



HAL
open science

The 4-way deterministic Periodic Domino Problem is undecidable

Bastien Le Gloannec

► **To cite this version:**

Bastien Le Gloannec. The 4-way deterministic Periodic Domino Problem is undecidable. 2014. hal-00985482

HAL Id: hal-00985482

<https://hal.science/hal-00985482>

Preprint submitted on 29 Apr 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The 4-way deterministic Periodic Domino Problem is undecidable

Bastien Le Gloannec

Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, FR-45067 Orléans, France

Abstract

The most fundamental undecidable question on tilings is the Domino Problem that asks whether a Wang tileset tiles the discrete plane. Lükkarila proved in 2009 that it remains undecidable when restricting the input to the class of 4-way deterministic tilesets. Due to the existence of aperiodic tilesets, the most natural distinct variant of this problem is the Periodic Domino Problem which asks whether a Wang tileset admits a periodic tiling of the plane. This problem is also undecidable. Jeandel recently discovered a new and elegant proof for this result. Inspired by this new proof technique and some ingredients from Lükkarila's construction, we prove that it remains undecidable when restricted to 4-way deterministic tilesets.

Keywords: wang tiles, deterministic tilesets, domino problem, undecidability

Wang tiles are unit square tiles with a color on each edge. Wang tiles and their purely combinatorial and syntactical formalism of discrete tilings were introduced by Wang [1] in relation to the decision question of the $\forall\exists\forall$ fragment of the first-order logic. The most common associated decision problem is the Domino Problem which consists in deciding whether a Wang tileset (given as input) tiles the discrete plane. This problem was proved undecidable by Berger [2]. A simpler construction was later introduced by Robinson [3]. Several recent more synthetic approaches to this problem also exist in the literature [4, 5, 6].

The undecidability of the Domino problem is tightly related to the existence of aperiodic tilesets, i.e. tilesets that tile the plane but never in a periodic way. Indeed, a classical variant of the Domino Problem is the Periodic Domino Problem that asks whether a Wang tileset given as input admits a periodic tiling. As the Domino Problem is co-recursively enumerable while its periodic variant is recursively enumerable, both problems would be equal and decidable in a world without aperiodicity. The Periodic Domino Problem was proved undecidable by Gurevich and Koryakov [7]. This result is obtained by a slight modification of Berger's construction [2] for the Domino Problem. Similarly, Robinson's construction from [3] can be adapted to prove this result as detailed in [8] for instance. It can also easily be seen that the undecidability of the Periodic Domino Problem is equivalent to the recursive inseparability of the aperiodic tilesets from the periodic ones (tilesets that admit a periodic tiling). A new and in many ways simpler proof technique was recently introduced by Jeandel [9]. This method does not rely anymore on a previous proof of the undecidability of the Domino Problem, but simply takes as input any arbitrary aperiodic tileset.

A Wang tileset is 4-way deterministic if any of its tilings is uniquely determined and can be locally reconstructed from the tiles from any path crossing each line and column of the discrete plane. Deterministic tilesets were originally introduced by Kari [10] to prove the undecidability of the nilpotency problem of one-dimensional cellular automata by reduction of the Domino Problem with input restricted to one-way deterministic tilesets. This problem had actually already been considered and proved undecidable in a somehow forgotten proof of the undecidability of the Domino Problem by Aanderaa and Lewis [11]. A two-way deterministic aperiodic tileset also already existed in the literature: the 16 Wang tiles derived from the geometrical aperiodic set of Ammann A2 [12]. A 4-way deterministic aperiodic tileset was first built in 1999 by Kari and Papasoglu [13] and an isomorphic tileset was recently discovered starting from a different construction in [14]. The Domino Problem with input restricted to the class of 4-way deterministic

Email address: bastien.le-gloannec@univ-orleans.fr (Bastien Le Gloannec)

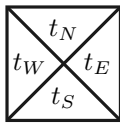


Figure 1: A Wang tile t

tilesets was proved undecidable ten years later by Lukkarila [15]. Although the Periodic Domino Problem was proved undecidable when restricted to one-way deterministic tilesets by Mazoyer and Rapaport [16] to derive the undecidability of the nilpotency of one-dimensional cellular automata over periodic configurations, the far more restrictive setting of 4-way deterministic tilesets remained untreated.

We prove in this article that the 4-way deterministic Periodic Domino Problem is undecidable. A quite natural strategy to obtain such a result would be to determinize a construction from an existing proof of undecidability in the non-deterministic case. Some constructions are however intrinsically non-deterministic. Moreover, determinism is a low-level, i.e. syntactical, property of the tileset, hence the determinization process requires a comprehensive knowledge and understanding of the construction at the level of the tiles and a careful treatment of non-determinism issues. Instead, we are inspired here by the general idea of the proof technique from Jeandel [9], which allows a rather neat high-level presentation and prevents us from a lot of purely technical annoying considerations. Our proof also builds upon the 4-way deterministic Turing machine simulation from Lukkarila [15].

1. Wang tiles and determinism

A *Wang tile* is a unit square tile with a color on each edge. Formally, it is a quadruple $t = (t_W, t_S, t_E, t_N) \in \mathcal{C}^4$ where \mathcal{C} is a finite set of colors, as depicted in figure 1. A *tileset* is a finite set of Wang tiles. Given a tileset τ , a *tiling* of the discrete plane is a map $c : \mathbb{Z}^2 \rightarrow \tau$ that associates to each point of the plane a tile of τ such that adjacent tiles share the same color on their common edge: for all $(x, y) \in \mathbb{Z}^2$, $t(x, y)_E = t(x + 1, y)_W$ and $t(x, y)_N = t(x, y + 1)_S$.

It is often convenient to describe Wang tiles by drawing simple patterns on them, instead of explicitly writing color letters. In that case, the color matching condition for tilings is represented by the continuity of pattern lines when crossing edges.

A tiling c is *periodic* if there exists a *periodicity vector* $p \in \mathbb{Z}^2$ such that for all $x \in \mathbb{Z}^2$, $c(x + p) = c(x)$. A tiling is *bi-periodic* if it admits two linearly independent periodicity vectors, or equivalently if it admits both a horizontal and a vertical period. The following lemma (folklore) links these notions regarding sets of tilings.

Lemma 1 (folklore). *The set of tilings by a tileset τ contains a periodic tiling if and only if it contains a bi-periodic tiling.*

A tileset is *aperiodic* if the set of tilings by τ is non-empty and does not contain any periodic tiling.

A tileset τ is *NE-deterministic* if each tile $t \in \tau$ is uniquely determined by its pair of colors (t_W, t_S) . Equivalently, for any pair of tiles $(t_W, t_S) \in \tau^2$, there is at most one tile $t \in \tau$ that is simultaneously compatible to its west side with t_W and to its south side with t_S . *{NW, SW, SE}-determinism* are defined symmetrically. A tileset is *4-way deterministic* if it is simultaneously deterministic in the four directions. In this article, we will essentially consider 4-way deterministic tilesets. As we will often make use of this without mentioning it, observe that it clearly follows from the definitions that any subset of a 4-way deterministic tileset remains 4-way deterministic.

The earliest and most common decision problem in the literature on tilings is the following.

Problem 1 (Domino Problem, [1]). *Given a tileset τ , decide whether there exists a tiling by τ .*

This problem was proved undecidable in [2].

The periodic variant of the Domino Problem is the following.



Figure 2: A tile by τ and the corresponding tiles in $\bar{\tau}^h$, $\bar{\tau}^v$ and $\bar{\tau}^c$

Problem 2 (Periodic Domino Problem). *Given a tiling set τ , decide whether there exists a periodic tiling by τ .*

This problem was introduced and proved undecidable in [7]. A new proof technique was recently discovered in [9].

In this article, we are interested in the 4-way deterministic restrictions of these problems where the input tiling set is furthermore required to be 4-way deterministic. The 4-way deterministic Domino Problem was proved undecidable in [15]. We prove in the following of this article that the 4-way deterministic Periodic Domino Problem is also undecidable.

2. Some transformations on tiling sets

In this section, we introduce some simple operations or transformations on tiling sets that preserve the determinism and will be useful in our construction.

Product. Given two tiling sets $\tau_1 \subseteq \mathcal{C}_1^4$ and $\tau_2 \subseteq \mathcal{C}_2^4$, the *product* tiling set $\tau = \tau_1 \times \tau_2$ is formally defined as the Wang tiling set

$$\tau = \{((w_1, w_2), (s_1, s_2), (e_1, e_2), (n_1, n_2)), (w_i, s_i, e_i, n_i) \in \tau_i, 1 \leq i \leq 2\}$$

over the set of colors $\mathcal{C}_1 \times \mathcal{C}_2$. Although the set τ is not strictly speaking the cartesian product of sets τ_1 and τ_2 , it is convenient to interpret it as such, seeing a product tiling set as a two-layered tiling set whose tiles hold a tile of τ_1 on the first layer and a tile of τ_2 on the second layer with a local matching condition requiring that the matching conditions of both layers are verified.

It is not difficult to see that if τ_1 and τ_2 are 4-way deterministic, then $\tau_1 \times \tau_2$ is also 4-way deterministic.

Disjoint mirror. Given a tiling set τ over a set of colors \mathcal{C} , the *horizontal disjoint mirror* tiling set of τ is the tiling set $\bar{\tau}^h$ obtained by duplicating all colors and swapping west and east colors of tiles from τ . More formally, we create a disjoint copy \mathcal{C}_h of \mathcal{C} whose colors are denoted a_h for each $a \in \mathcal{C}$ and we define $\bar{\tau}^h = \{(e_h, s_h, w_h, n_h) \in \mathcal{C}_h^4, (w, s, e, n) \in \tau\}$. We symmetrically define the *vertical disjoint mirror* $\bar{\tau}^v$ over a disjoint copy \mathcal{C}_v of \mathcal{C} (whose colors are denoted a_v for each $a \in \mathcal{C}$) by $\bar{\tau}^v = \{(w_v, n_v, e_v, s_v) \in \mathcal{C}_v^4, (w, s, e, n) \in \tau\}$. We finally define the *central disjoint mirror* $\bar{\tau}^c$ over a disjoint copy \mathcal{C}_c of \mathcal{C} (whose colors are denoted a_c for each $a \in \mathcal{C}$) by $\bar{\tau}^c = \{(e_c, n_c, w_c, s_c) \in \mathcal{C}_c^4, (w, s, e, n) \in \tau\}$. This last transformation corresponds to the combination of the two previous ones (hence to a central symmetry on the tiles). These three definitions are illustrated on figure 2.

It is clear that if a tiling set is 4-way deterministic, any of its disjoint mirrors is also 4-way deterministic (even though some determinism directions are interchanged in the transformation process). As their sets of colors are disjoint, any of the (disjoint) unions of a tiling set with some of its disjoint mirrors, in particular $\tau \sqcup \bar{\tau}^h \sqcup \bar{\tau}^v \sqcup \bar{\tau}^c$, is also 4-way deterministic.

Grouping. Given a tiling set τ , the 2×2 *grouping* of τ is the tiling set $\tau^{2 \times 2}$ over the set of colors τ^2 defined by

$$\tau^{2 \times 2} = \{(ac, cd, bd, ab) \in (\tau^2)^4, \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \text{ is a valid } 2 \times 2 \text{ pattern by } \tau\}$$

The tiling set $\tau^{2 \times 2}$ simply is a coding in Wang tiles of 2×2 patterns by τ with the local matching condition that two horizontally (resp. vertically) adjacent patterns overlap on one column (resp. row). Considering a tile of $\tau^{2 \times 2}$, we will use the notations $\begin{array}{|c|c|} \hline \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare \\ \hline \end{array}$ to identify the four underlying tiles of τ .

Observe that if τ is 4-way deterministic, then its 2×2 grouping $\tau^{2 \times 2}$ is also 4-way deterministic.

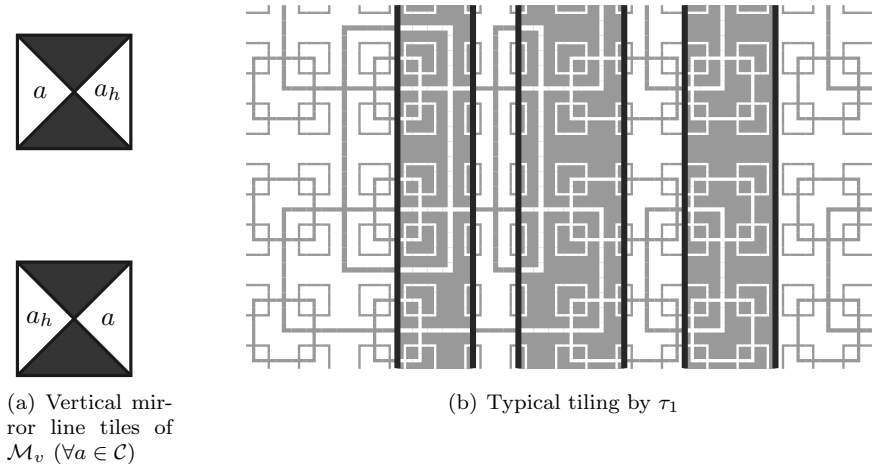


Figure 3: Description of tileset τ_1

3. The deterministic reduction

In this section, we prove the following result.

Theorem 1. *The Periodic Domino Problem restricted to 4-way deterministic tilesets is undecidable.*

We proceed by reduction of the Halting problem of Turing machines on the empty tape. Both problems are recursively enumerable. Given a Turing machine M , we build a 4-way deterministic tileset that always admits non-periodic tilings (some of them will actually not even contain Turing computations) but admits periodic tilings if and only if the computation of M halts from the empty tape.

The built tileset will be a product of several layers, each of which will be 4-way deterministic.

First layer: Aperiodic layer with vertical mirror lines. We consider here a 4-way deterministic aperiodic tileset τ_a on the set of colors \mathcal{C} . As already mentioned, such tilesets exist, some are built in [13, 14]. Let us stress that we do not need here to know anything about the way they are built or the structure of their tilings, the only required properties are 4-way determinism and aperiodicity. Let us consider the horizontal disjoint mirror $\bar{\tau}_a^h$ of tileset τ_a over a disjoint copy \mathcal{C}_h of \mathcal{C} (whose colors are denoted a_h for each $a \in \mathcal{C}$). It is clear by construction that $\bar{\tau}_a^h$ is aperiodic. We also add a set of vertical mirror line tiles by introducing a new color $\# \notin \mathcal{C} \cup \mathcal{C}_h$ and defining $\mathcal{M}_v = \{(a, \#, a_h, \#) \text{ and } (a_h, \#, a, \#), a \in \mathcal{C}\}$ as depicted in figure 3(a) (where the color $\#$ is painted black). We then define the tileset τ_1 as the disjoint union of the three previously considered tilesets $\tau_1 = \tau_a \sqcup \bar{\tau}_a^h \sqcup \mathcal{M}_v$.

A typical tiling by τ_1 is represented on figure 3(b). Due to the addition of mirror lines, this set is no longer aperiodic.

Lemma 2. τ_1 satisfies the following properties:

1. τ_1 is 4-way deterministic.
2. Every horizontally periodic tiling by τ_1 contains a vertical mirror line. Hence, by periodicity, it contains infinitely many such lines, the distance between two consecutive lines being bounded.

Proof. For the point 1, as τ_a is 4-way deterministic, $\bar{\tau}_a^h$ also has this property. As their sets of colors are disjoint, $\tau_a \sqcup \bar{\tau}_a^h$ is 4-way deterministic. The tileset \mathcal{M}_v is also 4-way deterministic since the color on each side of a mirror line is uniquely determined by the color on the other side. The color $\#$ introduced in \mathcal{M}_v being new, τ_1 is in turn 4-way deterministic.

For 2, if such a tiling does not contain any vertical mirror line, then it must be a tiling by one of the aperiodic tilesets τ_a or $\bar{\tau}_a^h$. It is hence non-periodic, a contradiction. \square

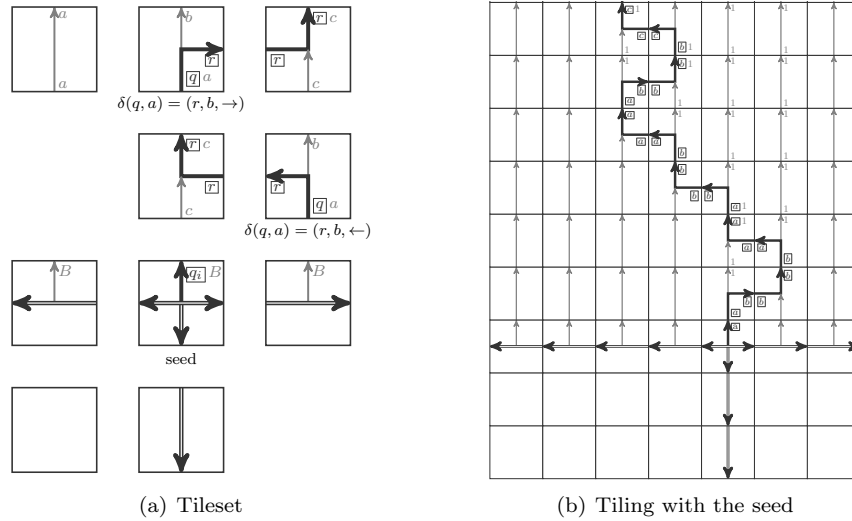


Figure 4: Computation layer of τ_M

Second layer: Aperiodic layer with horizontal mirror lines. Let us enrich the previous tileset by building, symmetrically to what we have just done¹, the vertical counterpart τ_v of τ_1 and define the two-layered tileset $\tau_2 = \tau_1 \times \tau_v$.

Lemma 3. τ_2 satisfies the following properties:

1. τ_2 is 4-way deterministic.
2. Every bi-periodic tiling by τ_2 contains a (irregular) grid (whose vertical and horizontal lines are to be read on the first and second layer respectively) with bounded rectangle size.

Proof. The point 1 is clear as τ_2 is a product of 4-way deterministic tilesets.

The point 2 is a direct consequence of the argument of lemma 2 (point 2) applied both horizontally on τ_1 and vertically on τ_v . \square

Third layer: Turing computation. We consider here the 4-way deterministic tileset introduced by Lukkarila in [15] (section 5, theorem 6) to simulate any given Turing machine in the tilings. This tileset is used to prove the undecidability of the 4-way deterministic Domino Problem *with a seed tile*, that asks whether a 4-way deterministic input tileset containing an identified seed tile admits a tiling of the plane in which the seed tile appears. It is generally easier to deal with that problem than the general Domino Problem as the seed tile can for instance be used to properly initialize an empty Turing machine tape with exactly one machine head. This tileset is a subset of a product of several layers, one of them (the first layer in [15], section 5.2) being a classical simulation of a Turing machine (whose head always moves at each transition) in Wang tiles using the tiles represented on figure 4(a) (where q_i is the initial state and B the blank letter). Such a tileset is obviously not itself 4-way deterministic, one of the difficulties overcome in [15] was to determinize it by coupling it with a particular structure on other layers. The built tileset draws the space-time diagram of the machine in the tilings, as depicted in figure 4(b). The lines of the tiling realize the transitions, each line of the diagram of the machine can be read at the edge between two lines of the tiling.

Consider a Turing machine M and the associated 4-way deterministic simulation tileset τ_M with unique seed tile t_0 (which simulation layer contains the indicated seed tile of figure 4(a)). We denote as $F \subseteq \tau_M$ the set of tiles in τ_M whose Turing machine simulation layer transmits a final state of M from its west or east

¹Both steps could actually have been done simultaneously combining directly the four mirrors of τ_a , we however believe that a two-step presentation is lighter and clearer.

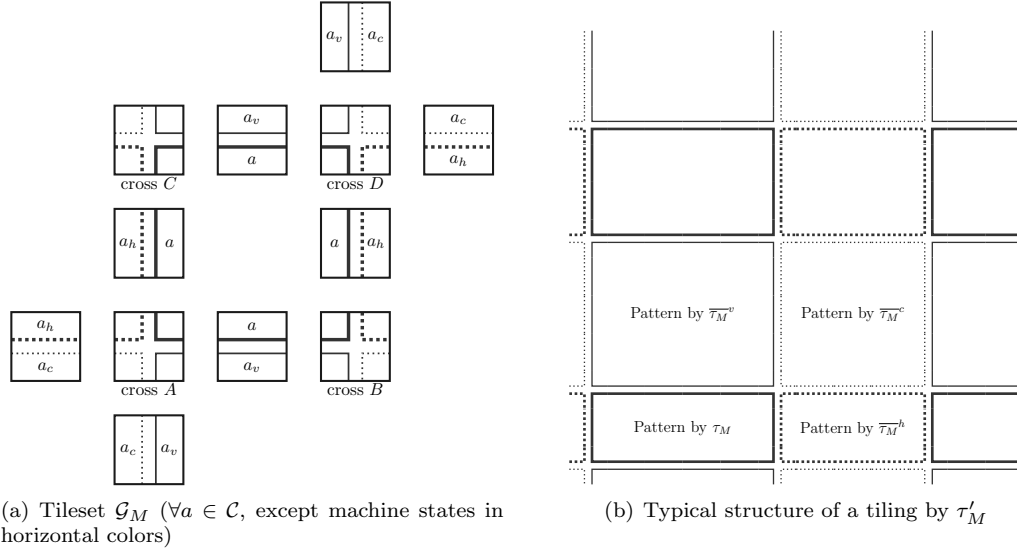


Figure 5: Description of tileset τ'_M

color to its north color. Tiles of F mark the end of the simulated Turing computation. Note that a tiling of the whole plane cannot contain both t_0 and a tile of F : Assuming that transitions from a final state are undefined in the Turing machine (which is the usual convention), there is no transition tile in the set with a final state as south color, hence the line directly on top of the line containing a tile of F cannot be tiled without error.

Let us consider the three disjoint mirrors of τ_M , over the set of colors \mathcal{C} , denoted as $\overline{\tau}_M^h, \overline{\tau}_M^v, \overline{\tau}_M^c$, over the respective sets of colors $\mathcal{C}_h, \mathcal{C}_v, \mathcal{C}_c$ (disjoint copies of \mathcal{C}). We assemble these four tilesets using a *parity grid* tileset \mathcal{G}_M whose tiles are represented on figure 5(a). Their purpose is to draw a grid with four types of rectangles alternating depending on the horizontal and vertical parities, as depicted in figure 5(b). The following constraint on \mathcal{G}_M will be crucial for our final result and cannot be read on the figure: Basically any color of τ_M (and its mirrors) can appear along grid lines of τ_M **except for** horizontal colors containing a machine state that cannot appear along vertical grid lines. The purpose of this alphabetical restriction is to forbid a Turing machine head from leaving its rectangle and complementarily to forbid any other Turing machine head from appearing unexpectedly in a rectangle. As the grid lines of \mathcal{G}_M play the role of mirror lines for the colors of the four mirror tilesets of τ_M , \mathcal{G}_M is easily seen 4-way deterministic. We define $\tau'_M = \tau_M \sqcup \overline{\tau}_M^h \sqcup \overline{\tau}_M^v \sqcup \overline{\tau}_M^c \sqcup \mathcal{G}_M$. Because the colors associated to the grid lines of \mathcal{G}_M do not appear in any of the four mirror tilesets of τ_M , it is not difficult to see that τ'_M is 4-way deterministic. The tilings by τ'_M are simply irregular parity grids (with possibly infinite rectangles) with patterns by each of the four mirror versions of τ_M in each of the four types of rectangles, as depicted in figure 5(b). Patterns are locally symmetrized along grid lines.

Let us assemble τ'_M with our previous construction τ_2 by defining a three-layered tileset τ_3 as a subset of $\tau_2 \times \tau'_M$ (as τ_2 has two layers, we consider here the tiles of τ'_M as the third layer) with the following restriction on tiles: A vertical (resp. horizontal) mirror line appears on the first (resp. second) layer (tiles of τ_1 , resp. τ_v) if and only if it is coupled with a vertical (resp. horizontal) grid line tile or a cross of \mathcal{G}_M on the third layer (tiles of τ'_M). Note that as a consequence, a vertical mirror line on the first layer is coupled with a horizontal mirror line on the second layer if and only if the third layer holds a cross of the grid. Moreover, a tile of $\tau_M \sqcup \overline{\tau}_M^h \sqcup \overline{\tau}_M^v \sqcup \overline{\tau}_M^c$ appears on the third layer if and only if it is coupled with tiles from the aperiodic sets on the first and second layers.

Lemma 4. τ_3 verifies the following properties:

1. τ_3 is 4-way deterministic.

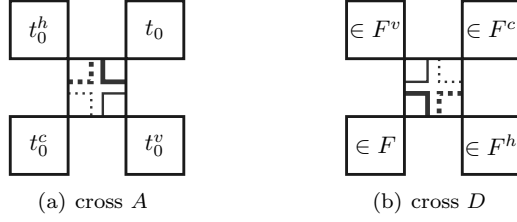


Figure 6: Alphabetical restrictions on the third layer of the grouped tileset τ_4

2. Every bi-periodic tiling by τ_3 contains a grid on its third layer with finite patterns (possibly empty) by one of each of the four mirror tilesets of τ_M in each of its four types of rectangles.

Proof. τ_3 is 4-way deterministic as a subset (restriction on tiles) of the product of the 4-way deterministic tilesets τ_2 and τ'_M .

By lemma 3 (point 2), the third layer of any bi-periodic tiling by τ_3 contains a grid of finite rectangles. By construction of τ'_M , each of the four types of rectangles contains a finite pattern by its associated mirror tileset of τ_M . Note however that any patterns can appear provided that they are correctly locally symmetrized along grid lines (and that they do not contain a machine head in their horizontal border colors due to the restriction on \mathcal{G}_M). \square

It only remains to enforce correctly initialized and terminating Turing computations in the rectangles of the third layer of τ_3 . We denote as t_0^h, t_0^v, t_0^c and F^h, F^v, F^c the mirror versions of the seed tile $t_0 \in \tau_M$ and the final set $F \subset \tau_M$ in the mirror tilesets $\tau_M^h, \tau_M^v, \tau_M^c$ respectively. Let us define τ_4 as a subset of the 2×2 grouping of τ_3 with the following restriction on the third layer of the 2×2 grouped tiles:

- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross A , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be t_0 ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross D , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be in F ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross A , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be t_0^h ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross D , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be in F^h ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross A , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be t_0^v ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross D , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be in F^v ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross A , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be t_0^c ;
- if the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ is the cross D , then the tile $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ must be in F^c .

These constraints are summarized on the figure 6.

Lemma 5. τ_4 is 4-way deterministic.

Proof. Clear as τ_4 is a subset of $\tau_3^{2 \times 2}$ which is 4-way deterministic as τ_3 is. \square

For the following result, we furthermore require the Turing machine M to verify the following properties:

- during the computation, the head never moves to a position situated to the left of its starting position;
- when the computation halts (if it halts), the position of the head is exactly the rightmost position that has been reached during the computation.

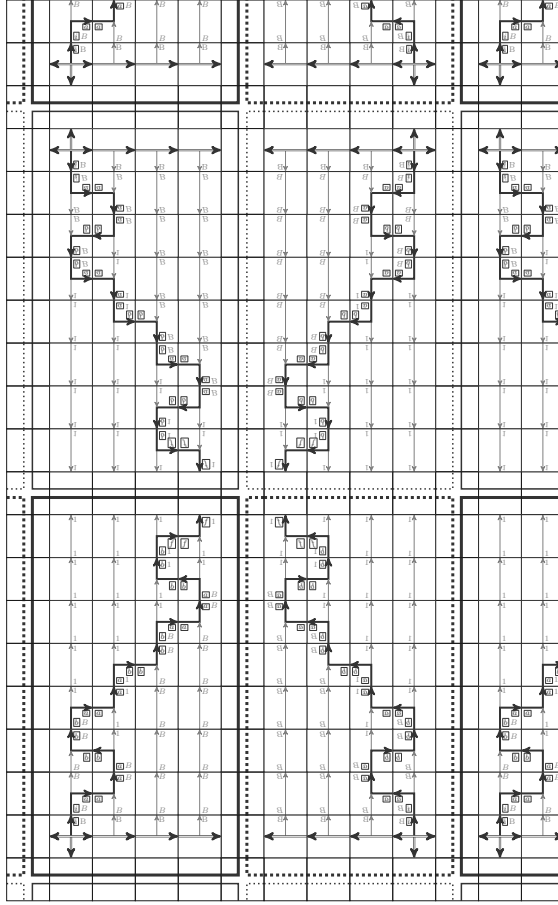


Figure 7: Computation layer of bi-periodic tilings by τ_4

These requirements can be made without loss of generality as any Turing machine is simulated by a machine satisfying these constraints (which can be recursively obtained from M).

The tileset τ_M always admits tilings that do not contain Turing computation (all blank on their computation layer), hence τ_4 always admits non-periodic tilings that do not embed any Turing computations on their third layer.

Proposition 1. τ_4 admits periodic tilings if and only if the Turing machine M halts from the empty tape.

Proof. If the machine halts, then it is easy to describe a bi-periodic tiling by τ_4 . Indeed, there exists a rectangle pattern P by τ_M that fully describes the space-time diagram of the computation of M from the empty tape, with the tile t_0 in the bottom-left corner and a tile of F in the top-right corner (due to the previous additional requirements on M). Then this pattern can be embedded in a rectangle of the grid of the third layer of τ_4 and the three mirror patterns by τ_M^h , τ_M^v and τ_M^c can be embedded in the three neighboring rectangles (east, north and north-east) of the grid. The obtained pattern of τ_4 can be repeated bi-periodically. Such a tiling is represented on figure 7.

Conversely, if τ_4 admits periodic tilings, then it admits bi-periodic tilings by lemma 1. By lemma 4 (point 2), the third layer of such a tiling draws a grid whose rectangles contain finite patterns by τ_M and its three mirror copies. Let us consider the case of such a pattern by τ_M (one can also observe that the three other cases would be symmetrical). Due to the restrictions imposed on the grouping, the tile at the bottom-left corner of this pattern is the seed tile t_0 . This tile enforces the first (bottom) line of the pattern to be an empty Turing machine tape with a (unique) head on the first cell. Recall that, due to the restrictions on the

grid tiles \mathcal{G}_M , no other machine head can appear in the rectangle and the original machine head cannot leave the pattern. Hence the local rules of the tileset τ_M (whose computation layer is depicted in figure 4) enforce the following rows to contain exactly the successive configurations of the Turing machine tape. As the head cannot leave without tiling error, the pattern width is necessarily sufficiently large for the head not to leave the pattern at the right border. Because of the additional requirements on the machine, its computation can be simulated without the head leaving the pattern at the left border. Hence the computation of the machine starting on the empty tape is correctly simulated in the pattern. As a final state is required in the top-right corner, the computation is required to halt. Thus the simulated machine M halts from the empty tape, which is the requested result. Moreover, the height of the pattern is exactly the halting time of the machine and its width is exactly the distance between the starting cell and the cell where the head halts, which exactly corresponds to the space used by the computation (due to the additional requirements on the machine). Thus, as considering patterns by the three mirror tilesets of τ_M would lead to symmetrical descriptions, the third layer of a bi-periodic tiling must be of the kind of the figure 7. \square

This concludes the reduction and the proof of theorem 1.

References

- [1] H. Wang, Proving theorems by Pattern Recognition II, Bell Systems technical journal 40 (1961) 1–41.
- [2] R. Berger, The Undecidability of the Domino Problem, Ph.D. thesis, Harvard University (1964).
- [3] R. Robinson, Undecidability and nonperiodicity for tilings of the plane, *Inventiones mathematicae* 12 (3) (1971) 177–209.
- [4] J. Kari, The tiling problem revisited, in: J. O. Durand-Lose, M. Margenstern (Eds.), *MCU*, Vol. 4664 of *Lecture Notes in Computer Science*, Springer, 2007, pp. 72–79.
- [5] N. Ollinger, Two-by-two substitution systems and the undecidability of the domino problem, in: A. Beckmann, C. Dimitracopoulos, B. Löwe (Eds.), *CiE*, Vol. 5028 of *Lecture Notes in Computer Science*, Springer, 2008, pp. 476–485.
- [6] B. Durand, A. E. Romashchenko, A. Shen, Fixed point and aperiodic tilings, in: M. Ito, M. Toyama (Eds.), *Developments in Language Theory*, Vol. 5257 of *Lecture Notes in Computer Science*, Springer, 2008, pp. 276–288.
- [7] Y. S. Gurevich, I. Koryakov, Remarks on Berger’s paper on the domino problem, *Siberian Mathematical Journal* 13 (2) (1972) 319–321.
- [8] C. Allauzen, B. Durand, Tiling Problems, in: *The Classical Decision Problem, Perspectives in Mathematical Logic*, Springer, 2001, Ch. A, pp. 407–420.
- [9] E. Jeandel, The periodic domino problem revisited, *Theor. Comput. Sci.* 411 (44-46) (2010) 4010–4016.
- [10] J. Kari, The nilpotency problem of one-dimensional cellular automata, *SIAM J. Comput.* 21 (3) (1992) 571–586.
- [11] S. O. Aanderaa, H. R. Lewis, Linear Sampling and the $\forall\exists\forall$ Case of the Decision Problem, *Journal of Symbolic Logic* 39 (3) (1974) 519–548.
- [12] B. Grunbaum, G. Shephard, *Tilings and Patterns*, W.H. Freeman, 1987.
- [13] J. Kari, P. Papasoglu, Deterministic aperiodic tile sets, *Geometric And Functional Analysis* 9 (1999) 353–369.
- [14] B. Le Gloannec, N. Ollinger, Substitutions and strongly deterministic tilesets, in: S. B. Cooper, A. Dawar, B. Löwe (Eds.), *CiE*, Vol. 7318 of *Lecture Notes in Computer Science*, Springer, 2012, pp. 462–471.
- [15] V. Lukkarila, The 4-way deterministic tiling problem is undecidable, *Theor. Comput. Sci.* 410 (16) (2009) 1516–1533.
- [16] J. Mazoyer, I. Rapaport, Global fixed point attractors of circular cellular automata and periodic tilings of the plane: Undecidability results, *Discrete Mathematics* 199 (1-3) (1999) 103–122.