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Connectivity Labeling Scheme for Graphs on Surface

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Abstract

We consider the problem of assigning each vertex of a graph G a (short) label such that, for every subset X of vertices of G and every vertex u in $G \setminus X$, the connected component of u in $G \setminus X$ can be (quickly) identified from the labels of u and those of all the vertices of X .

For planar graphs with n vertices we present a new labeling scheme with $O(\log n)$ -bit labels and $O(\log \log n)$ query time after an almost linear time preprocessing of X . The scheme is significantly simpler than previous solution due to Courcelle et al. [3] and the query time is exponentially faster. Our solution generalized to Euler genus- g graphs, and more generally to graphs having some constrained k -page embedding.

Keywords: connectivity, labeling scheme, forbidden sets, planar graphs, higher-genus graphs

1 Introduction

A natural question in graph theory is to ask whether two vertices u, v are in the same connected component of a graph G . This can trivially be done in linear time. Observe that in linear time one can also assign to each vertex of G a *label*, so that the subsequent query between u and v can be answered in constant time by inspecting the label of u and the label of v only.

Understanding the evolution of connected components of a graph when some vertices or edges are removed (or forbid) is of great importance for many applications, in particular for *Emergency Planning* (cf. [4]). Connectivity with forbidden set of edges has been solved almost optimally by [4] in general graphs. However, identifying the connected components in a graph G with a forbidden-set X becomes a challenging question whenever X contains more than two vertices, even if we are allowed to preprocess the graph.

In this paper we consider the problem of labeling the vertices of G such that, for every forbidden subset X of vertices of G , and every vertex u in $G \setminus X$, one can identify the component of u in $G \setminus X$ given the labels of u and the label of each vertex of X .

For this problem and for G planar, Courcelle et al. [3] have proposed a solution using $O(\log n)$ -bit labels and $O(\log n)$ query time, after preprocessing X in time $O(|X|^2)$. We propose an extension of this result to graphs embedded on a surface of Euler genus g (the surface can be orientable or not).

2 Our contribution

Theorem 1 *Every 3-connected n -vertex Euler genus- g map G admits a $O(g \log n)$ -bit labeling for forbidden-set connectivity queries. All the labels can be computed in time $O(n + g)$. Given a forbidden-set query $X \subset V(G)$, one can preprocess X in time $O(\text{SORT}(|X| + g, n + g))$ so that one can subsequently identify in time $O(\text{PRED}(|X| + g, n + g))$ the connected component of u in $G \setminus X$, for every vertex u of $G \setminus X$.*

Here $\text{SORT}(k, n)$ stands for the time complexity of sorting k integers ranging in $\{1, \dots, n\}$, and $\text{PRED}(k, n)$ for the time complexity for predecessor search in a sequence of k sorted integers in $\{1, \dots, n\}$. It is known that $\text{SORT}(k, n) = O(k\sqrt{\log \log k})$ and $\text{PRED}(k, n) = O(\log \log(n/k))$ in the $\log n$ -bit word RAM model.

Our scheme is only based on combinatorial embedding (map) of the graph. For the planar case ($g = 0$), our scheme is simpler than the one of Courcelle et al. [3], since it does not require any computational geometry reduction and so avoids the PLANAR POINT LOCATION bottleneck [2]. This result in an exponential improvement of the query time. These time bounds are comparable to those of the recent work of [1]. However, to the best of our knowledge, this latter scheme, restricted to planar graphs, does not convert into a labeling scheme.

3 Outline of our construction

First, we compute in $O(n + g)$ time a *cut-graph* C of the surface embedding of the input graph G . Recall that a cut-graph is a subgraph of G such that the surface obtained by cutting along the edges of C is homeomorphic to a disc. We can ensure that the cut-graph C is composed of a spanning tree of bounded degree plus $O(g)$ edges.

By definition of C , cutting along its edges results in a map M whose all the vertices lie on the border of C , i.e., an outerplanar map. Each vertex of G is associated with at most $O(g)$ vertices of M .

We construct a specific forbidden-set connectivity labeling for M and C . For the outerplanar map M we obtain labels of $O(\log(gn)) = O(\log n)$ bits with a scheme much simpler and efficient than the general planar scheme of [3]. For the cut-graph C we develop an ad-hoc scheme with $O(g \log n)$ -bit labels.

Our construction then uses a meta-scheme that allows us to identify the connected component of any vertex u in $G \setminus X$ by querying the components of u in $M \setminus X$ and in $C \setminus X$. The preprocessing of X by our meta-scheme costs $O(|X| + g)$ once X has been sorted.

Acutally, our meta-scheme applies to a wider class of graphs, typically the graphs having a *k-page embedding* (cf. [5] for definition) whose spine is constrained to be covered by a forest of at most k trees of degree bounded by k whose edges belong to only one page of the embedding. Labels are of size $O(k \log n)$ bits.

4 Conclusion

Our solution provides a query time of $O(\text{PRED}(|X|, n))$ (for fixed g), which can be proved optimal even on a path with n vertices by a straightforward reduction to predecessor query. Concerning the label length, our $O(g \log n)$ -bit solution can conceivably be improved. In fact, one can show that $\Omega(\sqrt{g} + \log n)$ -bit labels are required. In the light of our meta-scheme, we leave open the problem of determining the optimal label length for genus- g graphs.

References

- [1] G. Borradaile, S. Pettie, and C. Wulff-Nilsen. Connectivity oracles for planar graphs. In *13th Scandinavian Workshop on Algorithm Theory (SWAT)*, volume 7357 of Lecture Notes in Computer Science, pages 316–327. Springer, Jul. 2012.
- [2] T. M. Chan and M. Pătraşcu. Transdichotomous results in computational geometry, I: Point location in sublogarithmic time. *SIAM Journal on Computing*, 39(2):703–729, June 2009.
- [3] B. Courcelle, C. Gavaille, Mamadou Mustapha Kanté, and D. A. Twigg. Connectivity check in 3-connected planar graphs with obstacles. In *Int'l Conference on Topological & Geometric Graph Theory*, volume 31, pages 151–155. Electronic Notes in Discrete Mathematics, Aug. 2008.
- [4] M. Pătraşcu and M. Thorup. Planning for fast connectivity updates. In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 263–271. Oct. 2007.
- [5] D. R. Wood. Degree constrained book embeddings. *Journal of Algorithms*, 45(2):144–154, 2002.