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L₁-DISTANCE FOR ADDITIVE PROCESSES WITH TIME-HOMOGENEOUS LÉVY MEASURES

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ABSTRACT. We give an explicit bound for the L_1 -distance between two additive processes of local characteristics $(f_j(\cdot), \sigma^2(\cdot), \nu_j)$, $j = 1, 2$. The cases $\sigma = 0$ and $\sigma > 0$ are both treated. We allow ν_1 and ν_2 to be equivalent time-homogeneous Lévy measures, possibly with infinite variation. Some examples of possible applications are discussed.

1. INTRODUCTION AND MAIN RESULT

In this note we give an upper bound for the L_1 -distance between two additive processes of local characteristics $(f_j(\cdot), \sigma^2(\cdot), \nu_j)$, $j = 1, 2$. Let $\{x_t\}$ be the canonical process on the Skorokhod space (D, \mathcal{D}) and denote by $P^{(f, \sigma^2, \nu)}$ the law induced on (D, \mathcal{D}) by an additive process having local characteristics $(f(\cdot), \sigma^2(\cdot), \nu)$. We will denote such a process by $(\{x_t\}, P^{(f, \sigma^2, \nu)})$ (see Section 2 for the precise definitions).

In the case where $\nu_1 = \nu_2 = 0$, i.e. where there are no jumps, an explicit formula for this distance is well known. Indeed, denoting by ϕ the cumulative distribution function of a normal random variable $\mathcal{N}(0, 1)$, we have:

$$(1) \quad L_1(P^{(f_1, \sigma_1^2, 0)}, P^{(f_2, \sigma_2^2, 0)}) = \begin{cases} 2 \left(1 - 2\phi \left(-\frac{1}{2} \sqrt{\int_0^T \frac{(f_1(t) - f_2(t))^2}{\sigma_1^2(t)} dt} \right) \right) & \text{if } \sigma_1 = \sigma_2 \\ 2 & \text{otherwise,} \end{cases}$$

whenever the right-hand side term makes sense (see [1] and recall that the measures $P^{(f, \sigma_1^2, 0)}$ and $P^{(f, \sigma_2^2, 0)}$ are mutually absolutely singular when $\sigma_1 \neq \sigma_2$).

Suppose now that the Lévy measures ν_1 and ν_2 are non zero. Because of (1), we will also assume that $\sigma_1^2 = \sigma_2^2$. We will allow ν_1 and ν_2 to be possibly different, but equivalent Lévy measures. Then the main result is as follows:

Theorem 1.1. *Let $(\{x_t\}, P^{(f_1, \sigma^2, \nu_1)})$ and $(\{x_t\}, P^{(f_2, \sigma^2, \nu_2)})$ be two additive processes on $[0, T]$ with ν_1 and ν_2 equivalent Lévy measures such that*

$$(2) \quad \int_{\mathbb{R}} \left(\sqrt{\frac{d\nu_1}{d\nu_2}}(y) - 1 \right)^2 \nu_2(dy) < \infty.$$

Fix the following notation

$$\gamma^{\nu_j} = \int_{|y| \leq 1} y \nu_j(dy), \quad j = 1, 2; \quad \xi^2 = \int_0^T \frac{(f_2(r) - f_1(r) - (\gamma^{\nu_2} - \gamma^{\nu_1}))^2}{\sigma^2(r)} dr$$

and observe that, thanks to (2),

$$L_1(\nu_1, \nu_2) := \int_{\mathbb{R}} \left| 1 - \frac{d\nu_1}{d\nu_2}(y) \right| \nu_2(dy) < \infty.$$

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The following upper bounds hold:

If $\sigma^2 > 0$ then

$$L_1\left(P^{(f_1, \sigma^2, \nu_1)}, P^{(f_2, \sigma^2, \nu_2)}\right) \leq 2 \sinh\left(TL_1(\nu_1, \nu_2)\right) + 2\left[1 - 2\phi\left(-\frac{\xi}{2}\right)\right].$$

If $\sigma^2 = 0$ and $f_1 - f_2 \equiv \gamma^{\nu_1} - \gamma^{\nu_2}$, then

$$L_1\left(P^{(f_1, 0, \nu_1)}, P^{(f_2, 0, \nu_2)}\right) \leq 2 \sinh\left(TL_1(\nu_1, \nu_2)\right).$$

This result is proved in Section 3. In Section 2 we collect some preliminary results about additive processes that will play a role in the proof. Before that, we give some examples of situations where our result can be applied. The choice of these examples are inspired by the models exhibited in [2].

Example 1.2. (L_1 -distance between compound Poisson processes)

Let $\{X_t^1\}$ and $\{X_t^2\}$ be two compound Poisson processes on $[0, T]$ with intensities $\lambda_j > 0$, $j = 1, 2$ and jump size distributions G_j ; i.e. $\{X_t^j\}$ is a Lévy process of characteristic triplet $(\lambda_j \int_{|y| \leq 1} y G_j(dy), 0, \lambda_j G_j)$. Furthermore, let A be a subset of \mathbb{R} and suppose that G_j is equivalent to the Lebesgue measure restricted to A . Denote by g_j the density $\frac{dG_j}{d\text{Leb}|_A}$; then, an application of Theorem 1.1 yields:

$$L_1(X^1, X^2) \leq 2 \sinh\left(T \int_A |\lambda_1 g_1(y) - \lambda_2 g_2(y)| dy\right).$$

Example 1.3. (L_1 -distance between additive processes of jump-diffusion type)

An additive process of jump-diffusion type on $[0, T]$ has the following form:

$$X_t = \int_0^t f(r) dr + \int_0^t \sigma(r) dW_r + \sum_{i=1}^{N_t} Y_i, \quad t \in [0, T],$$

where $\{W_t\}$ is a standard brownian motion, $\{N_t\}$ is the Poisson process counting the jumps of $\{X_t\}$, and Y_i are jumps sizes (i.i.d. random variables). Consider now the additive processes of jump-diffusion type $\{X_t^j\}$ having local characteristics $(f_j(\cdot) + \lambda_j \int_{|y| \leq 1} y G_j(dy), \sigma^2(\cdot), \lambda_j G_j)$, $j = 1, 2$ and suppose again that G_j is equivalent to the Lebesgue measure restricted to some $A \subseteq \mathbb{R}$. Letting g_j denote the density of G_j as above, we have:

$$L_1(X^1, X^2) \leq 2 \sinh\left(T \int_A |\lambda_1 g_1(y) - \lambda_2 g_2(y)| dy\right) + 2\left(1 - 2\phi\left(-\sqrt{\int_0^T \frac{(f_1(t) - f_2(t))^2}{4\sigma^2(t)} dt}\right)\right).$$

Example 1.4. (L_1 -distance between tempered stable processes)

Let $\{X_t^1\}$ and $\{X_t^2\}$ be two tempered stable processes, i.e. Lévy processes on \mathbb{R} with no gaussian component and such that their Lévy measures ν_j have densities of the form

$$\frac{d\nu_j}{d\text{Leb}}(y) = \frac{C_-}{|y|^{1+\alpha}} e^{-\lambda_-^j |y|} \mathbb{1}_{y < 0} + \frac{C_+}{y^{1+\alpha}} e^{-\lambda_+^j y} \mathbb{1}_{y > 0}, \quad j = 1, 2,$$

for some parameters $C_{\pm} > 0$, $\lambda_{\pm}^j > 0$ and $\alpha < 2$. Then the hypothesis (2) is satisfied and Theorem 1.1 bounds the L_1 -distance by:

$$2 \sinh\left(T \left[C_+ \int_0^{\infty} \left| \frac{e^{-\lambda_+^1 y} - e^{-\lambda_+^2 y}}{y^{1+\alpha}} \right| dy + C_- \int_{-\infty}^0 \left| \frac{e^{-\lambda_-^1 |y|} - e^{-\lambda_-^2 |y|}}{|y|^{1+\alpha}} \right| dy \right]\right).$$

2. PRELIMINARY RESULTS

2.1. Additive processes.

Definition 2.1. A stochastic process $\{X_t\} = \{X_t : t \in [0, T]\}$ on \mathbb{R} defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is an *additive process* if the following conditions are satisfied.

- (1) $X_0 = 0$ \mathbb{P} -a.s.
- (2) For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \dots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
- (3) There is $\Omega_0 \in \mathcal{A}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.
- (4) It is stochastically continuous.

Thanks to the *Lévy-Khintchine formula*, the characteristic function of any additive process $\{X_t\}$ can be expressed, for all u in \mathbb{R} , as:

$$(3) \quad \mathbb{E}[e^{iuX_t}] = \exp\left(iu \int_0^t f(r)dr - \frac{u^2}{2} \int_0^t \sigma^2(r)dr - t \int_{\mathbb{R}} (1 - e^{iuy} + iuy\mathbb{I}_{|y| \leq 1})\nu(dy)\right),$$

where $f(\cdot), \sigma^2(\cdot)$ are functions on $L_1[0, T]$ and ν is a measure on \mathbb{R} satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} (|y|^2 \wedge 1)\nu(dy) < \infty.$$

In the sequel we shall refer to $(f(\cdot), \sigma^2(\cdot), \nu)$ as the local characteristics of the process $\{X_t\}$ and ν will be called *Lévy measure*. This data characterises uniquely the law of the process $\{X_t\}$. In the case in which $f(\cdot)$ and $\sigma(\cdot)$ are constant functions, a process $\{X_t\}$ satisfying (3) is said a *Lévy process* of characteristic triplet (f, σ^2, ν) .

Let $D = D([0, T], \mathbb{R})$ be the space of mappings ω from $[0, T]$ into \mathbb{R} that are right-continuous with left limits. Define the *canonical process* $x : D \rightarrow D$ by

$$\forall \omega \in D, \quad x_t(\omega) = \omega_t, \quad \forall t \in [0, T].$$

Let \mathcal{D} be the smallest σ -algebra of parts of D that makes x_s, s in $[0, T]$, measurable. Further, for any $t \in [0, T]$ let \mathcal{D}_t be the smallest σ -algebra that makes x_s, s in $[0, t]$, measurable (here, we use the same notations as in [3]).

Let $\{X_t\}$ be an additive process defined on $(\Omega, \mathcal{A}, \mathbb{P})$ having local characteristics $(f(\cdot), \sigma^2(\cdot), \nu)$. It is well known that it induces a probability measure $P^{(f, \sigma^2, \nu)}$ on (D, \mathcal{D}) such that $(\{x_t\}, P^{(f, \sigma^2, \nu)})$ is an additive process identical in law with $(\{X_t\}, \mathbb{P})$ (that is the local characteristics of $\{x_t\}$ under $P^{(f, \sigma^2, \nu)}$ is $(f(\cdot), \sigma^2(\cdot), \nu)$). In the case where $\int_{|y| \leq 1} |y|\nu(dy) < \infty$, we set $\gamma^\nu := \int_{|y| \leq 1} y\nu(dy)$. Note that, if ν is a finite Lévy measure, then the process $(\{x_t\}, P^{(\gamma^\nu, 0, \nu)})$ is a compound Poisson process.

Here and in the sequel we will denote by Δx_r the jump of process $\{x_t\}$ at the time r :

$$\Delta x_r = x_r - \lim_{s \uparrow r} x_s.$$

Definition 2.2. Consider $(\{x_t\}, P^{(f, \sigma^2, \nu)})$ and define the *jump part* of $\{x_t\}$ as

$$(4) \quad x_t^{d, \nu} = \lim_{\varepsilon \rightarrow 0} \left(\sum_{r \leq t} \Delta x_r \mathbb{I}_{|\Delta x_r| > \varepsilon} - t \int_{\varepsilon < |y| \leq 1} y\nu(dy) \right) \quad \text{a.s.}$$

and its *continuous part* as

$$(5) \quad x_t^{c, \nu} = x_t - x_t^{d, \nu} \quad \text{a.s.}$$

We now recall the *Lévy-Itô decomposition*, i.e. the decomposition in continuous and discontinuous parts of an additive process.

Theorem 2.3 (See [3], Theorem 19.3). *Consider $(\{x_t\}, P^{(f, \sigma^2, \nu)})$ and define $\{x_t^{d, \nu}\}$ and $\{x_t^{c, \nu}\}$ as in 4 and 5, respectively. Then the following hold.*

- (i) There is $D_1 \in \mathcal{D}$ with $P^{(f, \sigma^2, \nu)}(D_1) = 1$ such that, for any $\omega \in D_1$, $x_t^{d, \nu}(\omega)$ is defined for all $t \in [0, T]$ and the convergence is uniform in t on any bounded interval, $P^{(f, \sigma^2, \nu)}$ -a.s. The process $\{x_t^{d, \nu}\}$ is a Lévy process on \mathbb{R} with characteristic triplet $(0, 0, \nu)$.
- (ii) There is $D_2 \in \mathcal{D}$ with $P^{(f, \sigma^2, \nu)}(D_2) = 1$ such that, for any $\omega \in D_2$, $x_t^{c, \nu}(\omega)$ is continuous in t . The process $\{x_t^{c, \nu}\}$ is an additive process on \mathbb{R} with local characteristics $(f(\cdot), \sigma^2(\cdot), 0)$.
- (iii) The two processes $\{x_t^{d, \nu}\}$ and $\{x_t^{c, \nu}\}$ are independent.

2.2. Change of measure for additive processes. For the proof of Theorem 1.1 we also need some results on the equivalence of measures for additive processes. We will use the notation $P|_{\mathcal{D}_t}$ for the restriction of the probability P to \mathcal{D}_t .

2.2.1. *Case $\sigma^2 = 0$.*

Theorem 2.4 (See [3], Theorems 33.1–33.2). *Let $(\{x_t\}, P^{(0, 0, \tilde{\nu})})$ and $(\{x_t\}, P^{(\eta, 0, \nu)})$ be two Lévy processes on \mathbb{R} , where*

$$(6) \quad \eta = \int_{|y| \leq 1} y(\nu - \tilde{\nu})(dy)$$

is supposed to be finite. Then $P^{(\eta, 0, \nu)}$ is locally equivalent to $P^{(0, 0, \tilde{\nu})}$ if and only if ν and $\tilde{\nu}$ are equivalent and the density $\frac{d\nu}{d\tilde{\nu}}$ satisfies

$$(7) \quad \int \left(\sqrt{\frac{d\nu}{d\tilde{\nu}}}(y) - 1 \right)^2 \tilde{\nu}(dy) < \infty.$$

Remark that the finiteness in (7) implies that in (6). When $P^{(\eta, 0, \nu)}$ is locally equivalent to $P^{(0, 0, \tilde{\nu})}$, the density is

$$\frac{dP^{(\eta, 0, \nu)}}{dP^{(0, 0, \tilde{\nu})}} \Big|_{\mathcal{D}_t}(x) = \exp(U_t(x)),$$

with

$$(8) \quad U_t(x) = \lim_{\varepsilon \rightarrow 0} \left(\sum_{r \leq t} \ln \frac{d\nu}{d\tilde{\nu}}(\Delta x_r) \mathbb{1}_{|\Delta x_r| > \varepsilon} - \int_{|y| > \varepsilon} t \left(\frac{d\nu}{d\tilde{\nu}}(y) - 1 \right) \tilde{\nu}(dy) \right), P^{(0, 0, \tilde{\nu})}\text{-a.s.}$$

The convergence in (8) is uniform in t on any bounded interval, $P^{(0, 0, \tilde{\nu})}$ -a.s. Besides, $\{U_t(x)\}$ defined by (8) is a Lévy process satisfying $\mathbb{E}_{P^{(0, 0, \tilde{\nu})}}[e^{U_t(x)}] = 1, \forall t \in [0, T]$.

2.2.2. *Case $\sigma^2 > 0$.*

Lemma 2.5. *Let ν_1 and ν_2 be equivalent Lévy measures such that*

$$(9) \quad \int_{\mathbb{R}} \left(\sqrt{\frac{d\nu_1}{d\nu_2}}(y) - 1 \right)^2 \nu_2(dy) < \infty.$$

Define

$$(10) \quad \eta = \int_{|y| \leq 1} y(\nu_1 - \nu_2)(dy),$$

which is finite thanks to (9), and consider real functions f_1, f_2 and $\sigma > 0$ such that

$$(11) \quad \int_0^T \left(\frac{f_1(r) - f_2(r) - \eta}{\sigma(r)} \right)^2 dr < \infty.$$

Then, under $P^{(f_2, \sigma^2, \nu_2)}$,

$$(12) \quad M_t(x) = \exp(C_t(x) + D_t(x))$$

is a (\mathcal{D}_t) -martingale for all t in $[0, T]$, where

$$C_t(x) := \int_0^t \frac{f_1(r) - f_2(r) - \eta}{\sigma^2(r)} (dx_r^{c, \nu_2} - f_2(r) dr) - \frac{1}{2} \int_0^t \left(\frac{f_1(r) - f_2(r) - \eta}{\sigma(r)} \right)^2 dr,$$

$$(13) \quad D_t(x) := \lim_{\varepsilon \rightarrow 0} \left(\sum_{r \leq t} \ln \frac{d\nu_1}{d\nu_2} (\Delta x_r) \mathbb{1}_{|\Delta x_r| > \varepsilon} - t \int_{|y| > \varepsilon} (\nu_1 - \nu_2)(dy) \right).$$

The convergence in (13) is uniform in t on any bounded interval, $P^{(f_2, \sigma^2, \nu_2)}$ -a.s.

Proof. The existence of the limit in (13) is guaranteed by (9) (see Theorem 2.4). Since $\int_0^t \frac{1}{\sigma(r)} (dx_r^{c, \nu_2} - f_2(r) dr)$ is a standard Brownian motion under $P^{(f_2, \sigma^2, 0)}$, we have that $\int_s^t \frac{f_1(r) - f_2(r) - \eta}{\sigma^2(r)} (dx_r^{c, \nu_2} - f_2(r) dr)$ has normal law $\mathcal{N}\left(0, \int_s^t \left(\frac{f_1(r) - f_2(r) - \eta}{\sigma(r)} \right)^2 dr\right)$, hence $\mathbb{E}_{P^{(f_2, \sigma^2, 0)}}[\exp((C_t - C_s)(x))] = 1$. Theorem 2.3 entails that $\{x_t^{c, \nu_2}\}$ and $\{x_t^{d, \nu_2}\}$ are independent under $P^{(f_2, \sigma^2, \nu_2)}$. Moreover, the law of $\{C_t(x)\}$ (resp. $\{D_t(x)\}$) is the same under $P^{(f_2, \sigma^2, \nu_2)}$ or $P^{(f_2, \sigma^2, 0)}$ (resp. $P^{(f_2, 0, \nu_2)}$ or $P^{(0, 0, \nu_2)}$). Further, using Theorem 2.4, we know that $\{D_t(x)\}$ is a Lévy process such that $\mathbb{E}_{P^{(0, 0, \nu_2)}}[\exp(D_{t-s}(x))] = 1$ for all $s < t$. These facts together with the independence of the increments of $(\{x_t\}, P^{(f_2, \sigma^2, \nu_2)})$ and the stationarity of $\{D_t(x)\}$ imply:

$$\begin{aligned} \mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}}[M_t(x) | \mathcal{D}_s] &= \mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}} \left[M_s(x) \exp\left((C_t - C_s)(x) + (D_t - D_s)(x)\right) | \mathcal{D}_s \right] \\ &= M_s(x) \mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}}[\exp((C_t - C_s)(x) + (D_t - D_s)(x))] \\ &= M_s(x) \mathbb{E}_{P^{(f_2, \sigma^2, 0)}}[\exp((C_t - C_s)(x))] \mathbb{E}_{P^{(0, 0, \nu_2)}}[\exp((D_t - D_s)(x))] \\ &= M_s(x) \mathbb{E}_{P^{(0, 0, \nu_2)}}[\exp(D_{t-s}(x))] \\ &= M_s(x). \end{aligned}$$

□

Lemma 2.6. *Suppose that the hypothesis (9) and (11) of Lemma 2.5 are satisfied. Then, using the same notations as above, $P^{(f_1, \sigma^2, \nu_1)}|_{\mathcal{D}_t}$ and $P^{(f_2, \sigma^2, \nu_2)}|_{\mathcal{D}_t}$ are equivalent for all t and the density is given by:*

$$(14) \quad \frac{dP^{(f_1, \sigma^2, \nu_1)}}{dP^{(f_2, \sigma^2, \nu_2)}}(x) = M_t(x).$$

Proof. For $s < t$, we prove that $\mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}}[\exp(iu(x_t - x_s)) \frac{M_t(x)}{M_s(x)} | \mathcal{D}_s] = \mathbb{E}_P^{(f_1, \sigma^2, \nu_1)}[\exp(iu(x_t - x_s))]$. To that aim remark that, thanks again to Theorem 2.3:

$$(15) \quad \begin{aligned} \mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}} \left[e^{iu(x_t - x_s)} \frac{M_t(x)}{M_s(x)} | \mathcal{D}_s \right] &= \mathbb{E}_{P^{(f_2, \sigma^2, \nu_2)}} \left[e^{iu(x_t^{c, \nu_2} - x_s^{c, \nu_2} + x_t^{d, \nu_2} - x_s^{d, \nu_2})} \frac{M_t(x)}{M_s(x)} | \mathcal{D}_s \right] \\ &= \mathbb{E}_{P^{(f_2, \sigma^2, 0)}} \left[e^{iu(x_t - x_s)} e^{(C_t - C_s)(x)} \right] \mathbb{E}_{P^{(0, 0, \nu_2)}} \left[e^{iu(x_t - x_s)} e^{(D_t - D_s)(x)} \right]. \end{aligned}$$

Let us now compute the first factor of (15):

$$\begin{aligned} \mathbb{E}_{P^{(f_2, \sigma^2, 0)}} \left[e^{iu(x_t - x_s)} e^{(C_t - C_s)(x)} \right] &= \mathbb{E}_{P^{(f_1 - \eta, \sigma^2, 0)}} \left[e^{iu(x_t - x_s)} \right] \\ &= \exp\left(iu \int_s^t (f_1(r) - \eta) dr - \frac{u^2}{2} \int_s^t \sigma^2(r) dr\right). \end{aligned}$$

In the first equality we used the Girsanov theorem, thanks to the fact that $\int_0^t \frac{1}{\sigma(r)} (dx_r - f_2(r) dr)$ is a Brownian motion under $P^{(f_2, \sigma^2, 0)}$, while the second one follows from (3).

We compute the second factor of (15) by means of Theorem 2.4 and another application of (3):

$$\begin{aligned}\mathbb{E}_{P(0,0,\nu_2)} \left[e^{iu(x_t-x_s)} e^{(D_t-D_s)(x)} \right] &= \mathbb{E}_{P(0,0,\nu_2)} \left[e^{iux_{t-s}} e^{D_{t-s}(x)} \right] \\ &= \mathbb{E}_{P(\eta,0,\nu_1)} \left[e^{iux_{t-s}} \right] \\ &= \exp \left((t-s) \left[iu\eta - \int_{\mathbb{R}} (1 - e^{iuy} + iuy\mathbb{I}_{|y|\leq 1}) \nu_1(dy) \right] \right).\end{aligned}$$

Consequently:

$$(16) \quad \mathbb{E}_{P(f_2,\sigma^2,\nu_2)} \left[e^{iu(x_t-x_s)} \frac{M_t(x)}{M_s(x)} \middle| \mathcal{D}_s \right] = \mathbb{E}_{P(f_1,\sigma^2,\nu_1)} [e^{iu(x_t-x_s)}] \quad \forall 0 \leq s \leq t.$$

Fix t and define a probability measure P_t on \mathcal{D}_t by $P_t(B) = \mathbb{E}_{P(f_2,\sigma^2,\nu_2)} [M_t \mathbb{I}_B]$ for $B \in \mathcal{D}_t$. As a consequence of Lemma 2.5 and the Bayes rule, the two processes given by $(\{x_s : 0 \leq s \leq t\}, P^{(f_1,\sigma^2,\nu_1)} |_{\mathcal{D}_t})$ and $(\{x_s : 0 \leq s \leq t\}, P_t)$ are identical. Indeed, by (16), both have independent increments and the prescribed characteristic function. Consequently, (14) holds. \square

3. PROOF OF THEOREM 1.1

For the proof we will need the following three calculus lemmas.

Lemma 3.1. *Let X be a random variable with normal law $\mathcal{N}(m, \sigma^2)$. Then*

$$\mathbb{E} \left| 1 - e^X \right| = 2 \left[\phi \left(-\frac{m}{\sigma} \right) - \phi \left(-\frac{m}{\sigma} - \sigma \right) \right],$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$.

Proof. By definition we have

$$\begin{aligned}\mathbb{E} \left| 1 - e^X \right| &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |1 - e^x| e^{-\frac{(x-m)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^0 (1 - e^x) e^{-\frac{(x-m)^2}{2\sigma^2}} dx + \int_0^{\infty} (e^x - 1) e^{-\frac{(x-m)^2}{2\sigma^2}} dx \right).\end{aligned}$$

To conclude, just split the sums inside the integrals and use the change of variables $(y = \frac{x-m}{\sigma} - \sigma)$, resp. $(y = \frac{x-m}{\sigma})$. \square

Lemma 3.2. *For all x, y in \mathbb{R} we have:*

$$(17) \quad |1 - e^{x+y}| \leq \frac{1+e^x}{2} |1 - e^y| + \frac{1+e^y}{2} |1 - e^x|.$$

Proof. By symmetry we restrict to $x \geq 0$.

- $x, y \geq 0$: In this case we have that $|1 - e^{x+y}|$ is exactly equal to $\frac{1+e^x}{2} |1 - e^y| + \frac{1+e^y}{2} |1 - e^x|$.
- $x \geq 0, y \leq 0, x+y \geq 0$: Then the member on the right hand side of (17) is equal to $e^x - e^y \geq e^x - 1 \geq e^{x+y} - 1$.
- $x \geq 0, y \leq 0, x+y \leq 0$: In this case the member on the right of (17) is equal to $e^x - e^y \geq 1 - e^y \geq 1 - e^{x+y}$.

\square

Lemma 3.3. *With the same notations as in Theorem 1.1 and Lemma 2.5, we have:*

$$(18) \quad \mathbb{E}_{P(0,0,\nu_2)} \left[|1 - \exp(D_T(x))| \right] = \mathbb{E}_{P(\gamma^{\nu_2}, 0, \nu_2)} \left[|1 - \exp(D_T(x))| \right] \leq 2 \sinh \left(T \int_{\mathbb{R}} L_1(\nu_1, \nu_2) \right).$$

Proof. Because of Theorem 2.3 it is clear that $\mathbb{E}_{P(0,0,\nu_2)} [|1 - \exp(D_T(x))|] = \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} [|1 - \exp(D_T(x))|]$. In order to simplify the notations let us write

$$A^\pm(x) := \lim_{\varepsilon \rightarrow 0} \left(\sum_{r \leq T} \ln h^\pm(\Delta x_r) \mathbb{1}_{|\Delta(x_r)| > \varepsilon} - T \int_{|y| > \varepsilon} (h^\mp(y) - 1) \nu_2(dy) \right)$$

with $h^+ = \left(\frac{d\nu_1}{d\nu_2}\right)^{\mathbb{1}_{\frac{d\nu_1}{d\nu_2} \geq 1}}$ and $h^- = \left(\frac{d\nu_1}{d\nu_2}\right)^{\mathbb{1}_{\frac{d\nu_1}{d\nu_2} < 1}}$, so that

$$D_T(x) = \exp(A^+(x) + A^-(x)).$$

Then, using Lemma 3.2 and the fact that $A^+(x) \geq 0$ and $A^-(x) \leq 0$ we get:

$$\begin{aligned} \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} [|1 - D_T(x)|] &= \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} |1 - \exp(A^+(x) + A^-(x))| \\ &\leq \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} \left[\frac{1 + e^{A^+(x)}}{2} |1 - e^{A^-(x)}| + \frac{1 + e^{A^-(x)}}{2} |1 - e^{A^+(x)}| \right] \\ &= \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} [e^{A^+(x)} - e^{A^-(x)}]. \end{aligned}$$

In order to compute the last quantity we apply Theorem 2.4 and the fact that both $A^+(x)$ and $A^-(x)$ have the same law under $P(\gamma^{\nu_2},0,\nu_2)$ and $P(0,0,\nu_2)$:

$$\begin{aligned} \mathbb{E}_{P(\gamma^{\nu_2},0,\nu_2)} [e^{A^+(x)} - e^{A^-(x)}] &= \exp \left(T \int_{\mathbb{R}} (h^+(y) - h^-(y)) \nu_2(dy) \right) \\ &\quad - \exp \left(T \int_{\mathbb{R}} (h^-(y) - h^+(y)) \nu_2(dy) \right) \\ &= 2 \sinh \left(T \int_{\mathbb{R}} (h^+(y) - h^-(y)) \nu_2(dy) \right) \\ &= 2 \sinh \left(T \int_{\mathbb{R}} \left| 1 - \frac{d\nu_1}{d\nu_2}(y) \right| \nu_2(dy) \right). \end{aligned}$$

□

Proof of Theorem 1.1. Case $\sigma^2 > 0$: With the same notations as in Lemma 2.5 and by means of Lemma 2.6 one can write

$$L_1(P(f_1, \sigma^2, \nu_1), P(f_2, \sigma^2, \nu_2)) = \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} |1 - \exp(C_T(x) + D_T(x))|.$$

Now, using Lemma 3.2 and the independence between $C_T(x)$ and $D_T(x)$ (Theorem 2.3), we obtain

$$\begin{aligned} L_1(P(f_2, \sigma^2, \nu_2), P(f_1, \sigma^2, \nu_1)) &\leq \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} \left(\frac{1 + e^{C_T(x)}}{2} \right) \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} |1 - e^{D_T(x)}| \\ &\quad + \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} \left(\frac{1 + e^{D_T(x)}}{2} \right) \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} |1 - e^{C_T(x)}|. \end{aligned}$$

We conclude the proof using Lemmas 3.3 and 3.1 together with the fact that $\mathbb{E}_{P(f_2, \sigma^2, \nu_2)} e^{C_T(x)} = 1 = \mathbb{E}_{P(f_2, \sigma^2, \nu_2)} e^{D_T(x)}$.

Case $\sigma^2 = 0$: If $f_1 - f_2 \equiv \gamma^{\nu_1} - \gamma^{\nu_2}$, notice that, as the drift component of $(\{x_t\}, P(f_1, 0, \nu_1))$ and $(\{x_t\}, P(f_2, 0, \nu_2))$ is deterministic, we have

$$\frac{dP(f_1, 0, \nu_1)}{dP(f_2, 0, \nu_2)}(x) = \frac{dP(f_1 - f_2, 0, \nu_1)}{dP(0, 0, \nu_2)}(x) = D_T(x)$$

with $D_T(x)$ as in (13). Theorem 2.4 allows us to write the L_1 -distance between $P(f_1, 0, \nu_1)$ and $P(f_2, 0, \nu_2)$ as $\mathbb{E}_{P(f_2, 0, \nu_2)} |1 - D_T(x)|$. We then obtain the bound $2 \sinh(TL_1(\nu_1, \nu_2))$ by means of Lemma 3.3. □

REFERENCES

- [1] Brown, Lawrence D. and Low, Mark G. Asymptotic equivalence of nonparametric regression and white noise *Ann. Statist.*, 24(6):2384–2398, 1996.
- [2] Cont, Rama and Tankov, Peter. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.
- [3] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999. Translated from the 1990 Japanese original, Revised by the author.