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# Rigidity results for semilinear elliptic equations with exponential nonlinearities and Moser-Trudinger-Onofri inequalities on two-dimensional Riemannian manifolds

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**Abstract.** This paper is devoted to the Moser-Trudinger-Onofri inequality on smooth compact connected Riemannian manifolds. We establish a rigidity result for the Euler-Lagrange equation and deduce an estimate of the optimal constant in the inequality on two-dimensional closed Riemannian manifolds. Compared to existing results, we provide a non-local criterion which is well adapted to variational methods, introduce a nonlinear flow along which the evolution of a functional related with the inequality is monotone and get an integral remainder term which allows to discuss optimality issues. As an important application of our method, we also consider the non-compact case of the Moser-Trudinger-Onofri inequality on the two-dimensional Euclidean space, with weights. The standard weight is the one that is computed when projecting the two-dimensional sphere using the stereographic projection, but we also give more general results which are of interest, for instance, for the Keller-Segel model in chemotaxis.

In this paper we assume that  $(\mathfrak{M}, g)$  is a smooth compact connected Riemannian manifold of dimension  $d \geq 1$ , without boundary. We denote by  $\Delta_g$  the Laplace-Beltrami operator on  $\mathfrak{M}$ . For simplicity, we assume that the volume of  $\mathfrak{M}$ , is chosen equal to 1 and use the notation  $dv_g$  for the volume element. We shall also denote by  $\mathfrak{R}$  the Ricci tensor, by  $H_g u$  the Hessian of  $u$  and by

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. Let us denote by  $M_g u$  the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2.$$

We define

$$(1) \quad \lambda_* := \inf_{u \in H^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}} \left[ \|L_g u - \frac{1}{2} M_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] e^{-u/2} dv_g}{\int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g}.$$

If  $A = (a_{ij})_{i,j=1,2}$  and  $B = (b_{ij})_{i,j=1,2}$  are two matrices, then we use the convention that  $A : B = \sum_{i,j=1,2} a_{ij} b_{ij}$  and  $\|A\|^2 = A : A$ .  $I_2$  denotes the  $2 \times 2$  identity matrix.

**Theorem 1.** *Assume that  $d = 2$  and  $\lambda_* > 0$ . If  $u$  is a smooth solution to*

$$(2) \quad -\frac{1}{2} \Delta_g u + \lambda = e^u,$$

*then  $u$  is a constant function if  $\lambda \in (0, \lambda_*)$ .*

Next, let us consider the Moser-Trudinger-Onofri inequality on  $\mathfrak{M}$  written as

$$(3) \quad \frac{1}{4} \|\nabla u\|_{L^2(\mathfrak{M})}^2 + \lambda \int_{\mathfrak{M}} u dv_g \geq \lambda \log \left( \int_{\mathfrak{M}} e^u dv_g \right) \quad \forall u \in H^1(\mathfrak{M}),$$

for some constant  $\lambda > 0$ . Let us denote by  $\lambda_1$  the first positive eigenvalue of  $-\Delta_g$ .

**Corollary 2.** *If  $d = 2$ , then (3) holds with  $\lambda = \Lambda := \min\{4\pi, \lambda_*\}$ . Moreover, if  $\Lambda$  is strictly smaller than  $\lambda_1/2$ , then the optimal constant in (3) is strictly larger than  $\Lambda$ .*

In the case of the normalized sphere,  $\lambda_* = 4\pi$  is optimal but (2) has non-constant solutions because of the conformal invariance: see [22, 16] for a review of the Moser-Trudinger-Onofri inequality on the sphere, and references therein. The interested reader is invited to refer to the historical papers [32, 27, 30] and to [20, 19] for recent results on functionals related to the inequality, that have been obtained by variational methods. These two papers solve the question, in any dimension, of whether the *first best constant* can be reached. This is equivalent to showing that the difference of the two terms in (3) is bounded from below. Earlier results have been obtained by T. Aubin in [1], in the case of the sphere  $\mathbb{S}^n$ , and P. Cherrier in [13] for general 2-manifolds. The present paper focuses on the value of the *second best constant*, defined as the largest value of  $\lambda$  such that (3) holds. The value of the *first best constant* is of little concern to us, as it appears as the  $\frac{1}{4}$  coefficient in front of  $\|\nabla u\|_{L^2(\mathfrak{M})}^2$  and can be factored out into  $\lambda$ . The method of Z. Faget relies on a blow-up analysis which is reminiscent for  $d = 2$  of [29]. It generalizes some results contained in [2, Theorem 2.50 page 68]. Other references of general interest in the context of the Moser-Trudinger-Onofri inequality are [12, 4, 31, 22]. More references will be given within the text, whenever needed.

At this point, we should emphasize that in most of the literature the Moser-Trudinger-Onofri inequality in dimension  $d = 2$  is not written as in (3), but in the form

$$e^{\mu_2 \|\nabla u\|_{L^2(\mathfrak{M})}^2} \geq C \int_{\mathfrak{M}} e^u dv_g$$

for all functions  $u \in H^1(\mathfrak{M})$  such that  $\int_{\mathfrak{M}} u dv_g = 0$ , for some constant  $C$  which is in general non-explicit. In dimension  $d = 2$ , the optimal constant is  $\mu_2 = \frac{1}{16\pi}$ . This amounts to write that the functional

$$u \mapsto \mu_2 \|\nabla u\|_{L^2(\mathfrak{M})}^2 + \int_{\mathfrak{M}} u dv_g - \log \left( \int_{\mathfrak{M}} e^u dv_g \right)$$

is bounded from below by  $\log C$ . The issue of the *first best constant* is to prove that  $\mu_2$  cannot be replaced by a smaller constant unless the functional is unbounded from below. This problem is not the same as Inequality (3), except when  $C = 1$  and  $\lambda = 1$ . E. Onofri proved in [30] that this is the case, with optimal values for both  $C$  and  $\lambda$ , when  $\mathfrak{M} = \mathbb{S}^2$ , up to a factor  $4\pi$  that comes from the normalization

of  $\text{vol}_g(\mathfrak{M})$ . Except for the sphere we are aware of only one occurrence in the literature of the form (3) of the inequality, that has been derived by E. Fontenas in [21, Théorème 2] under more restrictive conditions on  $\mathfrak{M}$ . This result will be commented in more detail in Remark 4.

The proof of Theorem 1 is a *rigidity* method inspired by the one of [7] for the equation

$$-\Delta_g u + \lambda u = u^{p-1},$$

which is the Euler-Lagrange equation corresponding to the optimality case in the interpolation inequality

$$(4) \quad \|\nabla v\|_{L^2(\mathfrak{M})}^2 \geq \frac{\lambda}{p-2} \left[ \|v\|_{L^p(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right] \quad \forall v \in H^1(\mathfrak{M}).$$

See [7, 26, 3, 14, 17] for further results on this problem and [2, 24] for general accounts on Sobolev's inequality on Riemannian manifolds. Concerning spectral issues, a standard textbook is [6].

The case of the exponential nonlinearity in (2) has been much less considered in the literature, except when  $\mathfrak{M}$  is the two dimensional sphere  $\mathbb{S}^2$ . In [21], and in [5] in the case of the ultraspherical operator, the result is achieved by considering the interpolation inequalities (4) and then, as in [7] or [4] (in the case of the sphere), by taking the limit as  $p \rightarrow \infty$ . Here we consider a direct approach, based on rigidity methods and an associated nonlinear flow. As far as we know, this is an entirely new approach which has the interest of providing explicit estimates on the optimal constant in (3).

One may wonder if rigidity results can be achieved for dimensions  $d > 2$  by our method. We will give a negative answer in Section 1. Corollary 2 is established in Section 2 using a nonlinear flow that has already been considered on the sphere in [16]. The case  $d = 1$  is very simple and will be considered for completeness in Section 3. An important application of our method is the case of the Euclidean space with weights, with applications to chemotaxis. Section 4 is devoted to this issue with a main result in this direction stated in Theorem 8, that raises difficult questions of symmetry breaking.

## 1. PROOF OF THEOREM 1

In this section we consider a smooth solution to (2) and perform a computation to prove the rigidity result of Theorem 1. There is no *a priori* reason to assume that  $d = 2$  and so we shall do the computations for any dimension  $d \geq 1$ , which raises no special additional difficulties. However, due to restrictions that are inherent to the method and will be explicitly exposed, only  $d = 2$  can be covered. On several occasions, one has to divide by  $(d - 1)$ , so the case  $d = 1$  has to be excluded and will be handled directly in Section 3.

In the case of (4), it is well known (see [26, 3, 18]) that an interpolation depending on a parameter  $\theta \in (0, 1)$  between an estimate based on the Ricci curvature and another one based on the first eigenvalue of the Laplace-Beltrami operator can be used to obtain some improvements. Here we apply the same technique and realize in the end that only  $\theta = 1$  is admissible in dimension  $d = 2$ . However, when  $d$  is considered as a real parameter in the range  $(1, 2)$ , it is possible to optimize on  $\theta$  when  $0 \leq \theta \leq 1$ . We will comment this and possible improvements at the end of this section.

**Preliminaries.** A simple expansion of the square shows that

$$\|\mathbf{H}_g u\|^2 = \|\mathbf{L}_g u\|^2 + \frac{1}{d} (\Delta_g u)^2.$$

The Bochner-Lichnerovicz-Weitzenböck formula asserts that

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\mathbf{H}_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

where  $\mathfrak{R}$  denotes the Ricci tensor and, as a consequence,

$$(5) \quad \frac{1}{2} \Delta |\nabla u|^2 = \|\mathbf{L}_g u\|^2 + \frac{1}{d} (\Delta_g u)^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u).$$

**An identity based on integrations by parts.** Using integrations by parts, we may notice that

$$\begin{aligned} & - \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g \\ &= -\frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g + 2 \int_{\mathfrak{M}} \mathbf{H}_g u : \nabla u \otimes \nabla u e^{-u/2} dv_g \\ &= -\frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g + 2 \int_{\mathfrak{M}} (\mathbf{L}_g u + \frac{g}{d} \Delta_g u) : \nabla u \otimes \nabla u e^{-u/2} dv_g \\ &= -\frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g + 2 \int_{\mathfrak{M}} \mathbf{L}_g u : \mathbf{M}_g u e^{-u/2} dv_g \\ & \quad + \frac{2}{d} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g, \end{aligned}$$

with  $\mathbf{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2$ , which proves that

$$\begin{aligned} & \frac{d+2}{d} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g \\ &= \frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g - 2 \int_{\mathfrak{M}} \mathbf{L}_g u : \mathbf{M}_g u e^{-u/2} dv_g \end{aligned}$$

and finally

$$(6) \quad \begin{aligned} & \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g \\ &= \frac{1}{2} \frac{d}{d+2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g - \frac{2d}{d+2} \int_{\mathfrak{M}} \mathbf{L}_g u : \mathbf{M}_g u e^{-u/2} dv_g. \end{aligned}$$

**An identity based on the Bochner-Lichnerovicz-Weitzenböck formula.**

By expanding  $\Delta_g(e^{-u/2}) = (\frac{1}{4} |\nabla u|^2 - \frac{1}{2} \Delta_g u) e^{-u/2}$ , we have that

$$\int_{\mathfrak{M}} |\nabla u|^2 \Delta_g(e^{-u/2}) dv_g = \frac{1}{4} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g - \frac{1}{2} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g$$

so that, if we multiply (5) by  $e^{-u/2}$  and integrate by parts, then we get

$$\begin{aligned}
& \frac{1}{8} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g - \frac{1}{4} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g \\
&= \int_{\mathfrak{M}} \|\mathbb{L}_g u\|^2 e^{-u/2} dv_g + \frac{1}{d} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g \\
&\quad + \int_{\mathfrak{M}} (\nabla(\Delta_g u) \cdot \nabla u) e^{-u/2} dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g \\
&= \int_{\mathfrak{M}} \|\mathbb{L}_g u\|^2 e^{-u/2} dv_g - \frac{d-1}{d} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g \\
&\quad + \frac{1}{2} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g,
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
(7) \quad & \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g \\
&= \frac{3}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g - \frac{1}{8} \frac{d}{d-1} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \\
&\quad + \frac{d}{d-1} \int_{\mathfrak{M}} \|\mathbb{L}_g u\|^2 e^{-u/2} dv_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g.
\end{aligned}$$

**A Poincaré inequality.** Since

$$4 \Delta_g(e^{-u/4}) = \frac{1}{4} |\nabla u|^2 e^{-u/4} - \Delta_g u e^{-u/4},$$

we get that

$$\begin{aligned}
16 \int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2 dv_g &= \frac{1}{16} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g - \frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^2 \Delta_g u e^{-u/2} dv_g \\
&\quad + \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g.
\end{aligned}$$

On the other hand, a Poincaré inequality applied to  $\nabla(e^{-u/4})$ , as in [18, Lemma 7], shows that

$$\int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2 dv_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla(e^{-u/4})|^2 dv_g = \frac{\lambda_1}{16} \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g,$$

so that

$$\begin{aligned}
(8) \quad & \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g - \frac{1}{16} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \\
&\quad + \frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^2 \Delta_g u e^{-u/2} dv_g.
\end{aligned}$$

**An identity based on the equation.** By expanding  $\Delta_g(e^{-u/2}) = (\frac{1}{4} |\nabla u|^2 - \frac{1}{2} \Delta_g u) e^{-u/2}$ , we have that

$$\int_{\mathfrak{M}} (-\frac{1}{2} \Delta_g u) \Delta_g(e^{-u/2}) dv_g = \frac{1}{4} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g - \frac{1}{8} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g$$

so that, if we multiply (2) by  $\Delta_g(e^{-u/2}) - \frac{1}{2}|\nabla u|^2 e^{-u/2}$ , then we get

$$\frac{1}{4} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g + \frac{1}{8} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g - \frac{\lambda}{2} \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g = 0.$$

Let us introduce the parameter  $\theta \leq 1$ , to be chosen later, and use (8) and (7) to estimate  $\frac{1-\theta}{4} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g$  and  $\frac{\theta}{4} \int_{\mathfrak{M}} (\Delta_g u)^2 e^{-u/2} dv_g$  respectively. We get

$$\begin{aligned} & \frac{1-\theta}{4} \left[ \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g - \frac{1}{16} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \right. \\ & \quad \left. + \frac{1}{2} \int_{\mathfrak{M}} |\nabla u|^2 \Delta_g u e^{-u/2} dv_g \right] \\ & + \frac{\theta}{4} \left[ \frac{3}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g - \frac{1}{8} \frac{d}{d-1} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \right. \\ & \quad + \frac{d}{d-1} \int_{\mathfrak{M}} \|L_g u\|^2 e^{-u/2} dv_g + \frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) e^{-u/2} dv_g \left. \right] \\ & \quad + \frac{1}{8} \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g - \frac{\lambda}{2} \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g \leq 0. \end{aligned}$$

Collecting terms, we get

$$\begin{aligned} & \frac{\theta}{4} \frac{d}{d-1} \left[ \int_{\mathfrak{M}} \|L_g u\|^2 e^{-u/2} dv_g + \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) e^{-u/2} dv_g \right] \\ & - \frac{1}{64} \left( 1 - \theta + 2\theta \frac{d}{d-1} \right) \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \\ & + \left( \frac{1-\theta}{8} + \frac{3\theta}{16} \frac{d}{d-1} + \frac{1}{8} \right) \int_{\mathfrak{M}} \Delta_g u |\nabla u|^2 e^{-u/2} dv_g \\ & + \left( \frac{1-\theta}{4} \lambda_1 - \frac{\lambda}{2} \right) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g \leq 0 \end{aligned}$$

and can now use (6) to obtain

$$\begin{aligned} & \frac{\theta}{4} \frac{d}{d-1} \left[ \int_{\mathfrak{M}} \|L_g u\|^2 e^{-u/2} dv_g + \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) e^{-u/2} dv_g \right] \\ & - \frac{1}{64} \left( 1 - \theta + 2\theta \frac{d}{d-1} \right) \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \\ & + \left( \frac{1-\theta}{8} + \frac{3\theta}{16} \frac{d}{d-1} + \frac{1}{8} \right) \left[ \frac{1}{2} \frac{d}{d+2} \int_{\mathfrak{M}} |\nabla u|^4 e^{-u/2} dv_g \right. \\ & \quad \left. - \frac{2d}{d+2} \int_{\mathfrak{M}} L_g u : M_g u e^{-u/2} dv_g \right] \\ & + \left( \frac{1-\theta}{4} \lambda_1 - \frac{\lambda}{2} \right) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g \leq 0. \end{aligned}$$

Recall that  $M_g u$  denotes the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2.$$

We observe that

$$\|M_g u\|^2 = \left\| \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2 \right\|^2 = \left( 1 - \frac{1}{d} \right) |\nabla u|^4.$$

Altogether we end up with

$$\begin{aligned} \int_{\mathfrak{M}} \left( \mathbf{a} \|L_g u\|^2 + \mathbf{b} (L_g u : M_g u) + \mathbf{c} \|M_g u\|^2 \right) e^{-u/2} dv_g \\ + \frac{\theta}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g \\ + \left( \frac{1-\theta}{4} \lambda_1 - \frac{\lambda}{2} \right) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g \leq 0 \end{aligned}$$

with

$$\begin{aligned} \mathbf{a} &= \frac{\theta}{4} \frac{d}{d-1}, \\ \mathbf{b} &= - \left( \frac{1-\theta}{8} + \frac{3\theta}{16} \frac{d}{d-1} + \frac{1}{8} \right) \frac{2d}{d+2}, \\ \mathbf{c} &= \left[ \left( \frac{1-\theta}{8} + \frac{3\theta}{16} \frac{d}{d-1} + \frac{1}{8} \right) \frac{1}{2} \frac{d}{d+2} - \frac{1}{64} \left( 1-\theta + 2\theta \frac{d}{d-1} \right) \right] \frac{d}{d-1}. \end{aligned}$$

**Remark 3.** By the Lichnerowicz' theorem (see [25] or [18, Section 2])

$$\frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g \leq \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g$$

so that the largest possible value of  $\lambda$  for which we a priori know that

$$\frac{\theta}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g + \left( \frac{1}{4} \lambda_1 (1-\theta) - \lambda \right) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g$$

is nonnegative corresponds to the smallest possible value of  $\theta$ , i.e.  $\theta = \theta_0(d)$ .

**Discussion.** A simple but tedious commutation shows that the discriminant  $\delta := \mathbf{b}^2 - 4\mathbf{a}\mathbf{c}$  has the sign of

$$16(d-1)^2 - (6-d)(d+2)\theta.$$

If we denote by  $\theta_0 = \theta_0(d)$  the value of  $\theta$  for which  $\delta = 0$ , then we have

$$\theta_0 = \frac{16(d-1)^2}{(6-d)(d+2)}.$$

Altogether we can rewrite our estimate as

$$\begin{aligned} \mathbf{a} \int_{\mathfrak{M}} \left\| L_g u + \frac{\mathbf{b}}{2\mathbf{a}} M_g u \right\|^2 e^{-u/2} dv_g + \left( \mathbf{c} - \frac{\mathbf{b}^2}{4\mathbf{a}} \right) \int_{\mathfrak{M}} \|M_g u\|^2 e^{-u/2} dv_g \\ + 4(1-\theta) \int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2 dv_g + \frac{\theta}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g \\ - \frac{\lambda}{2} \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g = 0 \end{aligned}$$

and use the Poincaré inequality, that is,

$$4 \int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2 dv_g \geq \frac{1}{4} \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g,$$



to establish that

$$(9) \quad 0 \geq \mathbf{a} \int_{\mathfrak{M}} \left\| \mathbf{L}_g u + \frac{\mathbf{b}}{2\mathbf{a}} \mathbf{M}_g u \right\|^2 e^{-u/2} dv_g - \frac{\delta}{4\mathbf{a}} \int_{\mathfrak{M}} \|\mathbf{M}_g u\|^2 e^{-u/2} dv_g \\ + \left( \frac{1}{4} \lambda_1 (1 - \theta) - \frac{\lambda}{2} \right) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g + \frac{\theta}{4} \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g.$$

Our goal is to show that  $u$  has to be a constant, that is,  $\int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g = 0$ . We assume that  $d \geq 2$  is an integer. The discriminant  $\delta$  is nonpositive if and only if  $d < 6$  and  $\theta \geq \theta_0(d)$ . This is compatible with the condition  $\theta \leq 1$  only if  $d = 2$ ; in that case,  $\theta = \theta_0(2) = 1$  and, as a consequence of (9), we get that

$$0 \geq \int_{\mathfrak{M}} \left\| \mathbf{L}_g u - \frac{1}{2} \mathbf{M}_g u \right\|^2 e^{-u/2} dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) e^{-u/2} dv_g \\ - \lambda \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g \\ \geq (\lambda_* - \lambda) \int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g.$$

Hence we have shown that  $\nabla u \equiv 0$  for any  $\lambda < \lambda_*$ , which concludes the proof of Theorem 1.

**Remark 4.** *In order to compare our results with the results deduced from the curvature-dimension method, we can consider the case where  $\mathfrak{R}$  is uniformly bounded from below by some positive constant  $\rho$  and formally assume that  $d$  takes real values. This can be made precise for instance in the setting of the ultra-spherical operator (see for instance [5]), with exactly the same conditions as above. See [21] and [16, Section 7.1] for more details. If  $1 < d \leq 2$ , we find that rigidity holds if*

$$\lambda \leq \max_{\theta \in [\theta_0(d), 1]} \left( \frac{1}{2} \lambda_1 (1 - \theta) + \frac{\theta}{2} \frac{d}{d-1} \rho \right) = \frac{1}{2} \lambda_1 (1 - \theta_0(d)) + \frac{\theta_0(d)}{2} \frac{d}{d-1} \rho$$

according to Remark 3. Let  $x = \frac{d}{d-1} \frac{\rho}{\lambda_1} \leq 1$ . We have found that that rigidity holds if

$$2 \frac{\lambda}{\lambda_1} \leq 1 - \theta_0(d) + \theta_0(d) x =: f_1(x)$$

Quite surprisingly, a better condition has been obtained in [21, Théorème 2], when  $1 < d < 2$ , which amounts to

$$2 \frac{\lambda}{\lambda_1} \leq d(2-d) + (d-1)^2 x =: f_2(x),$$

by taking the limit as  $p \rightarrow \infty$  in (4). We may indeed check that  $f_2(x) - f_1(x) = \frac{(d-1)^2 (d-2)^2}{(6-d)(d+2)} (1-x) \geq 0$

Without assuming the positivity of  $\rho$ , one gets a similar result with our approach. In the range  $d \in (1, 2)$ , our computations show that rigidity holds for any  $\lambda$  at most equal to the infimum on  $u \in \mathbf{H}^2(\mathfrak{M}) \setminus \{0\}$  of

$$2 \int_{\mathfrak{M}} \left[ \mathbf{a} \left\| \mathbf{L}_g u + \frac{\mathbf{b}}{2\mathbf{a}} \mathbf{M}_g u \right\|^2 + 4(1-\theta) |\Delta_g(e^{-u/4})|^2 + \frac{\theta}{4} \frac{d}{d-1} \mathfrak{R}(\nabla u, \nabla u) \right] e^{-u/2} dv_g$$

under the condition that  $\int_{\mathfrak{M}} |\nabla u|^2 e^{-u/2} dv_g = 1$ ,  $\mathbf{a} = \frac{4d(d-1)}{(6-d)(d+2)}$ ,  $\mathbf{b} = -\frac{d(3d+2)}{2(6-d)(d+2)}$  and  $\theta = \theta_0(d)$ . However, in the same spirit as above, a passage to the limit as  $p \rightarrow \infty$  in the inequality obtained in [18, Theorem 4] gives a better result.

Let us emphasize that these considerations are essentially formal because  $d$  is restricted to the interval  $(1, 2)$  but can be entirely justified, as it is currently done in the curvature-dimension approach.

## 2. PROOF OF COROLLARY 2

Let us minimize the functional

$$\mathcal{F}_\lambda[u] := \frac{1}{4} \|\nabla u\|_{L^2(\mathfrak{M})}^2 + \lambda \int_{\mathfrak{M}} u \, dv_g - \lambda \log \left( \int_{\mathfrak{M}} e^u \, dv_g \right)$$

on  $H^1(\mathfrak{M})$ . According to [19, 20],  $\mathcal{F}_\lambda[u]$  has no minimizer if  $\lambda > 4\pi$ . Let us assume that

$$\lambda < 4\pi$$

(we shall take care of the equality case later). It is then standard that there is a well-defined minimizer. If it is smooth and  $\lambda < \lambda_*$ , we can apply the result of Theorem 1. Then the minimizer  $u$  can be any constant, for instance  $u \equiv 1$ , so that

$$\mathcal{F}_\lambda[u] \geq \mathcal{F}_\lambda[1] = 0 \quad \forall u \in H^1(\mathfrak{M}).$$

Notice that we can get rid of any smoothness requirement by considering the flow below. By passing to the limit as  $\lambda \nearrow \lambda_*$ , we get that the inequality also holds true if  $\lambda = \lambda_*$ .

Using a Taylor expansion of

$$\mathcal{F}_\lambda[1 + \varepsilon\varphi] = \left[ \frac{1}{4} \|\nabla\varphi\|_{L^2(\mathfrak{M})}^2 + \frac{\lambda}{2} \int_{\mathfrak{M}} \varphi^2 \, dv_g \right] \varepsilon^2 + o(\varepsilon^2),$$

as  $\varepsilon \rightarrow 0$ , where  $\varphi$  is an eigenfunction associated with the first positive eigenvalue  $\lambda_1$  of  $-\Delta_g$ , it is straightforward to see that the best constant  $\lambda$  in (3) is such that

$$\lambda \leq \frac{\lambda_1}{2}.$$

To complete the proof of Corollary 2, it remains to consider the case  $\lambda_* < \lambda_1/2$  and show that the optimal constant cannot be equal to  $\lambda_*$ . This can be done in the same spirit as in [18, Corollary 2]. Let us consider the evolution equation defined by

$$(10) \quad \frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2},$$

with initial datum  $u \in H^1(\mathfrak{M})$ . Let us define

$$\begin{aligned} \mathcal{G}_\lambda[f] := & \int_{\mathfrak{M}} \|\mathbf{L}_g f - \frac{1}{2} \mathbf{M}_g f\|^2 e^{-f/2} \, dv_g + \int_{\mathfrak{M}} \Re(\nabla f, \nabla f) e^{-f/2} \, dv_g \\ & - \lambda \int_{\mathfrak{M}} |\nabla f|^2 e^{-f/2} \, dv_g. \end{aligned}$$

Then for any  $\lambda \leq \lambda_*$  we have

$$\frac{d}{dt} \mathcal{F}_\lambda[f(t, \cdot)] = \int_{\mathfrak{M}} \left( -\frac{1}{2} \Delta_g f + \lambda \right) \left( \Delta_g(e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \right) \, dv_g = -\mathcal{G}_\lambda[f(t, \cdot)]$$

Since  $\mathcal{F}_\lambda$  is nonnegative and  $\lim_{t \rightarrow \infty} \mathcal{F}_\lambda[f(t, \cdot)] = 0$ , we obtain that

$$\mathcal{F}_\lambda[u] \geq \int_0^\infty \mathcal{G}_\lambda[f(t, \cdot)] \, dt$$

for any solution  $f$  to (10) with initial datum  $u \in L^1(\mathfrak{M})$  is such that  $\nabla u \in L^2(\mathfrak{M})$ . We have an equality if the solution is smooth for any  $t \geq 0$ . Otherwise we have to regularize and then pass to the limit so that, with full generality, we can only expect for an inequality.

**Remark 5.** *One has to mention that the sphere  $\mathfrak{M} = \mathbb{S}^2$  is an important case of application of our method, for which other types of remainder terms can be produced. See [16] for more details. It has to be noted that on  $\mathbb{S}^2$  we have  $\lambda_* = \rho = \lambda_1/2 = 1$ . As noted by many authors, e.g., in [4, 5, 21, 16] (also see references in [16]), the Onofri inequality is a limit case of various Sobolev type inequalities, for which similar methods have been developed: see [17] for a review and some recent results.*

### 3. THE CASE $d = 1$

Since we assume that  $\mathfrak{M}$  is compact, connected, and without boundary, we only have to consider the case of the circle. Hence we identify  $\mathfrak{M}$  with the 1-periodic interval  $[0, 1) \approx \mathbb{R}/\mathbb{Z} \approx \mathbb{S}^1$ . Consider a solution of the ordinary differential equation

$$(11) \quad -\frac{1}{2} u'' + \lambda - e^u = 0$$

with periodic boundary conditions. If we multiply the equation by  $(e^{-u/2})'' - \frac{1}{2} |u'|^2 e^{-u/2}$ , then we get that

$$\int_0^1 \left( \frac{1}{4} |u''|^2 + \frac{1}{8} |u'|^2 u'' - \frac{\lambda}{2} |u'|^2 \right) e^{-u/2} dx = 0.$$

The middle term is easy to handle using one integration by parts:

$$(12) \quad \int_0^1 |u'|^2 u'' e^{-u/2} dx = \frac{1}{6} \int_0^1 |u'|^4 e^{-u/2} dx.$$

Hence we have

$$(13) \quad \int_0^1 \left( \frac{1}{4} |u''|^2 + \frac{1}{48} |u'|^4 - \frac{\lambda}{2} |u'|^2 \right) e^{-u/2} dx = 0.$$

On the other hand,

$$\int_0^1 \left| \left( e^{-u/4} \right)'' \right|^2 dx \geq 4\pi^2 \int_0^1 \left| \left( e^{-u/4} \right)' \right|^2 dx,$$

where  $4\pi^2$  is the first positive eigenvalue of  $-\frac{d^2}{dx^2}$  on the periodic interval of length 1. From (12) we derive

$$(14) \quad \int_0^1 \left( |u''|^2 - \frac{1}{48} |u'|^4 \right) e^{-u/2} dx - 4\pi^2 \int_0^1 |u'|^2 e^{-u/2} dx \geq 0.$$

Combining (13) and (14), we get

$$\frac{5}{96} \int_0^1 |u'|^4 e^{-u/2} dx + (2\pi^2 - \lambda) \int_0^1 |u'|^2 e^{-u/2} dx \leq 0.$$

Hence we have proven the following result.

**Proposition 6.** *Assume that  $d = 1$ . With the above notations, if  $u$  is a smooth solution to (11) on  $\mathbb{S}^1 \approx [0, 1)$ , then  $u$  is a constant function for any  $\lambda \in (0, 2\pi^2)$ .*

Exactly as in the case of a manifold of dimension two, a variational approach allows to deduce a Moser-Trudinger-Onofri inequality.

**Corollary 7.** *If  $d = 1$ , then the following inequality holds on  $\mathbb{S}^1 \approx [0, 1)$ :*

$$\frac{1}{8\pi^2} \int_0^1 |u'|^2 dx + \int_0^1 u dx \geq \log \left( \int_0^1 e^u dx \right) \quad \forall u \in H^1(\mathbb{S}^1).$$

Moreover  $8\pi^2$  is the optimal constant.

The only difference with Corollary 2 is that we can identify the optimal constant in the inequality by considering  $u = 1 + \varepsilon \varphi$  and by taking the limit as  $\varepsilon \rightarrow 0$ , with  $\varphi(x) = \cos(2\pi x)$ .

#### 4. WEIGHTED MOSER-TRUDINGER-ONOFRI INEQUALITIES ON THE TWO-DIMENSIONAL EUCLIDEAN SPACE

The Euclidean Onofri inequality (see [12, 16]) can be deduced from (3) when  $\mathfrak{M} = \mathbb{S}^2$  using the stereographic projection and reads

$$(15) \quad \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 dx \geq \log \left( \int_{\mathbb{R}^2} e^u d\mu \right) - \int_{\mathbb{R}^2} u d\mu.$$

Here  $d\mu = \mu(x) dx$  denotes the probability measure defined by  $\mu(x) = \frac{1}{\pi} (1+|x|^2)^{-2}$ ,  $x \in \mathbb{R}^2$ , and the inequality holds for any function  $u \in L^1(\mathbb{R}^2, d\mu)$  such that  $\nabla u \in L^2(\mathbb{R}^2)$ . The constant  $16\pi$  is optimal as can be shown by considering the inequality on  $\mathbb{S}^2$  and comparing with the value given when expanding around a constant, as was done in Section 2.

In this section, our goal is to give sufficient conditions on a general probability measure  $\mu$  so that the inequality

$$(16) \quad \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 dx \geq \lambda \left[ \log \left( \int_{\mathbb{R}^2} e^u d\mu \right) - \int_{\mathbb{R}^2} u d\mu \right]$$

holds for some  $\lambda > 0$  and get an estimate of the optimal value of  $\lambda$ . Here  $d\mu = \mu dx$  is a probability measure with density  $\mu$  with respect to the Lebesgue measure. All our computations are done without symmetry assumption, and our final estimate is (29). In practical applications (see Examples 1–4) the function  $\mu$  is radially symmetric *and* one has to assume that  $\lambda$  is in a range for which the solution to (16), or at least the optimal function for (15), is radially symmetric. This delicate issue of *symmetry breaking* will be illustrated in Example 3.

Since (16) does not change when adding some constant to  $u$ , we can look for minimizers satisfying the constraint  $\int_{\mathbb{R}^2} e^u d\mu = 1$ . These solve the Euler-Lagrange equation

$$(17) \quad -\frac{1}{8\pi} \Delta u + \lambda \mu - \lambda e^u \mu = 0.$$

We can multiply each term of (17) by  $\frac{1}{\mu} \Delta(e^{-u/2}) - \frac{1}{2\mu} |\nabla u|^2 e^{-u/2}$  and integrate, which gives the following identities

$$\begin{aligned} - \int_{\mathbb{R}^2} \Delta u \Delta(e^{-u/2}) \frac{1}{\mu} dx &= \int_{\mathbb{R}^2} \Delta u \left( \Delta u - \frac{1}{2} |\nabla u|^2 \right) e^{-u/2} \frac{1}{2\mu} dx, \\ \int_{\mathbb{R}^2} \mu \frac{1}{\mu} \Delta(e^{-u/2}) dx &= 0, \\ \int_{\mathbb{R}^2} e^u \mu \left( \frac{1}{\mu} \Delta(e^{-u/2}) - \frac{1}{2\mu} |\nabla u|^2 e^{-u/2} \right) dx &= 0. \end{aligned}$$

Defining  $\nu := e^{-u/2}/\mu = e^{-u/2-g}$  with  $g := \log \mu$  and  $d\nu := \nu dx$  we have

$$\mathcal{I}[u] = 2 \int_{\mathbb{R}^2} (\Delta u)^2 d\nu + \int_{\mathbb{R}^2} \Delta u |\nabla u|^2 d\nu - 16 \pi \lambda \int_{\mathbb{R}^2} |\nabla u|^2 e^{-u/2} dx = 0.$$

Let us introduce some notations, which are consistent with the ones on manifolds. Let us denote by  $\text{Hu} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,2}$  the Hessian of  $u$ ,  $\text{Lu} = \text{Hu} - \frac{1}{2} \Delta u \text{I}_2$  is the trace free Hessian and  $\text{Mu} := \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 \text{I}_2$ , where  $\nabla u \otimes \nabla u = \left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\right)_{i,j=1,2}$ . For the convenience of the reader, we split the computations in four steps.

1) Let us start with some preliminary computations. An integration by parts shows that

$$\begin{aligned} (18) \quad & 2 \int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla g d\nu - \int_{\mathbb{R}^2} |\nabla u|^2 \nabla u \cdot \nabla g d\nu \\ &= -2 \int_{\mathbb{R}^2} \text{Hu} : (\nabla u \otimes \nabla g) d\nu - 2 \int_{\mathbb{R}^2} (\text{Hg} - \nabla g \otimes \nabla g) : (\nabla u \otimes \nabla u) d\nu \\ &= -2 \int_{\mathbb{R}^2} \text{Hu} : (\nabla u \otimes \nabla g) d\nu - 2 \int_{\mathbb{R}^2} \text{Hg} : (\nabla u \otimes \nabla u) d\nu + 2 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 d\nu. \end{aligned}$$

By expanding  $\text{Lu} - \frac{1}{2} \text{Mu}$ , we also get that

$$\begin{aligned} (19) \quad & \int_{\mathbb{R}^2} \left( \text{Lu} - \frac{1}{2} \text{Mu} \right) : (\nabla u \otimes \nabla g) d\nu \\ &= \int_{\mathbb{R}^2} \text{Hu} : (\nabla u \otimes \nabla g) d\nu - \frac{1}{2} \int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla g d\nu - \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 \nabla u \cdot \nabla g d\nu. \end{aligned}$$

Recalling the definition of  $\nu = e^{-\frac{1}{2}u-g}$ , we also find that

$$\begin{aligned} (20) \quad & -\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \nabla u \cdot \nabla g d\nu = \int_{\mathbb{R}^2} |\nabla u|^2 \nabla g e^{-g} \cdot \nabla(e^{-\frac{1}{2}u}) dx \\ &= -2 \int_{\mathbb{R}^2} \text{Hu} : (\nabla u \otimes \nabla g) d\nu + \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu. \end{aligned}$$

Equations (18), (19) and (20) allow us to eliminate

$$\int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla g d\nu, \quad \int_{\mathbb{R}^2} |\nabla u|^2 \nabla u \cdot \nabla g d\nu \quad \text{and} \quad \int_{\mathbb{R}^2} \text{Hu} : (\nabla u \otimes \nabla g) d\nu$$

in terms of the other quantities:

$$(21) \quad \int_{\mathbb{R}^2} \Delta u \nabla u \cdot \nabla g \, d\nu = \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) \, d\nu \\ - 2 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) \, d\nu,$$

$$(22) \quad \int_{\mathbb{R}^2} |\nabla u|^2 \nabla u \cdot \nabla g \, d\nu = 4 \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) \, d\nu - 4 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu \\ - 8 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) \, d\nu \\ + 6 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) \, d\nu,$$

$$(23) \quad \int_{\mathbb{R}^2} Hu : (\nabla u \otimes \nabla g) \, d\nu = \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) \, d\nu - \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu \\ - 2 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) \, d\nu \\ + 2 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) \, d\nu.$$

Moreover, the reader is invited to check that

$$(24) \quad \|Lu\|^2 = \|Hu\|^2 - \frac{1}{2} (\Delta u)^2$$

and

$$(25) \quad \|Mu\|^2 = \frac{1}{2} |\nabla u|^4.$$

2) On the one hand, integrating the second term in the expression of  $\mathcal{I}[u]$  by parts gives

$$\int_{\mathbb{R}^2} \Delta u |\nabla u|^2 \, d\nu \\ = - \int_{\mathbb{R}^2} \nabla u \cdot \nabla (|\nabla u|^2 \nu) \, dx \\ = - \int_{\mathbb{R}^2} \nabla u \cdot \left( 2Hu \nabla u - \frac{1}{2} \nabla u |\nabla u|^2 - |\nabla u|^2 \nabla g \right) \, d\nu \\ = - \int_{\mathbb{R}^2} \left( 2Hu : \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^4 - |\nabla u|^2 (\nabla u \cdot \nabla g) \right) \, d\nu \\ = - \int_{\mathbb{R}^2} \left( 2Lu : \nabla u \otimes \nabla u + \Delta u |\nabla u|^2 - \frac{1}{2} |\nabla u|^4 - |\nabla u|^2 (\nabla u \cdot \nabla g) \right) \, d\nu \\ = - \int_{\mathbb{R}^2} \left( 2Lu : Mu + \Delta u |\nabla u|^2 - \frac{1}{2} |\nabla u|^4 - |\nabla u|^2 (\nabla u \cdot \nabla g) \right) \, d\nu,$$

that is

$$\int_{\mathbb{R}^2} \Delta u |\nabla u|^2 \, d\nu = - \int_{\mathbb{R}^2} \left( Lu : Mu - \frac{1}{4} |\nabla u|^4 - \frac{1}{2} |\nabla u|^2 (\nabla u \cdot \nabla g) \right) \, d\nu.$$

According to (22) and (25), we obtain

$$\begin{aligned}
(26) \quad \int_{\mathbb{R}^2} \Delta u |\nabla u|^2 d\nu &= - \int_{\mathbb{R}^2} Lu : Mu d\nu + \frac{1}{2} \int_{\mathbb{R}^2} \|Mu\|^2 d\nu + 2 \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) d\nu \\
&\quad - 2 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 d\nu - 4 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad + 3 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu.
\end{aligned}$$

3) On the other hand, integrating by parts twice yields

$$\begin{aligned}
(27) \quad \int_{\mathbb{R}^2} \Delta |\nabla u|^2 d\nu &= \int_{\mathbb{R}^2} |\nabla u|^2 \Delta(e^{-u/2}) \frac{dx}{\mu} + \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu \\
&\quad + \int_{\mathbb{R}^2} |\nabla u|^2 (\nabla u \cdot \nabla g) d\nu, \\
&= -\frac{1}{2} \int_{\mathbb{R}^2} \Delta u |\nabla u|^2 d\nu + \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^4 d\nu \\
&\quad + \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu + \int_{\mathbb{R}^2} |\nabla u|^2 (\nabla u \cdot \nabla g) d\nu.
\end{aligned}$$

Integrating by parts again we have that

$$\int_{\mathbb{R}^2} \nabla \Delta u \cdot \nabla u d\nu = - \int_{\mathbb{R}^2} \Delta u \left( \Delta u - \frac{1}{2} |\nabla u|^2 - (\nabla u \cdot \nabla g) \right) d\nu,$$

which we can use along with the Bochner-Lichnerovicz-Weitzenböck formula on  $\mathbb{R}^2$  (with Ricci tensor identically equal to 0),

$$\Delta |\nabla u|^2 = 2 \|Lu\|^2 + (\Delta u)^2 + 2 \nabla \Delta u \cdot \nabla u,$$

to get

$$\int_{\mathbb{R}^2} \Delta |\nabla u|^2 d\nu = \int_{\mathbb{R}^2} \left( 2 \|Lu\|^2 - (\Delta u)^2 + \Delta u |\nabla u|^2 + 2 \Delta u (\nabla u \cdot \nabla g) \right) d\nu.$$

Combined with (27) this proves that

$$\begin{aligned}
\int_{\mathbb{R}^2} (\Delta u)^2 d\nu &= 2 \int_{\mathbb{R}^2} \|Lu\|^2 d\nu + \frac{3}{2} \int_{\mathbb{R}^2} \Delta u |\nabla u|^2 d\nu - \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^4 d\nu \\
&\quad - \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu + 2 \int_{\mathbb{R}^2} \Delta u (\nabla u \cdot \nabla g) d\nu - \int_{\mathbb{R}^2} |\nabla u|^2 (\nabla u \cdot \nabla g) d\nu.
\end{aligned}$$

Using (21), (22), (25) and (26), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} (\Delta u)^2 d\nu \\
&= 2 \int_{\mathbb{R}^2} \|Lu\|^2 d\nu \\
&\quad - \frac{3}{2} \int_{\mathbb{R}^2} Lu : Mu d\nu + \frac{3}{4} \int_{\mathbb{R}^2} \|Mu\|^2 d\nu + 3 \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) d\nu \\
&\quad - 3 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 d\nu - 6 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad + \frac{9}{2} \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^2} \|Mu\|^2 d\nu - \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu \\
&\quad + 2 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu - 4 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad - 4 \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) d\nu + 4 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 d\nu \\
&\quad + 8 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad - 6 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu.
\end{aligned}$$

Collecting terms, we arrive at

$$\begin{aligned}
(28) \quad \int_{\mathbb{R}^2} (\Delta u)^2 d\nu &= 2 \int_{\mathbb{R}^2} \|Lu\|^2 d\nu - \frac{3}{2} \int_{\mathbb{R}^2} Lu : Mu d\nu + \frac{1}{4} \int_{\mathbb{R}^2} \|Mu\|^2 d\nu \\
&\quad - 2 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad - \int_{\mathbb{R}^2} Hg : (\nabla u \otimes \nabla u) d\nu + \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 d\nu \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu.
\end{aligned}$$

4) By reinjecting (26) and (28) in the expression of  $\mathcal{I}$ , we get that

$$\begin{aligned}
0 = \mathcal{I}[u] &= 4 \int_{\mathbb{R}^2} \|Lu\|^2 d\nu - 4 \int_{\mathbb{R}^2} Lu : Mu d\nu + \int_{\mathbb{R}^2} \|Mu\|^2 d\nu \\
&\quad - 8 \int_{\mathbb{R}^2} \left( Lu - \frac{1}{2} Mu \right) : (\nabla u \otimes \nabla g) d\nu \\
&\quad + 2 \int_{\mathbb{R}^2} |\nabla u|^2 (|\nabla g|^2 - \Delta g) d\nu - 16 \pi \lambda \int_{\mathbb{R}^2} |\nabla u|^2 e^{-u/2} dx.
\end{aligned}$$

Since  $\|\nabla u \otimes \nabla g\|^2 = |\nabla u|^2 |\nabla g|^2$ , then we get for the corresponding trace free quantity  $Nu := \nabla u \otimes \nabla g - \frac{1}{2} (\nabla u \cdot \nabla g) I_2$  that

$$\|Nu\|^2 = |\nabla u|^2 |\nabla g|^2 - \frac{1}{2} (\nabla u \cdot \nabla g)^2$$



and hence obtain the identity

$$(29) \quad 0 = 4 \int_{\mathbb{R}^2} \left\| Lu - \frac{1}{2} Mu - Nu \right\|^2 d\nu \\ - 2 \int_{\mathbb{R}^2} [(\Delta g + |\nabla g|^2 - (\nabla g \cdot \omega)^2) e^{-g} + 8\pi\lambda] |\nabla u|^2 e^{-u/2} dx$$

where  $\omega := \nabla u / |\nabla u|$ . If we assume that  $\int_{\mathbb{R}^2} |\nabla u|^2 dx e^{-u/2} \neq 0$  and define

$$\Lambda := \frac{1}{8\pi} \frac{\int_{\mathbb{R}^2} [(\nabla u \cdot \nabla g)^2 - (\Delta g + |\nabla g|^2) |\nabla u|^2] e^{-u/2-g} dx}{\int_{\mathbb{R}^2} |\nabla u|^2 e^{-u/2} dx},$$

then we get a contradiction if  $\lambda < \Lambda$ . To keep maximal generality, one could even include the term  $\int_{\mathbb{R}^2} \left\| Lu - \frac{1}{2} Mu - Nu \right\|^2 d\nu$  in the definition of  $\Lambda$ , as it was done in the case of manifolds. However, this is a quite complicated criterion to verify since it involves the solution to (17) itself. Hence it makes sense to consider the simpler case where  $\mu$  has radial symmetry. In that case it is also known from [23] that  $u$  is radially symmetric if  $\mu$  is a monotone non-increasing function of  $|x|$ . Let

$$\Lambda_\star := -\frac{1}{8\pi} \inf_{x \in \mathbb{R}^2} (\Delta g e^{-g}) = \inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi \mu}.$$

**Theorem 8.** *Assume that  $\mu$  is a radially symmetric function. Then any radially symmetric solution to (17) is a constant if  $\lambda < \Lambda_\star$  and the inequality (16) holds with  $\lambda = \Lambda_\star$  if equality is achieved among radial functions.*

**Example 1.** *The Euclidean Onofri inequality corresponds to*

$$\mu(x) = \frac{1}{\pi(1+|x|^2)^2} \quad \forall x \in \mathbb{R}^2.$$

Since  $-\Delta \log \mu = 8\pi \mu$ , it is known that  $\Lambda_\star = 1$  is the optimal constant. See [12] for further details. Let us notice that the analysis of the equation

$$Lu - \frac{1}{2} Mu - Nu = 0$$

in the case  $\Lambda_\star = 1$  provides a proof of the uniqueness of the radial solution to (17), which is alternative to the result of [9].

**Example 2.** *It is straightforward to deduce a perturbation result from Theorem 8, that goes as follows. Let*

$$\mu(x) = \frac{e^{-h(x)}}{Z(1+|x|^2)^2} \quad \forall x \in \mathbb{R}^2,$$

where  $h$  is a radial function and  $Z$  a normalization constant so that  $\mu$  is a probability measure. We shall assume that  $h$  has a bounded variation and is such that  $|x|^4 \Delta h$  is bounded from below. Then we have the estimate

$$\inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi \mu} \geq e^{-\text{Var}(h)} \left[ 1 + \frac{1}{8} \inf_{x \in \mathbb{R}^2} (1+|x|^2)^2 \Delta h \right].$$

**Example 3.** *The subcritical Onofri inequality has been studied in [11]. It plays an important role for the study of the subcritical Keller-Segel model and its asymptotics*

for large times, and goes as follows. Let  $\mu = n/M$  where  $n$  is given as the unique (up to constants) radial solution to

$$-\Delta c = n = M \frac{e^{c-\frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c-\frac{1}{2}|x|^2} dx}$$

and the mass  $M$  is taken in the interval  $M \in (0, 8\pi)$ . According to [23],  $n$  is radially symmetric as a consequence of the moving plane method. It is straightforward to check that

$$\inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi\mu} = \frac{M}{8\pi} + \inf_{x \in \mathbb{R}^2} \frac{1}{4\pi\mu}$$

but the symmetry of the solution to (16) is not true for  $\lambda > \frac{M}{8\pi}$  and  $\frac{M}{8\pi}$  turns out to be the value of the optimal constant in (16). See [10, 11] for further details.

**Example 4.** The parabolic-parabolic Keller-Segel model has global in time solutions with mass larger than  $8\pi$  for some values of its parameters, according to [8]. The stationary solutions in self-similar variables can be written as

$$-\Delta c = \varepsilon x \cdot \nabla c + n \quad \text{with} \quad n = M \frac{e^{c-\frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c-\frac{1}{2}|x|^2} dx}$$

(where  $\varepsilon > 0$  is a given parameter) and have been shown to be radially symmetric in [28]. To prove that a weighted Onofri inequality holds with  $\mu = n/M$ , it is therefore sufficient to establish the range of  $\lambda \in (0, \Lambda_*)$  such that the minimizer of  $u \mapsto \int_{\mathbb{R}^2} |\nabla u|^2 dx - 16\pi\lambda [\log(\int_{\mathbb{R}^2} e^u d\mu) - \int_{\mathbb{R}^2} u d\mu]$  is radially symmetric, where

$$\inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi\mu} = \frac{M}{8\pi} + \inf_{x \in \mathbb{R}^2} \frac{\varepsilon x \cdot \nabla c + 2}{8\pi\mu}.$$

Such symmetry breaking issues are however known to be difficult: see for instance [15] for a discussion of a related problem.

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