inequality w.r.t. μ, it follows that

$$\int_{\mathbb{R}^m \times E \times A} l_{\gamma'}\left(y,a\right)\mu\left(dy d\gamma' da\right) \ge \eta.$$

Hence, $\Lambda^\delta\left(x,\gamma\right) \ge \Lambda^{\delta,*}\left(x,\gamma\right)$. To complete the proof, one needs to prove $\Lambda^{\delta,*}\left(x,\gamma\right) \ge \delta v^\delta\left(x,\gamma\right)$. We use v_ε^δ given in Subsubsection 6.6.1 to infer

$$\delta v_\varepsilon^\delta\left(x,\gamma\right) \le \mathcal{U}^a v_\varepsilon^\delta\left(y,\gamma'\right) + l_{\gamma'}\left(y,a\right) - \delta\left[v_\varepsilon^\delta\left(y,\gamma'\right) - v_\varepsilon^\delta\left(x,\gamma\right)\right],$$

for all $\left(y,\gamma',a\right) \in \overline{\mathcal{G} \times E \times A}$. Hence, $\delta v_\varepsilon^\delta\left(x,\gamma\right) \le \Lambda^{\delta,*}\left(x,\gamma\right)$. The proof is completed by taking the limit as $\varepsilon \to 0$ and recalling that (17) holds true. ∎

6.7 Conclusion and comments

The previous result can be interpreted in connection to Perron's method. If φ is a regular subsolution of (10) for $\rho = 0$, $\varepsilon = 0$ on $\overline{\mathcal{G}}$ (i.e. such that

$$\mathcal{U}^a\varphi\left(y,\gamma'\right) + l_{\gamma'}\left(y,a\right) - \delta\varphi\left(y,\gamma'\right) \ge 0,$$

for all $(y, \gamma', a) \in \overline{\mathcal{G} \times E \times A)}$, then $\delta\varphi(x, \gamma) \leq \Lambda^{\delta,*}(x, \gamma) = \delta v^\delta(x, \gamma)$. Since we have exhibited a family $\left(\left(v_\varepsilon^\delta(x, \gamma)\right)_{\varepsilon>0}\right)$ converging to $v^\delta(x, \gamma)$, it follows that v^δ is the pointwise supremum over such regular subsolutions, hence giving Perron's solution to the Hamilton-Jacobi integrodifferential system.

This implies a weak form of uniqueness for our solution. This approach has a couple of advantages. First, it provides an approximating scheme for the value function v^δ in the spirit of [29], [6] or [8]. However, the speed of convergence is given by the estimates in Lemma 23 and are less explicit than the Hölder ones exhibited in the cited papers. Second, having stated the equivalent problem on a linear space of measures should prove useful for optimality issues (see [20] or, more recently, [19] in a general Markovian framework or [25] in a Brownian one).

In a deterministic framework, there is an increasing literature dealing with stronger forms of uniqueness based on comparison principles. Some of the papers deal with frameworks similar to ours (e.g. [1]) and use the geodetic distance in the doubling variable approach . These results have been generalized and simplified in [2]. Another approach consists in introducing "vertex test" functions. This allows to treat a generalized quasi-convex case in the recent preprint [27]. A nice comparison between the different notions of solution (corresponding to [1], [28] and [33]) is also provided in [11]. Finally, let us note that, in our setting, the test functions only need to be substituted in the gradient and, hence, adapting the comparison methods of the previous papers should work rather smoothly.

With the assumptions of this section (in particular **(C))**, it follows that any regular subsolution in the sense of Definition 20 is also a regular subsolution in the sense of Definition 16. It follows that v^δ cannot exceed the supremum over regular subsolutions in the sense of Definition 16. Equality is obtained whenever a classical comparison principle is available.

7 Appendix

7.1 Proof of Lemma 6

Proof of Lemma 6. We will consider several cases and prove (i) in each case. We provide the construction for (ii) only in the first case (a) and hint what is needed for the remaining cases.

(a) (i) Let us assume that $x = O$. If $y = O$, then $\mathcal{P}_{O,y}(\alpha) = \alpha$. Otherwise, we let $t_{y,O} := \inf\left\{t \geq 0 : y_\gamma\left(t; y, a_{\gamma,1}^-\right) = O\right\}$. Obviously,

$$t_{y,O} \leq \frac{|y|^{1-\kappa}}{(1-\kappa)\beta} \leq \frac{\rho_\varepsilon^2}{(1-\kappa)\beta}.$$

(These estimates are for the "inactive" case; for the "active" one, one can consider $\kappa = 0$). For ε small enough, one can assume, without loss of generality that $\frac{\rho_\varepsilon}{(1-\kappa)\beta} < t_\varepsilon$. We define

$$\mathcal{P}_{O,y}(\alpha)(t) := a_{\gamma,1}^- \mathbf{1}_{[0,t_{y,O}]}(t) + \alpha(t - t_{y,O})\mathbf{1}_{(t_{y,O},\infty)}(t),$$

for all $t \geq 0$. Then, one gets

$$|y_\gamma(t; y, \mathcal{P}_{O,y}(\alpha)) - y_\gamma(t; O, \alpha)| \leq |y_\gamma(t; y, \mathcal{P}_{O,y}(\alpha)) - y| + |y| + |y(t; O, \alpha)|$$
$$\leq \left(\frac{2|f|_0}{(1-\kappa)\beta} + 1\right)|y|^{1-\kappa} \leq \left(\frac{2|f|_0}{(1-\kappa)\beta} + 1\right)\rho_\varepsilon^2,$$

if $t \in [0, t_{y,O}]$ and

$$|y_\gamma(t; y, \mathcal{P}_{O,y}(\alpha)) - y_\gamma(t; O, \alpha)| = |y_\gamma(t; O, \alpha) - y_\gamma(t - t_{y,O}; O, \alpha)|$$
$$\leq \frac{|f|_0}{(1-\kappa)\beta}|y|^{1-\kappa} \leq \frac{|f|_0}{(1-\kappa)\beta}\rho_\varepsilon^2,$$

if $t > t_{y,O}$. Moreover, for every $T \geq 0$,

$$\left| \int_0^T e^{-\delta t} l_\gamma \left(y_\gamma \left(t; y, \mathcal{P}_{O,y}\left(\alpha\right)\right), \mathcal{P}_{O,y}\left(\alpha\right)\left(t\right)\right) dt - \int_0^T e^{-\delta t} l_\gamma \left(y_\gamma \left(t; O, \alpha\right), \alpha\left(t\right)\right) dt \right|$$

$$\leq \int_0^{t_{y,O}} e^{-\delta t} \left| l_\gamma \left(y_\gamma \left(t; y, \mathcal{P}_{O,y}\left(\alpha\right)\right), \mathcal{P}_{O,y}\left(\alpha\right)\left(t\right)\right)\right| dt + \int_0^{t_{y,O}} e^{-\delta t} \left| l_\gamma \left(y_\gamma \left(t; O, \alpha\right), \alpha\left(t\right)\right)\right| dt$$

$$+ \mathbf{1}_{T > t_{y,O}} \left(1 - e^{-\delta t_{y,O}}\right) \int_0^{T - t_{y,O}} e^{-\delta t} \left| l_\gamma \left(y_\gamma \left(t; O, \alpha\right), \alpha\left(t\right)\right)\right| dt$$

$$+ \mathbf{1}_{T > t_{y,O}} \int_{T - t_{y,O}}^T e^{-\delta t} \left| l_\gamma \left(y_\gamma \left(t; O, \alpha\right), \alpha\left(t\right)\right)\right| dt$$

$$\leq 2 \left| l \right|_0 \frac{\left| y \right|^{1-\kappa}}{\left(1 - \kappa\right)\beta} + \frac{1}{\delta} \left| l \right|_0 \left(1 - e^{-\delta \frac{\left| y \right|^{1-\kappa}}{(1-\kappa)\beta}} \right) + \left| l \right|_0 \frac{\left| y \right|^{1-\kappa}}{\left(1 - \kappa\right)\beta}$$

$$\leq 4 \left| l \right|_0 \frac{\left| y \right|^{1-\kappa}}{\left(1 - \kappa\right)\beta} \leq \frac{4 \left| l \right|_0}{\left(1 - \kappa\right)\beta} \rho_\varepsilon^2.$$

(ii) If $\alpha \in \mathcal{A}_{ad}$, then we set

$$\mathcal{P}_{(O,\gamma)}\left(\alpha\right)\left(t; y, \eta\right) = \begin{cases} \mathcal{P}_{O,y}\left(\alpha\left(t; O, \gamma\right)\right) \text{ if } \eta = \gamma, \left| y \right| \leq \rho_\varepsilon^{\frac{2}{1-\kappa}}, \\ \alpha\left(t; y, \eta\right), \text{ otherwise.} \end{cases}$$

One only needs to notice that $y \mapsto t_{y,O}$ is Borel measurable to deduce that $\mathcal{P}_{O,\gamma}\left(\alpha\right) \in \mathcal{A}_{ad}$. In the other cases, the construction is similar. We will just hint the measurability properties needed to insure that the constructed function $\mathcal{P}_{(x,\gamma)}\left(\alpha\right)$ is Borel measurable in (t, y, η).

(b) If $y = O$, we distinguish two cases :

(b1) The road is "inactive". Then, we introduce $t_{x,O}\left(\alpha\right) := \inf \left\{ t > 0 : y_\gamma\left(t; x, \alpha\right) = O \right\}$ and define, if it is finite

$$\mathcal{P}_{x,y}\left(\alpha\right)\left(t\right) := a_{\gamma,1}^0 \mathbf{1}_{\left[0, t_{x,O}\left(\alpha\right)\right]}\left(t\right) + \alpha\left(t\right) \mathbf{1}_{\left(t_{x,O}\left(\alpha\right), \infty\right)}\left(t\right),$$

where $a_{\gamma,1}^0$ is given by (**Ab**). Then, due to (**Ab**), it is clear that

$$\left| y_\gamma \left(t; y, \mathcal{P}_{x,y}\left(\alpha\right)\right) - y_\gamma\left(t; x, \alpha\right)\right| \leq \left| x - y \right| \leq \rho_\varepsilon^2,$$

if $t \leq t_{x,O}\left(\alpha\right)$ and

$$\left| y_\gamma \left(t; y, \mathcal{P}_{x,y}\left(\alpha\right)\right) - y_\gamma\left(t; x, \alpha\right)\right| = 0,$$

otherwise. We note that $y_\gamma\left(t; y, \mathcal{P}_{x,y}\left(\alpha\right)\right) = O$, for $t \leq t_{x,O}\left(\alpha\right)$. Thus, the assumption (**Ac**) yields

$$\left| \int_0^T e^{-\delta t} l \left(y_\gamma \left(t; y, \mathcal{P}_{x,y}\left(\alpha\right)\right), \mathcal{P}_{x,y}\left(\alpha\right)\left(t\right)\right) dt - \int_0^T e^{-\delta t} l \left(y_\gamma\left(t; x, \alpha\right), \alpha\left(t\right)\right) dt \right|$$

$$\leq \int_0^T e^{-\delta t} Lip(l) \left| x - y \right| dt \leq \frac{Lip(l)}{\delta} \left| x - y \right| \leq \frac{Lip(l)}{\delta} \rho_\varepsilon^2.$$

(b2) The road is "active". Then, we introduce $t_{y,x} := \inf \left\{ t > 0 : y_\gamma\left(t; y, a_{\gamma,1}^+\right) = x \right\}$. Similar to (a), one easily proves that $t_{y,x} \leq \frac{\rho_\varepsilon^2}{\beta}$. In this case, we define

$$\mathcal{P}_{x,y}\left(\alpha\right)\left(t\right) := a_{\gamma,1}^+ \mathbf{1}_{\left[0, t_{y,x}\right]}\left(t\right) + \alpha\left(t - t_{y,x}\right) \mathbf{1}_{\left(t_{y,x}, \infty\right)}\left(t\right),$$

and get the same kind of estimates as in (a).

(c) We assume that $x \in J_1 \cup \{e_1\}$ and $y \in J_1$. Then, $\alpha \in \mathcal{A}_{\gamma,x}$ is admissible for y (at least for some small time). We define $t_y^*(\alpha) = \inf\{t > 0 : y_\gamma(t; y, \alpha) \in \partial J_1\} \wedge \inf\{t > 0 : y_\gamma(t; x, \alpha) = 0\} \wedge t_\varepsilon$. One notices, as before, that $y \mapsto t_y^*(\alpha)$ is Borel measurable.

(c1) If $t_y^*(\alpha) \geq t_\varepsilon$, then we let $\mathcal{P}_{x,y}(\alpha)(t) := \alpha(t)\mathbf{1}_{[0,t_\varepsilon)}(t) + \alpha_0(t; y_\gamma(t_\varepsilon; y, \alpha), \gamma)\mathbf{1}_{[t_\varepsilon,\infty)}(t)$, where $\alpha_0 \in \mathcal{A}_{ad}$ and have

$$|y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)) - y_\gamma(t; x, \alpha)| \leq e^{Lip(f)t}|x - y| \leq \sqrt{|x - y|} \leq \rho_\varepsilon^{\frac{1}{1-\kappa}},$$

for all $t \leq t_\varepsilon$. Also, one easily gets, for every $T \leq t_\varepsilon$,

$$\left| \int_0^T e^{-\delta t} l\left(y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)), \mathcal{P}_{x,y}(\alpha)(t)\right) dt - \int_0^T e^{-\delta t} l\left(y_\gamma(t; x, \alpha), \alpha(t)\right) dt \right|$$
$$\leq \frac{Lip(l)}{\delta}\sqrt{|x - y|} \leq \frac{Lip(l)}{\delta}\rho_\varepsilon^{\frac{1}{1-\kappa}}.$$

Since $\alpha_0 \in \mathcal{A}_{ad}$, it follows that $(t, y) \mapsto \mathcal{P}_{x,y}(\alpha)(t)1_{t_y^*(\alpha) \geq t_\varepsilon}$ is Borel-measurable.

(c2) If $t_y^*(\alpha) < t_\varepsilon$ and $y_\gamma(t_y^*(\alpha); y, \alpha) = e_1$, then, in particular, $\left|y_\gamma(t_y^*(\alpha); x, \alpha) - e_1\right| < \sqrt{|x - y|} \leq \rho_\varepsilon^{\frac{1}{1-\kappa}}$. Of course, this case is only interesting if α is no longer admissible. In particular, when $A_{\gamma,e_1} \neq A^{\gamma,1}$. Then, we introduce $t_{e_1, y_\gamma(t_y^*(\alpha); x, \alpha)} := \inf\{t \geq 0 : y_\gamma(t; e_1, a_{\gamma,1}) = y_\gamma(t_y^*(\alpha); x, \alpha)\}$. One has $t_{e_1, y_\gamma(t_y^*(\alpha); x, \alpha)} \leq \frac{\sqrt{|x-y|}}{\beta}$. We define

$$\mathcal{P}_{x,y}(\alpha)(t) := \alpha(t)\mathbf{1}_{[0,t_y^*(\alpha))}(t) + a_{\gamma,1}\mathbf{1}_{\left[t_y^*(\alpha), t_y^*(\alpha) + t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}\right]}(t)$$
$$+ \alpha\left(t - t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}\right)\mathbf{1}_{\left(t_y^*(\alpha) + t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)},\infty\right)}(t).$$

The functions $y \mapsto t_y^*(\alpha)$, $y \mapsto y_\gamma(t_y^*(\alpha); y, \alpha)$ are Borel measurable. Hence, so is $y \mapsto t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}$. It follows that

$$(t, y) \mapsto \mathcal{P}_{x,y}(\alpha)(t)\mathbf{1}_{t_y^*(\alpha) < t_\varepsilon,\, y_\gamma(t_y^*(\alpha);y,\alpha)=e_1}$$

is also Borel-measurable. One has

$$|y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)) - y_\gamma(t; x, \alpha)| \leq \sqrt{|x - y|},$$

if $t \leq t_y^*(\alpha)$,

$$|y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)) - y_\gamma(t; x, \alpha)| \leq \left|y_\gamma\left(t - t_y^*(\alpha); e_1, a_{\gamma,1}\right) - e_1\right| + \left|e_1 - y_\gamma(t_y^*(\alpha); x, \alpha)\right|$$
$$+ \left|y_\gamma(t_y^*(\alpha); x, \alpha) - y_\gamma(t; x, \alpha)\right|$$
$$\leq \left(\frac{2|f|_0}{\beta} + 1\right)\sqrt{|x - y|},$$

if $t \in \left[t_y^*(\alpha), t_y^*(\alpha) + t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}\right]$. Finally, if $t > t_y^*(\alpha) + t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}$, then

$$|y_\gamma(t; y, \mathcal{P}_{x,y}(\alpha)) - y_\gamma(t; x, \alpha)|$$
$$= \left| \begin{array}{c} y_\gamma\left(t - t_y^*(\alpha) + t_{e_1,y_\gamma(t_y^*(\alpha);x,\alpha)}; y_\gamma(t_y^*(\alpha); x, \alpha), \alpha(t_y^*(\alpha) + \cdot)\right) \\ -y_\gamma\left(t - t_y^*(\alpha); y_\gamma(t_y^*(\alpha); x, \alpha), \alpha(t_y^*(\alpha) + \cdot)\right) \end{array} \right|$$
$$\leq |f|_0 \frac{\sqrt{|x - y|}}{\beta}.$$

Moreover, if $T \leq t_\varepsilon$, one gets (similar to (a)),

$$\left| \int_0^T e^{-\delta t} l\left(y_\gamma\left(t; y, \mathcal{P}_{x,y}\left(\alpha \right) \right), \mathcal{P}_{x,y}\left(\alpha \right)\left(t \right) \right) dt - \int_0^T e^{-\delta t} l\left(y_\gamma\left(t; x, \alpha \right), \alpha\left(t \right) \right) dt \right|$$

$$\leq \int_0^{t_y^*(\alpha)} e^{-\delta t} Lip\left(l \right) \sqrt{|x - y|} dt + \frac{4 |l|_0}{\beta} \sqrt{|x - y|}.$$

(c3) The case $t_y^*\left(\alpha \right) < t_\varepsilon$ and $y_\gamma\left(t_y^*\left(\alpha \right); y, \alpha \right) = O$: In particular, one gets $\left| y_\gamma\left(t_y^*\left(\alpha \right); x, \alpha \right) \right| \leq \sqrt{|x - y|} \leq \rho_\varepsilon^{\frac{1}{1-\kappa}}$.

(c3.1) In the "active case", we consider $t_{O, y_\gamma\left(t_y^*(\alpha); x, \alpha \right)} = \inf \left\{ t > 0 : y_\gamma\left(t; O, a_{\gamma,1}^+ \right) = y_\gamma\left(t_y^*\left(\alpha \right); x, \alpha \right) \right\}$ and define

$$\mathcal{P}_{x,y}\left(\alpha \right)\left(t \right) := \alpha\left(t \right) \mathbf{1}_{\left[0, t_y^*(\alpha) \right)}\left(t \right) + a_{\gamma,1}^+ \mathbf{1}_{\left[t_y^*(\alpha), t_y^*(\alpha) + t_{O, y_\gamma\left(t_y^*(\alpha); x, \alpha \right)} \right]}\left(t \right)$$

$$+ \alpha\left(t - t_{O, y_\gamma\left(t_y^*(\alpha); x, \alpha \right)} \right) \mathbf{1}_{\left(t_y^*(\alpha) + t_{O, y_\gamma\left(t_y^*(\alpha); x, \alpha \right)}, \infty \right)}\left(t \right).$$

One gets the same estimates (and measurability properties) as in (c2).

(c3.2) The "inactive case" is similar to (b1). We consider

$$\mathcal{P}_{x,y}\left(\alpha \right)\left(t \right) := \alpha\left(t \right) \mathbf{1}_{\left[0, t_y^*(\alpha) \right)}\left(t \right) + a_{\gamma,1}^0 \mathbf{1}_{\left[t_y^*(\alpha), t_y^*(\alpha) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O} \right]}\left(t \right)$$

$$+ \alpha\left(t - t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O} \right) \mathbf{1}_{\left(t_y^*(\alpha) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O}, \infty \right)}\left(t \right),$$

for all $t \geq 0$. The functions $y \mapsto t_y^*\left(\alpha \right)$, $y \mapsto y_\gamma\left(t_y^*\left(\alpha \right); x, \alpha \right)$ are Borel measurable. Hence, so is $y \mapsto t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O}\left(a_{\gamma,1}^0 \right)$. It follows that

$$\left(t, y \right) \mapsto \mathcal{P}_{x,y}\left(\alpha \right)\left(t \right) \mathbf{1}_{t_y^*(\alpha) < t_\varepsilon, \, y_\gamma\left(t_y^*(\alpha); y, \alpha \right) = O}$$

is also Borel-measurable.

One easily notices that

$$\left| y_\gamma\left(t; y, \mathcal{P}_{x,y}\left(\alpha \right) \right) - y_\gamma\left(t; x, \alpha \right) \right| \leq \sqrt{|x - y|} \leq \rho_\varepsilon, \text{ if } 0 \leq t \leq t_y^*\left(\alpha \right) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O},$$

and $y_\gamma\left(t; y, \mathcal{P}_{x,y}\left(\alpha \right) \right) = y_\gamma\left(t; x, \alpha \right)$ if $t > t_y^*\left(\alpha \right) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O}$. Using the assumption (**Ac**) on $\left[t_y^*\left(\alpha \right), t_y^*\left(\alpha \right) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O} \right]$, one gets

$$\left| \int_0^T e^{-\delta t} l\left(y_\gamma\left(t; y, \mathcal{P}_{x,y}\left(\alpha \right) \right), \mathcal{P}_{x,y}\left(\alpha \right)\left(t \right) \right) dt - \int_0^T e^{-\delta t} l\left(y_\gamma\left(t; x, \alpha \right), \alpha\left(t \right) \right) dt \right|$$

$$\leq \int_0^{t_y^*(\alpha)} e^{-\delta t} Lip\left(l \right) \sqrt{|x - y|} dt + \int_{t_y^*(\alpha)}^{\left(t_y^*(\alpha) + t_{y_\gamma\left(t_y^*(\alpha); x, \alpha \right), O} \right) \wedge T} e^{-\delta t} Lip(l) \sqrt{|x - y|} dt$$

$$\leq \frac{1}{\delta} Lip(l) \sqrt{|x - y|}.$$

(c4) If $t_y^*\left(\alpha \right) < t_\varepsilon$ and $y_\gamma\left(t_y^*\left(\alpha \right); x, \alpha \right) = O$, then we proceed as in (a). We let

$$t_{y_\gamma\left(t_y^*(\alpha); y, \alpha \right), O} := \inf \left\{ t \geq 0 : y_\gamma\left(t; y_\gamma\left(t_y^*\left(\alpha \right); y, \alpha \right), a_{\gamma,1}^- \right) = O \right\}.$$

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Obviously, $t_{y_\gamma(t_y^*(\alpha);y,\alpha),O} \le \frac{\sqrt{|x-y|}^{1-\kappa}}{(1-\kappa)\beta}$. We set

$$\mathcal{P}_{x,y}(\alpha)(t) := \alpha(t)\,\mathbf{1}_{[0,t_y^*(\alpha))}(t) + a_{\gamma,1}^-\mathbf{1}_{\left[t_y^*(\alpha),t_y^*(\alpha)+t_{y_\gamma(t_y^*(\alpha);y,\alpha),O}\right]}(t)$$

$$+ \alpha\left(t - t_{y_\gamma(t_y^*(\alpha);y,\alpha),O}\right)\mathbf{1}_{\left(t_y^*(\alpha)+t_{y_\gamma(t_y^*(\alpha);y,\alpha),O},\infty\right)}(t),$$

for all $t \ge 0$ and the estimates follow. The measurability properties hold as before.

(d) Finally, we assume that $y = e_1$. Again, we only modify α if $A_{\gamma,e_1} \ne A^{\gamma,1}$. In this eventuality, we define $t_{e_1,x} := \inf\{t \ge 0 : y_\gamma(t; y, a_{\gamma,1}) = x\}$, where $a_{\gamma,1}$ appears in $(\mathbf{Aa})$. Then $t_{e_1,x} \le \frac{|x-y|}{\beta}$. We let

$$\mathcal{P}_{x,e_1}(\alpha)(t) := a_{\gamma,1}\mathbf{1}_{[0,t_{e_1,x}]}(t) + \alpha(t - t_{e_1,x})\mathbf{1}_{(t_{e_1,x},\infty)}(t).$$

and get the conclusion.

The proof of our lemma is now complete. ∎

7.2 Proof of Lemma 23.

For any $y \in [O, (1+\varepsilon)e_i]$ (with $\gamma \in E_i^{active}$), we set

$$a_{\gamma,i}^{\mathrm{opt},+}(y) = \operatorname*{argmax}_{a \in A_{\gamma,y}}\langle f_\gamma(y,a), e_i\rangle.$$

It is clear that

$$(20) \quad \begin{aligned} \left\langle f_\gamma(y',a) - f_\gamma\left(y, a_{\gamma,i}^{\mathrm{opt},+}(y)\right), e_i\right\rangle &\le \sup_{a' \in A_{\gamma,e_i}} |f_\gamma(y',a') - f_\gamma(y,a')| \le Lip(f)\,|y'-y|, \\ \left\langle f_\gamma(y,a) - f_\gamma\left(y, a_{\gamma,i}^{\mathrm{opt},+}(y)\right), e_i\right\rangle &\le 0, \end{aligned}$$

for all $y, y' \in [O, (1+\varepsilon)e_i]$. We also let

$$d_{geo}(x,y) := \left\{ \begin{array}{ll} |x - y|, & \text{if } x, y \in [-1-\varepsilon, 1+\varepsilon]\,e_i, \\ |x| + |y|, & \text{if } x \in [-1-\varepsilon, 1+\varepsilon]\,e_i, y \in [-1-\varepsilon, 1+\varepsilon]\,e_j,\ i \ne j \end{array} \right. .$$

Proof of Lemma 23.. We will prove only the estimates on the trajectory. The estimates on the partial cost follow from the construction $\mathcal{P}_x(\alpha)$ which coincides with α except at the end points (where $(\mathbf{C})$ applies; see also the similar condition (Ac) and the proof of Lemma 6). The assertion (ii) follows similar patterns to Lemma 6.

We aim at constructing $\tilde{\alpha} := \mathcal{P}_x(\alpha)$. We let $r_0 \le \varepsilon$ (to be specified later on). We can assume, without loss of generality, that $x \ne O$. (Should this not be the case, see Case 3). Then α is locally admissible. We set

$$\tau_0 := \inf\left\{t \ge 0 \ : \ d_{geo}\left(y_\gamma(t; x, \alpha), y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha})\right) \ge r_0\right\}.$$

If $\tau_0 \ge t_\varepsilon$, the conclusion follows. Otherwise, the time where y_γ meets again our target y^{ρ_ε} will be referred to as "renewal time". We give the construction of $\tilde{\alpha}$ on $[\tau_0, t_\varepsilon]$ prior to renewal time. We let τ_O^ε be the exit time of the target from the branch,

$$\tau_O^\varepsilon = \inf\left\{t \ge \tau_0 \ : \ y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha}) = O\right\}.$$

(Hence, $\tau_O^\varepsilon > \tau_0$). Let us assume that $\tilde{\alpha}$ has been constructed up to some time $\tau_0 \le t^* \le \tau_O^\varepsilon$ before the renewal time such that

$$(R) \quad\quad\quad\quad d_{geo}\left(y_\gamma^*, y_\gamma^{\rho_\varepsilon,*}\right) \le \omega_\varepsilon(t^*, r_0),$$

where we used the notation $y_\gamma^* = y_\gamma\left(t^*; x, \widetilde{\alpha}\right)$ and $y_\gamma^{\rho_\varepsilon,*} = y_\gamma^{\rho_\varepsilon}\left(t^*; x, \overline{\alpha}\right)$. Even if this is not crucial for the rest of the proof, remark that renewal cannot occur before $\tau_0 + \frac{r_0}{2|f|_0}$, so that this iterative procedure will be applied only a finite number of times.

Case 1: y_γ and $y_\gamma^{\rho_\varepsilon}$ are on the same branch (say $[O, (1+\varepsilon)e_1]$; the case when y_γ and $y_\gamma^{\rho_\varepsilon}$ are on a "new" branch $[O, -\varepsilon e_1]$ is similar), and y_γ lies between the junction O and $y_\gamma^{\rho_\varepsilon}$ (i.e. $0 \leq \langle y_\gamma^*, e_1 \rangle < \langle y_\gamma^{\rho_\varepsilon,*}, e_1 \rangle$). We let

$$t_{out} = \inf\left\{ t \geq 0 : y_\gamma\left(t; y_\gamma^*, \alpha(t^* + \cdot)\right) = (1+\varepsilon)\, e_1 \right\}, \quad t_{out}^{\rho_\varepsilon} = \inf\left\{ t \geq 0 : y_\gamma^{\rho_\varepsilon}\left(t; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}(t^* + \cdot)\right) = (1+\varepsilon)\, e_1 \right\},$$
$$t_0^{\rho_\varepsilon} = \inf\left\{ t \geq 0 : y_\gamma^{\rho_\varepsilon}\left(t; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}(t^* + \cdot)\right) = O \right\}, \quad t_0 = \inf\left\{ t \geq 0 : y_\gamma\left(t; y_\gamma^*, \alpha(t^* + \cdot)\right) = O \right\}.$$

Let us introduce $t_{act} = \min\left(t_{out}, t_{out}^{\rho_\varepsilon}, t_0, t_0^{\rho_\varepsilon}\right)$. Obviously, prior to the renewal time, only t_0 is relevant (since $t_{out}, t_0^{\rho_\varepsilon}$ cannot occur without renewal and if $t_{out}^{\rho_\varepsilon} < t_0$, then α is still locally admissible for the follower y_γ). We distinguish between the cases

(a1) If $t_{act} > 0$, we extend $\tilde{\alpha}$ by setting $\tilde{\alpha}(t) = \alpha(t)$, if $t^* < t \leq t^* + t_{act}$. Gronwall's inequality yields
$$\left| y_\gamma\left(t; x, \tilde{\alpha}\right) - y_\gamma^{\rho_\varepsilon}\left(t; x, \overline{\alpha}\right) \right| \leq \omega_\varepsilon\left(t - t^*; |y_\gamma^* - y_\gamma^{\rho_\varepsilon,*}|\right),$$
for all $t^* < t \leq t^* + t_{act}$.

(a2) If $t_{act} = t_0 = 0$, then we necessarily have that $t_0^{\rho_\varepsilon} > 0$. In this case $y_\gamma^* = O$ and $\langle y_\gamma^{\rho_\varepsilon}\left(t^*; x, \overline{\alpha}\right), e_1 \rangle > 0$.

(a2.1) The active case (by far the most complicated) $\gamma \in E_1^{active}$. In order to simplify our notations, denote, in this case, $a_{\gamma,O}^+ = a_{\gamma,1}^{\text{opt},+}(O)$. We introduce

$$t_{control} = \inf\{ t > 0 \ : \ y_\gamma\left(t; , y_\gamma^*, a_{\gamma,O}^+\right) = r_\varepsilon' e_1 \}$$
$$t_{collision} = \inf\{ t > 0 \ : \ y_\gamma\left(t; , y_\gamma^*, a_{\gamma,O}^+\right) = y_\gamma^{\rho_\varepsilon}\left(t^* + t; , y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}(t^* + \cdot)\right) \}$$

Note that because of the continuity of the trajectories and since $r_\varepsilon' > 0$, we have $t_{control} > 0$ and $t_{collision} > 0$. We extend naturally $\tilde{\alpha}$ by setting

$$\tilde{\alpha}\left(t + t^*\right) = a_{\gamma,O}^+, \text{ if } t \in \left(0, t_{collision} \wedge t_{control}\right].$$

With this extension, our assumptions guarantee that $\langle y_\gamma\left(t + t^*; x, \tilde{\alpha}\right), e_1 \rangle \geq Lip(f)\beta > 0$ and the junction O is now a reflecting barrier for $t \mapsto y_\gamma(t; y_\gamma^*, a_{\gamma,O}^+)$. Note also that for any $t \leq t_{collision} \wedge t_{control}$, we have $\langle y_\gamma^{\rho_\varepsilon}\left(t + t^*; x, \overline{\alpha}\right), e_1 \rangle > 0$. For every $0 < t \leq t_{collision} \wedge t_{control}$, one uses (20) to get

$$\left| y_\gamma^{\rho_\varepsilon}\left(t + t^*; x, \overline{\alpha}\right) - y_\gamma\left(t + t^*; x, \tilde{\alpha}\right) \right| = \left\langle y_\gamma^{\rho_\varepsilon}\left(t; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}\left(t^* + \cdot\right)\right) - y_\gamma\left(t; y_\gamma^*, a_{\gamma,O}^+\right), e_1 \right\rangle$$

$$= \langle (y_\gamma^{\rho_\varepsilon,*} - y_\gamma^*), e_1 \rangle + \int_0^t \left\langle f_\gamma^{\rho_\varepsilon}\left(y_\gamma^{\rho_\varepsilon}\left(s; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}\left(t^* + \cdot\right)\right), \overline{\alpha}\left(t^* + \cdot\right)\right) - f_\gamma\left(y_\gamma\left(s; y_\gamma^*, a_{\gamma,O}^+\right), a_{\gamma,O}^+\right), e_1 \right\rangle ds$$

$$\leq \langle (y_\gamma^{\rho_\varepsilon,*} - y_\gamma^*), e_1 \rangle + \int_0^t Lip\left(f\right)\left(\rho_\varepsilon + \left| y_\gamma^{\rho_\varepsilon}\left(s + t^*; x, \overline{\alpha}\right) - y_\gamma\left(s + t^*; x, \tilde{\alpha}\right) \right|\right) ds$$

$$+ \int_0^t \left[\left\langle \begin{array}{c} f_\gamma\left(y_\gamma\left(s; y_\gamma^*, a_{\gamma,O}^+\right), \alpha\left(t^* + \cdot\right)\right) - f_\gamma\left(O, a_{\gamma,O}^+\right) \\ + f_\gamma\left(O, a_{\gamma,O}^+\right) - f_\gamma\left(y_\gamma\left(s; y_\gamma^*, a_{\gamma,O}^+\right), a_{\gamma,O}^+\right) \end{array}, e_1 \right\rangle \right] ds$$

$$\leq \left| y_\gamma^{\rho_\varepsilon,*} - y_\gamma^* \right| + Lip\left(f\right)\left[\left(\rho_\varepsilon + 2r_\varepsilon'\right) t + \int_0^t \left| y_\gamma^{\rho_\varepsilon}\left(s + t^*; x, \overline{\alpha}\right) - y_\gamma\left(s + t^*; x, \tilde{\alpha}\right) \right| ds \right].$$

Using Gronwall's inequality and our assumptions on r_ε', we deduce that for any $0 < t \leq t_{control} \wedge t_{collision}$,
$$\left| y_\gamma\left(t + t^*; x, \tilde{\alpha}\right) - y_\gamma^{\rho_\varepsilon}\left(t + t^*; x, \overline{\alpha}\right) \right| \leq \omega_\varepsilon\left(t; \left| y_\gamma^{\rho_\varepsilon,*} - y_\gamma^* \right|\right).$$

Thus, we have constructed an extension of $t \mapsto \tilde{\alpha}(t)$ satisfying (R) during an increment of some strictly positive time $t_{control} \wedge t_{collision}$.

(a2.2) In the inactive case, it suffices to continue with the control α (since, in this case, $f_\gamma(O, a) = 0$, for all $a \in A^{\gamma,1}$) up till $t_{collision}$ (or t_ε).

Case 2 : We use the same notations as in the first case and aim at giving the control when $\tilde{\alpha}$ has been constructed up to some time $\tau_0 \leq t^* \leq \tau_O^\varepsilon$ such that renewal does not occur at t^* and both motions are at time t^* on the same active branch (say $[O, (1+\varepsilon)e_1]$). Contrary to Case 1, in this case we are assuming that $0 < \langle y_\gamma^{\rho_\varepsilon,*}, e_1 \rangle < \langle y_\gamma^*, e_1 \rangle$. We distinguish the following cases

(b1) If $t_{act} > 0$. In this case we proceed exactly as in case (a1) and get the same conclusion.

(b2) If $t_{act} = t_{out} = 0$ then $y_\gamma^* = (1+\varepsilon)e_1$ and we have $t_{out}^{\rho_\varepsilon} > 0$. This case is completely symmetric to case (a2.1) but with motions starting at t^* near $(1+\varepsilon)e_1$. The conclusion is similar.

(The case when $y_\gamma^* = -\varepsilon e_1$ is similar to (a2.1) if $\gamma \in E_1^{active}$ and to (a2.2) in the inactive case.)

Case 3 : control when $y_\gamma^{\rho_\varepsilon}(t^*; x, \overline{\alpha}) \in [O, (1+\varepsilon)e_j]$ and $y_\gamma(t^*; x, \alpha^*) \in [O, (1+\varepsilon)e_i]$ with $i \neq j$. In particular, the two points may be at the intersection or the target is at the intersection and the follower is not. We can assume, without loss of generality, that $\gamma \in E_j^{active}$. (Otherwise, recalling that we start at the same initial point, this situation can only happen if $y_\gamma^{\rho_\varepsilon,*} = O$ and no active branch exists. Then, whatever the control, y_γ can only get closer to O.) In this case, we introduce

$$\hat{t}_O = \inf\{t > 0 \ : \ y_\gamma\left(t; y_\gamma^*, a_{\gamma,i}^-\right) = O\}$$

$$\hat{t}_{collision} = \inf\{t > 0 \ : \ y_\gamma\left(t; y_\gamma^*, a_{\gamma,i}^-\right) = y_\gamma^{\rho_\varepsilon}(t^* + t; x, \overline{\alpha})\}$$

and we extend $t \mapsto \tilde{\alpha}(t)$ up to time $t^* + \hat{t}_O \wedge \hat{t}_{collision}$ by setting

$$\tilde{\alpha}(t) = a_{\gamma,i}^-, \text{ for } t^* < t < t^* + \hat{t}_O \wedge \hat{t}_{collision}.$$

Since by assumption $d_{geo}\left(y_\gamma^*, y_\gamma^{\rho_\varepsilon,*}\right) \leq \omega_\varepsilon(t^*; r_0)$, we have that

$$0 < \hat{t}_O \wedge \hat{t}_{collision} \leq \frac{(\omega_\varepsilon(t^*; r_0))^{1-\kappa}}{(1-\kappa)\beta}.$$

Hence, with such a construction we have that

$$d_{geo}\left(y_\gamma\left(t; y_\gamma^*, \tilde{\alpha}\right), y_\gamma^{\rho_\varepsilon}\left(t; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}\right)\right)| \leq \left(\frac{|f|_0}{(1-\kappa)\beta} + 1\right)(\omega_\varepsilon(t^*; r_0))^{1-\kappa},$$

for all $t < \hat{t}_O \wedge \hat{t}_{collision}$. If $\hat{t}_O = \hat{t}_O \wedge \hat{t}_{collision}$, we arrive at $y_\gamma\left(\hat{t}_O; y_\gamma^*, \tilde{\alpha}\right) = O$. If every road is inactive, we continue to stay at O.

(c1) If $y_\gamma^{\rho_\varepsilon}\left(\hat{t}_O; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}\right) \neq O$ we are back to case 1 but with r_0 now replaced by r_0' lower than $\left(\frac{|f|_0}{(1-\kappa)\beta} + 1\right)(\omega_\varepsilon(t^*; r_0))^{1-\kappa}$: even if there has been a deterioration of the distance between y_γ and y^{ρ_ε} (not exceeding $\left(\frac{|f|_0}{(1-\kappa)\beta} + 1\right)(\omega_\varepsilon(t^*; r_0))^{1-\kappa}$ because we are back to case 1, the situation of case 3 (and also the situation of (b2)) will never happen before some renewal time occurs. Consequently, in the situation of case 3 we are always allowed to take in (R) the same value for r_0 (and we choose $r_0 = r_\varepsilon'$).

(c2) Finally, we assume $y_\gamma^{\rho_\varepsilon}\left(\hat{t}_O; y_\gamma^{\rho_\varepsilon,*}, \overline{\alpha}\right) = O$. If every road is inactive, then y_γ stays at O and $y_\gamma^{\rho_\varepsilon}$ cannot go further than ρ_ε. Otherwise, let us assume that some j' is active. Then, we take $\tilde{\alpha}(t) = a_{\gamma,j'}^+$ for some very small (yet strictly positive) time $t^* + \hat{t}_O < t \leq t^* + \hat{t}_O + \frac{r_\varepsilon'}{2|f|_0}$ and get

$$d_{geo}\left(y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha}), y_\gamma(t; x, \tilde{\alpha})\right) \leq r_\varepsilon',$$

which allows one to iterate.

Conclusion Gathering all these results together, the constructed strategy $\tilde{\alpha}$ is such that

$$\left|y_\gamma(t; x, \tilde{\alpha}) - y_\gamma^{\rho_\varepsilon}(t; x, \overline{\alpha})\right| \leq \omega_\varepsilon(t_\varepsilon; \Phi(\varepsilon)).$$

for any $t \leq t_\varepsilon$ and the lemma is proved. ∎

7.3 Some hints on the proof of Lemma 24

The reader is invited to note that, if (C) holds true, then $l(y,a) = l\left(\Pi_{\overline{\mathcal{G}}}(y), a\right)$, for all $y \in \overline{\mathcal{G}}^{+,\varepsilon}$. Hence, the same kind of cost can be reached by :

- hurrying to O when the target is at O, then wait for collision by

- staying at O when the target enters a fictive road from the intersection if a control a such that $f(O,a) = 0$ exists (for example, in the inactive case).

- or mimic staying at O by making very small trips (see case (c2) of the previous Lemma);

- at e_1 :

- if $\langle f(e_1,a), e_1 \rangle \leq 0$, for all a, we are done, since the target will never enter $(1, 1+\varepsilon] e_1$ (recall we start from $\overline{\mathcal{G}}$).

- otherwise, there exists $\langle f(e_1, \widetilde{a}), e_1 \rangle > \beta' > 0$ and, by our assumption, we also have $\langle f(e_1, a_{\gamma,1}), e_1 \rangle < -\beta$. Then, again, we mimic staying at e_1 by making very small trips until collision.

The same kind of assertion are valid for λ and Q (notice the definition of these terms on "fictive" roads). The trajectories around O are close due to the ε distance from $\overline{\mathcal{G}}^{+,\varepsilon}$ to $\overline{\mathcal{G}}$ and as in the previous argument, coming around the intersection can only occur once before collision.

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