Higher order approximations via Stein's method

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HIGHER ORDER EXPANSIONS VIA STEIN METHOD

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Abstract. This paper is a sequel of [3]. We show how to establish a functional Edgeworth expansion of any order thanks to the Stein method. We apply the procedure to the Brownian approximation of compensated Poisson process and to the linear interpolation of the Brownian motion. It is then apparent that these two expansions are of rather different form.

1. Introduction

For $(\mu_n,\, n \geq 1)$, a sequence of probability measures which satisfies a central limit theorem, i.e. μ_n converges weakly to a Gaussian measure, it may be natural to ponder how this limit could be refined. That means, can we find an alternative distribution μ so that the speed of convergence of μ_n towards μ is faster than the convergence of μ_n to the Gaussian measure of the CLT ? For instance, for a sequence of i.i.d. centered random variables $(X_n,\, n \geq 1)$ with unit variance, if we consider $S_n = n^{-1/2}\sum_{j=1}^n X_j$, a Taylor expansion of the characteristic function of S_n yields the expansion:

$$\mathbb{E}\left[e^{itS_n}\right] = e^{-t^2/2}\left[1 + \frac{(it)^3\gamma}{6\sqrt{n}} + \frac{(it)^4(\tau-3)}{24n} + \frac{(it)^6\gamma^2}{72n}\right] + o\left(\frac{1}{n}\right).$$

where $\gamma = \mathbb{E}\left[X_1^3\right]$ and $\tau = \mathbb{E}\left[X_1^4\right]$. This can be interpreted as the distribution of S_n to be close to the measure with density g_n given by

$$g_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\left(1 + \frac{\gamma}{6\sqrt{n}}\,\mathcal{H}_3(x) + \frac{(\tau-3)}{24n}\,\mathcal{H}_4(x) + \frac{\gamma^2}{72n}\,\mathcal{H}_6(x)\right),$$

where $\mathcal{H}_n$ is the n-th Hermite polynomial. As the comparison of the characteristic functions of two probability measures does not give easily quantitative estimates regarding probability of events, moments and so on; it is necessary to investigate alternative distances between distribution of random variables.

One of the most natural distance is the so-called Kolmogorov distance defined, for measures supported on $\mathbf{R}$, by

$$\mathrm{d}_{\mathrm{Kol}}(\mu,\,\nu) = \sup_{x\in\mathbf{R}}\Big|\mu(-\infty,x] - \nu(-\infty,x]\Big|,$$

or more generally on $\mathbf{R}^k$,

$$\mathrm{d}_{\mathrm{Kol}}(\mu,\,\nu) = \sup_{(x_1,\cdots,x_k)\in\mathbf{R}^k}\Big|\mu(\times_{j=1}^k(-\infty,x_j]) - \nu(\times_{j=1}^k(-\infty,x_j])\Big|.$$

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This definition is hardly usable for probability measures on more abstract spaces like Hilbert spaces, space of continuous functions, etc. On the contrary, the Rubinstein distance can be defined in great generality. Assume that μ and ν are two probability measures on a metric space (X, d). The space of d-Lipschitz functions is the set of functions f such that there exists $c > 0$, depending only on f, satisfying

$$|f(x) - f(y)| \leq c\, d(x, y), \text{ for all } x, y \in X.$$

We denote by Lip_1 the set of d-Lipschitz functions for which c can be taken equal to 1. Then, the Rubinstein distance is defined by

$$d_R(\mu, \nu) = \sup_{F \in \mathrm{Lip}_1} \int_X F \,\mathrm{d}\mu - \int_X F \,\mathrm{d}\nu.$$

It is well known (see [4]) that if (X, d) is separable, $(\mu_n, n \geq 1)$ converges weakly to μ if and only if $d_R(\mu_n, \mu)$ tends to 0 as n goes to infinity. Moreover, according to [2],

$$d_{\mathrm{Kol}}(\mu, \nu) \leq 2\sqrt{d_R(\mu, \nu)}. \tag{1}$$

The Stein method, which dates back to the seventies, is one approach to evaluate such distances. Since [1, 5], it is well known that Stein method can also lead to expansions of higher order by pursuing the development. In a previous paper [3], we proved quantitative versions of some well known theorems: the Donsker Theorem, convergence of Poisson processes of increasing intensity towards a Brownian motion and approximation of a Brownian motion by increasingly refined linear interpolations. We now want to show that the same framework can be used to derive higher order expansions, even in functional spaces.

Before going deeply into technicalities, let us just show how this works on a simple 1 dimensional example. Imagine that we want to precise the speed of convergence of the well-known limit in distribution:

$$\frac{1}{\sqrt{\lambda}}(X_\lambda - \lambda) \xrightarrow{\lambda \to \infty} \mathcal{N}(0, 1),$$

where X_λ is a Poisson random variable of parameter λ. We consider the Rubinstein distance between the distribution of $\tilde{X}_\lambda = \lambda^{-1/2}(X_\lambda - \lambda)$ and $\mathcal{N}(0, 1)$, which is defined as

$$\mathrm{d}_(\tilde{X}_\lambda, \mathcal{N}(0, 1)) = \sup_{F \in \mathrm{Lip}_1} \mathbb{E}\left[F(\tilde{X}_\lambda)\right] - \mathbb{E}\left[F(\mathcal{N}(0, 1))\right]. \tag{2}$$

The well known Stein Lemma stands that for any $F \in \mathrm{Lip}_1$, there exists $\psi_F \in \mathcal{C}_b^2$ such that for all $x \in \mathbf{R}$,

$$F(x) - \mathbb{E}\left[F(\mathcal{N}(0, 1))\right] = x\,\psi_F(x) - \psi_F'(x).$$

$$\|\psi_F'\|_\infty \leq 1, \ \|\psi_F''\|_\infty \leq 2.$$

Hence, instead of the right-hand-side of (2), we are lead to estimate

$$\sup_{\|\psi'\|_\infty \leq 1, \ \|\psi''\|_\infty \leq 2} \mathbb{E}\left[\tilde{X}_\lambda \psi(\tilde{X}_\lambda) - \psi'(\tilde{X}_\lambda)\right]. \tag{3}$$

This is where the Malliavin-Stein approach differs from the classical line of thought. In order to transform the last expression, instead of constructing a coupling, we

resort to the integration by parts formula for functionals of Poisson random variable. The next formula is well known or can be viewed as a consequence of (12):

$$\mathbb{E}\left[\tilde{X}_\lambda G(\tilde{X}_\lambda)\right] = \sqrt{\lambda}\,\mathbb{E}\left[G(\tilde{X}_\lambda + 1/\sqrt{\lambda}) - G(\tilde{X}_\lambda)\right].$$

Hence, (3) is transformed into

$$\sup_{\|\psi'\|_\infty \le 1,\ \|\psi''\|_\infty \le 2} \mathbb{E}\left[\sqrt{\lambda}(\psi(\tilde{X}_\lambda + 1/\sqrt{\lambda}) - \psi(\tilde{X}_\lambda)) - \psi'(\tilde{X}_\lambda)\right]. \tag{4}$$

According to the Taylor formula

$$\psi(\tilde{X}_\lambda + 1/\sqrt{\lambda}) - \psi(\tilde{X}_\lambda) = \frac{1}{\sqrt{\lambda}}\psi'(\tilde{X}_\lambda) + \frac{1}{2\lambda}\psi''(\tilde{X}_\lambda + \theta/\sqrt{\lambda}),$$

where $\theta \in (0,1)$. If we plug this expansion into (4), the term containing ψ' is miraculously vanishing and we are left with only the second order term. This leads to the estimate

$$\mathrm{d}_R\left(\tilde{X}_\lambda,\ \mathcal{N}(0,\,1)\right) \le \frac{1}{\sqrt{\lambda}}.$$

We now want to precise the expansion. For, we go one step further in the Taylor formula (assuming ψ has enough regularity)

$$\psi(\tilde{X}_\lambda + 1/\sqrt{\lambda}) - \psi(\tilde{X}_\lambda) = \frac{1}{\sqrt{\lambda}}\psi'(\tilde{X}_\lambda) + \frac{1}{2\lambda}\psi''(\tilde{X}_\lambda) + \frac{1}{6\lambda^{3/2}}\psi^{(3)}(\tilde{X} + \theta/\sqrt{\lambda}).$$

Hence,

$$\mathbb{E}\left[\tilde{X}_\lambda\psi(\tilde{X}_\lambda) - \psi'(\tilde{X}_\lambda)\right] = \frac{1}{2\sqrt{\lambda}}\mathbb{E}\left[\psi''(\tilde{X}_\lambda)\right] + \frac{1}{6\lambda}\mathbb{E}\left[\psi^{(3)}(\tilde{X} + \theta/\sqrt{\lambda})\right]. \tag{5}$$

If F is twice differentiable with bounded derivatives then ψ_F is three time differentiable with bounded derivatives, hence the last term of (5) is bounded by $\lambda^{-1}\|\psi_F^{(3)}\|_\infty/6$. Moreover, the first part of the reasoning shows that

$$\mathbb{E}\left[\psi_F''(\tilde{X}_\lambda)\right] = \mathbb{E}\left[\psi_F''(\mathcal{N}(0,\,1))\right] + O(\lambda^{-1/2}).$$

Combining the last two results, we obtain that for F twice differentiable

$$\begin{aligned}\mathbb{E}\left[F(\tilde{X}_\lambda)\right] - \mathbb{E}\left[F(\mathcal{N}(0,\,1))\right] &= \mathbb{E}\left[\tilde{X}_\lambda\psi_F(\tilde{X}_\lambda) - \psi_F'(\tilde{X}_\lambda)\right] \\ &= \frac{1}{2\sqrt{\lambda}}\mathbb{E}\left[\psi_F''(\mathcal{N}(0,\,1))\right] + O(\lambda^{-1}).\end{aligned}$$

This line of thought can be pursued at any order provided that F is assumed to have sufficient regularity and we get an Edgeworth expansion up to any power of $\lambda^{-1/2}$. Using the properties of Hermite polynomials, this leads to the expansion:

$$\mathbb{E}\left[F(\tilde{X}_\lambda)\right] - \mathbb{E}\left[F(\mathcal{N}(0,\,1))\right] = \frac{1}{6\sqrt{\lambda}}\mathbb{E}\left[(F\mathcal{H}_3)(\mathcal{N}(0,\,1))\right] + O(\lambda^{-1}).$$

The paper is organized as follows. In Section 2, we recall the functional structure on which the computations are made. In Section 3, we establish the Edgeworth expansion for the Poisson approximation of the Brownian motion. In Section 4, we apply the same procedure to derive an Edgeworth expansion for the linear approximation of the Brownian motion, which turns to be of a very different flavor. In [3], we computed the first order term in the Donsker Theorem, we could as well pursue the expansion. It would be a mixture of the two previous kinds of expansion.

2. Gaussian structure on l^2

2.1. **Wiener measure.** For the three examples mentioned above, we seek to compare quantitatively the distribution of a piecewise differentiable process with that of a Brownian motion, hence we need to consider a functional space to which the sample-paths of both processes belong to. It has been established in [3] that a convenient space is the space of β-differentiable functions for any $\beta < 1/2$, which we describe now. We refer to [8] for details on fractional calculus. For $f \in \mathcal{L}^2([0,1];\, dt)$, (denoted by $\mathcal{L}^2$ for short) the left and right fractional integrals of f are defined by :

$$(I^{\alpha}_{0+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1}\, \mathrm{d}t\ ,\ x \geq 0,$$
$$(I^{\alpha}_{1-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 f(t)(t-x)^{\alpha-1}\, \mathrm{d}t\ ,\ x \leq 1,$$

where $\alpha > 0$ and $I^0_{0+} = I^0_{1-} = \mathrm{Id}$. For any $\alpha \geq 0$, any f, $g \in \mathcal{L}^2$ and $g \in \mathcal{L}^2$, we have :

$$\int_0^1 f(s)(I^{\alpha}_{0+} g)(s)\, \mathrm{d}s = \int_0^1 (I^{\alpha}_{1-} f)(s) g(s)\, \mathrm{d}s. \tag{6}$$

The Besov-Liouville space $I^{\alpha}_{0+}(\mathcal{L}^2) := \mathcal{I}^{+}_{\alpha,2}$ is usually equipped with the norm :

$$\|I^{\alpha}_{0+} f\|_{\mathcal{I}^{+}_{\alpha,2}} = \|f\|_{\mathcal{L}^2}. \tag{7}$$

Analogously, the Besov-Liouville space $I^{\alpha}_{1-}(\mathcal{L}^2) := \mathcal{I}^{-}_{\alpha,2}$ is usually equipped with the norm :

$$\|I^{\alpha}_{1-} f\|_{\mathcal{I}^{-}_{\alpha,2}} = \|f\|_{\mathcal{L}^2}.$$

Both spaces are Hilbert spaces included in $\mathcal{L}^2$ and if $(e_n,\, n \in \mathbf{N})$ denote a complete orthonormal basis of $\mathcal{L}^2$, then $(k^{\alpha}_n := I^{\alpha}_{1-} e_n,\, n \in \mathbf{N})$ is a complete orthonormal basis of $\mathcal{I}^{-}_{\alpha,2}$. Moreover, we have the following Theorem, proved in [3].

Theorem 2.1. *The canonical embedding κ_α from $\mathcal{I}^{-}_{\alpha,2}$ into $\mathcal{L}^2$ is Hilbert-Schmidt if and only if $\alpha > 1/2$. Moreover,*

$$c_\alpha := \|\kappa_\alpha\|_{HS} = \|I^{\alpha}_{0+}\|_{HS} = \|I^{\alpha}_{1-}\|_{HS} = \frac{1}{2\Gamma(\alpha)} \left(\frac{1}{\alpha(\alpha - 1/2)} \right)^{1/2}. \tag{8}$$

To construct the Wiener measure on $\mathcal{I}_{\beta,\,2}$, we start from the Itô-Nisio theorem. Let $(X_n,\, n \geq 1)$ be a sequence of independent centered Gaussian random variables of unit variance defined on a common probability space $(\Omega,\, \mathcal{A},\, \mathrm{P})$. Then,

$$B(t) := \sum_{n \geq 1} X_n I^1_{0+}(e_n)(t)$$

converges almost-surely for any $t \in [0,\, 1]$. Moreover, the convergence holds in $L^2(\Omega;\, \mathcal{I}_{\beta,\,2})$, so that, μ_β, the Wiener measure on $\mathcal{I}_{\beta,2}$ is the image measure of P by the map B. Thus, μ_β is a Gaussian measure on $\mathcal{I}_{\beta,\,2}$ of covariance operator given by

$$V_\beta = I^{\beta}_{0+} \circ I^{1-\beta}_{0+} \circ I^{1-\beta}_{1-} \circ I^{-\beta}_{0+}.$$

This means that

$$\mathbb{E}_{\mu_\beta}\left[\exp(i\langle \eta,\, \omega\rangle_{\mathcal{I}_{\beta,\,2}})\right] = \exp(-\frac{1}{2}\langle V_\beta \eta,\, \eta\rangle_{\mathcal{I}_{\beta,\,2}}).$$

We could thus in principle make all the computations in $\mathcal{I}_{\beta,2}$. It turns out that we were not able to be explicit in the computations of some traces of some involved operators, the expressions of which turned to be rather straightforward in $l^2(\mathbf{N})$ (where $\mathbf{N}$ is the set of positive integers). This is why we transfer all the structure to $l^2(\mathbf{N})$, denoted henceforth by l^2 for short. This is done at no loss of generality nor precision since there exists a bijective isometry between $\mathcal{I}_{\beta,2}$ and l^2.

Actually, the canonical isometry is given by the Fourier expansion of the β-th *derivative* of an element of $\mathcal{I}_{\beta,2}$: for $f \in \mathcal{I}_{\beta,2}$, we denote by $\partial_\beta f$ the unique element of $\mathcal{L}^2$ such that $f = I^\beta_{0+}\partial_\beta f$. We denote by $(x_n,\, n \geq 1)$ a complete orthonormal basis of l^2. In what follows, we adopt the usual notations regarding scalar product on l^2:

$$\|x\|^2_{l^2} = \sum_{n=1}^{\infty} |x_n|^2 \text{ and } x.y = \sum_{n=1}^{\infty} x_n y_n, \text{ for all } x,\, y \in l^2.$$

For the sake of simplicity, we also denote by a dot the scalar product in $(l^2)^{\otimes k}$ for any integer k.

Consider the map $\mathfrak{J}_\beta$ defined by:

$$\begin{aligned}\mathfrak{J}_\beta \,:\, \mathcal{I}_{\beta,2} &\longrightarrow l^2\\ f &\longmapsto \sum_{n\geq 1}\left(\int_0^1 \partial_\beta f(s) e_n(s)\,\mathrm{d}s\right)\, x_n.\end{aligned}$$

According to the properties of Gaussian measure (see [6]), we have the following result.

Theorem 2.2. *Let μ_β denote the Wiener measure on $\mathcal{I}_{\beta,2}$. Then $\mathfrak{J}^*_\beta \mu_\beta = m_\beta$, where m_β is the Gaussian measure on l^2 of covariance operator given by*

$$S_\beta = \sum_{n,m\geq 1}\left(\int_0^1 k_n^{1-\beta}(s) k_m^{1-\beta}(s)\, ds\right)\, x_n \otimes x_m.$$

2.2. **Dirichlet structure.** By $\mathcal{C}^k_b(l^2;\, X)$, we denote the space of k-times Fréchet differentiable functions from l^2 into an Hilbert space X with bounded derivatives: A function F belongs to $\mathcal{C}^k_b(l^2;\, X)$ whenever

$$\|F\|_{\mathcal{C}^k_b(l^2;\, X)} := \sup_{j=1,\cdots,k}\, \sup_{x\in l^2} \|\nabla^{(j)} F(x)\|_{X\otimes (l^2)^{\otimes j}} < \infty.$$

Definition 1. *The Ornstein-Uhlenbeck semi-group on $(l^2,\, m_\beta)$ is defined for any $F \in L^2(l^2, \mathcal{F}, m_\beta;\, X)$ by*

$$P^\beta_t F(u) = \int_{l^2} F(e^{-t}u + \sqrt{1-e^{-2t}}\, v)\, dm_\beta(v),$$

where the integral is a Bochner integral.

The following properties are well known.

Lemma 2.1. *The semi-group P^β is ergodic in the sense that for any $u \in l^2$,*

$$P^\beta_t F(u) \xrightarrow{t\to\infty} \int f\, dm_\beta.$$

Moreover, if F belongs to $\mathcal{C}_b^k(l^2; X)$, then, $\nabla^{(k)}(P_t^\beta F) = \exp(-kt)P_t^\beta(\nabla^{(k)}F)$ so that we have

$$\int_0^\infty \sup_{u\in l^2} \|\nabla^{(k)}(P_t^\beta F)(u)\|_{(l^2)^{\otimes(k)}\otimes X}\, dt \leq \frac{1}{k}\|F\|_{\mathcal{C}_b^k(l^2;\, X)}.$$

We recall that for X an Hilbert space and A a linear continuous map from X into itself, A is said to be trace-class whenever the series $\sum_{n\geq 1}|(Af_n,\, f_n)_X|$ is convergent for one (hence any) complete orthonormal basis $(f_n,\, n\geq 1)$ of X. When A is trace-class, its trace is defined as $\mathrm{trace}(A) = \sum_{n\geq 1}(Af_n,\, f_n)_X$. For $x,\, y \in X$, the operator $x\otimes y$ can be seen either as an element of $X\otimes X$ or as a continuous map from X into itself via the identification : $x\otimes y(f) = (y,\, f)_X\, x$. It is thus straightforward that such an operator is trace-class and that $\mathrm{trace}(x\otimes y) = \sum_{n\geq 1}(y,\, f_n)_X(x,\, f_n)_X = (x,\, y)_X$ according to the Parseval formula. We also need to introduce the notion of partial trace. For any vector space X, $\mathrm{Lin}(X)$ is the set of linear operator from into itself. For X and Y two Hilbert spaces, the partial trace operator along X can be defined as follows: it is the unique linear operator

$$\mathrm{trace}_X \,:\, \mathrm{Lin}(X\otimes Y) \longrightarrow \mathrm{Lin}(Y)$$

such that for any $R\in\mathrm{Lin}(Y)$, for any trace class operator S on X,

$$\mathrm{trace}_X(S\otimes R) = \mathrm{trace}_X(S)\, R.$$

For Hilbert valued functions, we define A^β as follows.

Definition 2. *Let A^β denote the linear operator defined for $F\in\mathcal{C}_b^2(l^2;\, X)$ by:*

$$(A^\beta F)(u) = u.(\nabla F)(u) - \mathrm{trace}_{l^2}(S_\beta \nabla^2 F(u)),\ \textit{for all } u\in l^2.$$

We still denote by A^β the unique extension of A^β to its maximal domain.

The map A^β is the infinitesimal generator of P^β in the sense that for $F\in\mathcal{C}_b^2(l^2;\, X)$: for any $u\in l^2$,

$$P_t^\beta F(u) = F(u) + \int_0^t A^\beta P_s^\beta F(u)\,\mathrm{d}s. \tag{9}$$

As a consequence of the ergodicity of P^β and of (9), we have the Stein representation formula: For any sufficiently integrable function $F : l^2\to\mathbf{R}$,

$$\int_{l^2} F\,\mathrm{d}m_\beta - \int_{l^2} F\,\mathrm{d}\nu = \int_{l^2}\int_0^\infty A^\beta P_t^\beta F(x)\,\mathrm{d}t\,\mathrm{d}\nu(x). \tag{10}$$

3. Normal approximation of Poisson processes

Consider the process

$$N_\lambda(t) = \frac{1}{\sqrt{\lambda}}\,(N(t)-\lambda t) = \frac{1}{\sqrt{\lambda}}\left(\sum_{n\geq 1}\mathbf{1}_{[T_n,\,1]}(t) - \lambda t\right)$$

where $(T_n,\, n\geq 1)$ are the jump times of N, a Poisson process of intensity λ. It is well known that N_λ converges in distribution on $\mathfrak{D}$ (the space of cadlag functions) to a Brownian motion as λ goes to infinity. Since

$$\begin{aligned} I_{0^+}^\alpha\left((.-\tau)_+^{-\beta}\right) &= \Gamma(\alpha)^{-1}\int_\tau^t (s-\tau)^{-\beta}(t-s)^{\alpha-1}\,\mathrm{d}s \\ &= \Gamma(1-\beta)\ (t-\tau)_+^{\alpha-\beta}, \end{aligned}$$

we have $I^{\beta}_{0^+}((.-\tau)_+^{-\beta}) = \Gamma(1-\beta)\mathbf{1}_{[\tau,1]}(t)$. Hence, the sample-paths of N_λ belong to $\mathcal{I}_{\beta,2}$ for any $\beta < 1/2$. It also follows that

$$\mathfrak{J}_\beta N_\lambda = \sum_{n\geq 1} \frac{1}{\sqrt{\lambda}} \int_0^1 k_n^{1-\beta}(s)(\,\mathrm{d}N(s) - \lambda\,\mathrm{d}s)\; x_n,$$

where, for any integer n,

$$k_n^{1-\beta}(t) = \frac{1}{\Gamma(1-\beta)} \int_t^1 (s-t)^{-\beta} e_n(s)\,\mathrm{d}s.$$

For the sake of notations, we introduce

$$K_\lambda = \frac{1}{\sqrt{\lambda}} \sum_{n\geq 1}\; k_n^{1-\beta} \otimes x_n = \frac{1}{\sqrt{\lambda}}\, K_1.$$

The following theorem has been established in [3].

Theorem 3.1. *For any $\lambda > 0$, for any $F \in \mathcal{C}_b^3(l^2; \mathbf{R})$,*

$$\left| \mathbb{E}\left[F(N_\lambda)\right] - \int_{l^2} F\; dm_\beta \right| \leq \frac{1}{6\sqrt{\lambda}}\; c^3_{1-\beta}\; \|F\|_{\mathcal{C}_b^3(l^2;\,\mathbf{R})}. \tag{11}$$

The proofs uses a few basic notions of Malliavin calculus with respect to the Poisson process N which we recall rapidly now (for details, see [3, 7]). It is customary to define the discrete gradient as

$$D_\tau F(N) = F(N + \delta_\tau) - F(N), \text{ for any } \tau \in [0,\,1],$$

where $N + \delta_\tau$ is the point process N with an extra atom at time τ. We denote by $\mathbb{D}_{2,1}$ the set of square integrable functionals F such that $\mathbb{E}\left[\int_0^1 |D_\tau F(N)|^2\,\mathrm{d}\tau\right]$ is finite. We then have the following relationship:

$$\mathbb{E}\left[F(N)\int_0^1 g(\tau)(\,\mathrm{d}N(\tau) - \lambda\,\mathrm{d}\tau)\right] = \lambda\;\mathbb{E}\left[\int_0^1 D_\tau F(N)\; g(\tau)\,\mathrm{d}\tau\right], \tag{12}$$

for any $g \in \mathcal{L}^2([0,1])$ and any $F \in \mathbb{D}_{2,1}$. Moreover,

$$D_\tau\left(\int_0^1 g(s)(\,\mathrm{d}N(s) - \lambda\,\mathrm{d}s)\right) = g(\tau).$$

so that $D_\tau \mathfrak{J}_\beta N_\lambda = K_\lambda(\tau)$. In what follows, we make the convention that a sum like $\sum_{r=1}^0 \ldots$ is zero. For any integers $r \geq k \geq 1$, consider $\mathcal{T}_r^k$ the set of all k-tuples (ordered lists of length k) of integers $(a_1, \cdots, a_k)$ such that $\sum_{i=1}^k a_i = r$ and $a_i \geq 1$ for any $i \in \{1, \cdots, k\}$. We denote by () the empty list and for any r, $\mathcal{T}_r^0 = \{()\}$ and $\mathcal{T}_r = \cup_{j=1}^r \mathcal{T}_r^j$. For $a \in \mathcal{T}_r$, we denote by $|a|$ its length, i.e. the unique index j (necessarily less than r) such that $a \in \mathcal{T}_r^j$. For two tuples $a = (a_1, \cdots, a_k)$ and $b = (b_1, \cdots, b_n)$, their concatenation $a \oplus b$ is the $(k+n)$-tuple $(a_1, \cdots, a_n, b_1, \cdots, b_n)$. We define by induction the following constants :

$$\Xi_{()} = 1,\; \Xi_{(j)} = \frac{1}{(j+2)!} \text{ and } \Xi_{a\oplus(j)} = \frac{\Xi_a}{(j+1)!(r+j+2k+2)} \text{ for any } j \in \mathbf{N}, a \in \mathcal{T}_r^k.$$

For instance, we have

$$\Xi_{(1)} = \frac{1}{6},\; \Xi_{(2)} = \frac{1}{24},\; \Xi_{(1,1)} = \frac{1}{72}.$$

For any tuple $a = (a_1, \cdots, a_k) \in \mathcal{T}_r$, we set

$$\mathcal{K}^a = \bigotimes_{l=1}^{k} \int_0^1 K_1^{\otimes(a_l+2)}(\tau)\,\mathrm{d}\tau \in (l^2)^{\otimes(r+2|a|)}.$$

Consider also the sequence $(\xi_s,\, s \geq 0)$ given by the recursion formula:

$$\xi_s = \sum_{j=2}^{s+1} \frac{1}{(j+1)!}\, \frac{\xi_{s+1-j}}{3s+7-2j}\, c_{1-\beta}^{j+1} + \frac{\eta_{s+3}}{(s+3)!}, \tag{13}$$

and

$$\eta_s = \frac{2}{2+s(1-2\beta)}\, \frac{1}{(1-2\beta)^{s/2}\Gamma(1-\beta)^{s/2}}.$$

Theorem 3.2. *Let s be a non negative integer. We denote by ν_λ^* the distribution of $\mathfrak{J}_\beta N_\lambda$ on l^2. For $F \in \mathcal{C}_b^{3s+3}(l^2; \mathbf{R})$, we have*

$$\begin{aligned}\int_{l^2} F(u)\, d\nu_\lambda^*(u) &= \int_{l^2} F(u)\, dm_\beta(u) \\ &\quad + \sum_{r=1}^{s} \lambda^{-r/2} \sum_{a\in\mathcal{T}_r} \Xi_a \int_{l^2} \nabla^{(r+2|a|)} F(u).\mathcal{K}^a\, dm_\beta(u) + \mathrm{Rem}(s,\, F,\, \lambda)\end{aligned} \tag{14}$$

where the remainder term can be bounded as

$$|\mathrm{Rem}(s,\, F,\, \lambda)| \leq \xi_s\ \lambda^{-(s+1)/2} \|F\|_{\mathcal{C}_b^{3s+3}(l^2;\mathbf{R})}.$$

For $s = 0$, this means that

$$\int_{l^2} F(u)\, \mathrm{d}\nu_\lambda^*(u) = \int_{l^2} F(u)\, \mathrm{d}m_\beta(u) + \mathrm{Rem}(0,\, F,\, \lambda)$$

where the remainder is bounded by $\lambda^{-1/2} c_{1-\beta}^3\, \|F\|_{\mathcal{C}_b^3(l^2;\mathbf{R})}/6$. That is to say that it is the exact content of Theorem [3.1]. For $s = 1$, we obtain

$$\begin{aligned}\int_{l^2} F(u)\, \mathrm{d}\nu_\lambda^*(u) &= \int_{l^2} F(u)\, \mathrm{d}m_\beta(u) \\ &\quad + \frac{\lambda^{-1/2}}{6} \int_{l^2} \nabla^{(3)} F(u).\mathcal{K}^{(1)}\, \mathrm{d}m_\beta(u) + \mathrm{Rem}(1,\, F,\, \lambda)\end{aligned}$$

where $\mathrm{Rem}(1,\, F,\, \lambda) = O(\lambda^{-1})$ and for $s = 2$, we get

$$\begin{aligned}\int_{l^2} F(u)\, \mathrm{d}\nu_\lambda^*(u) &= \int_{l^2} F(u)\, \mathrm{d}m_\beta(u) + \frac{\lambda^{-1/2}}{6} \int_{l^2} \nabla^{(3)} F(u).\mathcal{K}^{(1)}\, \mathrm{d}m_\beta(u) \\ &\quad + \lambda^{-1} \left[\frac{1}{72} \int_{l^2} \nabla^{(6)} F(u).\mathcal{K}^{(1,1)}\, \mathrm{d}m_\beta(u) + \frac{1}{24} \int_{l^2} \nabla^{(4)} F(u).\mathcal{K}^{(2)}\, \mathrm{d}m_\beta(u) \right] \\ &\quad + \mathrm{Rem}(2,\, F,\, \lambda),\end{aligned}$$

with $\mathrm{Rem}(2,\, F,\, \lambda) = O(\lambda^{-3/2})$.

Proof. As said before, for $s = 0$, the proof reduces to that of Theorem [3.1]. Let $s \geq 1$ and assume that (14) holds up to rank $s-1$. Let $F \in \mathcal{C}_b^{3s+2}(l^2; \mathbf{R})$ and $x \in l^2$.

Denoting by $G(y) = F(y)x$ for $y \in l^2$, we have

$$\begin{aligned}\mathbb{E}\left[\mathfrak{J}_\beta N_\lambda . G(\mathfrak{J}_\beta N_\lambda)\right] &= \frac{1}{\sqrt{\lambda}} \sum_{n\geq 1} \mathbb{E}\left[\int_0^1 k_n^{1-\beta}(s)(\,\mathrm{d}N(s) - \lambda\,\mathrm{d}s)\; F(\mathfrak{J}_\beta N_\lambda)\right]\; x_n.x \\ &= \frac{1}{\sqrt{\lambda}} \sum_{n\geq 1} \mathbb{E}\left[\int_0^1 k_n^{1-\beta}(\tau) D_\tau F(\mathfrak{J}_\beta N_\lambda)\lambda\,\mathrm{d}\tau\right]\; x_n.x \\ &= \sqrt{\lambda}\,\mathbb{E}\left[\int_0^1 D_\tau F(\mathfrak{J}_\beta N_\lambda).K_1(\tau)\,\mathrm{d}\tau\right].\end{aligned}$$

According to the Taylor formula at order $s+1$,

$$\begin{aligned}&D_\tau F(\mathfrak{J}_\beta N_\lambda) = F(\mathfrak{J}_\beta N_\lambda + K_\lambda(\tau)) - F(\mathfrak{J}_\beta N_\lambda) \\ &= \sum_{j=1}^{s+1} \frac{\lambda^{-j/2}}{j!} \nabla^{(j)} F(\mathfrak{J}_\beta N_\lambda).K_1(\tau)^{\otimes j} \\ &\qquad + \frac{\lambda^{-(s+2)/2}}{(s+1)!} \int_0^1 (1-r)^{s+1}\; \nabla^{(s+2)} F(\mathfrak{J}_\beta N_\lambda + rK_\lambda(\tau)).K_1(\tau)^{\otimes(s+2)}\,\mathrm{d}r.\end{aligned}$$

Thus, we get

$$\begin{aligned}(15)\quad \mathbb{E}\left[\mathfrak{J}_\beta N_\lambda . G(\mathfrak{J}_\beta N_\lambda)\right] &= \sum_{j=1}^{s+1} \frac{\lambda^{-(j+1)/2}}{j!} \int_0^1 \mathbb{E}\left[\nabla^{(j)} F(\mathfrak{J}_\beta N_\lambda).K_1(\tau)^{\otimes(j+1)}\right]\,\mathrm{d}\tau \\ + \frac{\lambda^{-(s+1)/2}}{(s+1)!} \int_0^1 \int_0^1 (1-r)^{s+1}\;\mathbb{E}\left[\nabla^{(s+2)} F(\mathfrak{J}_\beta N_\lambda + rK_\lambda(\tau)).K_1(\tau)^{\otimes(s+3)}\right]\,\mathrm{d}r\,\mathrm{d}\tau.\end{aligned}$$

By linearity and density, (15) holds for any $G \in C_b^{3s+2}(l^2; l^2)$. According to the Stein representation formula (10), we get:

$$\begin{aligned}&\mathbb{E}\left[F(N_\lambda)\right] = \int_{l^2} F(u)\,\mathrm{d}m_\beta(u) \\ &\qquad + \sum_{j=2}^{s+1} \frac{\lambda^{-(j-1)/2}}{j!} \int_0^\infty \int_{l^2} \nabla^{(j+1)} P_t^\beta F(u).\Big(\int_0^1 K_1(\tau)^{\otimes(j+1)}\,\mathrm{d}\tau\Big)\,\mathrm{d}\nu_\lambda^*(u)\,\mathrm{d}t \\ &+ \frac{\lambda^{-(s+1)/2}}{(s+1)!} \times \\ &\int_{l^2} \int_0^\infty \int_0^1 \int_0^1 (1-\theta)^{s+1} \nabla^{(s+3)} P_t F(u + \theta K_1(\tau)).K_1(\tau)^{\otimes(s+3)}\,\mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}\nu_\lambda^*(u) \\ &\qquad\qquad = \int_{l^2} F(u)\,\mathrm{d}m_\beta(u) + A_1 + A_2.\end{aligned}$$

For any j, we apply the recursion hypothesis of rank $s+1-j$ to the functional

$$F_j : u \mapsto \int_0^\infty \nabla^{(j+1)} P_t^\beta F(u).\mathcal{K}^{(j-1)}\,\mathrm{d}t.$$

Thus, we have:

$$A_1 = \sum_{j=2}^{s+1} \frac{\lambda^{-(j-1)/2}}{j!} \sum_{r=0}^{s-j+1} \lambda^{-r/2} \sum_{a\in\mathcal{T}_r} \Xi_a \int_0^\infty \int_{l^2} \nabla^{(2|a|+r+j+1)} P_t^\beta F(u).\mathcal{K}^{a\oplus(j-1)} \,\mathrm{d}m_\beta(u)\,\mathrm{d}t + \sum_{j=2}^{s+1} \frac{\lambda^{-(j-1)/2}}{j!} \mathrm{Rem}(s+1-j,\, F_j,\, \lambda) = B_1 + B_2 \tag{16}$$

According to the commutation relationship between ∇ and P^β and since m_β is P_t^β-invariant, it follows that

$$B_1 = \sum_{j=2}^{s+1} \sum_{r=0}^{s-j+1} \frac{\lambda^{-(r+j-1)/2}}{j!} \times \sum_{a\in\mathcal{T}_r} \frac{\Xi_a}{2|a|+r+j+1} \int_{l^2} \nabla^{(2|a|+r+j+1)} F(u).\mathcal{K}^a \otimes \mathcal{K}^{(j-1)} \,\mathrm{d}m_\beta(u).$$

We now proceed to the change of variables $r \leftarrow r+j-1$, $j \leftarrow j-1$ so that we have

$$B_1 = \sum_{r=1}^{s} \lambda^{-r/2} \sum_{j=1}^{r} \times \sum_{a\in\mathcal{T}_{r-j}} \frac{\Xi_a}{(j+1)!(2|a|+r+2)} \int_{l^2} \nabla^{(2|a|+r+2)} F(u).\mathcal{K}^a \otimes \mathcal{K}^{(j)} \,\mathrm{d}m_\beta(u).$$

If a belongs to $\mathcal{T}_{r-j}$ then $b = a\oplus(j)$ belongs to $\mathcal{T}_r$ and $2|a|+r+2 = 2|b|+2$. In the reverse direction, for $b\in\mathcal{T}_r$, by construction, the last component (say on the right) of b belongs to $\{1,\cdots,r\}$. Let j denote the value of this component, it uniquely determines a such that $b = a\oplus(j)$ with $a\in\mathcal{T}_{r-j}$ thus $\cup_{j=1}^r \mathcal{T}_{r-j} = \mathcal{T}_r$ (where the union is a disjoint union) and B_1 can be written as

$$B_1 = \sum_{r=1}^{s} \lambda^{-r/2} \sum_{a\in\mathcal{T}_r} \Xi_a \int_{l^2} \nabla^{(r+2|a|)} F(u).\mathcal{K}^a \,\mathrm{d}m_\beta(u).$$

Now, we estimate the remainder at rank s. According to the previous expansions,

$$\begin{aligned}
&\mathrm{Rem}(s,\, F,\, \lambda) = A_2 + B_2 = \sum_{j=2}^{s+1} \frac{\lambda^{-(j-1)/2}}{(j+1)!} \mathrm{Rem}(s+1-j,\, F_j,\, \lambda) \\
&+ \frac{\lambda^{-(s+1)/2}}{(s+1)!} \int_{l^2} \int_0^\infty \int_0^1 (1-\theta)^{s+1} \nabla^{(s+3)} P_t F(u+\theta K_1(\tau)).K_1(\theta)^{(s+3)} \,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\nu_\lambda^*(u) \\
&\le \sum_{j=2}^{s+1} \frac{\lambda^{-(j-1)/2}}{(j+1)!} \xi_{s+1-j} \lambda^{-(s+2-j)/2} \|F_j\|_{\mathcal{C}_b^{3(s+1-j)+3}(l^2;\mathbf{R})} + \frac{\lambda^{-(s+1)/2}}{(s+1)!} \times \\
&\int_{l^2} \int_0^\infty e^{-(s+3)t} \int_0^1 (1-\theta)^{s+1} P_t \nabla^{(s+3)} F(u+\theta K_1(\tau)).K_1(\theta)^{(s+3)} \,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\nu_\lambda^*(u)
\end{aligned}$$

Since $\nabla^{(s+3)}F$ is bounded and P^β is a Markovian semi-group, for any $t \geq 0$, $P_t^\beta \nabla^{(s+3)}F$ is bounded by $\|F\|_{\mathcal{C}_b^{s+3}(l^2;\mathbf{R})}$. Hence,

$$\begin{aligned}\mathrm{Rem}(s,\, F,\, \lambda) &\leq \lambda^{-(s+1)/2} \sum_{j=2}^{s+1} \frac{\xi_{s+1-j}}{(j+1)!} \|F\|_{\mathcal{C}_b^{3s+6-3j}(l^2;\mathbf{R})} \\ &\qquad + \frac{\lambda^{-(s+1)/2}}{(s+3)!}\; c_{1-\beta}^{s+1} \|F\|_{\mathcal{C}_b^{s+3}(l^2;\mathbf{R})} \\ &\leq \lambda^{-(s+1)/2} \sum_{j=2}^{s+1} \frac{\xi_{s+1-j}\; c_{1-\beta}^{j+1}}{(j+1)!\;(3s+7-2j)}\; \|F\|_{\mathcal{C}_b^{3s+7-2j}(l^2;\mathbf{R})} \\ &\qquad + \frac{\lambda^{-(s+1)/2}}{(s+3)!} c_{1-\beta}^{s+1} \|F\|_{\mathcal{C}_b^{s+3}(l^2;\mathbf{R})}.\end{aligned}$$

Since $\sup_{j=2,\cdots,s+1} 3s+7-2j = 3s+3$ and $s+3 \leq 3s+3$, the result follows. □

4. Linear interpolation of the Brownian motion

For $m \geq 1$, the linear interpolation $B_m^\dagger$ of a Brownian motion $B^\dagger$ is defined by

$$B_m^\dagger(0) = 0 \text{ and } \mathrm{d}B_m^\dagger(t) = m \sum_{i=0}^{m-1} (B^\dagger(i+1/m) - B^\dagger(i/m)) \mathbf{1}_{[i/m,\,(i+1)/m)}(t)\,\mathrm{d}t.$$

Thus, $\mathfrak{J}_\beta B_m^\dagger$ is given by

$$\mathfrak{J}_\beta B_m^\dagger = \left(m \sum_{i=0}^{m-1} (B^\dagger(i+1/m) - B^\dagger(i/m)) \int_{i/m}^{(i+1)/m} k_n^{1-\beta}(t)\,\mathrm{d}t,\; n \geq 1 \right).$$

Consider the $\mathcal{L}^2([0,\,1])$-orthonormal functions

$$e_j^m(s) = \sqrt{m}\, \mathbf{1}_{[j/m,\,(j+1)/m)}(s),\; j = 0,\, \cdots,\, m-1,\; s \in [0,\,1]$$

and $F_m^\dagger = \mathrm{span}(e_j^m,\; j = 0,\, \cdots,\, m-1)$. We denote by $p_{F_m^\dagger}$ the orthogonal projection over $F_m^\dagger$. Since $B_m^\dagger$ is constructed as a function of a standard Brownian motion, we work on the canonical Wiener space $(\mathcal{C}^0([0,\,1];\,\mathbf{R}),\; \mathcal{I}_{1,\,2},\; m^\dagger)$. The gradient we consider, $D^\dagger$, is the derivative of the usual gradient on the Wiener space and the integration by parts formula reads as:

$$\mathbb{E}_{m^\dagger}\left[F \int_0^1 u(s)\,\mathrm{d}B^\dagger(s) \right] = \mathbb{E}_{m^\dagger}\left[\int_0^1 D_s^\dagger F\; u(s)\,\mathrm{d}s \right] \tag{17}$$

for any $u \in \mathcal{L}^2([0,\,1])$. We need to introduce some constants which already appeared in [3]. For any $\alpha \in (0,1]$, let

$$d_\alpha = \max\left(\sup_{z \geq 0} \int_0^z s^{\alpha-1} \cos(\pi s)\,\mathrm{d}s,\; \sup_{z \geq 0} \int_0^z s^{\alpha-1} \sin(\pi s)\,\mathrm{d}s \right).$$

Moreover,

Theorem 4.1 (cf. [3])**.** *Let $\nu_m^\dagger$ be the law of $\mathfrak{J}_\beta B_m^\dagger$ on l^2 and let*

$$H_m^\dagger = (p_{F_m^\dagger} k_n^{1-\beta},\; n \geq 1).$$

For any $F \in \mathcal{C}_b^2(l^2; \mathbf{R})$,

$$\left| \int_{l^2} F \, d\nu_m^\dagger - \int_{l^2} F \, dm_\beta \right| \leq \frac{\gamma_{m,\beta}^\dagger}{2} \|F\|_{\mathcal{C}_b^2(l^2;\mathbf{R})},$$

where, for any $0 < \varepsilon < 1/2 - \beta$, *for any* p *such that* $p(1/2 - \beta - \varepsilon) > 1$,

$$\gamma_{m,\beta}^\dagger \leq \frac{d_{1/2+\varepsilon} c_{1-\beta} \zeta_{1/2+\varepsilon,\, p}}{\Gamma(1/2+\varepsilon)} \left(\sum_{n \geq 1} \frac{1}{n^{1+2\varepsilon}} \right)^{1/2} m^{-(1/2-\beta-\varepsilon)},$$

with for any $p \geq 1$, $\alpha \in (1/p,\, 1]$,

$$\zeta_{\alpha,\, p} = \sup_{\|f\|_{\mathcal{I}_{\alpha,\, p}}=1} \|f\|_{\mathrm{Hol}_0(\alpha - 1/p)}.$$

Theorem 4.2. *For any integer* s, *for any* $F \in \mathcal{C}_b^{2s+2}(l^2; \mathbf{R})$, *we have the following expansion:*

$$\mathbb{E}_{\nu_m^\dagger}[F] = \sum_{j=0}^{s} \frac{1}{2^j\, j!} \int_{l^2} \langle \nabla^{(2j)} F(u),\ (S_m^\dagger - S_\beta)^{\otimes j} \rangle_{(l^2)^{\otimes 2j}} \, dm_\beta(u) + \mathrm{Rem}^\dagger(s,\, F,\, m),$$

where $S_m^\dagger = \mathrm{trace}_{\mathcal{L}^2([0,\,1])}(K_m^\dagger \otimes K_m^\dagger)$ *and* $\mathrm{Rem}^\dagger(s,\, F,\, m)$ *can be bounded by*

$$\left| \mathrm{Rem}^\dagger(s,\, F,\, m) \right| \leq \frac{(\gamma_{m,\beta}^\dagger)^{s+1}}{2^{s+1}} \ \|F\|_{\mathcal{C}_b^{2s+2}(l^2;\mathbf{R})}.$$

Proof. For $s = 0$, the result boils down to Theorem [4.1]. We proceed by induction on s. According to the induction hypothesis and to the Stein representation formula [3], for F sufficiently regular,

$$\begin{aligned}
&\int_{l^2} F(u)\, \mathrm{d}\nu_m^\dagger(u) - \int_{l^2} F(u)\, \mathrm{d}m_\beta(u) \\
&= \int_{l^2} \int_0^\infty \nabla^{(2)} P_t^\beta F(u).\, (S_m^\dagger - S_\beta)\, \mathrm{d}t\, \mathrm{d}\nu_m^\dagger(u) \\
&= \sum_{j=0}^{s} \frac{1}{2^j\, j!} \int_{l^2} \nabla^{(2)} \Big(\int_0^\infty \nabla^{(2j)} P_t^\beta F(u).\, (S_m^\dagger - S_\beta)^{\otimes(j)}\, \mathrm{d}t \Big).(S_m^\dagger - S_\beta)\, \mathrm{d}m_\beta(u) \\
&\qquad + \mathrm{Rem}^\dagger\Big(s,\, \int_0^\infty \nabla^{(2)} P_t^\beta F.(S_m^\dagger - S_\beta)\, \mathrm{d}t,\, m\Big) \\
&= \sum_{j=0}^{s} \frac{1}{2^j\, j!} \int_{l^2} \int_0^\infty \nabla^{(2j+2)} P_t^\beta F(u).\, (S_m^\dagger - S_\beta)^{\otimes(j+1)}\, \mathrm{d}t\, \mathrm{d}m_\beta(u) \\
&\qquad + \mathrm{Rem}^\dagger\Big(s,\, \int_0^\infty \nabla^{(2)} P_t^\beta F.(S_m^\dagger - S_\beta)\, \mathrm{d}t,\, m\Big).
\end{aligned}$$

Since $\nabla^{(j)} P_t^\beta F(u) = e^{-jt} P_t^\beta \nabla^{(j)} F(u)$ and since m_β is invariant under the action of P^β, we obtain

$$\int_{l^2} F(u)\,\mathrm{d}\nu_m^\dagger(u) = \int_{l^2} F(u)\,\mathrm{d}m_\beta(u) + \sum_{j=0}^{s} \frac{1}{2^{j+1}} \int_{l^2} \nabla^{(2j+2)} F(u).\,(S_m^\dagger - S_\beta)^{\otimes(j+1)}\,\mathrm{d}m_\beta(u) + \mathrm{Rem}^\dagger(s,\, \int_0^\infty \nabla^{(2)} P_t^\beta F.\,(S_m^\dagger - S_\beta)\,\mathrm{d}t,\, m).$$

By a change of index in the sum, we obtain the main part of the expansion for the rank $s+1$. Moreover,

$$|\,\mathrm{Rem}^\dagger(s+1,\, F,\, m)| \leq \frac{(\gamma_{m,\,\beta}^\dagger)^s}{2^{s+1}(s+1)!}\, \| \int_0^\infty \nabla^{(2)} P_t^\beta F.(S_m^\dagger - S_\beta)\,\mathrm{d}t\|_{\mathcal{C}_b^{2(s+1)}(l^2)} \leq \frac{(\gamma_{m,\,\beta}^\dagger)^s}{2^{s+1}} \frac{\|S_m^\dagger - S_\beta\|_{l^2}}{2}\, \|F\|_{\mathcal{C}_b^{2(s+2)}(l^2)}.$$

Since the norm of a bounded operator is bounded by its trace provided that the latter exists, we have $\|S_m^\dagger - S_\beta\|_{l^2} \leq \gamma_{m,\,\beta}^\dagger$, hence the result. □

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