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ON THE RATE OF CONVERGENCE TO EQUILIBRIUM FOR THE HOMOGENEOUS LANDAU EQUATION WITH SOFT POTENTIALS

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Abstract. We investigate in this work the rate of convergence to equilibrium of solutions to the spatially homogeneous Landau equation with soft potentials. Firstly, we prove a polynomial in time convergence using an entropy method with some new a priori estimates. Finally, we prove an exponential in time convergence towards the equilibrium with the optimal rate, given by the spectral gap of the associated linearised operator, combining new decay estimates for the semigroup generated by the linearised Landau operator in weighted L^p-spaces together with the polynomial decay described above.

1. Introduction

The Landau equation is a fundamental model in kinetic theory that describes the evolution of the density of particles in a plasma in the phase space of all positions and velocities. We consider in this work the case of spatially homogeneous density functions, which verifies the *spatially homogeneous Landau equation* given by

$$\left\{ \begin{array}{rcl} \partial_t f & = & Q(f,f) \\ f_{|t=0} & = & f_0, \end{array} \right. \tag{1.1}$$

where $f = f(t,v) \geq 0$ is the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The Landau collision operator Q is a bilinear operator acting only on the variable v and given by

$$Q(g,f) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v-v_*) \left[g_* \partial_j f - f \partial_j g_* \right] dv_*, \tag{1.2}$$

where here and below we shall use the convention of implicit summation over repeated indices and the usual shorthand $g_* = g(v_*)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $f = f(v)$ and $\partial_j f = \partial_{v_j} f(v)$.

The matrix-valued function a is nonnegative, symmetric and depends on the interaction between particles. One usually assumes that particles interact by binary relation through a potential proportional to $1/r^s$, where r denotes their distance. In this case a is given by (see for instance [23])

$$a_{ij}(z) = |z|^{\gamma+2} \Pi_{ij}(z), \quad \Pi_{ij}(z) = \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right), \tag{1.3}$$

with $\gamma = (s-4)/s$. One usually calls hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, soft potentials if $\gamma = (-3,0)$ and Coulombian potential if $\gamma = -3$. One also separates the soft potentials into two categories: *moderately* soft potentials when $\gamma \in (-2,0)$ and *very* soft potentials if $\gamma \in (-3,-2]$. In this paper we are interested in the case of moderately soft potentials.

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We also define the following quantities

$$b_i(z) = \partial_j a_{ij}(z) = -2\,|z|^\gamma\, z_i, \quad c(z) = \partial_{ij} a_{ij}(z) = -2(\gamma+3)\,|z|^\gamma, \tag{1.4}$$

from which we are able to rewrite the Landau operator in the following way

$$\begin{aligned} Q(g,f) &= \nabla \cdot \{(a*g)\nabla f - (b*g) f\} \\ &= (a_{ij} * g)\partial_{ij} f - (c*g) f. \end{aligned} \tag{1.5}$$

Let us present some important properties of the Landau equation. First of all, it conserves mass, momentum and energy. Indeed, at least formally, for any test function φ we have (see e.g. [21])

$$\int_{\mathbb{R}^3} Q(f,f)\varphi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*) f f_* \left(\frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*}\right)(\partial_j\varphi - \partial_j\varphi_*)\,dv\,dv_*,$$

from which we deduce, for any $t \geq 0$,

$$\frac{d}{dt}\int f\varphi\,dv = \int Q(f,f)\varphi\,dv = 0 \qquad \text{for} \qquad \varphi(v) = 1, v, |v|^2. \tag{1.6}$$

Another important property of this equation is the Landau version of the celebrated H-Theorem of Boltzmann: The entropy $H(f) := \int f\log f$ is nonincreasing and any equilibrium is a Maxwellian distribution (Gaussian distribution). Indeed, at least formally, the entropy-dissipation functional defined as

$$D(f) := -\int Q(f,f)\log f, \tag{1.7}$$

verifies the following inequality

$$D(f) = -\frac{d}{dt}H(f) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*)\left(\frac{\partial_i f}{f} - \frac{\partial_{i*} f_*}{f_*}\right)\left(\frac{\partial_j f}{f} - \frac{\partial_{j*} f_*}{f_*}\right) f f_*\,dv\,dv_* \geq 0, \tag{1.8}$$

and we also have

$$H(f(t)) + \int_0^t D(f(\tau))\,d\tau = H(f_0). \tag{1.9}$$

From this, it also follows that any equilibrium is a Maxwellian distribution

$$\mu_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}},$$

for some $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$.

It is then expected that any solution $f(t,\cdot)$ converges towards the Maxwellian equilibrium μ_{ρ_f,u_f,T_f} when $t \to +\infty$, where ρ_f is the density of the gas, u_f the mean velocity and T_f the temperature, defined by

$$\rho_f = \int f(v), \quad u_f = \frac{1}{\rho}\int v f(v), \quad T_f = \frac{1}{3\rho}\int |v-u|^2 f(v),$$

and these quantities are defined by the initial datum f_0 thanks to the conservation properties of the Landau operator (1.6).

We shall always assume that f_0 is a nonnegative function with finite mass, energy and entropy, more precisely

$$\int f_0 = M_0 < \infty, \quad \int |v|^2 f_0 = E_0 < \infty, \quad \int f_0 \log f_0 = H_0 < \infty,$$

and it is classical that this implies

$$f_0 \in L^1_2 \cap L\log L, \qquad L\log L := \left\{ f \in L^1 \mid \int |f|\log(|f|) < \infty \right\}. \tag{1.10}$$

Furthermore, we may only consider the case of initial datum f_0 satisfying

$$f_0 \in L^1_{1,0,1} := \{f \in L^1 \mid \rho_f = 1,\ u_f = 0,\ T_f = 1\}, \tag{1.11}$$

the general case being reduced to (1.11) by a simple change of coordinates. We shall then denote $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ the standard Gaussian distribution in $\mathbb{R}^3$, which corresponds to the Maxwellian with same mass, momentum and energy of f_0.

We can linearise the Landau equation around the equilibrium μ, with the perturbation $f(t,v) = \mu(v) + h(t,v)$, which satisfies at the first order the *linearised Landau equation*

$$\left\{ \begin{array}{rcl} \partial_t h & = & \mathcal{L} h \\ h_{|t=0} & = & h_0, \end{array} \right. \tag{1.12}$$

where the initial datum is defined by $h_0 = f_0 - \mu$, and where the linearised Landau operator $\mathcal{L}$ is given by

$$\mathcal{L} h = Q(\mu, h) + Q(h, \mu). \tag{1.13}$$

Furthermore, from the conservation properties (1.6), we observe that the null space of $\mathcal{L}$ has dimension 5 and is given by (see e.g. [5, 12, 2, 16, 18])

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}. \tag{1.14}$$

Consider the weighted Hilbert space $L^2(\mu^{-1/2})$ associated with the following scalar product and norm

$$\langle h, g \rangle_{L^2(\mu^{-1/2})} := \int h g\, \mu^{-1} \quad \text{and} \quad \|h\|^2_{L^2(\mu^{-1/2})} := \int |h|^2\, \mu^{-1}.$$

A simple computation gives

$$\begin{aligned} &\langle \mathcal{L} h, h \rangle_{L^2(\mu^{-1/2})} \\ &\quad = -\frac{1}{2} \iint a_{ij}(v - v_*) \{\partial_i(\mu^{-1} h) - \partial_{*i}(\mu_*^{-1} h_*)\} \{\partial_j(\mu^{-1} h) - \partial_{*j}(\mu_*^{-1} h_*)\}\, \mu_*\, \mu\, dv_*\, dv \\ &\quad \leq 0, \end{aligned}$$

which implies that $\mathcal{L}$ is self-adjoint on $L^2(\mu^{-1/2})$ and, moreover, that the spectrum of $\mathcal{L}$ in $L^2(\mu^{-1})$ is included in $\mathbb{R}_-$.

1.1. **Existing results.** Let us mention known results concerning the long-time behaviour of solutions to the Landau equation (and for a more detailed presentation we refer to [4]).

In the Maxwellian molecules case $\gamma = 0$, Villani [22] proves an exponential in time convergence to equilibrium. For hard potentials $\gamma \in (0, 1]$, Desvillettes and Villani [8] obtain a polynomial in time convergence to equilibrium, and more recently we prove in [4] an optimal exponential decay to equilibrium. Moreover, Toscani and Villani [19] also prove a decay to equilibrium polynomially in time, in the case of *mollified* soft potentials $\gamma \in (-3, 0)$, which corresponds to replace $|z|^{\gamma+2}$ in (1.3) by a mollified function $\Psi(z)$ truncating the singularity at the origin (see Section 4.1 for more details). It is worth mentioning that all the results from [22, 8, 19] above are purely nonlinear and based on an entropy method.

Another approach for studying the long-time behaviour consists in considering the linearised equation around the equilibrium (1.12), which has been investigated by several authors. Summarising results of Degond and Lemou [5], Guo [12], Baranger and Mouhot [2], Mouhot [16], Mouhot and Strain [18], we have the following proposition:

Proposition 1.1. *Let $\gamma \in [-2, 1]$. There exists a constructive constant $\lambda_0 > 0$ (spectral gap) such that, for any $h \in L^2(\mu^{-1/2})$ with $h \in \mathcal{N}(\mathcal{L})^\perp$,*

$$\langle \mathcal{L} h, h \rangle_{L^2(\mu^{-1/2})} \leq -\lambda_0 \|h\|^2_{L^2(\mu^{-1/2})}.$$

As a consequence we obtain an exponential decay for the linearised Landau equation (1.12)*: for any* $t \geq 0$ *and* $h \in L^2(\mu^{-1/2})$*, there holds*

$$\|e^{t\mathcal{L}}h - \Pi_0 h\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda_0 t}\|h - \Pi_0 h\|_{L^2(\mu^{-1})},$$

where Π_0 *is the projection onto* $\mathcal{N}(\mathcal{L})$*.*

1.2. **Main results and strategy.** Let us define the notion of weak solution we consider in this paper.

Definition 1.2 (Weak solutions [21])**.** Let $\gamma \in [-2, 1]$ and consider a nonnegative $f_0 \in L^1_2 \cap L\log L$. We say that f is a weak solution of the Cauchy problem (1.1) if the following conditions are fulfilled:

(*i*) $f \geq 0$, $f \in C([0,\infty); \mathcal{D}') \cap L^\infty([0,\infty); L^1_2 \cap L\log L) \cap L^1_{loc}([0,\infty); L^1_{2+\gamma})$;
(*ii*) $f(0) = f_0$;
(*iii*) for any $t \geq 0$

$$\int f(t)\varphi = \int f_0\varphi \quad \text{for} \quad \varphi(v) = 1, v, |v|^2; \quad \text{and} \quad H(f(t)) + \int_0^t D(f(\tau)) \leq H(f_0);$$

(*iv*) f verifies (1.1) in the distributional sense: for any $\varphi \in C([0,\infty); C^\infty_c)$, for any $t \geq 0$,

$$\int f(t)\varphi(t) - \int f_0\varphi(0) - \int_0^t \int f(\tau)\partial_t\varphi(\tau) = \int_0^t \int Q(f(\tau), f(\tau))\varphi(\tau),$$

where the last integral in the right-hand side is defined by

$$\int Q(f,f)\varphi = \frac{1}{2}\iint a_{ij}(v - v_*)(\partial_{ij}\varphi + \partial_{ij}\varphi_*)\, f_*f + \iint b_i(v - v_*)(\partial_i\varphi - \partial_i\varphi_*)\, f_*f.$$

It is observed in [21] that these formulae make sense as soon as f satisfies (*i*) and $\varphi \in W^{2,\infty}(\mathbb{R}^3)$.

In the case of moderately soft potentials $\gamma \in (-2, 0)$, it is proven in [21] that if $f_0 \in L^1_2 \cap L\log L$ there exists a global weak solution. If moreover we assume $f_0 \in L^1_k$, with $k > \gamma^2/(2+\gamma)$, then the weak solution is unique [10, Corollary 4].

We can now state our main results on the rate of convergence to equilibrium: a polynomial convergence in Theorem 1.3 and then an exponential convergence in Theorem 1.4.

Theorem 1.3 (Polynomial convergence)**.** *Let* $\gamma \in (-2, 0)$ *and* $f_0 \in L^1_{k+8-3\gamma/4} \cap L\log L$ *with* $k > 7|\gamma|/2$*. Then there exists a weak solution* f *to the Landau equation associated to* f_0 *such that*

$$\forall\, t \geq 0, \qquad H(f(t)|\mu) \leq C(1+t)^{-\frac{k}{|\gamma|}+\frac{7}{2}},$$

for some constructive constant $C > 0$ *and where* $H(f|\mu) := \int f\log(f/\mu)$ *is the relative entropy of* f *with respect to* μ*.*

The proof of Theorem 1.3 follows the strategy introduced by Toscani and Villani [19] (see Section 4.1 for more details), in which, as already explained, a polynomial in time convergence to equilibrium for mollified soft potentials is proven. This strategy was developed in order to treat the trend to equilibrium issue for kinetic equations with relatively bad control of the distribution tails (as for Boltzmann and Landau-type equations with soft potentials) and they compensate the lack of uniform in time estimates by some precise logarithmic Sobolev inequalities. In order to use this strategy, we prove some new *a priori* estimates for the evolution of weighted L^1 and Sobolev norms in Section 3. Then we prove Theorem 1.3 in Section 4.1 using these *a priori* estimates together with a functional inequality relying entropy and entropy-dissipation from [19].

Theorem 1.4 (Exponential convergence)**.** *Let $\gamma \in (-1,0)$ and $f_0 \in L\log L \cap L^1(e^{\kappa\langle v\rangle^s})$ with $\kappa > 0$ and $-\gamma < s < 2+\gamma$. Then the unique weak solution f to the Landau equation associated to f_0 satisfies*

$$\forall\, t \geq 0, \qquad \|f(t) - \mu\|_{L^1} \leq Ce^{-\lambda_0 t},$$

for some constructive constant $C > 0$ and where $\lambda_0 > 0$ is the spectral gap of the associated linearised operator.

Remark 1.5. The restriction $\gamma \in (-1,0)$ comes from the fact that we need $s+\gamma > 0$ in order to prove the "spectral gap/semigroup decay" extension theorem for the linearised equation (see Theorem 2.1) and $s < \gamma + 2$ to prove the propagation of stretched exponential moments (see Lemma 3.6).

The strategy to prove this theorem is based on:

(1) New exponential decay estimates (with sharp rate) for the semigroup generated by the linearised Landau operator $\mathcal{L}$ in various L^p-spaces with stretched exponential weight, using a method developed in [11]. This question is addressed in Section 2.
(2) New *a priori* estimates for the nonlinear equation proved in Section 3 and the convergence to equilibrium from Theorem 1.3 proven in Section 4.1.
(3) A "coupling method" in order to connect the linearised theory with the nonlinear one: for small times we use the polynomial convergence from Theorem 1.3; then for large times we use (2) to prove that the solution enters in a suitable neighbourhood of the equilibrium, in which the linear part is dominant, and we have an optimal exponential decay from (1). This is proven in Section 4.2.

It is worth mentioning that this strategy has been used by several authors and for different equations in order to prove an exponential in time convergence to equilibrium. It was first introduced by Mouhot [17] for the homogeneous Boltzmann equation for hard potentials with Grad's cut-off. This same approach was later used by Gualdani, Mischler and Mouhot [11] for the inhomogeneous Boltzmann equation for hard spheres on the torus and for the Fokker-Planck equation, and also by Mischler and Mouhot [14] for Fokker-Planck equations. More recently, the author [4] used it for the homogeneous Landau equation with hard potentials, and Tristani [20] for the homogeneous Boltzmann equation for hard potentials without cut-off.

1.3. **Notations.** Let $m : \mathbb{R}^d \to \mathbb{R}^+$ be a weight function. For any $1 \leq p \leq \infty$ we define the weighted space $L^p(m)$ associated with the norm

$$\|f\|_{L^p(m)} := \|mf\|_{L^p}.$$

We also define higher-order weighted Sobolev spaces $W^{\ell,p}(m)$ associated with the norm

$$\|f\|^p_{W^{\ell,p}(m)} := \sum_{|\alpha|\leq\ell} \|\partial^\alpha f\|^p_{L^p(m)}, \qquad 1 \leq p < \infty,$$

with the usual modification for $p = \infty$ and for homogeneous spaces $\dot{W}^{\ell,p}(m)$. When $m = \langle v\rangle^k := (1+|v|^2)^{k/2}$ is a polynomial weight, we denote $W^{\ell,p}_k := W^{\ell,p}(\langle v\rangle^k)$.

Let X, Y be Banach spaces and consider a linear operator $\Lambda : X \to Y$. We shall denote by $\mathcal{S}_\Lambda(t) = e^{t\Lambda}$ the semigroup generated by Λ. Moreover we denote by $\mathscr{B}(X,Y)$ the space of bounded linear operators from X to Y and by $\|\cdot\|_{\mathscr{B}(X,Y)}$ its norm operator, with the usual simplification $\mathscr{B}(X) = \mathscr{B}(X,X)$.

2. The linearised operator

In this section we shall denote

$$\bar{a}_{ij}(v) = a_{ij} * \mu, \quad \bar{b}_i(v) = b_i * \mu, \quad \bar{c}(v) = c * \mu. \tag{2.1}$$

Let us now make our assumptions on the weight function $m = m(v)$:

(W) Stretched exponential weight. We consider a weight function $m = \exp(\kappa\langle v\rangle^s)$ with $\kappa > 0$, $0 < s < 2$ and $s + \gamma > 0$.

We are now able to state the main result of this section, which extends to various weighted L^p spaces the decay of the semigroup $\mathcal{S}_{\mathcal{L}}(t)$ generated by the operator $\mathcal{L}$, known to hold in $L^2(\mu^{-1/2})$ by Proposition 1.1.

Theorem 2.1. *Let $\gamma \in (-2, 0)$, $1 \le p \le 2$ and a weight function m satisfying (W). Then there exists a constant $C > 0$ such that, for all $t \ge 0$ and any $h \in L^p(m)$, there holds*

$$\|\mathcal{S}_{\mathcal{L}}(t)h - \Pi_0 h\|_{L^p(m)} \le Ce^{-\lambda_0 t}\|h - \Pi_0 h\|_{L^p(m)},$$

where Π_0 is the projection onto $\mathcal{N}(\mathcal{L})$ and $\lambda_0 > 0$ is the spectral gap of $\mathcal{L}$ on $L^2(\mu^{-1/2})$.

In order to prove this theorem we shall use the method of *enlargement of the functional space of semigroup decay* developed by Gualdani, Mischler and Mouhot [11]. Roughly speaking, if one knows some quantitative information on the semigroup decay associated with an operator $\mathcal{L}$ in some *small* space E, this method enables one to deduce this quantitative estimate on a *larger* space $\mathcal{E} \supset E$, when the operator $\mathcal{L}$ satisfies some properties. In order to do that, we need to factorise $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and to prove some properties for these operators, namely that $\mathcal{B}$ has a well localised spectrum (see Section 2.2) and $\mathcal{A}$ is regularising in some sense (see Section 2.3).

2.1. Factorisation of the operator.

Using the form (1.5) of the operator Q, we decompose the linearised Landau operator $\mathcal{L}$ defined in (1.13) as $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where we define

$$\begin{aligned} \mathcal{A}_0 h &:= Q(h, \mu) = (a_{ij} * h)\partial_{ij}\mu - (c * h)\mu, \\ \mathcal{B}_0 h &:= Q(\mu, h) = (a_{ij} * \mu)\partial_{ij} h - (c * \mu)h. \end{aligned} \tag{2.2}$$

Consider a smooth nonnegative function $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \le \chi(v) \le 1$, $\chi(v) \equiv 1$ for $|v| \le 1$ and $\chi(v) \equiv 0$ for $|v| > 2$. For any $R \ge 1$ we define $\chi_R(v) := \chi(R^{-1}v)$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$. Then, we make the final decomposition of the operator $\mathcal{L}$ as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A} := \mathcal{A}_0 + M\chi_R, \qquad \mathcal{B} := \mathcal{B}_0 - M\chi_R, \tag{2.3}$$

where M and R will be chosen later.

2.2. Dissipativity properties.

We investigate in this section dissipativity properties of the operator $\mathcal{B}$.

First of all, we state the following results concerning $\bar{a}_{ij}(v)$ (see [5, Propositions 2.3 and 2.4, Corollary 2.5] and [12, Lemma 3]) that will be useful.

Lemma 2.2. *The following properties hold:*

(a) The matrix $\bar{a}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace $v^\perp$. Moreover,

$$\ell_1(v) = \int_{\mathbb{R}^3} \left(1 - \left(\frac{v}{|v|} \cdot \frac{w}{|w|}\right)^2\right) |w|^{\gamma+2}\mu(v - w)\, dw$$

$$\ell_2(v) = \int_{\mathbb{R}^3} \left(1 - \frac{1}{2}\left|\frac{v}{|v|} \times \frac{w}{|w|}\right|^2\right) |w|^{\gamma+2}\mu(v - w)\, dw.$$

When $|v| \to +\infty$ *we have*

$$\ell_1(v) \sim 2\langle v\rangle^{\gamma}$$
$$\ell_2(v) \sim \langle v\rangle^{\gamma+2}.$$

If $\gamma \in (0,1]$ *there exists* $\ell_0 > 0$ *such that, for all* $v \in \mathbb{R}^3$, $\min\{\ell_1(v), \ell_2(v)\} \geq \ell_0$.

(b) The function $\bar{a}_{ij}$ *is smooth, for any multi-index* $\beta \in \mathbb{N}^3$

$$|\partial^\beta \bar{a}_{ij}(v)| \leq C_\beta \langle v\rangle^{\gamma+2-|\beta|}$$

and

$$\bar{a}_{ij}(v)\xi_i\xi_j = \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I-P_v)\xi|^2,$$
$$\bar{a}_{ij}(v)v_iv_j = \ell_1(v)|v|^2,$$

where P_v *is the projection on* v, *i.e.*

$$P_v\xi = \left(\xi \cdot \frac{v}{|v|}\right)\frac{v}{|v|}.$$

(c) We have

$$\bar{a}_{ii}(v) = 2\int_{\mathbb{R}^3} |v-v_*|^{\gamma+2}\mu(v_*)\,dv_* \qquad \text{and} \qquad \bar{b}_i(v) = -\ell_1(v)\,v_i.$$

Let us we define

$$\varphi_{m,p}(v) := \frac{1}{m}\,\bar{a} : \nabla^2 m + \frac{(p-1)}{m^2}\,\bar{a}\nabla m \cdot \nabla m + \frac{2}{m}\,\bar{b}\cdot\nabla m + (1/p-1)\bar{c}. \tag{2.4}$$

Before proving the desired result in Lemma 2.5, we give the following elementary lemma to be used in the sequel.

Lemma 2.3. *Let* $J_\alpha(v) := \int_{\mathbb{R}^3} |v-v_*|^\alpha \mu(v_*)\,dv_*$ *for* $-3 < \alpha \leq 2$. *Then it holds:*

(i) If $0 \leq \alpha \leq 2$ *then* $J_\alpha(v) \leq |v|^\alpha + C_\alpha$ *for some constant* $C_\alpha > 0$.

(ii) If $-3 < \alpha < 0$ *then* $J_\alpha(v) \leq C\langle v\rangle^\alpha$ *for some constant* $C > 0$.

Proof. Point (i) can be found in [4, Lemma 2.5]. For point (ii) we observe that the result easily follows if $|v| \leq 1$. On the other hand if $|v| > 1$ we write

$$\begin{aligned} J_\alpha(v) &= \int_{|v_*|\leq 1} |v_*|^\alpha \mu(v-v_*)\,dv_* + \int_{|v_*|\geq 1} |v_*|^\alpha \mu(v-v_*)\,dv_* \\ &\leq \sup_{|v_*|\leq 1} \mu(v-v_*) \int_{|v_*|\leq 1} |v_*|^\alpha\,dv_* + C\int_{|v_*|\geq 1} \langle v_*\rangle^\alpha \mu(v-v_*)\,dv_*. \end{aligned}$$

Using that $\sup_{|v_*|\leq 1}\mu(v-v_*) \leq Ce^{-|v|^2/4} \leq C$ and that $\langle v_*\rangle^\alpha \leq C\langle v\rangle^\alpha\langle v-v_*\rangle^{|\alpha|}$ by Peetre's inequality we conclude to

$$J_\alpha(v) \leq C + C\langle v\rangle^\alpha \int \langle v-v_*\rangle^{|\alpha|}\mu(v-v_*)\,dv_* \leq C\langle v\rangle^\alpha.$$

□

Lemma 2.4. *Let* m *satisfy assumption (W). Then for all* $\lambda > 0$ *we can choose* M *and* R *large enough such that, for all* $v \in \mathbb{R}^3$,

$$\varphi_{m,p}(v) - M\chi_R(v) \leq -\lambda.$$

Proof. Let $m = \exp(\kappa\langle v\rangle^s)$. We easily compute

$$\frac{\nabla m}{m} = \kappa s v\langle v\rangle^{s-2}$$

and

$$\frac{(\nabla^2 m)_{ij}}{m} = \kappa s\langle v\rangle^{s-2}\delta_{ij} + \kappa s(s-2)v_i v_j\langle v\rangle^{s-4} + \kappa^2 s^2 v_i v_j\langle v\rangle^{2s-4}.$$

It follows then

$$\begin{aligned}\bar a : \frac{\nabla^2 m}{m} &= (\delta_{ij}\bar a_{ij})\kappa s\langle v\rangle^{s-2} + (\bar a_{ij}v_i v_j)\kappa s(s-2)\langle v\rangle^{s-4} + (\bar a_{ij}v_i v_j)\kappa^2 s^2\langle v\rangle^{2s-4}\\ &= 2\kappa s J_{\gamma+2}(v)\langle v\rangle^{s-2} + \kappa s(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + \kappa^2 s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4},\end{aligned}$$

where we have used Lemma 2.2. Moreover, using again Lemma 2.2, we obtain

$$\bar a\,\frac{\nabla m}{m}\,\frac{\nabla m}{m} = \bar a v v\kappa^2 s^2\langle v\rangle^{2s-4} = \kappa^2 s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4}$$

and

$$\bar b\cdot\frac{\nabla m}{m} = -\kappa s\ell_1(v)|v|^2\langle v\rangle^{s-2}.$$

Putting together the above estimates, we obtain

$$\begin{aligned}\varphi_{m,p}(v) &= 2\kappa s J_{\gamma+2}(v)\langle v\rangle^{s-2} + \kappa s(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + p\kappa^2 s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4}\\ &\quad - 2\kappa s\ell_1(v)|v|^2\langle v\rangle^{s-2} + 2(\gamma+3)(1-1/p)J_\gamma(v).\end{aligned}\tag{2.5}$$

From the asymptotic behaviour of ℓ_1, $J_{\gamma+2}$ and J_γ, the dominant terms of $\varphi_{m,p}$ in (2.5) when $|v|\to\infty$ are the first and the fourth one, both of order $\langle v\rangle^{\gamma+s}$. Using Lemma 2.3 to bound $J_{\gamma+2}(v) \le \tilde J_{\gamma+2}(v) \underset{|v|\to\infty}{\sim} \langle v\rangle^{\gamma+2}$ and $\ell_1(v) \underset{|v|\to\infty}{\sim} 2\langle v\rangle^\gamma$ from Lemma 2.2, we obtain that $\varphi_{m,p}(v) \le \tilde\varphi_{m,p}(v)$ with

$$\tilde\varphi_{m,p}(v) \underset{|v|\to\infty}{\sim} -2\kappa s\langle v\rangle^{s+\gamma} \xrightarrow[|v|\to\infty]{} -\infty,\tag{2.6}$$

because $s+\gamma>0$ from assumption (W).

Let us fix $\lambda>0$. Then, thanks to (2.6), we can choose R large enough such that

$$\forall\,|v|>R,\quad \varphi_{m,p}(v) - M\chi_R(v) \le -\lambda.$$

Finally, we choose $M \ge \sup_{|v|\le R}\varphi_{m,p}(v) + \lambda$ so that

$$\forall\,|v|\le R,\quad \varphi_{m,p}(v) - M\chi_R(v) = \varphi_{m,p}(v) - M \le -\lambda,$$

from which we conclude. □

With the help of the result above, we are able to state a result on the dissipativity of $\mathcal{B}$. Recall that

$$\mathcal{B} = \mathcal{B}_0 - M\chi_R,\qquad \mathcal{B}_0 f = \nabla\cdot\{\bar a\nabla f - \bar b f\}.$$

Lemma 2.5. *Let $\gamma\in(-2,0)$, $p\in[1,+\infty)$ and m be a weight function satisfying assumption (W). Then for any $\lambda>0$, we can choose M and R large enough such that the operator $(\mathcal{B}+\lambda)$ is dissipative in $L^p(m)$.*

Lemma 2.6. *Let $\gamma\in(-2,0)$. Then for any $\lambda>0$, we can choose M and R large enough such that the operator $(\mathcal{B}+\lambda)$ is dissipative in $L^2(\mu^{-1/2})$.*

Proof of Lemma 2.5. We denote $\Phi'(x) = |x|^{p-1}\text{sign}(x)$ and consider the equation

$$\partial_t h = \mathcal{B}h = \mathcal{B}_0 h - M\chi_R h.$$

For all $p \in [1, \infty)$, we compute

$$\frac{1}{p}\frac{d}{dt}\|h\|^p_{L^p(m)} = \int (\mathcal{B}_0 h)\Phi'(h)m^p - \int (M\chi_R)|h|^p m^p.$$

For the first term, we perform integration by parts to obtain

$$\begin{aligned}\int (\mathcal{B}_0 h)\Phi'(h)m^p &= \int \nabla \cdot \{\bar{a}\nabla h - \bar{b}h\}\Phi'(h)m^p \\ &= -\int \bar{a}\nabla h \nabla(\Phi'(h))m^p - \int \bar{a}\nabla h \Phi'(h)\nabla(m^p) \\ &\quad + \int \bar{b}h\nabla(\Phi'(h))m^p + \int \bar{b}h\Phi'(h)\nabla(m^p).\end{aligned}$$

Using that $\nabla(\Phi'(h)) = (p-1)|h|^{p-2}\nabla h$, $\Phi'(h)\nabla h = p^{-1}\nabla(|h|^p)$ and $h\nabla(\Phi'(h)) = (1-1/p)\nabla(|h|^p)$, and integrating by parts, we finally get

$$\begin{aligned}\int (\mathcal{B}_0 h)\Phi'(h)m^p &= -(p-1)\int \bar{a}\nabla h \nabla h |h|^{p-2}m^p \\ &\quad + \frac{1}{p}\int \left\{\bar{a} : \frac{\nabla^2(m^p)}{m^p} + 2\,\bar{b}\cdot\frac{\nabla(m^p)}{m^p} - (p-1)\bar{c}\right\}|h|^p m^p.\end{aligned}$$

We can rewrite

$$\nabla(m^p) = pm^{p-1}\nabla m$$

and

$$\nabla^2(m^p) = (\partial_{ij}m^p)_{1\le i,j\le 3} = p(p-1)m^{p-2}\partial_i m\partial_j m + pm^{p-1}\partial_{ij}m$$

to obtain

$$\frac{1}{p}\frac{d}{dt}\|h\|^p_{L^p(m)} = -(p-1)\int \bar{a}\nabla h\nabla h|h|^{p-2}m^p + \int(\varphi_{m,p} - M\chi_R)|h|^p m^p, \tag{2.7}$$

where $\varphi_{m,p}$ is defined in (2.4).

From Lemma 2.4, for all $\lambda \ge 0$, we can choose M and R large enough such that $\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda$. Hence, it follows that the operator $(\mathcal{B}+\lambda)$ is dissipative in $L^p(m)$. Indeed, from (2.7) we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|h\|^p_{L^p(m)} &= -(p-1)\int \bar{a}\,\nabla h\nabla h\,|h|^{p-2}m^p + \int(\varphi_{m,p} - M\chi_R)|h|^p m^p \\ &\le -\lambda\|h\|^p_{L^p(m)},\end{aligned}$$

since the matrix $\bar{a}$ is positive, and it follows that

$$\|\mathcal{S}_{\mathcal{B}}(t)h\|_{L^p(m)} \le e^{-\lambda t}\|h\|_{L^p(m)}. \tag{2.8}$$

□

Proof of Lemma 2.6. Arguing as in the proof above and denoting $\varphi_\mu := \varphi_{\mu^{-1/2},2}$, that satisfies from (2.5)

$$\varphi_\mu(v) = J_{\gamma+2}(v) - \frac{1}{2}\ell_1(v)|v|^2 + (\gamma+3)J_\gamma(v),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2(\mu^{-1/2})} = -\int \bar{a}\nabla h\nabla h\,\mu^{-1} + \int(\varphi_\mu - M\chi_R)h^2\mu^{-1}.$$

Remark that here we can not conclude as in the proof of Lemma 2.5 because the coefficient of order $\langle v\rangle^{\gamma+2}$ in φ_μ vanishes in the asymptotic $|v| \to \infty$.

From Lemma 2.2, there exists $K>0$ such that $\bar{a}_{ij}\xi_i\xi_j \geq K\langle v\rangle^\gamma|\xi|^2$. We obtain then

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2(\mu^{-1/2})} \leq -K\int\langle v\rangle^\gamma|\nabla h|^2\,\mu^{-1} + \int(\varphi_\mu - M\chi_R)h^2\mu^{-1}$$

and, by integration by parts, we also have

$$\begin{aligned}
&\int|\nabla(\langle v\rangle^{\gamma/2}\mu^{-1/2}h)|^2\\
&\quad= \int\langle v\rangle^\gamma\mu^{-1}|\nabla h|^2 + \frac{\gamma^2}{4}|v|^2\langle v\rangle^{\gamma-4}\mu^{-1}h^2 + \frac{1}{4}|v|^2\langle v\rangle^\gamma h^2\mu^{-1}\\
&\qquad+2\int\frac{\gamma}{2}h\nabla hv\langle v\rangle^{\gamma-2}\mu^{-1} + \frac{1}{2}h\nabla hv\langle v\rangle^\gamma\mu^{-1} + \frac{\gamma}{4}|v|^2\langle v\rangle^{\gamma-2}h^2\mu^{-1}\\
&\quad= \int\langle v\rangle^\gamma|\nabla h|^2\mu^{-1} + \int\left\{-\frac{1}{4}\langle v\rangle^{\gamma+2} - \left(\frac{5}{4}+\frac{\gamma}{2}\right)\langle v\rangle^\gamma - \frac{\gamma^2}{4}\langle v\rangle^{\gamma-2} - \frac{\gamma}{4}(4-\gamma)\langle v\rangle^{\gamma-4}\right\}h^2\mu^{-1}.
\end{aligned}$$

Finally, it follows that

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2(\mu^{-1/2})} \leq -K\int|\nabla(\langle v\rangle^\gamma\mu^{-1/2}h)|^2 + \int(\tilde{\varphi}_\mu - M\chi_R)h^2\mu^{-1}, \tag{2.9}$$

where

$$\begin{aligned}
\tilde{\varphi}_\mu(v) &= \varphi_\mu(v) - \frac{1}{4}\langle v\rangle^{\gamma+2} + C\langle v\rangle^\gamma\\
&= -\frac{1}{4}\langle v\rangle^{\gamma+2} + J_{\gamma+2}(v) - \frac{1}{2}\ell_1(v)|v|^2 + (\gamma+3)J_\gamma(v) + C\langle v\rangle^\gamma.
\end{aligned}$$

Thanks to the asymptotic behaviour of ℓ_1, $J_{\gamma+2}$ and J_γ, and arguing as in Lemma 2.4, we easily get that

$$\varphi_\mu(v) \underset{|v|\to\infty}{\sim} -\frac{1}{4}\langle v\rangle^{\gamma+2} \xrightarrow[|v|\to\infty]{} -\infty.$$

Then, for any $\lambda>0$, we can choose M,R large enough such that $\varphi(v) - M\chi_R(v) \leq -\lambda$ for any $v\in\mathbb{R}^3$. We conclude the proof as in the previous lemma. □

2.3. **Regularisation properties.** We are now interested in regularisation properties of the operator $\mathcal{A}$ and the iterated convolutions of $\mathcal{A}\mathcal{S}_\mathcal{B}$. Let us recall the operator $\mathcal{A}$ defined in (2.3),

$$\mathcal{A}g = \mathcal{A}_0 g + M\chi_R g = (a_{ij}*g)\partial_{ij}\mu - (c*g)\mu + M\chi_R g,$$

for M and R large enough chosen before. Thanks to the smooth cut-off function χ_R, for any $q\in[1,+\infty)$, $p\geq q$ and any weight function m satisfying (W), we easily observe that

$$\|M\chi_R g\|_{L^q(\mu^{-1/2})} \leq C\|\chi_R\mu^{-1/2}m^{-1}\|_{L^{pq/(p-q)}}\|g\|_{L^p(m)} \leq C\|g\|_{L^p(m)}, \tag{2.10}$$

from which we deduce that $M\chi_R \in \mathscr{B}(L^p(m), L^q(\mu^{-1/2}))$.

Let us now focus on the operator $\mathcal{A}_0$.

Lemma 2.7. *Let $\gamma\in(-2,0)$ and $q\in[1,2]$.*

(i) If $1\leq q<3/|\gamma|$ then

$$\|\mathcal{A}_0 g\|_{L^q(\mu^{-1/2})} \lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})} + \|g\|_{L^1} + \|g\|_{L^q}.$$

(ii) If $\gamma\in(-2,-3/2]$ and $3/|\gamma|\leq q\leq 2$ then

$$\|\mathcal{A}_0 g\|_{L^q(\mu^{-1/2})} \lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})} + \|g\|_{L^{\frac{3}{4+\gamma}}}.$$

As a consequence, for any $1\leq p\leq 2$ and m satisfying (W1) there hold:

- $\mathcal{A}\in\mathscr{B}(L^2(\mu^{-1/2}))$;
- $\mathcal{A}\in\mathscr{B}(L^p(m))$ *and moreover* $\mathcal{A}\in\mathscr{B}(L^p(m), L^p(\mu^{-1/2}))$.

Proof. For any $1 \le q \le 2$ we write

$$\|\mathcal{A}_0 g\|_{L^q(\mu^{-1/2})} \le \|(a_{ij} * g)\partial_{ij}\mu\|_{L^q(\mu^{-1/2})} + \|(c * g)\mu\|_{L^q(\mu^{-1/2})},$$

and we estimate each term separately. For the first term, since $|a_{ij}(v - v_*)| \le C\langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}$ and $|\partial_{ij}\mu(v)| \le C\langle v\rangle^2\mu$, we easily obtain

$$\|(a_{ij} * g)\partial_{ij}\mu\|^q_{L^q(\mu^{-1/2})} \lesssim \|g\|^q_{L^1(\langle v\rangle^{\gamma+2})} \int \langle v\rangle^{(\gamma+4)q}\,\mu^{q/2} \lesssim \|g\|^q_{L^1(\langle v\rangle^{\gamma+2})}.$$

For the second term we separate into two cases.

(i) Suppose $1 \le q < 3/|\gamma|$. We decompose $c = c_- + c_+$ with $c_- = c\mathbf{1}_{|\cdot|\le 1}$ and $c_+ = c\mathbf{1}_{|\cdot|>1}$. We easily bound

$$|(c_+ * g)(v)| \lesssim \int_{v_*} \mathbf{1}_{|v-v_*|>1}\,|v - v_*|^\gamma\,|g_*| \lesssim \|g\|_{L^1},$$

hence

$$\|(c_+ * g)\mu\|_{L^q(\mu^{-1/2})} = \|(c_+ * g)\mu^{1/2}\|_{L^q} \lesssim \|g\|_{L^1}.$$

For the other term, we get

$$\begin{aligned}
\|(c_- * g)\mu\|^q_{L^q(\mu^{-1/2})} &\lesssim \int_v \left| \int_{v_*} \mathbf{1}_{|v-v_*|\le 1}\,|v - v_*|^\gamma g_*\,dv_* \right|^q \mu^{q/2} \\
&\lesssim \int_v \int_{v_*} \mathbf{1}_{|v-v_*|\le 1}\,|v - v_*|^{\gamma q}|g_*|^q\,\mu^{q/2} \\
&\lesssim \int_{v_*} \left(\int_v \mathbf{1}_{|v-v_*|\le 1}\,|v - v_*|^{\gamma q}\mu^{q/2}\,dv \right) |g_*|^q \\
&\lesssim C_\mu\,\|g\|^q_{L^q},
\end{aligned}$$

where we have used Jensen's inequality at the first line and, in the last line, the integral in v is bounded since $q < 3/|\gamma|$. This concludes the proof of point (i).

(i) Now suppose $\gamma \in (-2, -3/2]$ and $3/|\gamma| \le q \le 2$. We write then

$$\begin{aligned}
\|(c * g)\mu\|_{L^q(\mu^{-1/2})} = \|(c * g)\mu^{1/2}\|_{L^q} &\le \|(c * g)\|_{L^3}\,\|\mu^{1/2}\|_{L^{\frac{3q}{3-q}}} \\
&\lesssim \|g\|_{L^{\frac{3}{4+\gamma}}}\,\|\mu^{1/2}\|_{L^{\frac{3q}{3-q}}},
\end{aligned}$$

where we have used Hölder's inequality in first line and Hardy-Littlewood-Sobolev inequality in the second one. This gives point (ii).

The conclusion of the lemma is a easy consequence of the above estimates and (2.10), observing that in the case (i) we have

$$\|g\|_{L^1(\langle v\rangle^{\gamma+2})} + \|g\|_{L^1} + \|g\|_{L^q} \lesssim \|g\|_{L^q(m)}$$

and in the case (ii)

$$\|g\|_{L^1(\langle v\rangle^{\gamma+2})} + \|g\|_{L^{\frac{3}{4+\gamma}}} \lesssim \|g\|_{L^q(m)},$$

for any weight function m satisfying (W). □

We prove now a regularisation estimate for the convolution of $\mathcal{AS}_\mathcal{B}(t)$. Let $m_0 := \exp(\kappa_0\langle v\rangle^s)$ and $m_1 := \exp(\kappa_1\langle v\rangle^s)$ be weight functions satisfying (W) with $\kappa_1 > \kappa_0$, so that $m_0 \le Cm_1$.

Lemma 2.8. *Let $\gamma \in (-2, 0)$. Consider $1 \le p \le 2$, then there exists $C > 0$ such that*

$$\forall\, t \ge 0, \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^p(m_1),L^2(m_0))} \le C\,t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}\,e^{-\lambda t}. \tag{2.11}$$

As a consequence, for all $1 \le p \le 2$ and m satisfying assumption (W), for any $\lambda' < \lambda$ ($\lambda > 0$ fixed in Lemma 2.5) we have

$$\forall\, t \ge 0, \qquad \|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)\|_{\mathscr{B}(L^p(m),L^2(\mu^{-1/2}))} \le C\, e^{-\lambda' t}, . \tag{2.12}$$

Proof of Lemma 2.8. We split the proof into two steps.

Step 1. We first prove (2.11) for $p = 1$. Consider the equation $\partial_t f = \mathcal{B}f$. Then from (2.7) we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} = -\int \bar{a}\nabla f \nabla f m^2 + \int (\varphi_{m_0,2} - M\chi_R) m_0^2 f^2$$

From Lemma 2.2, there exists $K > 0$ such that $\bar{a}_{ij}\xi_i\xi_j \ge K\langle v\rangle^\gamma |\xi|^2$, which yields

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} \le -K\int \langle v\rangle^\gamma |\nabla f|^2 m_0^2 + \int (\varphi_{m_0,2} - M\chi_R) m_0^2 f^2,$$

and, from

$$|\nabla(\langle v\rangle^\gamma m_0 f)|^2 \le C\left\{\langle v\rangle^\gamma m_0^2 |\nabla f|^2 + C\langle v\rangle^{\gamma+2s-2} m_0^2 f^2\right\},$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} \le -K\int |\nabla(\langle v\rangle^\gamma m_0 f)|^2 + \int (\tilde{\varphi}_{m_0,2} - M\chi_R) m_0^2 f^2, \tag{2.13}$$

where

$$\tilde{\varphi}_{m_0,2}(v) = \varphi_{m_0,2}(v) + C\langle v\rangle^{\gamma+2s-2}.$$

From Lemma 2.4 we easily see that $\tilde{\varphi}_{m_0,2} \underset{|v|\to+\infty}{\sim} \varphi_{m_0,2}$, then for all $\lambda \ge 0$ we can chose M and R large enough such that $\tilde{\varphi}_{m_0,2}(v) - M\chi_R(v) \le -\lambda$, and moreover estimate (2.8) holds.

Applying the following inequality (which can be obtained by Hölder's inequality followed by Sobolev embedding in dimension $d = 3$):

$$\|\langle v\rangle^\alpha g\|_{L^2} \le c_1 \|\nabla g\|^{3/5}_{L^2} \|\langle v\rangle^{5\alpha/2} g\|^{2/5}_{L^1}$$

with $g = \langle v\rangle^{\gamma/2} m_0 f$ and $\alpha = -\gamma/2$ to (2.13), it follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} &\le -K\|f\|^{10/3}_{L^2(m_0)} \|\langle v\rangle^{-3\gamma/4} f\|^{-4/3}_{L^1(m_0)} - \lambda\|f\|^2_{L^2(m_0)} \\ &\le -K\|f\|^{10/3}_{L^2(m_0)} \|f\|^{-4/3}_{L^1(m_1)} - \lambda\|f\|^2_{L^2(m_0)}. \end{aligned} \tag{2.14}$$

Recall that the weight functions m_0 and m_1 satisfy assumption (W), then Lemma 2.5 holds, more precisely, for all $t \ge 0$,

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^p(m_0)} \le e^{-\lambda t}\|f\|_{L^p(m_0)} \quad \text{and} \quad \|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^p(m_1)} \le e^{-\lambda t}\|f\|_{L^p(m_1)}. \tag{2.15}$$

Let us denote now

$$X(t) := \|f(t)\|^2_{L^2(m_0)} \quad \text{and} \quad Y(t) := \|f(t)\|_{L^1(m_1)}.$$

For all $t \ge 0$ we have $Y(y) \le Y_0$ from (2.15), which together with (2.14) gives

$$\dot{X}(t) \le -2KX(t)^{1+2/3} Y_0^{-4/3} - 2\lambda X(t). \tag{2.16}$$

Arguing as [11, Lemma 3.9] we obtain that

$$\forall\, t \ge 0 \qquad X(t) \le C\, t^{-3/2}\, e^{-2\lambda t}\, Y_0^2,$$

which concludes the proof of (2.11) when $p = 1$. Then for any $1 < p < 2$ we use Riesz-Thorin interpolation theorem, with $\mathcal{S}_{\mathcal{B}} : L^2(m_0) \to L^2(m_0)$ and $\mathcal{S}_{\mathcal{B}} : L^1(m_1) \to L^2(m_0)$, to conclude to (2.11).

Step 2. Let us prove now (2.12). From Lemma 2.7 we have the following estimates, for any $p \in [1,2]$,

$$\|\mathcal{A}g\|_{L^2(\mu^{-1/2})} \lesssim \|g\|_{L^2(m_0)}, \qquad \|\mathcal{A}g\|_{L^p(m_0)} \lesssim \|g\|_{L^p(m)}. \tag{2.17}$$

Hence, by (2.17) and (2.11), for $1 \le p \le 2$, it follows

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(\mu^{-1/2})} \lesssim \|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(m_0)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{-\lambda t} \|f\|_{L^p(m_1)}. \tag{2.18}$$

Computing the convolution of $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)$ we have

$$\begin{aligned}
\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)f\|_{L^2(\mu^{-1/2})} &\lesssim \int_0^t \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^2(\mu^{-1/2})}\, ds \\
&\lesssim \int_0^t \|\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^2(m_0)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{-\lambda(t-s)} \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^p(m_1)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{-\lambda(t-s)} \|\mathcal{S}_{\mathcal{B}}(s)f\|_{L^p(m)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{-\lambda(t-s)}\, e^{-\lambda s} \|f\|_{L^p(m)}\, ds \\
&\lesssim t^{(\frac{7}{4}-\frac{3}{2p})}\, e^{-\lambda t} \|f\|_{L^p(m)} \\
&\lesssim e^{-\lambda' t} \|f\|_{L^p(m)},
\end{aligned}$$

where we have used successively (2.17), (2.11), (2.17) and Lemma 2.5 with $1 \le p \le 2$, which concludes the proof. □

2.4. **Proof of Theorem 2.1.** We are know able to prove Theorem 2.1 that extends to various weighted L^p-spaces the semigroup decay estimate known to hold on $L^2(\mu^{-1/2})$ as presented Proposition 1.1.

Let $E = L^2(\mu^{-1/2})$, in which space we already know that there is a spectral gap $\lambda_0 > 0$ from Proposition 1.1, and $\mathcal{E} = L^p(m)$, for any $p \in [1,2]$ and m satisfying assumption (W). We consider the decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ as in (2.3). For any $\lambda > 0$, the operator $\mathcal{B} + \lambda$ is hypo-dissipative in $\mathcal{E}$ from Lemma 2.5, moreover $\mathcal{A} \in \mathscr{B}(\mathcal{E})$ and $A \in \mathscr{B}(E)$ from Lemma 2.7. Finally, from Lemma 2.8 we have that $(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t) \in \mathscr{B}(\mathcal{E}, E)$ with an exponential decay rate $\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)\|_{\mathscr{B}(\mathcal{E},E)} \le C_{\lambda'}\, e^{-\lambda' t}$ for any $\lambda' < \lambda$. Then the result of Theorem 2.1 follows from [11, Theorem 2.13].

3. A priori estimates

The purpose of this section is to establish a priori estimates for the (nonlinear) Landau equation that will be of crucial importance in the proof of the main results in Section 4.

Let us recall the Landau equation that is given by

$$\partial_t f = Q(f,f)$$

with

$$Q(g,f) = \nabla \cdot \{(a * g)\nabla f - (b * g) f\} = (a_{ij} * g)\partial_{ij} f - (c * g) f.$$

3.1. **Preliminaries.** Denoting $\bar a_g = a * g$, $\bar b_g = b * g$, $\bar c_g = c * g$ and considering some weight function m, we easily compute

$$\begin{aligned}\int Q(g,f)\, f^{p-1}\, m^p &= \int \nabla\cdot\{\bar a_g \nabla f - \bar b f\} f^{p-1}\, m^p \\ &= -\int \bar a_g \nabla f \nabla (f^{p-1}) m^p - \int \bar a_g \nabla f \nabla m^p\, f^{p-1} \\ &\quad + \int \bar b_g f \nabla (f^{p-1}) m^p + \int \bar b_g \nabla m^p\, f^p.\end{aligned}$$

It follows that

$$\begin{aligned}\int Q(g,f)\, f^{p-1}\, m^p &= -\frac{4}{p}(1-1/p)\int_v \bar a_g \nabla (f^{p/2}) \nabla (f^{p/2})\, m^p \\ &\quad + \int_v\int_{v_*} \Theta_{m,p}(v,v_*)\, g_*\, f^p\, m^p \\ &\quad + (1/p-1)\int_v\int_{v_*} c(v-v_*)\, g_*\, f^p\, m^p\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\Theta_{m,p}(v,v_*) = a(v-v_*) : \frac{D^2 m}{m}(v) + (p-1)\, a(v-v_*)\, \frac{\nabla m}{m}(v)\, \frac{\nabla m}{m}(v) \\ + 2b(v-v_*)\cdot \frac{\nabla m}{m}(v).\end{aligned} \tag{3.2}$$

In the particular case of a polynomial weight $m = \langle v\rangle^k$, we have

$$\begin{aligned}\Theta_{m,p}(v,v_*) &= k|v-v_*|^\gamma \langle v\rangle^{-2}(-2\langle v\rangle^2 + 2\langle v_*\rangle^2) \\ &\quad + k(kp-2)|v-v_*|^\gamma \langle v\rangle^{-4}\left[|v|^2|v_*|^2 - (v\cdot v_*)^2\right].\end{aligned} \tag{3.3}$$

We recall the following elementary interpolation inequalities.

Lemma 3.1. *Let $k, \ell \in \mathbb{R}_+$. For all $\varepsilon > 0$ there is C_ε such that*

$$\|g\|^2_{\dot H^k} \le \varepsilon \|g\|^2_{\dot H^{k+\ell}} + C_\varepsilon \|g\|^2_{L^1},$$
$$\|g\|^2_{\dot H^k} \le \varepsilon \|g\|^2_{\dot H^{k+\ell}} + C_\varepsilon \|g\|^2_{L^2}.$$

Moreover, we have an interpolation inequality for weighted Sobolev spaces from [9]:

Lemma 3.2. *For any $\delta, \alpha \ge 0$ and $k \in \mathbb{R}$, there holds*

$$\|f\|^2_{H^k_l} \le C_\delta \|f\|_{H^{k-\delta}_{l+\alpha}}\, \|f\|_{H^{k+\delta}_{l-\alpha}}.$$

Now we state a technical lemma that will be useful in the estimates of weighted L^2-type norms.

Lemma 3.3. *Let $0 < \alpha < d$. Consider smooth nonnegative functions $f, g, h : \mathbb{R}^d \to \mathbb{R}$ and define*

$$K_\alpha(f,g,h) := \iint |v-v_*|^{-\alpha}\, f_*\, g\, h\, dv_*\, dv.$$

Let $\ell_0 \le \alpha$ and $\ell_1 + \ell_2 = -\ell_0$, then the following estimates hold:

(1) For any $\sigma \ge 0$ such that $2\sigma < d$ and $2(\alpha - \sigma) < d$ we have

$$K_\alpha(f,g,h) \lesssim \|\langle v\rangle^{\ell_0} f\|_{L^1}\, \|\langle v\rangle^{\ell_1} g\|_{L^2}\, \|\langle v\rangle^{\ell_2} h\|_{L^2} + \|\langle v\rangle^{\ell_0} f\|_{L^1}\, \|\langle v\rangle^{\ell_1} g\|_{\dot H^{\alpha-\sigma}}\, \|\langle v\rangle^{\ell_2} h\|_{\dot H^\sigma}.$$

(2) For any $0 < \sigma < \alpha$ we have

$$K_\alpha(f,g,h) \lesssim \|\langle v\rangle^{\ell_0} f\|_{L^1} \|\langle v\rangle^{\ell_1} g\|_{L^2} \|\langle v\rangle^{\ell_2} h\|_{L^2} + \|\langle v\rangle^{\ell_0} f\|_{L^{\frac{d}{d-\alpha+\sigma}}} \|\langle v\rangle^{\ell_1} g\|_{H^\sigma} \|\langle v\rangle^{\ell_2} h\|_{L^2}.$$

Proof. Denote $F_* = \langle v_*\rangle^{\ell_0}|f_*|$, $G = \langle v\rangle^{\ell_1}|g|$ and $H = \langle v\rangle^{\ell_2}|h|$ such that $\ell_1 + \ell_2 = -\ell_0$, and split the integral into two parts, $K_1 := \iint \mathbf{1}_{\{|v-v_*|\le 1\}}$ and $K_2 := \iint \mathbf{1}_{\{|v-v_*|>1\}}$. Then

$$\begin{aligned} K_2 &= \iint \mathbf{1}_{\{|v-v_*|>1\}} |v-v_*|^{-\alpha} \langle v_*\rangle^{-\ell_0} \langle v\rangle^{\ell_0} F_* G H \, dv_* \, dv \\ &\lesssim \iint F_* G H = \|\langle v\rangle^{\ell_0} f\|_{L^1} \|\langle v\rangle^{\ell_1} g\|_{L^2} \|\langle v\rangle^{\ell_2} h\|_{L^2}, \end{aligned}$$

where we have used, since $\ell_0 \le \alpha$,

$$|v-v_*|^{-\alpha} \mathbf{1}_{\{|v-v_*|>1\}} \langle v_*\rangle^{-\ell_0} \langle v\rangle^{\ell_0} \le 2^{\alpha/2} \mathbf{1}_{\{|v-v_*|>1\}} \frac{\langle v-v_*\rangle^{\ell_0}}{|v-v_*|^\alpha} \le C.$$

This gives the first term in the estimates above, both for points (1) and (2). For the term K_1 we split into two cases.

(1) Using that $\langle v_*\rangle^{-\ell_0} \langle v\rangle^{\ell_0} \mathbf{1}_{\{|v-v_*|\le 1\}} \le C$ we obtain

$$K_1 \lesssim \iint |v-v_*|^{-\alpha} F_* G H = \int_{v_*} F_* \left\{ \int_v |v-v_*|^{-\alpha} G H \right\}$$

and we need to estimate the integral in v. Using Pitt's inequality [3], for any $\sigma \ge 0$ such that $2\sigma < d$ and $2(\alpha - \sigma) < d$, we get

$$\begin{aligned} \int_v |v-v_*|^{-\alpha} G H &\le \left(\int |v-v_*|^{-2(\alpha-\sigma)} G^2 \right)^{1/2} \left(\int |v-v_*|^{-2\sigma} H^2 \right)^{1/2} \\ &\lesssim \left(\int |\xi|^{2(\alpha-\sigma)} |\hat{G}|^2 \right)^{1/2} \left(\int |\xi|^{2\sigma} |\hat{H}|^2 \right)^{1/2} \\ &\lesssim \|\langle v\rangle^{\ell_1} g\|_{\dot{H}^{\alpha-\sigma}} \|\langle v\rangle^{\ell_2} h\|_{\dot{H}^\sigma}. \end{aligned}$$

(2) Using Hardy-Littlewood-Sobolev inequality, for any $0 < \sigma < \alpha$, we get

$$K_1 \lesssim \iint |v-v_*|^{-\alpha} F_* G H \lesssim \|F\|_{L^{\frac{d}{d-\alpha+\sigma}}} \|GH\|_{L^{\frac{d}{d-\sigma}}}.$$

Using Hölder's inequality and the Sobolev embedding $H^\sigma(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2\sigma}}(\mathbb{R}^d)$, it follows that

$$\|GH\|_{L^{\frac{d}{d-\sigma}}} \le \|G\|_{L^{\frac{2d}{d-2\sigma}}} \|H\|_{L^2} \lesssim \|\langle v\rangle^{\ell_1} g\|_{H^\sigma} \|\langle v\rangle^{\ell_2} h\|_{L^2},$$

which completes the proof. □

We state next a result from [7, Proposition 4] (see also [1]) concerning ellipticity properties of the matrix $a * f$.

Lemma 3.4. *Let $\gamma \in [-2, 1]$ and $f \in L^1_2 \cap L\log L(\mathbb{R}^d)$. Then there exists $K > 0$ depending on $\|f\|_{L^1_2 \cap L\log L}$ such that*

$$(a * f)(v) \ge K \langle v\rangle^\gamma \, I_d.$$

The proof of this result is stated in [7] in the case $\gamma \in (0, 1]$, however we easily observe that the result is also valid for $\gamma \ge -2$ by following the proof.

3.2. **Moments estimates.** The moments of solutions to the Landau equation in the case of soft potentials is known to be propagated linearly in time, as is stated in [23, Section 2.4, p. 73]. We give however a proof of this fact for the sake of completeness and because we shall need a precise estimate in order to use it later for the stretched exponential moments in Lemma 3.6.

Lemma 3.5. *Let $\gamma \in (-2,0)$, $f_0 \in L^1_2 \cap L\log L$ and consider a weak solution $f \in L^\infty([0,\infty); L^1_2 \cap L\log L)$ to the Landau equation associated to f_0. Suppose further that $f_0 \in L^1_l$ for some $l > 2$. Then, at least formally, there exists a constant $C > 0$ depending on $\|f\|_{L^\infty([0,\infty);L^1_2)}$ and $\|f_0\|_{L^1_l}$ (but not on l) such that*

$$\forall\, t \geq 0 \qquad \|f(t)\|_{L^1_l} \leq C\,\alpha(l)\,(1+t)$$

with

$$\alpha(l) := \begin{cases} l^2, & l \leq 4, \\ \dfrac{l^{2-\frac{4}{\gamma+2}}}{l-4}\left(\dfrac{l-4}{l+\gamma-2}\right)^{\frac{l-4}{\gamma+2}} l^{\frac{l}{\gamma+2}}, & l > 4. \end{cases}$$

Proof. The equation for the moments is

$$\frac{d}{dt}\|f\|_{L^1_l} = \iint |v-v_*|^\gamma \left\{-2l + 2l\langle v\rangle^{-2}\langle v_*\rangle^2 + l(l-2)\langle v\rangle^{-4}[|v|^2|v_*|^2 - (v\cdot v_*)^2]\right\} f_*\, f\, \langle v\rangle^l.$$

Because of the singularity of $|v-v_*|^\gamma$, we split it into two parts $|v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\geq 1\}}$ and $|v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}$, denoting respectively T_1 and T_2 each associated term. Using that $|v|^2|v_*|^2 - (v\cdot v_*)^2 \leq \langle v\rangle^2\langle v_*\rangle^2$, we obtain for T_1 that

$$\begin{aligned} T_1 \leq\ & -2l \iint |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\geq 1\}}\, f_*\, f\, \langle v\rangle^l \\ & + l^2 \iint |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\geq 1\}}\, \langle v\rangle^{l-2}\langle v_*\rangle^2 f_*\, f, \end{aligned}$$

from which we get

$$T_1 \leq -Kl\|f\|_{L^1_{l+\gamma}} + Cl^2\|f\|_{L^1_{l-2}}, \tag{3.4}$$

for constants $K, C > 0$, using the conservation of mass and energy.

For the term T_2, we write

$$\begin{aligned} T_2 =\ & l \iint |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^{l-2} \left\{-2\langle v\rangle^2 + 2\langle v_*\rangle^2\right\} f_*\, f \\ & + l(l-2) \iint |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^{l-4} \left\{|v|^2|v_*|^2 - (v\cdot v_*)^2\right\} f_*\, f =: T_{21} + T_{22}. \end{aligned}$$

Using Hölder's inequality

$$\begin{aligned} & \iint f f_* |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^{l-2}\langle v_*\rangle^2 \\ & \quad \leq \left(\iint f f_* |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^l\right)^{(l-2)/l} \left(\iint f f_* |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v_*\rangle^l\right)^{2/l} \\ & \quad = \iint f f_* |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^l \end{aligned}$$

and this implies $T_{21} \leq 0$. Moreover, using the inequality $|v|^2|v_*|^2 - (v\cdot v_*)^2 \leq |v||v_*||v-v_*|^2$, we obtain

$$\begin{aligned} T_{22} & \leq Cl^2 \iint f f_* |v-v_*|^{\gamma+2} \mathbf{1}_{\{|v-v_*|\leq 1\}}\, \langle v\rangle^{l-3}\langle v_*\rangle \\ & \leq Cl^2 \|f\|_{L^1_1}\, \|f\|_{L^1_{l-3}} \leq Cl^2\|f\|_{L^1_{l-2}}, \end{aligned}$$

where we have used $|v-v_*|^{\gamma+2}\mathbf{1}_{\{|v-v_*|\le 1\}} \le 1$ and $\|f\|_{L^1_1}$ uniformly bounded. Gathering T_1 and T_2, it follows that

$$\frac{d}{dt}\|f\|_{L^1_l} \le -Kl\|f\|_{L^1_{l+\gamma}} + Cl^2\|f\|_{L^1_{l-2}}.$$

If $l \le 4$ then $\|f\|_{L^1_{l-2}}$ is uniformly bounded and we easily conclude.

Consider then $l > 4$. Since $\gamma > -2$, denoting $r = (l+\gamma-2)/(l-4) > 1$ and $r' = r/(r-1) = (l+\gamma-2)/(\gamma+2)$, it follows by Hölder and Young's inequality that

$$\|f\|_{L^1_{l-2}} \le \|f\|_{L^1_2}^{1/r'}\,\|f\|_{L^1_{l+\gamma}}^{1/r} \le \frac{1}{r'}\,\eta^{-\frac{l-4}{\gamma+2}}\|f\|_{L^1_2} + \frac{\eta}{r}\|f\|_{L^1_{l+\gamma}},$$

for all $\eta > 0$. We obtain

$$\frac{d}{dt}\|f\|_{L^1_l} \le -Kl\|f\|_{L^1_{l+\gamma}} + Cl^2\frac{\eta}{r}\|f\|_{L^1_{l+\gamma}} + C\frac{l^2}{r'}\,\eta^{-\frac{l-4}{\gamma+2}}\|f\|_{L^1_2} \le Cl^{2-\frac{4}{\gamma+2}}\,l^{\frac{l}{\gamma+2}}\,\frac{r^{-\frac{l-4}{\gamma+2}}}{r'}\,\|f\|_{L^1_2},$$

choosing $\eta = Kr/(Cl)$, from which we conclude to

$$\|f(t)\|_{L^1_l} \le C\,\alpha(l)\,(1+t).$$

□

As a consequence of the above result, we deduce a similar linearly growing estimate for some stretched exponential moments.

Lemma 3.6. *Let $\gamma \in (-2,0)$, $f_0 \in L^1_2 \cap L\log L$ and consider a weak solution $f \in L^\infty([0,\infty); L^1_2 \cap L\log L)$ to the Landau equation associated to f_0. Suppose further that $f_0 \in L^1(e^{\kappa\langle v\rangle^s})$ with $\kappa > 0$ and $0 < s < 2+\gamma$. Then, at least formally, there exists a constant $C > 0$ depending on $\|f\|_{L^\infty([0,\infty);L^1_2)}$, $\|f_0\|_{L^1(e^{\kappa\langle v\rangle^s})}$, κ and s such that*

$$\forall\, t \ge 0, \qquad \|f(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} \le C(1+t).$$

Proof. Write

$$e^{\kappa\langle v\rangle^s} = \sum_{j=0}^{\infty} \kappa^j \frac{\langle v\rangle^{js}}{j!}$$

and then, using Lemma 3.5, we have

$$\begin{aligned}\|f(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} &= \sum_{j=0}^{\infty}\frac{\kappa^j}{j!}\int f(t)\langle v\rangle^{js} \\ &\le \sum_{j=0}^{\infty}\frac{\kappa^j}{j!}\left\{C\,\alpha(sj)\,t + \int f_0\langle v\rangle^{js}\right\} = Ct\sum_{j=0}^{\infty}\frac{\kappa^j}{j!}\,\alpha(sj) + \|f_0\|_{L^1(e^{\kappa\langle v\rangle^s})},\end{aligned}$$

and we only need to prove that the sum is finite. Let $j_0 \in \mathbb{N}$ such that $sj_0 \le 4 < s(j_0+1)$. Then we have

$$\sum_{j=j_0+1}^{\infty}\frac{\kappa^j}{j!}\,\alpha(sj) = \sum_{j=j_0+1}^{\infty}\kappa^j\,\frac{(sj)^{2-\frac{4}{\gamma+2}}}{sj-4}\,\left(\frac{sj-4}{sj+\gamma-2}\right)^{\frac{sj-4}{\gamma+2}}\,s^{\frac{sj}{\gamma+2}}\,\frac{j^{(\frac{s}{\gamma+2})j}}{j!},$$

which is finite if $s < \gamma+2$. □

3.3. **Regularity estimates.** We shall establish coercivity estimates for the Landau operator Q, which are inspired by some similar estimates obtained by Wu [24] and Alexandre, Lao and Lin [1].

Lemma 3.7. *Let $\gamma \in (-2,0)$. Then for smooth functions f and g, there are constants $K, C > 0$ depending on $\|f\|_{L^1_2 \cap L\log L}$ such that:*

(i) If $0 \le k \le (\gamma+3)/2$ then

$$\big\langle Q(f,g), g\langle v\rangle^{2k}\big\rangle \le -K\|g\|^2_{\dot H^1_{k+\gamma/2}} + C\|g\|^2_{L^2_{k+\gamma/2}}.$$

(ii) If $k > (\gamma+3)/2$ then

$$\big\langle Q(f,g), g\langle v\rangle^{2k}\big\rangle \le -K\|g\|^2_{\dot H^1_{k+\gamma/2}} - K'\|g\|^2_{L^2_{k+\gamma/2}} + C\|g\|^2_{L^2_{k-1}}.$$

Proof. From (3.1) and (3.3) for $m = \langle v\rangle^k$, we obtain

$$\begin{aligned}
\big\langle Q(f,g), g\langle v\rangle^{2k}\big\rangle &= -\int_v (a*f)\nabla g \nabla g \,\langle v\rangle^{2k} \\
&\quad + (\gamma+3-2k)\int_v\int_{v_*} |v-v_*|^\gamma\, f_*\, g^2\,\langle v\rangle^{2k} \\
&\quad + 2k\int_v\int_{v_*} |v-v_*|^\gamma \langle v\rangle^{-2}\langle v_*\rangle^2\, f_*\, g^2\,\langle v\rangle^{2k} \\
&\quad + 2k(k-1)\int_v\int_{v_*} |v-v_*|^\gamma \langle v\rangle^{-4}[|v|^2|v_*|^2 - (v\cdot v_*)^2]\, f_*\, g^2\,\langle v\rangle^{2k} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.5}$$

For the first term I_1, we use the coercivity property of $\bar a$, since $f \in L^1_2 \cap L\log L$, we have from Lemma 3.4 that

$$\bar a(v) = (a*f)(v) \ge K\langle v\rangle^\gamma I_3.$$

Then we get

$$I_1 \le -K\|\langle v\rangle^{\gamma/2+k}\nabla g\|^2_{L^2} = -K\|g\|^2_{\dot H^1_{k+\gamma/2}},$$

which can also be written as

$$I_1 \le -K\|g\|^2_{\dot H^1_{k+\gamma/2}} \le -K\|\langle v\rangle^{\gamma/2+k} g\|^2_{\dot H^1} + C\|g\|^2_{L^2_{k+\gamma/2-1}}.$$

For the second term I_2, we split into two cases. If $k \le (\gamma+3)/2$ we have, from Lemma 3.3 and the interpolation inequality from Lemma 3.1, that

$$\begin{aligned}
|I_2| &\lesssim \iint |v-v_*|^\gamma\, f_*\, g^2\,\langle v\rangle^{2k} \\
&\lesssim \|\langle v\rangle^{-\gamma} f\|_{L^1}\big\{C_\epsilon\|\langle v\rangle^{\gamma/2+k} g\|^2_{L^2} + \epsilon\|\langle v\rangle^{\gamma/2+k} g\|^2_{\dot H^1}\big\},
\end{aligned}$$

for any $\epsilon > 0$. However, if $k > (\gamma+3)/2$, we get

$$I_2 \le -K'\int_v\int_{v_*} |v-v_*|^\gamma f_*\, g^2\,\langle v\rangle^{2k} \le -K\|\langle v\rangle^{\gamma/2+k} g\|^2_{L^2}.$$

Finally, using that $|v|^2|v_*|^2 - (v\cdot v_*)^2 \le \langle v\rangle^2\langle v_*\rangle^2$ we easily get

$$I_3 + I_4 \lesssim \int_v\int_{v_*} |v-v_*|^\gamma \langle v\rangle^{-2}\langle v_*\rangle^2\, f_*\, g^2\,\langle v\rangle^{2k}.$$

Then, arguing as in the proof of Lemma 3.3 (term K_1 in that lemma) and using again Lemma 3.1, it follows that

$$\begin{aligned} I_3 + I_4 &\lesssim \|\langle v\rangle^2 f\|_{L^1} \|\langle v\rangle^{k-1} g\|^2_{\dot H^{-\gamma/2}} \\ &\lesssim C_\epsilon \|\langle v\rangle^{k-1} g\|^2_{L^2} + \epsilon \|\langle v\rangle^{k-1} g\|^2_{\dot H^1} \\ &\lesssim C_\epsilon \|\langle v\rangle^{k-1} g\|^2_{L^2} + \epsilon \|\langle v\rangle^{\gamma/2+k} g\|^2_{\dot H^1}. \end{aligned}$$

for any $\epsilon > 0$. We then conclude gathering all previous estimates and taking $\epsilon > 0$ small enough. □

We also prove an upper bound for Q in the following lemma. It is worth mentioning that He [13] obtain similar estimates by a different method.

Lemma 3.8. *Let $\gamma \in (-2, 0)$ and consider smooth functions f, g and h. Then for any $\ell_1 + \ell_2 = \gamma + 2$ we have*

$$\big|\big\langle Q(f,g), h\langle v\rangle^{2k}\big\rangle\big| \lesssim \|f\|_{L^1_{\gamma+2}} \|g\|_{H^1_{\ell_1+k}} \|h\|_{H^1_{\ell_2+k}}.$$

Proof. We write

$$\big\langle Q(f,g), g\langle v\rangle^{2k}\big\rangle = \int \nabla\{(a*f)\nabla g\} h\langle v\rangle^{2k} - \int \nabla\{(b*f)g\} h\langle v\rangle^{2k} =: T_1 + T_2.$$

For the first term, we easily obtain, since $|a(v-v_*)| \lesssim |v-v_*|^{\gamma+2} \lesssim \langle v_*\rangle^{\gamma+2}\langle v\rangle^{\gamma+2}$, that

$$\begin{aligned} T_1 &\lesssim \iint |v-v_*|^{\gamma+2} |f_*| |\nabla g| |\nabla h| \langle v\rangle^{2k} + \iint |v-v_*|^{\gamma+2} |f_*| |\nabla g| |h| \langle v\rangle^{2k-1} \\ &\lesssim \|f\|_{L^1_{\gamma+2}} \Big\{ \|\nabla g\|_{L^2_{\ell_1+k}} \|\nabla h\|_{L^2_{\ell_2+k}} + \|\nabla g\|_{L^2_{\ell_1+k-1/2}} \|h\|_{L^2_{\ell_2+k-1/2}} \Big\}. \end{aligned}$$

Moreover, for the second term, it follows that

$$T_2 \lesssim \iint |v-v_*|^{\gamma+1} |f_*| |g| |\nabla h| \langle v\rangle^{2k} + \iint |v-v_*|^{\gamma+1} |f_*| |g| |h| \langle v\rangle^{2k-1}.$$

Now we investigate two different cases. If $\gamma + 1 \geq 0$, using $|v-v_*|^{\gamma+1} \lesssim \langle v_*\rangle^{\gamma+1}\langle v\rangle^{\gamma+1}$ we obtain

$$T_2 \lesssim \|f\|_{L^1_{\gamma+1}} \Big\{ \|g\|_{L^2_{\ell_1+k-1/2}} \|\nabla h\|_{L^2_{\ell_2+k-1/2}} + \|g\|_{L^2_{\ell_1+k-1}} \|h\|_{L^2_{\ell_2+k-1}} \Big\}.$$

On the other hand, if $\gamma + 1 < 0$, i.e. $-2 < \gamma < -1$, we use Lemma 3.3 to get

$$\begin{aligned} T_2 \lesssim \|f\|_{L^1_{-(\gamma+1)}} \Big\{ &\|\langle v\rangle^{\ell_1+k-1/2} g\|_{L^2} \|\langle v\rangle^{\ell_2+k-1/2} \nabla h\|_{L^2} + \|\langle v\rangle^{\ell_1+k-1/2} g\|_{\dot H^{-(\gamma+1)}} \|\langle v\rangle^{\ell_2+k-1/2} \nabla h\|_{L^2} \\ &+ \|\langle v\rangle^{\ell_1+k-1} g\|_{L^2} \|\langle v\rangle^{\ell_2+k-1} h\|_{L^2} + \|\langle v\rangle^{\ell_1+k-1} g\|_{\dot H^{-(\gamma+1)}} \|\langle v\rangle^{\ell_2+k-1} h\|_{L^2} \Big\}. \end{aligned}$$

We conclude gathering the above estimates. □

We prove now some estimates for weighted L^2 and Sobolev norms.

Proposition 3.9. *Let $\gamma \in (-2, 0)$, $f_0 \in L^1_2 \cap L\log L(\mathbb{R}^3)$ and consider a weak solution $f \in L^\infty([0,\infty); L^1_2 \cap L\log L)$ of the Landau equation associated to f_0. Then, at least formally, there holds:*

(1) Let $0 \leq k \leq 2 + 3\gamma/4$. Then for any $t_0 > 0$ there is $C = C(t_0) > 0$ such that

$$\sup_{t \geq t_0} \left\{ \|f(t)\|^2_{L^2_k} + \int_t^{t+1} \|f(\tau)\|^2_{H^1_{k+\gamma/2}} \, d\tau \right\} \leq C.$$

(2) Let $k > 2 + 3\gamma/4$ and suppose $f_0 \in L^1_{k-3\gamma/4}$. Then for any $t_0 > 0$ there is $C = C(t_0) > 0$ such that

$$\forall\, t \geq t_0, \quad \|f(t)\|^2_{L^2_k} + \int_0^t \|f(\tau)\|^2_{H^1_{k+\gamma/2}} \, d\tau \leq C(1+t).$$

Proof. (1) From Proposition 3.7, for $0 \le k \le 2 + 3\gamma/4$, we have

$$\begin{aligned} \frac{d}{dt}\|f\|_{L^2_k}^2 &\le -K\|f\|_{\dot H^1_{k+\gamma/2}}^2 + C\|f\|_{L^2_{k+\gamma/2}}^2 \\ &\le -K\|\langle v\rangle^{\gamma/2+k} f\|_{\dot H^1}^2 + C\|f\|_{L^2_{k+\gamma/2}}^2. \end{aligned} \tag{3.6}$$

Using the following inequality (obtained by Hölder and Sobolev's inequalities in dimension $d = 3$),

$$\|\langle v\rangle^{\alpha} u\|_{L^2} \le C\|\nabla u\|_{L^2}^{3/5} \, \|\langle v\rangle^{5\alpha/2} u\|_{L^1}^{2/5}, \tag{3.7}$$

we obtain that, choosing $\alpha = -\gamma/2$,

$$\frac{d}{dt}\|f\|_{L^2_k}^2 \le -K\|f\|_{L^1_{k-3\gamma/4}}^{-4/3} \|f\|_{L^2_k}^{2+4/3} + C\|f\|_{L^2_{k+\gamma/2}}^2. \tag{3.8}$$

Since $\|f(t)\|_{L^1_{k-3\gamma/4}} \le \|f(t)\|_{L^1_2}$ is uniformly bounded in time, we finally get, applying Young's inequality for the last term,

$$\frac{d}{dt}\|f\|_{L^2_k}^2 \le -K\|f\|_{L^2_k}^{2+4/3} + C,$$

from which we deduce by standard arguments that for any $t_0 > 0$ there exists $C = C(t_0) > 0$ such that

$$\sup_{t \ge t_0} \|f(t)\|_{L^2_k}^2 \le C.$$

Coming back to (3.6) we also obtain

$$\sup_{t \ge t_0} \int_t^{t+1} \|f(\tau)\|_{H^1_{k+\gamma/2}}^2 \, d\tau \le C.$$

(2) Remark that $k > 2 + 3\gamma/4 > (\gamma + 3)/2$, hence Proposition 3.7 yields

$$\begin{aligned} \frac{d}{dt}\|f\|_{L^2_k}^2 &\le -K\|f\|_{\dot H^1_{k+\gamma/2}}^2 - K\|f\|_{L^2_{k+\gamma/2}}^2 + C\|f\|_{L^2_{k-1}}^2 \\ &\le -K\|\langle v\rangle^{k+\gamma/2} f\|_{\dot H^1}^2 - K\|f\|_{L^2_{k+\gamma/2}}^2 + C\|f\|_{L^2_{k-1}}^2. \end{aligned} \tag{3.9}$$

Using the interpolation inequality, for any $\delta, \varepsilon > 0$,

$$\|f\|_{L^2_\alpha}^2 \le \varepsilon \|f\|_{L^2_{\alpha+\delta}}^2 + C_\varepsilon \|f\|_{L^2}^2,$$

for $\varepsilon > 0$ small enough and (3.7), we finally get

$$\begin{aligned} \frac{d}{dt}\|f\|_{L^2_k}^2 &\le -K\|\langle v\rangle^{\gamma/2+k} f\|_{\dot H^1}^2 - K\|f\|_{L^2_{k+\gamma/2}}^2 + C\|f\|_{L^2}^2 \\ &\le -K\|f\|_{L^1_{k-3\gamma/4}}^{-4/3} \|f\|_{L^2_k}^{2+4/3} - K\|f\|_{L^2_{k+\gamma/2}}^2 + C\|f\|_{L^2}^2. \end{aligned} \tag{3.10}$$

Now we fix some $t_0 > 0$. From point (1) we know that there exists $C > 0$ such that $\sup_{t \ge t_0/2} \|f(t)\|_{L^2}^2 \le C$. Moreover, since $f_0 \in L^1_{k-3\gamma/4}$, Lemma 3.5 implies $\|f(t)\|_{L^1_{k-3\gamma/4}} \le C(1+t)$ for any $t \ge 0$, so that $\sup_{[t_0/2, 3t_0/2]} \|f(t)\|_{L^1_{k-3\gamma/4}} < \infty$. Writing (3.10) for $t \in [t_0/2, 3t_0/2]$, we obtain by standard arguments that for any $t_1 > t_0/2$ we have $\sup_{[t_1, 3t_0/2]} \|f(t)\|_{L^2_k} < \infty$. Coming back to (3.10) and neglecting the negative terms, we obtain

$$\forall\, t \ge t_0, \qquad \frac{d}{dt}\|f\|_{L^2_k}^2 \le C,$$

from which we have

$$\|f(t)\|_{L^2_k}^2 \le C \int_{t_0}^t d\tau + \|f(t_0)\|_{L^2_k}^2 \le C(1+t).$$

We also deduce

$$\int_{t_0}^{t} \|f(\tau)\|^2_{H^1_{k+\gamma/2}}\, d\tau \le C(1+t)$$

coming back to (3.9) and using the previous bound. □

Proposition 3.10. *Let $\gamma \in (-2,0)$, $f_0 \in L^1_2 \cap L\log L(\mathbb{R}^3)$ and consider a weak solution $f \in L^\infty([0,\infty); L^1_2 \cap L\log L)$ of the Landau equation associated to f_0. Then, at least formally, there hold:*

(1) Suppose $f_0 \in L^1_{k-5\gamma/4}$. Then for any $t_0 > 0$ there exists $C = C(t_0) > 0$ such that, for all $t \ge t_0$,

$$\|f(t)\|^2_{H^1_k} + \int_{t_0}^{t} \|f(\tau)\|^2_{H^2_{k+\gamma/2}}\, d\tau \le C(1+t)^2.$$

(2) Suppose $f_0 \in L^1_{k-7\gamma/4} \cap L^1_{2l-k-7\gamma/4}$ with $l := \max(\gamma+4, k+\gamma/2+2)$. Then for any $t_0 > 0$ there exists $C = C(t_0) > 0$ such that, for all $t \ge t_0$,

$$\|f(t)\|_{H^2_k} \le C(1+t)^{7/2}.$$

(3) Suppose further that the weak solution satisfies $f \in L^\infty([0,\infty); L^1_l)$ for any $l \ge 0$. Then for all $t_1 > 0$, any $k \ge 0$ and $n \in \mathbb{N}$, there is $C = C(t_1) > 0$ such that

$$\sup_{t \ge t_1} \|f(t)\|_{H^n_k} \le C.$$

Proof. (1) Let $\alpha \in \mathbb{N}^3$ be a multi-index such that $|\alpha| = 1$ and denote $g = \partial^\alpha f$, which satisfies the equation

$$\partial_t g = Q(f,g) + Q(g,f),$$

and then we easily compute

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2_k} = \big\langle Q(f,g), g\langle v\rangle^{2k}\big\rangle + \big\langle Q(g,f), g\langle v\rangle^{2k}\big\rangle =: T_1 + T_2. \tag{3.11}$$

From Lemma 3.7 we observe that

$$\begin{aligned} T_1 &\le -K\|g\|^2_{\dot H^1_{k+\gamma/2}} + C\|g\|^2_{L^2_{k+\gamma/2}} \\ &\le -K\|\langle v\rangle^{\gamma/2+k} g\|^2_{\dot H^1} + C\|g\|^2_{L^2_{k+\gamma/2}}. \end{aligned} \tag{3.12}$$

For the second term, we write $T_2 = T_{22} + T_{21}$ with

$$T_{21} := \int \nabla\cdot\{(a*g)\nabla f\} g\langle v\rangle^{2k} \quad \text{and} \quad T_{22} := -\int \nabla\cdot\{(b*g) f\} g\langle v\rangle^{2k}.$$

Integrating by parts and using the symmetry of a, it follows that

$$\begin{aligned} T_{21} &= -\int (a_{ij}*g)\partial_j f \partial^\alpha(\partial_i f)\langle v\rangle^{2k} - \int (a_{ij}*g)\partial_j f \partial^\alpha f \partial_i\langle v\rangle^{2k} \\ &= \frac{1}{2}\int (\partial^\alpha\partial^\alpha a_{ij} * f)\partial_i f \partial_j f \langle v\rangle^{2k} + \frac{1}{2}\int (\partial^\alpha a_{ij} * f)\partial_i f \partial_j f \partial^\alpha\langle v\rangle^{2k} - \int (a_{ij}*g)\partial_j f\partial^\alpha f \partial_i\langle v\rangle^{2k} \\ &\lesssim \iint |v-v_*|^\gamma f_* |\nabla f|^2 \langle v\rangle^{2k} + \iint |v-v_*|^{\gamma+1} f_* |\nabla f|^2 \langle v\rangle^{2k-1}. \end{aligned}$$

Using Lemma 3.3-(1), it follows that

$$\iint |v-v_*|^\gamma f_* |\nabla f|^2 \langle v\rangle^{2k} \lesssim \|\langle v\rangle^{-\gamma} f\|_{L^1}\Big\{\|\langle v\rangle^{\gamma/2+k}\nabla f\|^2_{L^2} + \|\langle v\rangle^{\gamma/2+k}\nabla f\|^2_{\dot H^{-\gamma/2}}\Big\},$$

and also

$$\iint |v-v_*|^{\gamma+1} f_* \, |\nabla f|^2 \langle v\rangle^{2k-1}$$
$$\lesssim \begin{cases} \|\langle v\rangle^{\gamma+1} f\|_{L^1} \, \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{L^2}^2, & \text{if } \gamma+1 \geq 0; \\ \|\langle v\rangle^{-\gamma-1} f\|_{L^1} \left(\|\langle v\rangle^{\gamma/2+k} \nabla f\|_{L^2}^2 + \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{\dot{H}^{-(\gamma+1)/2}}^2 \right), & \text{if } \gamma+1 < 0. \end{cases}$$

Using the uniform in time bound of $\|f(t)\|_{L^1_2}$, the previous estimates yield

$$\begin{aligned} T_{21} &\lesssim \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{L^2}^2 + \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{\dot{H}^{-\gamma/2}}^2 \\ &\lesssim C(\epsilon) \|f\|_{\dot{H}^1_{k+\gamma/2}}^2 + \epsilon \|f\|_{\dot{H}^2_{k+\gamma/2}}^2, \end{aligned} \tag{3.13}$$

for any $\epsilon > 0$, thanks to the interpolation Lemma 3.1. For the term T_{22} we obtain

$$T_{22} \lesssim \iint |v-v_*|^{\gamma} f_* \, f \, |\nabla^2 f| \, \langle v\rangle^{2k} + \iint |v-v_*|^{\gamma} f_* \, f \, |\nabla f| \, \langle v\rangle^{2k-1} =: T_{221} + T_{222}.$$

Thanks to Lemma 3.3-(1) again, we get

$$\begin{aligned} T_{222} &\lesssim \|\langle v\rangle^{-\gamma} f\|_{L^1} \Big\{ \|\langle v\rangle^{\gamma/2+k} f\|_{L^2} \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{L^2} + \|\langle v\rangle^{\gamma/2+k} f\|_{\dot{H}^{-\gamma-1}} \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{\dot{H}^1} \Big\} \\ &\lesssim \|\langle v\rangle^{\gamma/2+k} f\|_{L^2}^2 + \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{L^2}^2 + C(\epsilon) \|\langle v\rangle^{\gamma/2+k} f\|_{\dot{H}^{-\gamma-1}}^2 + \epsilon \|\langle v\rangle^{\gamma/2+k} \nabla f\|_{\dot{H}^1}^2 \\ &\lesssim C(\epsilon) \|f\|_{\dot{H}^1_{k+\gamma/2}}^2 + \epsilon \|f\|_{\dot{H}^2_{k+\gamma/2}}^2, \end{aligned} \tag{3.14}$$

for any $\epsilon > 0$, where we have used Young's inequality and the interpolation Lemma 3.1.

For the last term T_{221}, we split into two different cases.

Case (i): $\gamma \in (-3/2, 0)$. Using again Lemma 3.3-(1) (remark that here we need $\gamma > -3/2$), it follows

$$\begin{aligned} T_{221} &\lesssim \|\langle v\rangle^{-\gamma} f\|_{L^1} \Big\{ \|\langle v\rangle^{\gamma/2+k} f\|_{L^2} \|\langle v\rangle^{\gamma/2+k} \nabla^2 f\|_{L^2} + \|\langle v\rangle^{\gamma/2+k} f\|_{\dot{H}^{-\gamma}} \|\langle v\rangle^{\gamma/2+k} \nabla^2 f\|_{L^2} \Big\} \\ &\lesssim C(\epsilon) \|\langle v\rangle^{\gamma/2+k} f\|_{L^2}^2 + C(\epsilon) \|\langle v\rangle^{\gamma/2+k} f\|_{\dot{H}^{-\gamma}}^2 + \epsilon \|\langle v\rangle^{\gamma/2+k} \nabla^2 f\|_{L^2}^2 \\ &\lesssim C(\epsilon) \|f\|_{\dot{H}^1_{k+\gamma/2}}^2 + \epsilon \|f\|_{\dot{H}^2_{k+\gamma/2}}^2. \end{aligned} \tag{3.15}$$

Now, coming back to (3.11), gathering the above estimates (3.12)-(3.13)-(3.14)-(3.15) and taking $\epsilon > 0$ small enough, we obtain

$$\begin{aligned} \frac{d}{dt} \|f\|_{\dot{H}^1_k}^2 &\leq -K \|f\|_{H^2_{k+\gamma/2}}^2 + C \|f\|_{H^1_{k+\gamma/2}}^2 \\ &\leq -K \|f\|_{H^2_{k+\gamma/2}}^2 + C \|f\|_{L^2_{k+\gamma/2}}^2, \end{aligned} \tag{3.16}$$

where we have used Lemma 3.1 again.

We fix some $t_0 > 0$. Since $f_0 \in L^1_{k-5\gamma/4}$, we can use Proposition 3.9 to get that there is $C > 0$ such that

$$\int_{t_0/2}^{t_0} \|f(\tau)\|_{H^1_k}^2 \, d\tau < C \quad \text{and} \quad \|f(t)\|_{L^2_{k-\gamma/2}}^2 \leq C(1+t), \quad \forall \, t \geq t_0/2,$$

from which we can choose some $t_1 \in [t_0/2, t_0]$ such that $\|f(t_1)\|_{H^1_k}^2 < \infty$. Now we integrate (3.16) from t_1 to t to obtain

$$\begin{aligned} \|f(t)\|_{H^1_k}^2 + K \int_{t_1}^t \|f(\tau)\|_{H^2_{k+\gamma/2}}^2 \, d\tau &\leq C \int_{t_1}^t \|f(\tau)\|_{L^2_{k+\gamma/2}}^2 \, d\tau + \|f(t_1)\|_{H^1_k}^2 \\ &\leq C(1+t)^2, \end{aligned}$$

which concludes the case $\gamma \in (-3/2, 0)$.

Case (ii): $\gamma \in (-2, -3/2]$. In this case, Lemma 3.3-(2) implies, for any $0 < \sigma < |\gamma|$, any $\ell_0 \leq -\gamma$ and $\ell_1 + \ell_2 = -\ell_0$, that

$$\begin{aligned} T_{221} &\lesssim \|\langle v\rangle^{-\gamma} f\|_{L^1} \|\langle v\rangle^{\gamma/2+k} f\|_{L^2} \|\langle v\rangle^{\gamma/2+k} \nabla^2 f\|_{L^2} \\ &\quad + \|\langle v\rangle^{\ell_0} f\|_{L^{\frac{3}{3+\gamma+\sigma}}} \|\langle v\rangle^{k+\ell_1} f\|_{H^\sigma} \|\langle v\rangle^{k+\ell_2} \nabla^2 f\|_{L^2} =: A + B. \end{aligned}$$

The first term A can be easily bounded by

$$\begin{aligned} A &\lesssim C(\epsilon) \|f\|^2_{L^1_{-\gamma}} \|f\|^2_{L^2_{k+\gamma/2}} + \epsilon \|f\|^2_{\dot{H}^2_{k+\gamma/2}} \\ &\lesssim C(\epsilon) \|f\|^2_{L^2_{k+\gamma/2}} + \epsilon \|f\|^2_{\dot{H}^2_{k+\gamma/2}}, \end{aligned} \tag{3.17}$$

for any $\epsilon > 0$, and it remains to estimate the last term B. We choose σ verifying $-3/2 - \gamma < \sigma$ so that $3/(3+\gamma+\sigma) < 2$. Moreover, we choose $\ell_2 = \gamma/2$ and $\ell_0 = 2 + 3\gamma/4$, which implies $\ell_1 = -2 - 5\gamma/4$. We interpolate $L^{\frac{3}{3+\gamma+\sigma}}$ between L^1 and L^2, which yields

$$\|\langle v\rangle^{2+3\gamma/4} f\|_{L^{\frac{3}{3+\gamma+\sigma}}} \leq \|\langle v\rangle^{2+3\gamma/4} f\|^{1+\frac{2}{3}(\gamma+\sigma)}_{L^1} \|\langle v\rangle^{2+3\gamma/4} f\|^{-\frac{2}{3}(\gamma+\sigma)}_{L^2}.$$

Since we have $-3/2 - \gamma < \sigma < -\gamma$ and $\gamma \in (-2, -3/2]$, we can choose $\sigma = 1/2$. Using the fact that $\|\langle v\rangle^{k-2-5\gamma/4} f\|_{H^{1/2}} \lesssim \|f\|_{H^{1/2}_{k-2-5\gamma/4}}$ and applying Lemma 3.2 twice, it follows

$$\|f\|_{H^{1/2}_{k-2-5\gamma/4}} \lesssim \|f\|^{3/4}_{L^2_{k-8/3-11\gamma/6}} \|f\|^{1/4}_{H^2_{k+\gamma/2}}.$$

This implies, using the uniform in time bound of $\|f(t)\|_{L^1_{2+3\gamma/4}}$ and Young's inequality, the following estimate

$$\begin{aligned} B &\lesssim \|f\|^{1+\frac{2}{3}(\gamma+\sigma)}_{L^1_{2+3\gamma/4}} \|f\|^{-\frac{2}{3}(\gamma+\sigma)}_{L^2_{2+3\gamma/4}} \|f\|^{3/4}_{L^2_{k-8/3-11\gamma/6}} \|f\|^{5/4}_{H^2_{k+\gamma/2}} \\ &\lesssim C(\epsilon) \|f\|^{1+\frac{8}{9}(2\gamma+1)}_{L^1_{2+3\gamma/4}} \|f\|^{-\frac{8}{9}(2\gamma+1)}_{L^2_{2+3\gamma/4}} \|f\|^2_{L^2_{k-8/3-11\gamma/6}} + \epsilon \|f\|^2_{H^2_{k+\gamma/2}} \\ &\lesssim C(\epsilon) \|f\|^{-\frac{8}{9}(2\gamma+1)}_{L^2_{2+3\gamma/4}} \|f\|^2_{L^2_{k-8/3-11\gamma/6}} + \epsilon \|f\|^2_{H^2_{k+\gamma/2}}. \end{aligned} \tag{3.18}$$

We can now come back to (3.11). Gathering the above estimates (3.12)-(3.13)-(3.14)-(3.17)-(3.18) and taking $\epsilon > 0$ small enough, we obtain

$$\begin{aligned} \frac{d}{dt} \|f\|^2_{\dot{H}^1_k} &\leq -K \|f\|^2_{H^2_{k+\gamma/2}} + C \|f\|^2_{H^1_{k+\gamma/2}} + C \|f\|^{-\frac{8}{9}(2\gamma+1)}_{L^2_{2+3\gamma/4}} \|f\|^2_{L^2_{k-8/3-11\gamma/6}} \\ &\leq -K \|f\|^2_{H^2_{k+\gamma/2}} + C \|f\|^2_{L^2_{k+\gamma/2}} + C \|f\|^{-\frac{8}{9}(2\gamma+1)}_{L^2_{2+3\gamma/4}} \|f\|^2_{L^2_{k-8/3-11\gamma/6}}, \end{aligned} \tag{3.19}$$

where we have used Lemma 3.1.

We fix some $t_0 > 0$ and argue in a similar way as in the previous case. First of all, thanks to Proposition 3.9 there holds $\sup_{t \geq t_0/2} \|f(t)\|_{L^2_{2+3\gamma/4}} \leq C$, hence we can rewrite (3.19) starting from $t_0/2$ as

$$\begin{aligned} \frac{d}{dt} \|f\|^2_{\dot{H}^1_k} &\leq -K \|f\|^2_{H^2_{k+\gamma/2}} + C \|f\|^2_{L^2_{k+\gamma/2}} + C \|f\|^2_{L^2_{k-8/3-11\gamma/6}}, \qquad \forall\, t \geq t_0/2, \\ &\leq -K \|f\|^2_{H^2_{k+\gamma/2}} + C \|f\|^2_{L^2_{k+\gamma/2}} + C \|f\|^2_{L^2_{k-\gamma/2}}, \end{aligned} \tag{3.20}$$

using the fact that $-8/3 - 11\gamma/6 \leq -\gamma/2$ because $\gamma > -2$. Since $f_0 \in L^1_{k-5\gamma/4}$, we can use Proposition 3.9 to deduce that there is $C > 0$ such that

$$\int_{t_0/2}^{t_0} \|f(\tau)\|^2_{H^1_k} \, d\tau \leq C \quad \text{and} \quad \|f(t)\|^2_{L^2_{k-\gamma/2}} \leq C(1+t), \quad \forall\, t \geq t_0/2,$$

from which we can choose some $t_1 \in [t_0/2, t_0]$ such that $\|f(t_1)\|^2_{H^1_k} < \infty$. Now we integrate (3.20) from t_1 to t, then we obtain

$$\begin{aligned}\|f(t)\|^2_{H^1_k} + K\int_{t_1}^t \|f(\tau)\|^2_{H^2_{k+\gamma/2}}\,d\tau &\le C\int_{t_1}^t \|f(\tau)\|^2_{L^2_{k-\gamma/2}}\,d\tau + \|f(t_1)\|^2_{H^1_k}\\ &\le C(1+t)^2,\end{aligned}$$

and the case $\gamma \in (-2, -3/2]$ is complete.

(2) Let $\beta \in \mathbb{N}^3$ be a multi-index with $|\beta| = 2$ and denote $g = \partial^\beta f$. Then g satisfies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2_k} &= \langle Q(f,g), g\langle v\rangle^{2k}\rangle + \sum_{\substack{\beta_1+\beta_2=\beta\\ 1\le|\beta_1|\le 2, 0\le|\beta_2|\le 1}} C^{\beta_1}_{\beta_2}\langle Q(\partial^{\beta_1} f, \partial^{\beta_2} f), g\langle v\rangle^{2k}\rangle\\ &=: I_1 + \sum I_2^{\beta_1,\beta_2}.\end{aligned}$$

For the first term, we have from Lemma 3.7 that

$$\begin{aligned}I_1 &\le -K\|g\|^2_{\dot H^1_{k+\gamma/2}} + C\|g\|^2_{L^2_{k+\gamma/2}}\\ &\le -K\|\langle v\rangle^{\gamma/2+k} g\|^2_{\dot H^1} + C\|g\|^2_{L^2_{k+\gamma/2}}.\end{aligned}$$

For the second one, we use Lemma 3.8 to obtain

$$\begin{aligned}I_2^{\beta_1,\beta_2} &\lesssim \|\partial^{\beta_1} f\|_{L^1_{\gamma+2}}\,\|\partial^{\beta_2} f\|_{H^1_{k+\gamma/2+2}}\,\|\partial^\beta f\|_{H^1_{k+\gamma/2}}\\ &\lesssim \|f\|_{H^{|\beta_1|}_{\gamma+4}}\,\|f\|_{H^{|\beta_2|+1}_{k+\gamma/2+2}}\,\|f\|_{H^3_{k+\gamma/2}},\end{aligned}$$

using Hölder's inequality. Then, denoting $l = \max(\gamma+4, k+\gamma/2+2)$ and using Lemma 3.2, it follows that

$$\begin{aligned}I_2^{\beta_1,\beta_2} &\lesssim \|f\|_{H^1_l}\,\|f\|_{H^2_l}\,\|f\|_{H^3_{k+\gamma/2}}\\ &\lesssim \|f\|_{H^1_l}\,\|f\|^{1/2}_{H^1_{2l-k-\gamma/2}}\,\|f\|^{3/2}_{H^3_{k+\gamma/2}}\\ &\lesssim C_\epsilon\|f\|^6_{H^1_{2l-k-\gamma/2}} + \epsilon\|f\|^2_{H^3_{k+\gamma/2}},\end{aligned}$$

for any $\epsilon > 0$. Gathering the above estimates and taking $\epsilon > 0$ small enough, we finally obtain the following differential inequality

$$\frac{d}{dt}\|f\|^2_{\dot H^2_k} \le -K\|f\|^2_{H^3_{k+\gamma/2}} + C\|f\|^2_{L^2_{k+\gamma/2}} + C\|f\|^6_{H^1_{2l-k-\gamma/2}}. \tag{3.21}$$

We fix some $t_0 > 0$. Since $f_0 \in L^1_{k-7\gamma/4}$ we obtain from point (1) that

$$\int_{t_0/2}^{t_0} \|f(\tau)\|^2_{H^2_k}\,d\tau < \infty,$$

from which we deduce that there is some $t_1 \in [t_0/2, t_0]$ such that $\|f(t_1)\|_{H^2_k} < \infty$. Moreover, since $f_0 \in L^1_{2l-k-7\gamma/4}$, we also get from point (1) that

$$\forall\, t > t_0/2, \quad \|f(t)\|^2_{H^1_{2l-k-\gamma/2}} \le C(1+t)^2.$$

Coming back to (3.21) and integrating from t_1, it yields, for any $t \ge t_1$,

$$\begin{aligned}\|f(t)\|^2_{H^2_k} + K\int_{t_1}^t \|f(\tau)\|^2_{H^3_{k+\gamma/2}}\,d\tau &\le C\int_{t_1}^t \|f(\tau)\|^2_{L^2_{k+\gamma/2}}\,d\tau + C\int_{t_1}^t \|f(\tau)\|^6_{H^1_{2l-k-\gamma/2}}\,d\tau + \|f(t_1)\|^2_{H^2_k}\\ &\le C\int_{t_1}^t (1+\tau)\,d\tau + C\int_{t_1}^t (1+\tau)^6\,d\tau + C\\ &\le C(1+t)^7,\end{aligned}$$

which concludes the proof.

(3) Suppose now that $f \in L^\infty([0,\infty); L^1_l)$ for any $l \geq 0$. We recall that we obtain in Proposition 3.9 (see equations (3.6) and (3.8)) the following differential inequality

$$\begin{aligned}\frac{d}{dt}\|f\|^2_{L^2_k} &\leq -K\|f\|^2_{H^1_{k+\gamma/2}} + C\|f\|^2_{L^2_{k+\gamma/2}} \\ &\leq -K\|f\|^{-4/3}_{L^1_{k-3\gamma/4}}\|f\|^{2+4/3}_{L^2_k} + C\|f\|^2_{L^2_{k+\gamma/2}}.\end{aligned}$$

Now since $\|f(t)\|_{L^1_{k-3\gamma/4}}$ is bounded uniformly in time, arguing as in Proposition 3.9, we obtain from last inequality that for any $t_1 > 0$, for any $k \geq 0$, there exists $C = C(t_1) > 0$ such that

$$\sup_{t\geq t_1}\left\{\|f(t)\|^2_{L^2_k} + \int_t^{t+1}\|f(\tau)\|^2_{H^1_k}\,d\tau\right\} \leq C. \tag{3.22}$$

Let us now investigate the $\dot{H}^1_k$-norm. Coming back to (3.16) if $\gamma \in (-3/2, 0)$ or to (3.20) if $\gamma \in (-2, -3/2]$, and using Lemma 3.2, we get

$$\begin{aligned}\frac{d}{dt}\|f\|^2_{\dot{H}^1_k} &\leq -K\|f\|^2_{H^2_{k+\gamma/2}} + C\|f\|^2_{L^2_{k-\gamma/2}} \\ &\leq -K\|f\|^{-2}_{L^2_{k-\gamma/2}}\|f\|^4_{H^1_k} + C\|f\|^2_{L^2_{k-\gamma/2}}.\end{aligned}$$

We fix some $t_1 > 0$. Thanks to (3.22) we have $\sup_{t\geq t_1/2}\|f(t)\|_{L^2_l} \leq C$ for any $l \geq 0$, then, arguing as before, there is $C = C(t_1)$ such that

$$\sup_{t\geq t_1}\|f(t)\|^2_{H^1_k} + \int_t^{t+1}\|f(\tau)\|^2_{H^2_k}\,d\tau \leq C.$$

We conclude the proof by induction. Assume that for some integer $n \geq 2$, for any $t_1 > 0$ and any $k \geq 0$ we have

$$\sup_{t\geq t_0}\|f(t)\|^2_{H^{n-1}_k} \leq C.$$

Arguing as in point (2) we obtain that

$$\frac{d}{dt}\|f\|^2_{\dot{H}^n_k} \lesssim -K\|f\|^2_{H^{n+1}_{k+\gamma/2}} + C\|f\|^2_{H^n_{k+\gamma/2}} + C\sum_{\substack{|\beta_1|+|\beta_2|=n\\ 1\leq|\beta_1|\leq n, 0\leq|\beta_2|\leq n-1}} I^{\beta_1,\beta_2}.$$

where

$$I^{\beta_1,\beta_2} \lesssim \|f\|_{H^{|\beta_1|}_{\gamma+4}}\|f\|_{H^{|\beta_2|+1}_{k+\gamma/2+2}}\|f\|_{H^{n+1}_{k+\gamma/2}}.$$

If $(|\beta_1|, |\beta_2|) = (1, n-1)$ or $(|\beta_1|, |\beta_2|) = (n, 0)$ then, using Lemma 3.2, it follows

$$\begin{aligned}I^{\beta_1,\beta_2} &\lesssim \|f\|_{H^1_l}\|f\|_{H^n_l}\|f\|_{H^{n+1}_{k+\gamma/2}} \\ &\lesssim \|f\|_{H^1_l}\|f\|^{1/2}_{H^{n-1}_{2l}}\|f\|^{3/2}_{H^{n+1}_{k+\gamma/2}} \lesssim C_\epsilon + \epsilon\|f\|^2_{H^{n+1}_{k+\gamma/2}},\end{aligned}$$

for any $\epsilon > 0$, using the induction hypothesis. In all the other cases, $2 \leq |\beta_1| \leq n-1$ and $1 \leq |\beta_2| \leq n-2$, we get

$$I^{\beta_1,\beta_2} \lesssim \|f\|^2_{H^{n-1}_l}\|f\|_{H^{n+1}_{k+\gamma/2}} \lesssim C_\epsilon + \epsilon\|f\|^2_{H^{n+1}_{k+\gamma/2}}.$$

Taking $\epsilon > 0$ small enough and iterating Lemma 3.2, we obtain the differential inequality, for some $l, \eta > 0$,

$$\begin{aligned}\frac{d}{dt}\|f\|^2_{H^n_k} &\lesssim -K\|f\|^2_{H^{n+1}_{k+\gamma/2}} + C\|f\|^2_{H^n_{k+\gamma/2}} + C \\ &\lesssim -K\|f\|^{-\eta}_{L^2_l}\|f\|^{2+\eta}_{H^n_k} + C\|f\|^2_{H^n_{k+\gamma/2}} + C, \\ &\lesssim -K\|f\|^{2+\eta}_{H^n_k} + C,\end{aligned}$$

using the induction hypothesis, from which it follows that for any $t_1 > 0$ and any $k \geq 0$ there exists $C = C(t_1) > 0$ such that

$$\sup_{t \geq t_1} \|f(t)\|_{H^n_k} \leq C.$$

□

4. Convergence to equilibrium

4.1. **Polynomial in time convergence.** Toscani and Villani [19] have proved a polynomial rate in the trend to equilibrium for the Landau equation for *mollified soft potentials*, i.e. replacing the function $a(z) = |z|^{\gamma+2}\Pi(z)$ by a mollified version truncating the singularity at the origin, given by

$$\widetilde{a}(z) = \widetilde{\Psi}(z)\Pi(z) \qquad \text{with} \qquad c_\Psi \langle z\rangle^\gamma \leq \frac{\widetilde{\Psi}(z)}{|z|^2} \leq C_\Psi \langle z\rangle^\gamma,$$

for some constants $c_\Psi, C_\psi > 0$. Their strategy was based on two ingredients: a functional inequality relating the entropy and the entropy dissipation functional stated in Lemma 4.1 (which is also valid in our case of *true* soft potentials), and a priori estimates for the evolution of moments and weighted Sobolev norms for the Landau equation associated with $\widetilde{a}(z)$ (which of course do not hold in our case and we shall use the new a priori estimates proven in Section 3).

The entropy - entropy dissipation inequality is given in the following result.

Lemma 4.1 ([19, Proposition 4])**.** *Let $a^\dagger(z) = \Psi^\dagger(z)\Pi(z)$ where $\Psi^\dagger$ verifies*

$$c_{\Psi^\dagger}\langle z\rangle^\gamma \leq \frac{\Psi^\dagger(z)}{|z|^2},$$

for some constant $c_{\Psi^\dagger} > 0$, and consider the associated entropy-dissipation functional

$$D_{a^\dagger} = \frac{1}{2}\iint \Psi^\dagger(|v-v_*|)\left|\Pi(v-v_*)\left(\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*}\right)\right|^2 f_* f \, dv_* \, dv.$$

Then, for all $k > 0$ and all f satisfying (1.11)*, there is $C_k(f) > 0$ depending on k and $H(f)$ such that*

$$D_{a^\dagger}(f) \geq C_k(f)\, H(f|\mu)^{1-\gamma/k}\left\{\|f\|_{L^1_{k+2}} + J_{k+2}(f)\right\}^{\gamma/k}$$

where

$$J_{k+2}(f) = \|\langle v\rangle^{k+2}\nabla\sqrt{f}\|^2_{L^2}.$$

As a consequence of this functional inequality and the fact that

$$\frac{d}{dt}H(f|\mu) = -D(f),$$

where $H(f|\mu) = \int f\log(f/\mu)$ is the relative entropy of f with respect to μ, we get the following result.

Corollary 4.2 ([19, Corollary 4.1]). *If for some $k > 0$ we have*

$$\|f(t)\|_{L^1_{k+2}} + J_{k+2}(f(t)) \leq C(1+t)^{\theta}, \qquad \theta < \frac{k}{|\gamma|},$$

then the following estimate holds true

$$H(f(t)|\mu) \leq C(1+t)^{-\frac{k}{|\gamma|}+\theta}.$$

In order to estimate the evolution of the quantity $J_{k+2}(f(t))$, it is proven in [19] that this quantity can be reduced to weighted Sobolev norms. More precisely, they first prove that

$$J_{k+2}(f) \lesssim I(\langle v\rangle^{k+2} f) + \|f\|_{L^1_k}$$

where $I(g)$ is the Fisher information define by

$$I(g) := \int \frac{|\nabla g|^2}{g} = 4\int |\nabla \sqrt{g}|^2.$$

Finally they prove the following inequality in [19, Lemma 1]: for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$I(g) \leq C_\epsilon \|g\|_{H^2_{3/2+\epsilon}},$$

so that at the end we get

$$\|f\|_{L^1_{k+2}} + J_{k+2}(f) \lesssim \|f\|_{H^2_{k+7/2+\epsilon}}.$$

Now we are in position to prove the polynomial in time convergence in Theorem 1.3.

Proof of Theorem 1.3. This theorem is a consequence of Proposition 3.10 and Corollary 4.2. Indeed, remark that Lemma 4.1 also holds in our case of true soft potentials with $a(z) = |z|^{\gamma+2}\Pi(z)$ given by (1.3). Then since $f_0 \in L^1_{k+8-3\gamma/4} \cap L\log L$ with $k > 7|\gamma|/2$, the a priori estimate in Proposition 3.10-(2) (here one should use approximate solutions of the Landau equation as in [21] in order to give a completely rigorous proof) implies that for any $t_0 > 0$ it holds

$$\|f(t)\|_{H^2_{k+4}} \leq C(1+t)^{7/2}.$$

We conclude the proof applying Corollary 4.2. □

As a consequence of Theorem 1.3 we can improve the slowly increasing a priori bounds for L^1 moments in Lemmas 3.5 and 3.6, obtaining uniform in time estimates, as done in [6] for the Boltzmann equation.

Proposition 4.3. *Let $\gamma \in (-2, 0)$ and $f_0 \in L^1_2 \cap L\log L$. Consider a global weak solution $f \in L^\infty([0,\infty); L^1_2 \cap L\log L)$ to the Landau equation.*

(1) Suppose that $f_0 \in L^1_{2\ell} \cap L^1_{k+8-3\gamma/4}$ with $\ell > 2$ and $k > 11|\gamma|/2$. Then

$$\sup_{t\geq 0} \|f(t)\|_{L^1_\ell} \leq C.$$

(2) Suppose that $f_0 \in L^1(e^{2\kappa\langle v\rangle^s})$ with $\kappa > 0$, $0 < s < 2$ with $s < \gamma + 2$. Then we have

$$\sup_{t\geq 0} \|f(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} \leq C.$$

Proof. (1) We write, using Lemma 3.5 and Theorem 1.3

$$\begin{aligned}\|f(t)\|_{L^1_\ell} &\le \|f(t)-\mu\|_{L^1_\ell} + \|\mu\|_{L^1_\ell}\\ &\le \|f(t)-\mu\|_{L^1}^{1/2}\,\|f(t)-\mu\|_{L^1_{2\ell}}^{1/2} + C\\ &\le C(1+t)^{-\frac{k}{4|\gamma|}+\frac{7}{8}}(1+t)^{\frac12} + C\\ &\le C.\end{aligned}$$

(2) Using Lemma 3.6 and Theorem 1.3, for some $k>0$ large enough we have

$$\begin{aligned}\|f(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} &\le \|f(t)-\mu\|_{L^1(e^{\kappa\langle v\rangle^s})} + \|\mu\|_{L^1(e^{2\kappa\langle v\rangle^s})}\\ &\le \|f(t)-\mu\|_{L^1}^{1/2}\,\|f(t)-\mu\|_{L^1(e^{2\kappa\langle v\rangle^s})}^{1/2} + C\\ &\le C(1+t)^{-\frac{k}{4|\gamma|}+\frac{7}{8}}(1+t)^{1/2} + C\\ &\le C.\end{aligned}$$

□

4.2. **Exponential in time convergence.** We are able now to conclude the proof of Theorem 1.4. Recall that in this setting we suppose $\gamma\in(-1,0)$ and $f_0\in L\log L\cap L^1(e^{\kappa\langle v\rangle^s})$ with $\kappa>0$ and $-\gamma<s<2+\gamma$. Let us denote $m=e^{\bar\kappa\langle v\rangle^s}$ with $\bar\kappa\le\kappa/10$, which satisfies assumption (W).

We write $h(t)=f(t)-\mu$ that satisfies

$$\begin{cases}\partial_t h = \mathcal{L}h + Q(h,h)\\ h_{|t=0}=h_0.\end{cases}$$

Since $\Pi_0 h_0=0$ and $\Pi_0 Q(h_0,h_0)=0$, for all $t\ge0$, we also have $\Pi_0 h(t)=0$ and $\Pi_0 Q(h(t),h(t))=0$, thanks to the conservation laws. By Duhamel's principle it follows

$$h(t)=S_{\mathcal{L}}(t)h_0+\int_0^t S_{\mathcal{L}}(t-\tau)Q(h(\tau),h(\tau))\,d\tau. \tag{4.1}$$

Before starting the proof of the main theorem, let us state a result that will be useful in the sequel.

Lemma 4.4. *Let $\gamma\in(-2,0)$, $p\in[1,+\infty)$ and m be a weight function. Then, if $1\le q<3/|\gamma|$, it holds*

$$\|Q(g,f)\|_{L^p(m)}\lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})}\,\|\nabla^2 f\|_{L^p(m\langle v\rangle^{\gamma+2})}+\|g\|_{L^1}\,\|f\|_{L^p(m)}+\|g\|_{L^{q/(q-1)}}\,\|f\|_{L^p(m)}.$$

Proof. Since $\gamma\in(-2,0)$, using $|v-v_*|^{\gamma+2}\lesssim\langle v_*\rangle^{\gamma+2}\langle v\rangle^{\gamma+2}$ we easily obtain

$$\begin{aligned}\|(a_{ij}*g)\partial_{ij}f\|_{L^p(m)}^p &\lesssim \int_v\left|\int_{v_*}|v-v_*|^{\gamma+2}g_*\,dv_*\right|^p|\partial_{ij}f(v)|^p\,m^p\,dv\\ &\lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})}^p\|\nabla^2 f\|_{L^p(m\langle v\rangle^{\gamma+2})}^p.\end{aligned} \tag{4.2}$$

Now let us denote $c_-=c\mathbf{1}_{\{|\cdot|\le1\}}$ and $c_+=c\mathbf{1}_{\{|\cdot|>1\}}$. We can also obtain

$$\begin{aligned}\|(c_+*g)f\|_{L^p(m)}^p &\lesssim \int\left|\int|v-v_*|^\gamma\mathbf{1}_{\{|v-v_*|>1\}}\,|g_*|\,dv_*\right|^p|f|^p\,m^p\,dv\\ &\lesssim \|g\|_{L^1}^p\,\|f\|_{L^p(m)}^p,\end{aligned}$$

and

$$\begin{aligned}\|(c_- * g)f\|^p_{L^p(m)} &\lesssim \int \left| \int |v-v_*|^\gamma \mathbf{1}_{\{|v-v_*|\le 1\}} |g_*|\, dv_* \right|^p |f|^p\, m^p\, dv \\ &\lesssim \int \left\{ \left(\int |v-v_*|^{\gamma q} \mathbf{1}_{\{|v-v_*|\le 1\}}\, dv_* \right)^{1/q} \left(\int |g_*|^{q'}\, dv_* \right)^{1/q'} \right\}^p |f|^p\, m^p\, dv \\ &\lesssim \|g\|^p_{L^{q'}}\, \|f\|^p_{L^p(m)}\end{aligned}$$

using Hölder's inequality and if $1 \le q < 3/|\gamma|$. □

Proof of Theorem 1.4. We split the proof into several steps.

Step 1. Since $f_0 \in L^1_2 \cap L\log L \cap L^1(e^{\kappa\langle v\rangle^s})$ we can apply Theorem 1.3 that implies

$$\forall\, t \ge 0, \quad \|h(t)\|_{L^1} = \|f(t)-\mu\|_{L^1} \le C(1+t)^{-\theta}, \quad \forall\, \theta > 0. \tag{4.3}$$

Moreover we get, using Lemma 3.6,

$$\begin{aligned}\|h(t)\|_{L^1(e^{\frac{\kappa}{2}\langle v\rangle^s})} &\le \|h(t)\|_{L^1}\, \|h(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} \\ &\le C(1+t)^{-\theta} \left(\|f(t)\|_{L^1(e^{\kappa\langle v\rangle^s})} + \|\mu\|_{L^1(e^{\kappa\langle v\rangle^s})} \right) \\ &\le C(1+t)^{-\theta} \left((1+t) + C_\mu \right) \le C(1+t)^{-\theta+1}.\end{aligned}$$

Step 2. Since $f_0 \in L^1_\ell \cap L\log L$ for any $\ell \ge 0$, Proposition 4.3 implies

$$f \in L^\infty([0,\infty); L^1_\ell) \quad \forall\, \ell \ge 0.$$

As a consequence, Proposition 3.10-(3) gives that

$$\forall t_0 > 0,\, \forall\, n, \ell \ge 0, \quad f \in L^\infty([t_0,\infty); H^n_\ell).$$

Step 3. Writing (4.1) starting from some time $t_* > 0$ to be chosen later and using Theorem 2.1 (since $\Pi_0 h(t) = \Pi_0 Q(h(t), h(t)) = 0$ for any $t \ge 0$) it follows, for any $t \ge t_*$, that

$$\begin{aligned}\|h(t)\|_{L^1(m)} &\le \|S_{\mathcal{L}}(t-t_*)h(t_*)\|_{L^1(m)} + \int_{t_*}^t \|S_{\mathcal{L}}(t-\tau)Q(h(\tau),h(\tau))\|_{L^1(m)}\, d\tau \\ &\le Ce^{-\lambda_0 t}\|h(t_*)\|_{L^1(m)} + C\int_{t_*}^t e^{-\lambda_0(t-\tau)}\|Q(h(\tau),h(\tau))\|_{L^1(m)}\, d\tau.\end{aligned}$$

From Lemma 4.4 we have

$$\|Q(h,h)\|_{L^1(m)} \lesssim \|h\|_{L^1_{\gamma+2}}\, \|\nabla^2 h\|_{L^1(\langle v\rangle^{\gamma+2}m)} + \|h\|_{L^1}\, \|h\|_{L^1(m)} + \|h\|_{L^2}\, \|h\|_{L^1(m)}.$$

Moreover, we have the following interpolation inequality from [15, Lemma B.1]

$$\|u\|_{H^n(m^{3/2})} \lesssim \|u\|^{1/2}_{L^1(m^3)}\, \|u\|^{1/2}_{H^{2n+1+3/2}}.$$

Gathering the above bounds we get

$$\begin{aligned}\|\nabla^2 h\|_{L^1(\langle v\rangle^{\gamma+2}m)} &\lesssim \|h\|_{H^2(\langle v\rangle^{\gamma+4}m)} \lesssim \|h\|_{H^2(m^{3/2})} \\ &\lesssim \|h\|^{1/2}_{L^1(m^3)}\, \|h\|^{1/2}_{H^{4+1+3/2}} \\ &\lesssim \|h\|^{1/4}_{L^1(m)}\, \|h\|^{1/4}_{L^1(m^5)}\, \|h\|^{1/2}_{H^{13/2}},\end{aligned}$$

where we have used Hölder's inequality in the last line. Moreover, using Nash's inequality we have

$$\|h\|_{L^2} \lesssim \|h\|^{3/5}_{\dot H^1}\, \|h\|^{2/5}_{L^1} \lesssim \|h\|^{3/5}_{\dot H^1}\, \|h\|^{2/5}_{L^1(m)}.$$

Putting together the previous estimates it yields

$$\begin{aligned}\|h(t)\|_{L^1(m)} \le Ce^{-\lambda t}\|h(t_*)\|_{L^1(m)} &+ C\int_{t_*}^t e^{-\lambda_0(t-\tau)}\|h(\tau)\|_{H^{13/2}}^{1/2}\,\|h(\tau)\|_{L^1(m^5)}^{1/4}\,\|h(\tau)\|_{L^1(m)}^{1+1/4}\,d\tau\\ &+ C\int_{t_*}^t e^{-\lambda_0(t-\tau)}\|h(\tau)\|_{L^1(m)}^2\,d\tau\\ &+ C\int_{t_*}^t e^{-\lambda_0(t-\tau)}\|h(\tau)\|_{H^1}^{3/5}\,\|h(\tau)\|_{L^1(m)}^{1+2/5}\,d\tau.\end{aligned}$$

Thanks to step 1, for any $\epsilon > 0$ we can choose $t_* = t_*(\epsilon)$ such that

$$\sup_{t\ge t_*}\|h(t)\|_{L^1(m)} \le \sup_{t\ge t_*}\|h(t)\|_{L^1(m^5)} \le \epsilon.$$

Also, from step 2 we get

$$\sup_{t\ge t_*}\|h(t)\|_{H^{13/2}} \le C_1.$$

Hence we obtain, for any $t \ge t_*$,

$$\|h(t)\|_{L^1(m)} \le Ce^{-\lambda_0 t}\|h(t_*)\|_{L^1(m)} + C\left(C_1^{1/2}\epsilon^{1/4} + \epsilon^{3/4} + C_1^{3/5}\epsilon^{9/20}\right)\int_{t_*}^t e^{-\lambda_0(t-\tau)}\|h(\tau)\|_{L^1(m)}^{1+1/4}\,d\tau.$$

From this differential inequality, we argue as in [17, Lemma 4.5] and choose $\epsilon > 0$ small enough to obtain

$$\forall\, t \ge t_*, \quad \|h(t)\|_{L^1(m)} \le C\,e^{-\lambda_0 t}\|h(t_*)\|_{L^1(m)} \le C\,e^{-\lambda_0 t},$$

from which, together with (4.3) for $t < t_*$, we conclude the proof. □

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