



**HAL**  
open science

# On Hodge Theory of Singular Plane Curves

Nancy Abdallah

► **To cite this version:**

| Nancy Abdallah. On Hodge Theory of Singular Plane Curves. 2014. hal-00984947v2

**HAL Id: hal-00984947**

**<https://hal.science/hal-00984947v2>**

Preprint submitted on 2 Apr 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON HODGE THEORY OF SINGULAR PLANE CURVES

NANCY ABDALLAH

ABSTRACT. The dimensions of the graded quotients of the cohomology of a plane curve complement  $U = \mathbb{P}^2 \setminus C$  with respect to the Hodge filtration are described in terms of simple geometrical invariants. The case of curves with ordinary singularities is discussed in detail. We also give a precise numerical estimate for the difference between the Hodge filtration and the pole order filtration on  $H^2(U, \mathbb{C})$ .

## 1. INTRODUCTION

The Hodge theory of the complement of projective hypersurfaces have received a lot of attention, see for instance Griffiths [10] in the smooth case, Dimca-Saito [5] and Sernesi [12] in the singular case. In this paper we consider the case of plane curves and continue the study initiated by Dimca-Sticlaru [7] in the nodal case and the author [1] in the case of plane curves with ordinary singularities of multiplicity up to 3.

In the second section we compute the Hodge-Deligne polynomial of a plane curve  $C$ , the irreducible case in Proposition 2.1 and the reducible case in Proposition 2.2. Using this we determine the Hodge-Deligne polynomial of  $U = \mathbb{P}^2 \setminus C$  and then we deduce in Theorem 2.7 the dimensions of the graded quotients of  $H^2(U)$  with respect to the Hodge filtration.

In section three we consider the case of arrangements of curves having ordinary singularities and intersecting transversely at smooth points and obtain a formula in Theorem 3.1 generalizing the formulas obtained in [7] and in [1] (for this type of curves). In fact, the results in [1] show that this formula holds in the more general case of plane curves with ordinary singularities of multiplicity up to 3 (without assuming transverse intersection).

In the fourth section we show that the case of plane curves with ordinary singularities of multiplicity up to 4 (without assuming transverse intersection) is definitely more complicated and the formula in Theorem 3.1 has to be replaced by the formula in Theorem 4.1 containing a correction term coming from triple points on one component through which another component of  $C$  passes.

In the final section we give some applications, we hope of general interest, expressing the difference between the Hodge filtration and the pole order filtration on  $H^2(U, \mathbb{C})$  in terms of numerical invariants easy to compute in given situations, see Theorem 5.1 and its corollaries. One example involving a free divisor concludes this note.

---

2000 *Mathematics Subject Classification.* Primary 32S35, 32S22; Secondary 14H50.

*Key words and phrases.* plane curves, Hodge and Pole order filtrations .

## 2. HODGE THEORY OF PLANE CURVE COMPLEMENTS

For the general theory of mixed Hodge structures we refer to [2] and [14]. Recall the definition of the Hodge-Deligne polynomial of a quasi-projective complex variety  $X$

$$P(X)(u, v) = \sum_{p, q} E^{p, q}(X) u^p v^q$$

where  $E^{p, q}(X) = \sum_s (-1)^s h^{p, q}(H_c^s(X))$ , with  $h^{p, q}(H_c^s(X)) = \dim Gr_F^p Gr_{p+q}^W H_c^s(X, \mathbb{C})$ , the mixed Hodge numbers of  $H_c^s(X)$ .

This polynomial is additive with respect to constructible partitions, i.e.  $P(X) = P(X \setminus Y) + P(Y)$  for a closed subvariety  $Y$  of  $X$ . In this section we determine  $P(C)$  for a (reduced) plane curve  $C$ .

Suppose first that the curve  $C$  is irreducible, of degree  $N$ . Denote by  $a_k$ ,  $k = 1, \dots, p$  the singular points of  $C$ , and let  $r(C, a_k)$  be the number of irreducible branches of the germ  $(C, a_k)$ . Let  $\nu : \tilde{C} \rightarrow C$  be the normalization mapping. Using the normalization map  $\nu$  and the additivity of the Hodge-Deligne polynomial, it follows that,

$$\begin{aligned} P(C) &= P(C \setminus (C)_{sing}) + P((C)_{sing}) = P(\tilde{C} \setminus (\cup_k \nu^{-1}(a_k))) + p = \\ &= P(\tilde{C}) - \sum_k P(\nu^{-1}(a_k)) + p = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1). \end{aligned}$$

Indeed, it is known that for the smooth curve  $\tilde{C}$ , the genus  $g = g(\tilde{C})$  is exactly the Hodge number  $h^{1,0}(\tilde{C}) = h^{0,1}(\tilde{C})$ . Moreover, it is known that one has the formula

$$(2.1) \quad g = \frac{(N-1)(N-2)}{2} - \sum_k \delta(C, a_k),$$

relating the genus, the degree and the local singularities of  $C$ , and the  $\delta$ -invariants can be computed using the formula

$$(2.2) \quad 2\delta(C, a_k) = \mu(C, a_k) + r(C, a_k) - 1,$$

where  $\mu(C, a_k)$  is the Milnor number of the singularity  $(C, a_k)$ . For both formulas above, see Milnor, p. 85. This proves the following result.

**Proposition 2.1.** *With the above notation and assumptions, we have the following for an irreducible plane curve  $C \subset \mathbb{P}^2$ .*

(i) *The Hodge-Deligne polynomial of  $C$  is given by*

$$P(C)(u, v) = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1),$$

*with  $g$  given by the formula (2.1).*

(ii)  *$H^0(C) = \mathbb{C}$  is pure of type  $(0, 0)$ .*

(iii)  *$H^2(C) = \mathbb{C}$  is pure of type  $(1, 1)$ .*

(iv) *The mixed Hodge numbers of the MHS on  $H^1(C)$  are given by*

$$h^{0,0}(H^1(C)) = \sum_k (r(C, a_k) - 1), \quad h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = g.$$

In particular, one has the following formulas for the first Betti number of  $C$ .

$$b_1(C) = \sum_k (r(C, a_k) - 1) + 2g = (N - 1)(N - 2) - \sum_k \mu(C, a_k).$$

Now we consider the case of a curve  $C$  having several irreducible components. More precisely, let  $C = \bigcup_{j=1, r} C_j$  be the decomposition of  $C$  as a union of irreducible components  $C_j$ , let  $\nu_j : \tilde{C}_j \rightarrow C_j$  be the normalization mappings and set  $g_j = g(\tilde{C}_j)$ . Suppose that the curve  $C_j$  has degree  $N_j$ , denote by  $a_k^j$  for  $k = 1, \dots, p_j$  be the singular points of  $C_j$  and let  $r(C_j, a_k^j)$  be the number of branches of the germ  $(C_j, a_k^j)$ . Then the formulas (2.1) and (2.2) can be applied to each irreducible curve  $C_j$ , as well as Proposition 2.1.

Let  $A$  be the union of the singular sets of the curves  $C_j$ . Let  $B$  be the set of points in  $C$  sitting on at least two distinct components  $C_i$  and  $C_j$ . For  $b \in B$ , let  $n(b)$  be the number of irreducible components  $C_j$  passing through  $b$ . By definition,  $n(b) \geq 2$ . Moreover, note that the sets  $A$  and  $B$  are not disjoint in general, and their union is precisely the singular set of  $C$ .

Using the additivity of Hodge-Deligne polynomials we get

$$P(C) = P(C_1 \cup \dots \cup C_r) = \sum_{j=1}^r P(C_j) + (-1)^{l-1} \sum_{0 \leq i_1 < \dots < i_l \leq r} P(C_{i_1} \cap \dots \cap C_{i_l}).$$

The first sum is easy to determine using Proposition 2.1.

$$\sum_{j=1}^r P(C_j) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - \sum_{j,k} ((r(C_j, a_k^j) - 1)).$$

Consider now the alternated sum, where  $l \geq 2$ . The only points of  $C$  that give a contribution to this sum are the points in  $B$ . Now, for a point  $b \in B$ , its contribution to the alternated sum is clearly given by

$$c(b) = -\binom{n(b)}{2} + \binom{n(b)}{3} - \dots + (-1)^{n(b)-1} \binom{n(b)}{n(b)} = -n(b) + 1.$$

**Proposition 2.2.** *With the above notation and assumptions, we have the following for a reducible plane curve  $C = \bigcup_{j=1, r} C_j$ .*

(i) *The Hodge-Deligne polynomial of  $C$  is given by*

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - \sum_{j,k} ((r(C_j, a_k^j) - 1) - \sum_{b \in B} (n(b) - 1)).$$

*with  $g_j$  given by the formula (2.1).*

(ii)  $H^0(C) = \mathbb{C}$  *is pure of type (0, 0).*

(iii)  $H^2(C) = \mathbb{C}^r$  *is pure of type (1, 1).*

(iv) *The mixed Hodge numbers of the MHS on  $H^1(C)$  are given by*

$$h^{0,0}(H^1(C)) = \sum_{j,k} ((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1,$$

$$h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = \sum_j g_j.$$

In particular, one has the following formula for the first Betti number of  $C$ .

$$b_1(C) = \sum_{j,k} ((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1 + 2 \sum_j g_j).$$

Note that a point in the intersection  $A \cap B$  will give a contribution to the last two sums in the above formula for  $P(C)$ .

**Example 2.3.** Suppose  $C$  is a nodal curve. Then for each singularity  $a_k^j \in A$  one has  $a_k^j \notin B$  (otherwise we get worse singularities than nodes) and  $r(a_k^j) = 2$ . Moreover, each point  $b \in B$  satisfies  $n(b) = 2$ . It follows that in this case we get

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - n_2,$$

with  $n_2$  the number of nodes of  $C$ . More precisely, in this case we have  $n_2 = n'_2 + n''_2$ , where  $n'_2$  (resp.  $n''_2$ ) is the number of nodes of  $C$  in  $A$  (resp. in  $B$ ) and one clearly has

$$n'_2 = S_1 := \sum_{j,k} ((r(C_j, a_k^j) - 1), \quad n''_2 = S_2 := \sum_{b \in B} (n(b) - 1).$$

**Example 2.4.** Suppose  $C$  has only nodes and ordinary triple points as singularities. Then let  $n_3$  be the number of triple points and note that we can write as above  $n_3 = n'_3 + n''_3$ , where  $n'_3$  (resp.  $n''_3$ ) is the number of triple points of  $C$  in  $A_0 = A \setminus B$  (resp. in  $B$ ). For a point  $a \in A_0$ , the contribution to the sum  $S_1$  is 2, while the contribution to the sum  $S_2$  is 0.

A point  $b \in B$  can be of two types. The first type, corresponding to the partition  $3 = 1 + 1 + 1$ , is when  $b$  is the intersection of three components  $C_j$ , all smooth at  $b$ . The contribution of such a point  $b$  is 0 to the sum  $S_1$  and 2 to the sum  $S_2$ .

The second type, corresponding to the partition  $3 = 2 + 1$ , is when  $b$  is the intersection of two components, say  $C_i$  and  $C_j$ , such that  $C_i$  has a node at  $b$ , and  $C_j$  is smooth at  $b$ . The contribution of such a point  $b$  is 1 to the sum  $S_1$  and 1 to the sum  $S_2$ .

It follows that the contribution of any triple point to the sum  $S_1 + S_2$  is equal to 2. Since the double points in  $C$  can be treated exactly as in Example 2.3, this yields the following.

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - n_2 - 2n_3.$$

When there are only triple points in  $B$  of the first type, then we obviously have the following additional relations

$$S_1 = n'_2 + 2n'_3, \quad S_2 = n''_2 + 2n''_3.$$

**Example 2.5.** Suppose  $C$  has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let  $n_4$  be the number of points of multiplicity 4 and note that we can write as above  $n_4 = n'_4 + n''_4$ , where  $n'_4$  (resp.  $n''_4$ ) is the number of points of multiplicity

4 of  $C$  in  $A_0 = A \setminus B$  (resp. in  $B$ ). For a point  $a \in A_0$  of multiplicity 4, the contribution to the sum  $S_1$  is 3, while the contribution to the sum  $S_2$  is 0.

A point  $b \in B$  can be of 4 types. The first type, corresponding to the partition  $4 = 1 + 1 + 1 + 1$ , is when  $b$  is the intersection of 4 components  $C_j$ , all smooth at  $b$ . The contribution of such a point  $b$  is 0 to the sum  $S_1$  and 3 to the sum  $S_2$ .

The second type, corresponding to the partition  $4 = 2 + 1 + 1$ , is when  $b$  is the intersection of 3 components, say  $C_i, C_j$  and  $C_k$ , such that  $C_i$  has a node at  $b$ , and  $C_j$  and  $C_k$  are smooth at  $b$ . The contribution of such a point  $b$  is 1 to the sum  $S_1$  and 2 to the sum  $S_2$ .

The third type, corresponding to the partition  $4 = 2 + 2$ , is when  $b$  is the intersection of 2 components, say  $C_i$  and  $C_k$ , such that  $C_i$  and  $C_k$  have a node at  $b$ . The contribution of such a point  $b$  is 2 to the sum  $S_1$  and 1 to the sum  $S_2$ .

The fourth type, corresponding to the partition  $4 = 3 + 1$ , is when  $b$  is the intersection of 2 components, say  $C_i$  and  $C_k$ , such that  $C_i$  has a triple point at  $b$ , and  $C_k$  is smooth at  $b$ . The contribution of such a point  $b$  is 2 to the sum  $S_1$  and 1 to the sum  $S_2$ .

It follows that the contribution of any point of multiplicity 4 to the sum  $S_1 + S_2$  is equal to 3. Since the double and triple points in  $C$  can be treated exactly as in Example 2.4, this yields the following.

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - n_2 - 2n_3 - 3n_4.$$

When there are only points of multiplicity 4 in  $B$  of the first type, then we obviously have the following additional relations

$$S_1 = n'_2 + 2n'_3 + 3n''_4, \quad S_2 = n''_2 + 2n''_3 + 3n''_4.$$

Let's look now at the cohomology of the smooth surface  $U = \mathbb{P}^2 \setminus C$ . By the additivity we get  $P(U) = P(\mathbb{P}^2) - P(C)$  where  $P(\mathbb{P}^2) = u^2v^2 + uv + 1$ . This yields the following consequence.

**Corollary 2.6.**

$$\begin{aligned} P(U)(u, v) &= u^2v^2 - (r-1)uv + \left( \sum_{j=1}^r g_j \right) u + \left( \sum_{j=1}^r g_j \right) v - (r-1) + \\ &\quad + \sum_{j,k} ((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1)). \end{aligned}$$

The contribution of  $H_c^4(U, \mathbb{C})$  to  $P(U)$  is the term  $u^2v^2$ , and that of  $H_c^3(U, \mathbb{C})$  is the term  $-(r-1)uv$ . Moreover, the dimension  $\dim Gr_F^1 H^2(U, \mathbb{C})$  is the number of independent classes of type (1,2), which correspond to classes of type (1,0) in  $H_c^2(U)$ , and hence to the terms in  $u$  in  $P(U)$ . For both statements see the proof of Theorem 2.1 in [1]. This proves the following result.

**Theorem 2.7.**

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j$$

and

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j + \sum_{j,k} ((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1.$$

In particular, all the components  $C_j$  of the curve  $C$  are rational if and only if  $H^2(U)$  is pure of type  $(2, 2)$ .

**Example 2.8.** Suppose  $C$  has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let  $n_k$  be the number of points of multiplicity  $k$ , for  $k = 2, 3, 4$ . Then using Example 2.5, we get the formula

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j - r + 1 + n_2 + 2n_3 + 3n_4.$$

### 3. ARRANGEMENTS OF TRANSVERSELY INTERSECTING CURVES

Recall that  $C = \bigcup_{j=1, r} C_j$  is the decomposition of  $C$  as a union of irreducible components  $C_j$ , and the curve  $C_j$  has degree  $N_j$ . In this section we assume that any curve  $C_j$  has only ordinary multiple points as singularities and let  $n_k(C_j)$  denote the number of ordinary points on  $C_j$  of multiplicity  $k$ . We also assume that the intersection of any two distinct components  $C_i$  and  $C_j$  is transverse, i.e. the points in  $C_i \cap C_j$  are nodes of the curve  $C_i \cup C_j$ . This implies in particular that  $A \cap B = \emptyset$ . The formulas (2.1) and (2.2) yield the equality.

$$(3.1) \quad g_j = \frac{(N_j - 1)(N_j - 2)}{2} - \frac{1}{2} \sum_k (\mu(C_j, a_k^j) + r(C, a_k^j) - 1),$$

Using this, Theorem 2.7 gives the formula

$$\begin{aligned} \dim Gr_F^2 H^2(U, \mathbb{C}) &= \sum_{j=1}^r \frac{(N_j - 1)(N_j - 2)}{2} - \frac{1}{2} \sum_{j,k} (\mu(C_j, a_k^j) - r(C, a_k^j) + 1) + \\ &\quad + \sum_{b \in B} (n(b) - 1) - r + 1. \end{aligned}$$

If  $a_k^j$  is an ordinary  $m$ -multiple point on the curve  $C_j$ , one has  $\mu(C_j, a_k^j) = (m - 1)^2$  and hence

$$\mu(C_j, a_k^j) - r(C, a_k^j) + 1 = (m - 1)(m - 2).$$

If we denote by  $n'_m$  (resp.  $n''_m$ ) the number of  $m$ -multiple points of  $C$  coming from just one component  $C_j$  (resp. from the intersection of several components  $C_j$ ), we see that we have

$$\sum_{j,k} (\mu(C_j, a_k^j) - r(C, a_k^j) + 1) = \sum_m (m - 1)(m - 2)n'_m.$$

This equality explains the contribution of the points in  $A$ . Now let  $b \in B$  such that  $n(b) = m$ . The number of such points is precisely  $n''_m$ . It follows that

$$\sum_{b \in B} (n(b) - 1) = \sum_m (m - 1)n''_m.$$

Let  $1 \leq i < j \leq r$  and consider the intersection  $C_i \cap C_j$ . It contains exactly  $N_i N_j$  points, since  $C_i$  and  $C_j$  intersects transversely. The sum  $S = \sum_{1 \leq i < j \leq r} N_i N_j$  represents the number of all such intersection points. Note that a point  $b \in B$  is counted in this sum exactly  $\binom{n(b)}{2}$  times. This yields the following formula

$$2S = \sum_m m(m-1)n_m''.$$

These formulas give the following result.

**Theorem 3.1.** *With the above assumptions and notation, one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,$$

with  $n_m = n_m' + n_m''$  the number of ordinary  $m$ -tuple points of  $C$ .

The following consequence of Theorem 2.7 and Theorem 3.1 applies in particular to any projective line arrangement.

**Corollary 3.2.** *Assume that  $C = \bigcup_{j=1,r} C_j$  is the decomposition of  $C$  as a union of irreducible components  $C_j$ , with any curve  $C_j$  having only ordinary multiple points as singularities and being rational, i.e.  $g_j = 0$ . If the intersection of any two distinct components  $C_i$  and  $C_j$  is transverse, i.e. the points in  $C_i \cap C_j$  are nodes of the curve  $C_i \cup C_j$ , then one has*

$$\dim H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,$$

with  $n_m$  the number of ordinary  $m$ -tuple points of  $C$ .

#### 4. CURVES WITH ORDINARY SINGULARITIES OF MULTIPLICITY $\leq 4$

Let  $C \subset \mathbb{P}^2$  be a curve of degree  $N$  having only ordinary singular points of multiplicity at most 4. Set  $U = \mathbb{P}^2 \setminus C$ , and let  $C = \bigcup_{j=1}^r C_j$  be the decomposition of  $C$  in irreducible components. Then,

$$\begin{aligned} P(C) &= \sum_{j=1}^r P(C_j) - \sum_{0 \leq i < j \leq r} P(C_i \cap C_j) + \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) \\ &\quad - \sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l). \end{aligned}$$

Let  $a_m^j$  denote the number of singular points of multiplicity  $m$  that belong to the component  $C_j$  (note that a point can be singular on two components, being a node on each of them).

Denote by  $b_3^k$  (respectively  $b_4^k$ ) the number of triple points (respectively points of multiplicity 4) of  $C$  that are intersection of exactly  $k$  components, for  $k = 2, 3$  (respectively  $k = 3, 4$ ). Let  $b_4^2$  (respectively  $\tilde{b}_4^2$ ) be the number of singular points  $p$  of multiplicity 4



in  $C$  representing the intersection of exactly 2 components, such that one of which has a triple point at  $p$  (respectively each one has a node at  $p$ ). Then one has

$$\sum_{0 \leq i < j \leq r} P(C_i \cap C_j) = \sum_{0 \leq i < j \leq r} N_i N_j - b_3^2 - 3\tilde{b}_4^2 - 2b_4^2 - 2b_4^3.$$

Indeed, a point of type  $b_3^2$  (resp.  $b_4^2$ , resp.  $\tilde{b}_4^2$ ) occurs only in one intersection  $C_i \cap C_j$ , and has the multiplicity 2 (resp. 3, resp. 4) in this intersection. A point of type  $b_4^3$  occurs in 3 intersections  $C_i \cap C_j$  with multiplicities 1, 2, 2, and this accounts for the correction term  $-2b_4^3$ . Then one has

$$\sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) = b_3^3 + b_4^3 + \binom{4}{3} b_4^4,$$

and

$$\sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l) = b_4^4.$$

Hence, by Proposition 2.1, we get the following.

$$\begin{aligned} P(C) &= ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v - \sum_{j=1}^r (a_2^j + 2a_3^j + 3a_4^j) - \sum N_i N_j \\ &\quad + b_3^2 + 3\tilde{b}_4^2 + 2b_4^2 + 3b_4^3 + b_3^3 + 3b_4^4. \end{aligned}$$

Therefore, as above, we obtain

$$\begin{aligned} P(U) &= u^2 v^2 - (r-1)uv + 1 - r + \left( \sum_{j=1}^r g_j \right) u + \left( \sum_{j=1}^r g_j \right) v + \sum_{j=1}^r (a_2^j + 3a_3^j + 6a_4^j) \\ &\quad - \sum_{j=1}^r (a_3^j + 3a_4^j) + \sum N_i N_j - b_3^2 - 3\tilde{b}_4^2 - 2b_4^2 - 3b_4^3 - b_3^3 - 3b_4^4. \end{aligned}$$

Finally we get

$$\begin{aligned} \dim Gr_F^2 H^2(U) &= \sum_{j=1}^r (g_j + a_2^j + 3a_3^j + 6a_4^j - 1) + \sum N_i N_j + 1 - \left( \sum_{j=1}^r a_3^j + b_3^2 + b_3^3 \right) \\ &\quad - 3 \left( \sum_{j=1}^r a_4^j + \tilde{b}_4^2 + b_4^2 + b_4^3 + b_4^4 \right) + b_4^2 \\ &= \frac{(N-1)(N-2)}{2} - n_3 - 3n_4 + b_4^2, \end{aligned}$$

with  $n_m$  the number of ordinary  $m$ -tuple points of  $C$ .

**Theorem 4.1.** *Let  $C \subset \mathbb{P}^2$  be a curve of degree  $N$  having only ordinary singular points of multiplicity at most 4. If  $U = \mathbb{P}^2 \setminus C$ , then one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_{m=3,4} \binom{m-1}{2} n_m + b_4^2,$$

with  $n_m$  the number of ordinary  $m$ -tuple points of  $C$  and  $b_4^2$  the number of singular points  $p$  of  $C$  which are smooth on one component  $C_i$  of  $C$  and have multiplicity 3 on the other component  $C_j$  of  $C$  passing through  $p$ .

## 5. POLE ORDER FILTRATION VERSUS HODGE FILTRATION FOR PLANE CURVE COMPLEMENTS

For any hypersurface  $V$  in a projective space  $\mathbb{P}^n$ , the cohomology groups  $H^*(U, \mathbb{C})$  of the complement  $U = \mathbb{P}^n \setminus V$  have a pole order filtration  $P^k$ , see for instance [8], and it is known by the work of P. Deligne, A. Dimca [3] and M. Saito [11] that one has

$$F^k H^m(U, \mathbb{C}) \subset P^k H^m(U, \mathbb{C})$$

for any  $k$  and any  $m$ . For  $m = 0$  and  $m = 1$ , the above inclusions are in fact equalities (the case  $m = 0$  is obvious and the case  $m = 1$  follows from the equality  $F^1 H^1(U, \mathbb{C}) = H^1(U, \mathbb{C})$ ). For  $m = 2$ , we have again  $F^k H^2(U, \mathbb{C}) = P^k H^2(U, \mathbb{C})$  for  $k = 0, 1$  for obvious reasons, but one may get strict inclusions

$$F^2 H^2(U, \mathbb{C}) \neq P^2 H^2(U, \mathbb{C})$$

already in the case when  $V = C$  is a plane curve, see [5], Remark 2.5 or [4]. However, to give such examples of plane curves was until now rather complicated. We give below a numerical condition which tells us exactly when the above strict inclusion holds.

We need first to recall some basic definitions. Let  $S = \bigoplus_r S_r = \mathbb{C}[x, y, z]$  be the graded ring of polynomials with complex coefficients, where  $S_r$  is the vector space of homogeneous polynomials of  $S$  of degree  $r$ . For a homogeneous polynomial  $f$  of degree  $N$ , define the Jacobian ideal of  $f$  to be the ideal  $J_f$  generated in  $S$  by the partial derivatives  $f_x, f_y, f_z$  of  $f$  with respect to  $x, y$  and  $z$ . The graded *Milnor algebra* of  $f$  is given by

$$M(f) = \bigoplus_r M(f)_r = S/J_f.$$

Note that the dimensions  $\dim M(f)_r$  can be easily computed in a given situation using some computer software e.g. Singular. Now we can state the main result of this section.

**Theorem 5.1.** *Let  $C : f = 0$  be a reduced curve of degree  $N$  in  $\mathbb{P}^2$  having only weighted homogeneous singularities and let  $C_i$  for  $i = 1, \dots, r$  be the irreducible components of  $C$ . If  $U = \mathbb{P}^2 \setminus C$ , then*

$$\dim P^2 H^2(U, \mathbb{C}) - \dim F^2 H^2(U, \mathbb{C}) = \tau(C) + \sum_{i=1, r} g_i - \dim M(f)_{2N-3},$$

where  $\tau(C)$  is the global Tjurina number of  $C$  (that is the sum of the Tjurina numbers of all the singularities of  $C$ ) and  $g_i$  is the genus of the normalization of  $C_i$  for  $i = 1, \dots, r$ .

In particular we get the following result, which yields in particular a new proof for Theorem 1.3 in [7].

**Corollary 5.2.** *If a reduced plane curve has only nodes as singularities, then one has*

$$\dim M(f)_{2N-3} = \tau(C) + \sum_{i=1, r} g_i.$$

*Proof.* Indeed, it is known that for a nodal curve one has the equality  $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$ , see [2] or [11]. □

Note that we have the following obvious consequence of Theorem 2.7.

**Corollary 5.3.** *For a reduced plane curve  $C$  one has*

$$\dim P^2H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) \leq \sum_{i=1,r} g_i.$$

*Proof.* Indeed, Theorem 2.7 can be restated as

$$\dim H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) = \sum_{i=1,r} g_i,$$

in view of the equality  $F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$ , see [4], proof of Corollary 1.32, page 185. □

**Remark 5.4.** If a reduced plane curve  $C$  has only rational irreducible components, i.e.  $g_i = 0$  for all  $i$ , then the above inequality implies  $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$ . This result can be regarded as an improvement of a part of the Remark 2.5 in [5], where the result is claimed only for curves with nodes and cusps as singularities.

The above discussion implies also the following result, which can be regarded as a generalization of Theorem 4.1 (A) in [1].

**Corollary 5.5.** *If a reduced plane curve  $C : f = 0$  has only weighted homogeneous singularities, then one has*

$$0 \leq \dim M(f)_{2N-3} - \tau(C) \leq \sum_{i=1,r} g_i.$$

*In particular, if in addition the curve  $C$  has only rational irreducible components, then one has*

$$\dim M(f)_{2N-3} = \tau(C).$$

Now we give the proof of Theorem 5.1. Corollary 1.3 in [8] implies that

$$\dim P^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) + \tau(C) - \dim M(f)_{2N-3}.$$

On the other hand, Theorem 2.7 and the fact  $\dim F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$  yield

$$\dim F^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) - \sum_{i=1,r} g_i,$$

which clearly completes the proof of Theorem 5.1.

**Example 5.6.** In this example we present a free divisor  $C : f = 0$ , whose irreducible components consist of 12 lines and one elliptic curve, and where  $F^2H^2(U, \mathbb{C}) \neq P^2H^2(U, \mathbb{C})$ . Let  $f = xyz(x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3]$ . If we consider the pencil of cubic curves  $(x^3 + y^3 + z^3, xyz)$ , then the curve  $C$  contains all the singular fibers of this pencil, and this accounts for the 12 lines given by

$$xyz[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3] = 0,$$

and the elliptic curve (hence of genus 1) given by  $x^3 + y^3 + z^3 = 0$ . Then  $C$  is a free divisor, see [13] or by a direct computation using Singular, which shows that  $I = J_f$ , where  $I$  is the saturation of the Jacobian ideal  $J_f$ , see Remark 4.7 in [6]. The direct computation by Singular also yields  $\tau(C) = 156$  and  $\dim M(f)_{2N-3} = \dim M(f)_{27} = 156$ . Moreover, applying Corollary 1.5 in [9], we see via a Singular computation that all singularities of the curve  $C$  are weighted homogeneous. Alternatively, there are 12 nodes, 3 in each of the 4 singular fibers of the pencils (which are triangles), and the 9 base points of the pencil, each an ordinary point of multiplicity 5. Each of the 12 lines contains exactly 3 of these base points, and they are exactly the intersection of the elliptic curve with the line. This description implies that there are no other singularities, in accord with

$$12 + 9 \times 16 = 156 = \tau(C).$$

It follows from Theorem 5.1 that  $\dim P^2 H^2(U, \mathbb{C}) - \dim F^2 H^2(U, \mathbb{C}) = 1$ . Hence the presence of a single irrational component of  $C$  leads to  $F^2 H^2(U, \mathbb{C}) \neq P^2 H^2(U, \mathbb{C})$ .

**Acknowledgment:** I gratefully acknowledge the support of the Lebanese National Council for Scientific Research, without which the present study could not have been completed.

#### REFERENCES

- [1] N. Abdallah: *On plane curves with double and triple points*, arXiv:1401.6032, to appear in *Mathematica Scandinavica*.
- [2] P. Deligne: *Théorie de Hodge II*, Publ. Math. IHES, 40 (1971), 5–58.
- [3] P. Deligne, A. Dimca: *Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières*, Ann. Sci. Ecole Norm. Sup. (4) 23(1990), 645–656.
- [4] A. Dimca: *Singularities and Topology Hypersurface* (Universitext, Springer-Verlag, 1992).
- [5] A. Dimca, M. Saito: *A Generalization on Griffiths' Theorem on Rational Integrals*, Duke Math. J. 135(2006),303-326.
- [6] A. Dimca, E. Sernesi: *Szygies and logarithmic vector fields along plane curves*, Journal de l'École polytechnique-Mathématiques 1(2014), 247-267.
- [7] A. Dimca, G. Sticlaru: *Chebyshev Curves, Free Resolutions and Rational Curve Arrangements*, Math. Proc. Camb. Phil. Soc. 1-13, 2012
- [8] A. Dimca, G. Sticlaru: *Koszul complexes and pole order filtrations*, Proc. Edinburgh Math. Soc. doi:10.1017/S0013091514000182.
- [9] A. Dimca, G. Sticlaru: *Hessian ideals of a homogeneous polynomial and generalized Tjurina algebras*, arXiv:1412.0632.
- [10] Ph. Griffiths: *On the Period of certain Rational Integrals I, II*, Ann. Math. 90(1969), 460-541.
- [11] M. Saito: *On b-function, spectrum and rational singularity*, Math. Ann. 295(1993), 51–74.
- [12] E. Sernesi: *The local cohomology of the jacobian ring*, Documenta Mathematica, 19 (2014), 541-565.
- [13] J. Vallès: *Free divisors in a pencil of curves*, arXiv:1502.02416
- [14] C. Voisin: *Théorie de Hodge et Géométrie algébrique complexe*, Société mathématique de France 2002.

UNIV. NICE SOPHIA ANTIPOLIS, CNRS, LJAD, UMR 7351, 06100 NICE, FRANCE.

*E-mail address:* Nancy.ABDALLAH@unice.fr